Probabilistic Numerical Methods 2024–2025

# Lecture 5: Ito's Calculus

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Now that we have constructed an integral with respect to Brownian trajectories, we need an associated differential calculus.

## 1 Ito's formula for the Brownian motion

If $(B_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-BM and $\Phi : \mathbb{R} \to \mathbb{R}$ is $C^2$ then for any $T > 0$,

$$\Phi(B_T) = \Phi(B_0) + \int_0^T \Phi'(B_t)\mathrm{d}B_t + \frac{1}{2}\int_0^T \Phi''(B_t)\mathrm{d}t, \text{almost surely.}$$

We rewrite this under the short-hand notation

$$\mathrm{d}\Phi(B_t) = \Phi'(B_t)\mathrm{d}B_t + \frac{1}{2}\Phi''(B_t)\mathrm{d}t.$$

We will call $\mathrm{d}\Phi(B_t)$ the *stochastic differential* of $\Phi(B_t)$.

## 2 Ito process

An Ito process is a process $(X_t)_{t\geq 0}$ which writes under the form

$$\forall T > 0, \qquad X_T = X_0 + \int_0^T K_t\mathrm{d}t + \int_0^T H_t\mathrm{d}B_t, \qquad \text{almost surely,}$$

with:

- $X_0$ $\mathcal{F}_0$-measurable;
- $(K_t)_{t\geq 0}$ progressively measurable and such that $\int_0^T |K_t|\mathrm{d}t < \infty$, a.s., for all $T > 0$;
- $(H_t)_{t\geq 0}$ progressively measurable and such that $\int_0^T H_t^2\mathrm{d}t < \infty$, a.s., for all $T > 0$.

We use the stochastic differential notation $\mathrm{d}X_t = K_t\mathrm{d}t + H_t\mathrm{d}B_t$.

An Ito process is adapted and a.s. continuous.

We want to prove that the decomposition of an Ito process into a stochastic integral and an absolutely continuous part is unique. The first step of the argument is given by the following statement.

**Lemme 2.1** (Quadratic variation)**.** *Let $(X_t)_{t\geq 0}$ be an Ito process. Then, for any $T > 0$,*

$$\lim_{\|\sigma\|\to 0} \sum_{i=0}^{n-1}(X_{t_i} - X_{t_{i-1}})^2 = \int_0^T H_t^2\mathrm{d}t, \qquad \text{in probability,}$$

*with $\sigma = (t_0, \dots, t_n)$ a subdivision of $[0, T]$.*

**Corollary 2.2** (Uniqueness of the decomposition of an Ito process)**.** *If there are $X_0'$, $(K_t')_{t\geq 0}$ and $(H_t')_{t\geq 0}$ such that*

$$\forall T > 0, \qquad X_0 + \int_0^T K_t\mathrm{d}t + \int_0^T H_t\mathrm{d}B_t = X_0' + \int_0^T K_t'\mathrm{d}t + \int_0^T H_t'\mathrm{d}B_t, \quad \text{almost surely,}$$

*then $X_0 = X_0'$, a.s., and $K_t = K_t'$, $H_t = H_t'$, $\mathbb{P} \otimes \mathrm{d}t$-a.e.*

*Proof.* For $X_0$, $X'_0$ just take $T = 0$. Then the lemma above allows to identify $H_t$ and $H'_t$, which finally yields the identification of $K_t$ and $K'_t$. □

From this corollary there is now no ambiguity in defining the quadratic variation of $(X_t)_{t\geq 0}$ as the process $(\langle X \rangle_t)_{t\geq 0}$ defined by

$$\forall T \geq 0, \qquad \langle X \rangle_T = \int_0^T H_t^2 \mathrm{d}t.$$

The BM $(B_t)_{t\geq 0}$ is an Ito process with quadratic variation $\langle B \rangle_t = t$. Any absolutely continuous process $X_t = X_0 + \int_0^t K_s \mathrm{d}s$ is an Ito process with quadratic variation $\langle X \rangle_t = 0$.

**Theorem 2.3** (Ito's formula for Ito processes)**.** *For any $C^2$ function $\Phi : \mathbb{R} \to \mathbb{R}$, $\mathrm{d}\Phi(X_t) = \Phi'(X_t)\mathrm{d}X_t + \frac{1}{2}\Phi''(X_t)\mathrm{d}\langle X \rangle_t$.*

Application: the Geometric Brownian motion.

**Proposition 2.4** (Lévy's characterisation of the BM)**.** *If $(H_t)_{t\geq 0}$ is progressively measurable and $H_t^2 = 1$, $\mathbb{P} \otimes \mathrm{d}t$-a.e., then $X_t := \int_0^t H_s \mathrm{d}B_s$ is a Brownian motion.*

This is in fact a particular case of the following statement.

**Theorem 2.5** (Dambis–Dubins–Schwarz)**.** *If $(H_t)_{t\geq 0} \in \mathbf{\Lambda}_{\mathrm{loc}}$ then there exists a Brownian motion $(\beta_r)_{r\geq 0}$ such that*

$$\forall T > 0, \qquad X_t = \int_0^t H_s \mathrm{d}B_s = \beta_{\langle X \rangle_t} = \beta_{\int_0^t H_s^2 \mathrm{d}s}.$$

It is useful when you want to study trajectories of the stochastic integral: they are time-changes of Brownian trajectories.

# 3 Multidimensional Ito's formula

We now take $d$ independent $(\mathcal{F}_t)_{t\geq 0}$-BM $B^1, \ldots, B^d$. An Itô process now writes

$$\mathrm{d}X_t = K_t \mathrm{d}t + \sum_{k=1}^d H_t^k \mathrm{d}B_t^k.$$

The extension of Lemma 2.1 then writes

$$\lim_{\|\sigma\|\to 0} \sum_{i=0}^{n-1} (X_{t_i} - X_{t_{i-1}})^2 = \sum_{k=1}^d \int_0^T (H_t^k)^2 \mathrm{d}t, \qquad \text{in probability.}$$

This allows to show again that if $\mathrm{d}X_t = K'_t \mathrm{d}t + \sum_{k=1}^d H_t'^k \mathrm{d}B_t^k$ then $K_t = K'_t$ and for all $k$, $H_t^k = H_t'^k$, $\mathbb{P} \otimes \mathrm{d}t$-a.e., and also to define

$$\langle X \rangle_T = \sum_{k=1}^d \int_0^T (H_t^k)^2 \mathrm{d}t.$$

By polarisation, for arbitrary Ito processes $X$, $X'$, we then have

$$\langle X, X' \rangle_T := \sum_{k=1}^d \int_0^T H_t^k H_t'^k \mathrm{d}t = \lim_{\|\sigma\|\to 0} \sum_{i=0}^{n-1} (X_{t_i} - X_{t_{i-1}})(X'_{t_i} - X'_{t_{i-1}}), \quad \text{in probability.}$$

**Theorem 3.1** (Multidimensional Ito's formula)**.** *Let* $X^1, \dots, X^n$ *be Itô processes and* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *be* $C^2$*. Then, with* $X_t = (X_t^1, \dots, X_t^n)$*,*

$$\mathrm{d}\Phi(X_t) = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(X_t)\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(X_t)\mathrm{d}\langle X^i, X^j\rangle_t.$$

Particular case of time-dependent $\Phi$.

Applications: integration by parts; recurrence and transience of the BM is dimension $d \geq 2$.