Fake Brownian motion and calibration of a Regime Switching Local Volatility model

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Plan

Outline

Fake Brownian motion

- A *fake* Brownian motion $(X_t)_{t \geq 0}$ is a continuous martingale that has the same marginal distributions as the Brownian motion $(W_t)_{t \geq 0}$ but is not a Brownian motion.
- Examples by Albin (2007) and Oleszkiewicz (2008)
- Hobson (2009): fake exponential Brownian motion and more general martingale diffusions.
- Stochastic processes matching given marginals is a question arising in mathematical finance.

Trying to match marginals

- The market gives the prices of European Calls $C(T,K)$ for $T,K>0$ (idealized situation; in practice only $(C(T_i,K_i))_{1\leq i\leq I}$).
- A model $(S_t)_{t\geq 0}$ is calibrated to European options if

$$\forall T,K\geq 0,\ \ C(T,K)=\mathbb{E}\left[e^{-rT}\left(S_T-K\right)^+\right].$$

- By Breeden and Litzenberger (1978), $\{$prices of European Call options for all $T,K>0\}$ $\Longleftrightarrow$ $\{$marginal distributions of $(S_t)_{t\geq 0}\}$.
- Dupire Local Volatility model (1992), matching market marginals:

$$dS_t = rS_t dt + \sigma_{Dup}(t,S_t)S_t dW_t$$

$$\sigma_{Dup}(T,K)=\sqrt{2\frac{\partial_T C(T,K)+rK\partial_K C(T,K)}{K^2\partial^2_{KK}C(T,K)}}$$

LSV models

- Motivation: get processes with richer dynamics (e.g. take into account volatility risk) and satisfying marginal constraints.
- Alexander and Nogueira (2004) and Piterbarg (2006): Local and Stochastic Volatility (LSV) model

$$dS_t = rS_t + f(Y_t)\sigma(t, S_t)S_t dW_t$$

- "Adding uncertainty" to LV models by a random multiplicative factor $f(Y_t)$, $(Y_t)_{t\geq 0}$ is a stochastic process.

Calibration of LSV Models

- By Gyongy's theorem (1988), the LSV model is calibrated to $C(T,K), \forall T, K > 0$ if

$$\mathbb{E}\left[f^2(Y_t)|S_t\right]\sigma^2(t,S_t) = \sigma^2_{Dup}(t,S_t)$$

$$\sigma(t,x) = \frac{\sigma_{Dup}(t,x)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t = x\right]}}$$

- The obtained SDE is nonlinear in the sense of McKean:

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}}\sigma_{Dup}(t,S_t)S_t dW_t.$$

Simulation results

- Madan and Qian, Ren (2007): solve numerically the associated Fokker-Planck PDE, and get the joint-law of (S_t, Y_t).
- Guyon and Henry-Labordère (2011): efficient calibration procedure based on kernel approximation of the conditional expectation.
  Subsequent extension to stochastic interest rates, stochastic dividends, multidimensional local correlation models,...
- However, calibration errors seem to appear when the range of $f(Y)$ is too large.

Theoretical results

- Abergel and Tachet (2010): perturbation of the constant f case (Dupire) $\longrightarrow$ existence for the restriction to a compact spatial domain of the associated Fokker-Planck equation when $\sup f - \inf f$ small.
- Global existence and uniquess to LSV models remain on open problem.

Outline

A simpler SDE

- Let Y be a r.v. with values in $\mathcal{Y} := \{y_1, ..., y_d\}$.
- We assume $\forall i \in \{1, ..., d\}$, $\alpha_i = \mathbb{P}(Y = y_i) > 0$.
- We study the SDE (FBM), with $f > 0$:

$$\begin{aligned} dX_t &= \frac{f(Y)}{\sqrt{\mathbb{E}\left[f^2(Y)|X_t\right]}} dW_t \\ X_0 &\sim \mu. \end{aligned}$$

- $X_0, Y, (W_t)_{t\geq 0}$ are independent.

Fake Brownian Motion

Lemma

If the positive function f is not constant on $\mathcal{Y}$, then any solution to the SDE

$$dX_t \quad = \quad \frac{f(Y)}{\sqrt{\mathbb{E}\left[f^2(Y)|X_t\right]}} dW_t, \; X_0 = 0$$

with Y and $(W_t)_{t\geq 0}$ indep. is a fake Brownian motion.

If $(X_t)_{t\geq 0}$ is a Brownian motion then a.s. $\forall t \geq 0$, $< X >_t = t$ i.e. ds a.e. $\frac{f^2(Y)}{\mathbb{E}[f^2(Y)|X_s]} = 1 = \frac{f(Y)}{\sqrt{\mathbb{E}[f^2(Y)|X_s]}}$ so that a.s. $\forall t \geq 0$, $X_t = W_t$.
Therefore $X_t \perp Y$, $\mathbb{E}\left[f^2(Y)|X_t\right] = \mathbb{E}\left[f^2(Y)\right]$ and $f^2(Y) = \mathbb{E}\left[f^2(Y)\right]$ is constant.

Existence to SDE (FBM) and fake Brownian motion

We define for $i \in \{1, ..., d\}$, $\lambda_i := f^2(y_i)$,
$\lambda_{min} := \min_i \lambda_i$, $\lambda_{max} := \max_i \lambda_i$.

Theorem

Under Condition (C):

$$(C) \; : \; \sum_i \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \vee \sum_i \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) < 2d + 4.$$

there exists a weak solution to the SDE (FBM) on $[0, T]$.

The associated Fokker Planck system

- For $i \in \{1, ..., d\}$, define p_i s.t., for $\phi \geq 0$ and measurable, $\mathbb{E}\left[\phi\left(X_t\right) 1_{\{Y=y_i\}}\right] = \int_{\mathbb{R}} \phi(x) p_i(t,x) dx$.
- The associated Fokker-Planck system is:

$$\forall i \in \{1, ..., d\}, \partial_t p_i = \frac{1}{2}\partial^2_{xx}\left(\frac{\sum_j p_j}{\sum_j \lambda_j p_j}\lambda_i p_i\right)$$
$$p_i(0) = \alpha_i \mu$$

- $\sum_j p_j$ is solution to the heat equation.

Rewriting into divergence form

The system can be rewritten in divergence form:

$$\begin{pmatrix} \partial_t p_1 \\ \cdot \\ \cdot \\ \partial_t p_d \end{pmatrix} = \frac{1}{2}\partial_x \left((I_d + M(p)) \begin{pmatrix} \partial_x p_1 \\ \cdot \\ \cdot \\ \partial_x p_d \end{pmatrix} \right).$$

$$\begin{aligned} M_{ii}(p) &= \frac{\sum_{j\neq i} \lambda_j p_j \sum_j (\lambda_i - \lambda_j) p_j}{\left(\sum_j \lambda_j p_j\right)^2}, \\ M_{ik}(p) &= \frac{\lambda_i p_i \sum_j (\lambda_j - \lambda_k) p_j}{\left(\sum_j \lambda_j p_j\right)^2}, \quad i \neq k. \end{aligned}$$

Computing standard energy estimates (S.E.E)

- Multiply the system by $(p_1, ..., p_d)$, and integrate in x:

$$\frac{1}{2}\partial_t \left(\int_{\mathbb{R}} \sum_{i=1}^{d} p_i^2 dx \right) = -\frac{1}{2} \int_{\mathbb{R}} (\partial_x p_1, ..., \partial_x p_d) \left(I_d + M(p) \right) \begin{pmatrix} \partial_x p_1 \\ \cdot \\ \cdot \\ \partial_x p_d \end{pmatrix} dx.$$

- Goal : S.E.E. in $L^2([0,T], H^1(\mathbb{R})) \cap L^\infty([0,T], L^2(\mathbb{R}))$.

- We want (coercivity property): for $(\mathbb{R}_+^d)^* = \mathbb{R}_+^d \setminus \{(0, ..., 0)\}$

$$\exists \epsilon > 0 \ s.t. \ \forall \rho \in (\mathbb{R}_+^d)^*, \ \forall y \in \mathbb{R}^d, \ y^* M(\rho) y \geq (\epsilon - 1) |y|^2.$$

$M(\rho)$ as a convex combination

- $\overline{\lambda} := \frac{\sum_j \lambda_j \rho_j}{\sum_j \rho_j}$, $w_j := \frac{\lambda_j \rho_j}{\sum_k \lambda_k \rho_k}$, $\sum_{j=1}^d w_j = 1$.
- $M_{ii}(\rho) = \sum_{j \neq i} w_j \left(\frac{\lambda_i}{\overline{\lambda}} - 1 \right)$, and if $j \neq k$, $M_{jk}(\rho) = w_j \left(1 - \frac{\lambda_k}{\overline{\lambda}} \right)$.
- Then $M(\rho) = \sum_{j=1}^d w_j M_j(\overline{\lambda})$, where

$$
M_j(\overline{\lambda}) := \begin{pmatrix}
\left(\frac{\lambda_1}{\overline{\lambda}} - 1\right) & & & & & & \\
 & \cdot & & & & & \\
 & & \left(\frac{\lambda_{j-1}}{\overline{\lambda}} - 1\right) & & & & \\
\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) & \cdot & \left(1 - \frac{\lambda_{j-1}}{\overline{\lambda}}\right) & 0 & \left(1 - \frac{\lambda_{j+1}}{\overline{\lambda}}\right) & \cdot & \left(1 - \frac{\lambda_d}{\overline{\lambda}}\right) \\
 & & & & \left(\frac{\lambda_{j+1}}{\overline{\lambda}} - 1\right) & & \\
 & & & & & \cdot & \\
 & & & & & & \left(\frac{\lambda_d}{\overline{\lambda}} - 1\right)
\end{pmatrix} \leftarrow \text{row } j.
$$

How to have $\forall \rho \in (\mathbb{R}^d_+)^*,\ \forall y \in \mathbb{R}^d,\ y^* M(\rho) y \geq -|y|^2$?

- Sufficient condition

$$\forall j, \forall \overline{\lambda} \in [\lambda_{min}, \lambda_{max}],\ y^* M_j(\overline{\lambda}) y \geq -|y|^2$$

- $a_i := \left(\frac{\lambda_i}{\overline{\lambda}} - 1\right) > -1$
- $y^* M_j(\overline{\lambda}) y = \sum_{i \neq j} a_i \left(y_i^2 - y_i y_j\right)$
- Young's inequality : $-a_i y_i y_j \geq -(1 + a_i) y_i^2 - \frac{a_i^2}{4(1+a_i)} y_j^2$
- $y^* M_j(\overline{\lambda}) y \geq -\left(\sum_{i \neq j} y_i^2\right) - \left(\sum_{i \neq j} \frac{(\lambda_i - \overline{\lambda})^2}{4 \lambda_i \overline{\lambda}}\right) y_j^2$
- Sufficient condition:

$$\max_j \max_{\overline{\lambda} \in [\lambda_{min}, \lambda_{max}]} \left(\sum_{i \neq j} \frac{(\lambda_i - \overline{\lambda})^2}{4 \lambda_i \overline{\lambda}}\right) \leq 1.$$

How to have $\forall \rho \in (\mathbb{R}^d_+)^*,\ \forall y \in \mathbb{R}^d,\ y^* M(\rho) y \geq -|y|^2$?

- Equivalent formulation:

$$\max_j \max_{\overline{\lambda} \in [\lambda_{min}, \lambda_{max}]} \sum_{i \neq j} \left(\frac{\lambda_i}{\overline{\lambda}} + \frac{\overline{\lambda}}{\lambda_i} \right) \leq 2d + 2.$$

- Convexity of $\overline{\lambda} \to \frac{\lambda_i}{\overline{\lambda}} + \frac{\overline{\lambda}}{\lambda_i}$ on $[\lambda_{min}, \lambda_{max}]$:

$$\max_j \sum_{i \neq j} \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) \vee \max_j \sum_{i \neq j} \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \leq 2d + 2.$$

- Sufficient condition:

$$\sum_i \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) \vee \sum_i \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \leq 2d + 4.$$

Coercivity

Lemma

The coercivity property:

$$\exists \epsilon > 0 \; s.t. \; \forall \rho \in (\mathbb{R}_+^d)^*, \; \forall y \in \mathbb{R}^d, \; y^* M(\rho) y \geq (\epsilon - 1) \, |y|^2.$$

is satisfied iff

$$(C) \; : \; \sum_i \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \vee \sum_i \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) < 2d + 4.$$

Step 1/3: Existence to an approximate PDS when $\mu \in L^2(\mathbb{R})$

- Assume that $\mu(dx) = p_0(x)dx$, $p_0 \in L^2(\mathbb{R})$.
- For $\epsilon > 0$, use Galerkin's method to solve an approximate PDE:

$$\begin{aligned}
\begin{pmatrix} \partial_t p_1^\epsilon \\ \cdot \\ \cdot \\ \partial_t p_d^\epsilon \end{pmatrix} &= \frac{1}{2}\partial_x \left((I_d + M^\epsilon(p)) \begin{pmatrix} \partial_x p_1^\epsilon \\ \cdot \\ \cdot \\ \partial_x p_d^\epsilon \end{pmatrix} \right) \\
(p_1^\epsilon(0), \dots, p_d^\epsilon(0)) &= (\alpha_1, \dots, \alpha_d)\, p_0
\end{aligned}$$

Step 1/3: Existence to an approximate PDS when $\mu \in L^2(\mathbb{R})$

$$M^{\epsilon}_{ii}(\rho) = \frac{\sum_{j\neq i}\lambda_j\rho_j^{+}\sum_j(\lambda_i-\lambda_l)\rho_l^{+}}{\left(\epsilon\vee\sum_j\lambda_j\rho_j^{+}\right)^2},$$

$$M^{\epsilon}_{ik}(\rho) = \frac{\lambda_i\rho_i^{+}\sum_j(\lambda_j-\lambda_k)\rho_j^{+}}{\left(\epsilon\vee\sum_j\lambda_j\rho_j^{+}\right)^2},\quad i\neq k.$$

- $\rho \mapsto M^{\epsilon}(\rho)$ locally Lipschitz and bounded $\rightarrow \exists!$ solution p^{ϵ}_m to a projection of the equation in dimension m.
- coercivity uniform in ϵ under (C) : $\exists$ solution p^{ϵ} satisfying uniform in ϵ SEE by taking the limit $m \to \infty$.
- Taking p_{ϵ}^{-} as test function, we show that $p_{\epsilon} \geq 0$.
- $\forall \epsilon, \forall i, \sum_j M^{\epsilon}_{ji} = 0 \implies \sum_j p^{\epsilon}_j$ solves the heat equation $\longrightarrow$ lower bound uniform in ϵ (but not t, x) for $\sum_j \lambda_j p^{\epsilon}_j$.
- $\epsilon \to 0$, existence of a solution to the original PDS.

Step 2/3: Existence to the PDS when $\mu \in \mathcal{P}(\mathbb{R})$

- By mollification of μ, we use the results of Step 1 to extract a solution to the PDS when $\mu \in \mathcal{P}(\mathbb{R})$.
- We use the fact that $\sum_j p_j$ is solution to the heat equation to control the rate of explosion of $t \mapsto \int_{\mathbb{R}} \sum_{i=1}^{d} p_i^2(t,x)dx$ as $t \to 0$ uniformly in the mollification parameter.

Step 3/3: Existence of a weak the SDE (FBM)

Theorem (Figalli (2008))

For $a : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ meas. and bounded let $L_t\varphi(x) = \frac{1}{2}a(t,x)\varphi''(x) + b(t,x)\varphi'(x)$.
If $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{R})$ is weakly continuous and solves the Fokker-Planck equation $\partial_t \mu_t = L_t^* \mu_t$ in the sense of distributions then there exists a probability measure P on $C([0,T],\mathbb{R})$ with marginals $(P_t = \mu_t)_{t\in[0,T]}$ such that
$\forall \varphi \in C_b^2(\mathbb{R})$, $\varphi(X_t) - \int_0^t L_s\varphi(X_s)ds$ is a P-martingale.

$\Rightarrow$for $i \in \{1,\dots,d\}$, there exists a probab. P^i on $C([0,T],\mathbb{R})$ with $P_0^i = \mu$ and $P_t^i = \frac{p_i(t,x)dx}{\alpha_i}$ for $t \in (0,T]$ and $\forall \varphi \in C_b^2(\mathbb{R})$,

$$\varphi(X_t) - \int_0^t \frac{f^2(y_i)\sum_{j=1}^d p_j}{\sum_{j=1}^d f^2(y_j)p_j}(s,X_s)\varphi''(X_s)ds \text{ is a } P^i\text{-martingale.}$$

Step 3/3: Existence of a weak the SDE (FBM)

For

$$P(dX, dY) = \sum_{i=1}^{d} \alpha_i P^i(dX) \otimes \delta_{y_i}(dY),$$

- Under P, $(X_0, Y) \sim \mu \otimes \sum_{i=1}^d \alpha_i \delta_{y_i}$ and for $t \in (0, T]$, $(X_t, Y) \sim \sum_{i=1}^d p_i(t,x)dx\delta_{y_i}$ so that $X_t \sim \sum_{i=1}^d p_i(t,x)dx$ and
$$\mathbb{E}^P[f^2(Y)|X_t] = \frac{\sum_{j=1}^d f^2(y_j)p_j(t,X_t)}{\sum_{j=1}^d p_j(t,X_t)}.$$
- $\forall \varphi \in C_b^2(\mathbb{R})$
$$\varphi(X_t) - \int_0^t \underbrace{\frac{f^2(Y)\sum_{j=1}^d p_j}{\sum_{j=1}^d f^2(y_j)p_j}(s,X_s)}_{=\frac{f^2(Y)}{\mathbb{E}^P[f^2(Y)|X_s]}} \varphi''(X_s)ds \text{ is a } P\text{-martingale.}$$

Outline

Presentation

- We consider the following dynamics (RSLV):

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t\right]}}\sigma_{Dup}(t, S_t)S_t dW_t,$$

where $(Y_t)_{t\geq 0}$ takes values in $\mathcal{Y}$, and

$$\mathbb{P}\left(Y_{t+dt} = y_j | Y_t = y_i, S_t = x\right) = q_{ij}(x)dt.$$

- Switching diffusion, special case of LSV model.

- Jump distributions and intensities are functions of the asset level.

Assumptions

- (C), (Coerc. 1): f satisfies condition (C).
- (HQ), (Bounded I) $\exists \overline{q} > 0, \; s.t. \; \forall x \in \mathbb{R}, |q_{ij}(x)| \leq \overline{q}$.

We define $\tilde{\sigma}_{Dup}(t,x) := \sigma_{Dup}(t, e^x)$.

- (H1), (Bounded vol.) $\tilde{\sigma}_{Dup} \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$.
- (H2), (Coerc. 2) $\exists \underline{\sigma} > 0$ s.t. $\underline{\sigma} \leq \tilde{\sigma}_{Dup}$ a.e. on $[0,T] \times \mathbb{R}$,.
- (H3), (Regul. 1) $\exists \eta \in (0,1], \exists H_0 > 0$, s.t. $\forall s,t \in [0,T], \forall x,y \in \mathbb{R}$,

$$|\tilde{\sigma}_{Dup}(s,x) - \tilde{\sigma}_{Dup}(t,y)| \leq H_0 \left(|x-y|^\eta + |t-s|^\eta \right).$$

(HQ), (H1) and (H2) permit to generalize the energy estimations to the Fokker-Planck system associated with $((\ln(S_t), Y_t))_{t \in [0,T]}$
With (H3), uniqueness and Aronson estimates for the Fokker-Planck equation associated with $(\ln(S_t^{Dup}))_{t \in [0,T]}$ where

$$dS_t^{Dup} = \sigma_{Dup}(t, S_t^{Dup}) S_t^{Dup} dW_t + r S_t^{Dup} dt, \; S_0^{Dup} = S_0.$$

$\longrightarrow$ replaces the heat equation

Main result

Theorem

Under Conditions (H1)-(H3), (HQ) and (C) there exists a weak solution to the SDE (RSLV). Moreover, it has the same marginals as the solution to the local volatility SDE

$$dS_t^{Dup} = \sigma_{Dup}(t, S_t^{Dup}) S_t^{Dup} dW_t + r S_t^{Dup} dt, \; S_0^{Dup} = S_0.$$

We generalize the results of Figalli to the regime switching case.

Thank you for your attention!