Méthodes Numériques Probabilistes

Exercices sur les équations différentielles stochastiques

1 An explosive SDE

Let $(B_t)_{t\geq 0}$ be a Brownian motion. We consider the Stochastic Differential Equation (SDE)

$$\begin{cases} \mathrm{d}X_t = \dfrac{1}{2}\exp(2X_t)\mathrm{d}t + \exp(X_t)\mathrm{d}B_t, \\ X_0 = 0. \end{cases} \tag{1}$$

1. Do the coefficients of this equation satisfy the conditions of Itô's theorem?
2. Show that the solution to the Ordinary Differential Equation (ODE)

$$\begin{cases} \dfrac{\mathrm{d}x_t}{\mathrm{d}t} = \dfrac{1}{2}\exp(2x_t), \\ x_0 = 0, \end{cases}$$

   blows up in finite time, that is to say that there exist $\overline{t} \in (0, +\infty)$ and a C^1 function $(x_t)_{t\in[0,\overline{t})}$ which satisfies this equation on $[0, \overline{t})$ and which goes to $+\infty$ when $t \to \overline{t}$.

The goal of this exercise is to construct a solution to the SDE (1), and to show that it blows up in finite time, almost surely.

1.1 Solving the SDE

For any $M > 0$ we consider the SDE

$$\begin{cases} \mathrm{d}X_t^M = \dfrac{1}{2}\exp(2(X_t^M \wedge M))\mathrm{d}t + \exp(X_t^M \wedge M)\mathrm{d}B_t, \\ X_0^M = 0, \end{cases}$$

where we recall the notation $X_t^M \wedge M = \min(X_t^M, M)$.

1. Show that this equation has a unique solution $(X_t^M)_{t\geq 0}$.
2. We set $\tau^M := \inf\{t > 0 : X_t^M \geq M\}$, and for any $x \in \mathbb{R}$, we define $\Phi(x) = 1 - \exp(-x)$. Using Itô's formula, show that for any $t \leq \tau^M$, $\Phi(X_t^M) = B_t$.
3. Deduce an expression of τ^M in terms of M and of the Brownian motion $(B_t)_{t\geq 0}$.
4. Show that, almost surely, the function $M \mapsto \tau^M$ is nondecreasing, and justify that when $M \to +\infty$, τ^M converges almost surely to

$$\overline{\tau} := \inf\{t \geq 0 : B_t \geq 1\} \in (0, +\infty]. \tag{2}$$

5. Deduce a process $(X_t)_{t\in[0,\overline{\tau})}$ such that, almost surely, for any $t \in [0, \overline{\tau})$,

$$X_t = \int_{s=0}^t \frac{1}{2}\exp(2X_s)\mathrm{d}s + \int_{s=0}^t \exp(X_s)\mathrm{d}B_s.$$

6. What can you say about X_t when $t \to \overline{\tau}$?

1.2 Study of the blow-up time

To complete the exercise, it remains to check that the blow-up time $\overline{\tau}$ given by (2) is finite, almost surely. To proceed, for any $x < 1$ we introduce the notation

$$B_t^x := x + B_t, \qquad \overline{\tau}^x := \inf\{t \geq 0 : B_t^x \geq 1\}.$$

1. Let $\lambda > 0$. Compute the unique solution $u_\lambda : \mathbb{R} \to \mathbb{R}$ which is C^2 on $\mathbb{R}$ and bounded on $(-\infty, 1]$ to the ODE
$$\begin{cases} \frac{1}{2}u_\lambda''(x) = \lambda u_\lambda(x), \text{ for any } x \in \mathbb{R}, \\ u_\lambda(1) = 1. \end{cases}$$
2. Show that, for any $t \geq 0$ and $x < 1$,
$$\exp(-\lambda t)u_\lambda(B_t^x) = u_\lambda(x) + \int_{s=0}^t \exp(-\lambda s)u_\lambda'(B_s^x)\mathrm{d}B_s.$$
3. Deduce that for any $t \geq 0$, and $x < 1$, $u_\lambda(x) = \mathbb{E}\left[\exp(-\lambda(t \wedge \overline{\tau}^x))u_\lambda(B_{t\wedge\overline{\tau}^x}^x)\right]$, and then that $u_\lambda(x) = \mathbb{E}\left[\exp(-\lambda\overline{\tau}^x)\mathbb{1}_{\{\overline{\tau}^x < +\infty\}}\right]$.
4. Show that $\lim_{\lambda\to 0} u_\lambda(x) = \mathbb{P}(\overline{\tau}^x < +\infty)$ and conclude.

2 Particle approximation of a nonlinear SDE

Let $\phi : \mathbb{R} \to \mathbb{R}$ a **bounded** function, such that there exists $L \in [0, +\infty)$ for which

$$\forall x, y \in \mathbb{R}, \qquad \max\{|\phi(x) - \phi(y)|, |x\phi(x) - y\phi(y)|\} \leq L|x - y|.$$

We consider the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = \phi(\mathbb{E}[X_t])X_t\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = \xi, \tag{$*$}$$

where $(B_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, and the random variable ξ is $\mathcal{F}_0$-measurable and in $\mathbf{L}^2(\mathbb{P})$.

The important point of the SDE ($*$) is that its drift not only depends on the value of the random variable X_t, but also on its *law*, through the expectation. Therefore this equation is not covered by the theoretic results seen in class. A solution to this SDE is an Itô process $(X_t)_{t\geq 0}$ such that, for any $t \geq 0$,

(i) $X_t \in \mathbf{L}^1(\mathbb{P})$;

(ii) $X_t = \xi + \int_0^t \phi(\mathbb{E}[X_s])X_s\mathrm{d}s + B_t$.

2.1 Preliminary results

1. Give an example of a (nonconstant) function ϕ which satisfies the assumptions made above.
2. Let $T > 0$ and $q : [0, T] \to \mathbb{R}$ a C^1 function, such that there exist $\alpha > 0$ and $C \in \mathbb{R}$ such that
$$\forall t \in [0, T], \qquad q(t) \leq \alpha \int_0^t q(s)\mathrm{d}s + C.$$
Show that, for any $t \in [0, T]$, $q(t) \leq C\mathrm{e}^{\alpha t}$.

2.2 Existence and uniqueness

1. Justify that the Ordinary Differential Equation (ODE)

$$m'(t) = \phi(m(t))m(t), \qquad m(0) = \mathbb{E}[\xi],$$

has a unique solution, which we denote by m in the sequel.

2. Let us assume that there exists a solution $(X_t)_{t\geq 0}$ to $(*)$, such that the function $t \mapsto \mathbb{E}[X_t]$ is continuous on $[0, +\infty)$. Show that the later function is a solution to the ODE of the previous question.

3. Show that the SDE

$$\mathrm{d}X_t = \phi(m(t))X_t \, \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = \xi,$$

has a unique solution, which moreover satisfies, for any $t \geq 0$,

$$\mathbb{E}\left[\int_0^t X_s^2 \mathrm{d}s\right] < +\infty.$$

4. Conclude on the existence and uniqueness of a solution to the equation $(*)$.

5. If $(X_t)_{t\geq 0}$ is a solution to $(*)$, show that for any $T \geq 0$,

$$\sup_{t\in[0,T]} \mathbb{E}[X_t^2] < +\infty.$$

2.3 Particle system

We now consider n particles, whose respective positions $Y_t^1, \ldots, Y_t^n$ evolve in $\mathbb{R}$ according to the system of SDEs

$$\forall i \in \{1, \ldots, n\}, \qquad \mathrm{d}Y_t^i = \phi\left(\overline{Y}_t^n\right) Y_t^i \, \mathrm{d}t + \mathrm{d}B_t^i, \qquad Y_0^i = \xi^i, \qquad (**)$$

where we write

$$\overline{Y}_t^n = \frac{1}{n}\sum_{j=1}^n Y_t^j.$$

The processes $(B_t^1)_{t\geq 0}, \ldots, (B_t^n)_{t\geq 0}$ are independent $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions. The random variables $\xi^1, \ldots, \xi^n$ are $\mathcal{F}_0$-measurable and iid with the same law as ξ.

1. Show that the process $(\overline{B}_t^n)_{t\geq 0}$ defined by $\overline{B}_t^n = (B_t^1 + \cdots + B_t^n)/\sqrt{n}$ is a Brownian motion.

2. We write $\overline{\xi}^n = (\xi^1 + \cdots + \xi^n)/n$. Show that the SDE

$$\mathrm{d}M_t = \phi(M_t)M_t \mathrm{d}t + \frac{1}{\sqrt{n}}\mathrm{d}\overline{B}_t^n, \qquad M_0 = \overline{\xi}^n,$$

has a unique solution, which we denote by $(M_t)_{t\geq 0}$ in the sequel.

3. Let $(\widetilde{M}_t)_{t\geq 0}$ be a process satisfying, for any $t \geq 0$,

$$\widetilde{M}_t = \overline{\xi}^n + \int_0^t \phi(M_s)\widetilde{M}_s \mathrm{d}s + \frac{1}{\sqrt{n}}\overline{B}_t^n.$$

Show that there exists a constant C such that for any $t \geq 0$,

$$|M_t - \widetilde{M}_t| \leq C\int_0^t |M_s - \widetilde{M}_s| \mathrm{d}s.$$

Deduce that $M_t = \widetilde{M}_t$ for any $t \geq 0$.

4. Adapting an argument seen in class, justify that the system of SDEs

$$\forall i \in \{1, \dots, n\}, \qquad \mathrm{d}Y_t^i = \phi(M_t) Y_t^i \mathrm{d}t + \mathrm{d}B_t^i, \qquad Y_0^i = \xi^i,$$

has a unique solution $(Y_t^1, \dots, Y_t^n)_{t \geq 0}$.

5. Show that $(Y_t^1, \dots, Y_t^n)_{t \geq 0}$ is the unique solution to the system of SDEs ($**$).

2.4 Coupling

For any $i \in \{1, \dots, n\}$, we denote by $(X_t^i)_{t \geq 0}$ the solution to the SDE ($*$) driven by the Brownian motion $(B_t^i)_{t \geq 0}$ and with initial condition ξ^i. The processes $(X_t^i)_{t \geq 0}$ and $(Y_t^i)_{t \geq 0}$ are therefore driven by the same Brownian motion and have the same initial condition.

1. Justify that the processes $(X_t^1)_{t \geq 0}, \dots, (X_t^n)_{t \geq 0}$ are independent. What can you say about the processes $(Y_t^1)_{t \geq 0}, \dots, (Y_t^n)_{t \geq 0}$?

2. Show that for any $T > 0$, and for any $t \in [0, T]$,

$$(M_t - m(t))^2 \leq 3(\overline{\xi}^n - \mathbb{E}[\xi])^2 + 3L^2 T \int_0^t (M_s - m(s))^2 \mathrm{d}s + \frac{3}{n}(\overline{B}_t^n)^2.$$

Deduce that for any $T > 0$, there exists a constant C_T such that

$$\sup_{t \in [0,T]} \mathbb{E}[(M_t - m(t))^2] \leq \frac{C_T}{n}.$$

3. Show that there is a constant C such that for any $t \geq 0$,

$$\frac{1}{n} \sum_{i=1}^n |X_t^i - Y_t^i| \leq C \int_0^t \left(\frac{1}{n} \sum_{i=1}^n |X_s^i - Y_s^i| \right) \mathrm{d}s + \int_0^t |\phi(M_s) - \phi(m(s))| \left(\frac{1}{n} \sum_{i=1}^n |X_s^i| \right) \mathrm{d}s.$$

4. Deduce that for any $T > 0$, there is a constant D_T such that

$$\sup_{t \in [0,T]} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n |X_t^i - Y_t^i| \right] \leq \frac{D_T}{\sqrt{n}},$$

and then that $\lim_{n \to \infty} \sup_{t \in [0,T]} \mathbb{E}|X_t^1 - Y_t^1| = 0$.

One may in fact show that when $n \to +\infty$, the particles defined by the system of SDEs ($**$) asymptotically behave as independent copies of the process $(X_t)_{t \geq 0}$ solution to the SDE ($*$): the interaction between the particles is encoded by the fact that the evolution of X_t depends on its law. This phenomenon is called *propagation of chaos*, it is characteristic of so-called *mean-field systems*.

5. For any $t \geq 0$, we denote by μ_t the law of the random variable X_t. Using Itô's formula, show that the Fokker–Planck equation satisfied by μ_t is nonlinear.

This justifies the fact that $(X_t)_{t \geq 0}$ is sometimes called a *nonlinear* diffusion process.