Probabilistic Numerical Methods 2024–2025

Lecture 3: Stochastic Processes and Brownian Motion

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This is essentially the Chapter 9 of the 2023/2024 notes.

1 Generalities on stochastic processes

Probability space $(\Omega, \mathcal{A}, \mathbb{P})$, set of indices $I = [0, T]$ or $I = [0, +\infty)$, state space $(E, \mathcal{E})$.

1.1 Stochastic processes

Definition of an E-valued stochastic process $(X_t)_{t \in I}$.

Notion of cylinder and definition of the product σ-field $\mathcal{E}^{\otimes I}$ on E^I. A stochastic process is a measurable function for the product σ-field. This allows to talk about the *law* of a stochastic process.

Two processes have the same law if they have the same finite-dimensional distributions. Independence and finite-dimensional distributions.

1.2 Gaussian processes

Gaussian vectors, characterisation of the law by expectation and covariance matrix.

Gaussian processes, characterisation of the law by expectation and covariance matrix.

2 The Brownian motion

2.1 Definition

Random walk with time- and space-steps $\Delta t > 0$, $\Delta x > 0$. Under the scaling $\Delta t = \Delta x^2 \to 0$, one can prove that $(B^{\Delta t, \Delta x}_{t_1}, \ldots, B^{\Delta t, \Delta x}_{t_k})$ converges in distribution to a centered Gaussian vector $(B_{t_1}, \ldots, B_{t_k})$ with covariance matrix $\mathrm{Cov}(B_{t_i}, B_{t_j}) = t_i \wedge t_j$. This motivates the definition of the Brownian motion.

Transformations of Brownian motion. The Markov property.

2.2 Trajectories of the Brownian motion

Notion of modification. Two processes which have modifications of each other have the same law.

Almost surely continuous process.

The Kolmogorov continuity criterion. Application to the Brownian motion: from now on we systematically work with an almost surely continuous modification.

2.3 Multidimensional Brownian motion

Definition and isotropy property.

3 Filtrations, stopping times and $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion

Sub-σ-field, measurability with respect to a sub-σ-field, sub-σ-field generated by a random variable.

Independence between two sub-σ-fields.

Filtration, adapted process, filtration generated by a process.

$(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Stopping time, example of the hitting time of a closed set for an almost surely continuous process.

4 The strong Markov property and the reflection principle

The sub-σ-field $\mathcal{F}_\tau$. Strong Markov property for the Brownian motion.

Reflection principle and application to the computation of the law of $\sup_{s\in[0,t]} B_s$.