Interpretation of some nonlinear partial differential equations thanks to diffusions processes associated with signed initial measures

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Abstract

For some nonlinear parabolic evolution equations, it is possible to construct probability measures on the continuous sample-paths space such that either the time marginals or, in the one-dimensional space case, the cumulative distribution functions of the time marginals give a weak solution of the Cauchy problem. The class of initial conditions concerned is naturaly restricted to probability measures in the first case and distribution functions of such measures in the second case. Here, we present on the examples of the McKean-Vlasov equation and of a viscous scalar conservation law an approach allowing to take into account bounded signed measures or their distributions functions as initial conditions. In both cases, we construct a probability measure P on the continuous sample-paths space linked with a weak solution of the Cauchy problem. We then prove a propagation of chaos result to P for a system of weakly interacting diffusions.

This paper is dedicated to the probabilistic interpretation of two nonlinear evolution equations. These equations, although both of parabolic type, are quite different.

The first one is the McKean-Vlasov equation. As the nonlinearity is nonlocal, this equation takes sense for a collection $t \to m_t \in \mathcal{M}(\mathbb{R}^d)$, where $\mathcal{M}(\mathbb{R}^d)$ denotes the space of bounded signed measures on $\mathbb{R}^d$. It is written :

$$\frac{\partial m_t}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}[x,m_t]m_t) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i[x,m_t]m_t), \quad m_0 = m \tag{0.1}$$

$$\text{where} \qquad \forall x \in \mathbb{R}^d, \forall \nu \in \mathcal{M}(\mathbb{R}^d),\; b[x,\nu] = \int_{\mathbb{R}^d} b(x,y)\nu(dy)$$
$$a[x,\nu] = \sigma\sigma^*[x,\nu]$$
$$\sigma[x,\nu] = \int_{\mathbb{R}^d} \sigma(x,y)\nu(dy)$$

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with $b(.,.)$ and $\sigma(.,.)$ bounded and Lipschitz continuous mappings on $\mathbb{R}^{2d}$ with values in $\mathbb{R}^d$ and the space of $d \times k$ real matrices respectively.
When $m \in \mathcal{P}(\mathbb{R}^d)$ is a probability measure, this equation has been studied from a probabilistic point of view by several authors (see for instance McKean [7], Sznitman [11] and the references cited in it). They have proved existence and trajectorial uniqueness for the nonlinear stochastic differential equation

$$\begin{cases} Y_t = Y_0 + \int_0^t \sigma[Y_s, P_s]dB_s + \int_0^t b[Y_s, P_s]ds \\ P \text{ with time marginals } (P_s)_{s\geq 0} \text{ is the distribution of } Y. \end{cases} \tag{0.2}$$

where B is a $\mathbb{R}^k$-valued Brownian motion and Y_0 is a random variable with law m independent of B. Applying Itô's formula for a test function and taking expectations, it is easy to check that $s \to P_s$ is a weak solution of (0.1).
Let $(B^i)_{i\in\mathbb{N}^*}$ be independent $\mathbb{R}^k$-valued Brownian motions, $(Y_0^i)_{i\in\mathbb{N}^*}$ be random variables I.I.D. with law m independent of the Brownian motions and $Y^i, i \in \mathbb{N}^*$ denote the solution of (0.2) for the Brownian motion B^i and the initial data Y_0^i. The convergence when $n \to +\infty$ of any fixed subsystem of the following interacting diffusion processes

$$\begin{cases} X_t^{i,n} = Y_0^i + \int_0^t \sigma[X_s^{i,n}, \mu_s^n]dB_s^i + \int_0^t b[X_s^{i,n}, \mu_s^n]ds, \;\; 1 \leq i \leq n \\ \text{where } \mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}} \in \mathcal{P}(C([0,+\infty), \mathbb{R}^d)) \text{ is the empirical measure} \end{cases} \tag{0.3}$$

to the corresponding subsystem of $(Y^i)_{i\in\mathbb{N}^*}$ is ensured by the classical estimate

$$\forall T > 0, \sup_i \sup_{n\geq i} \; n\, \mathbb{E}\Big(\sup_{s\leq T} |X_s^{i,n} - Y_s^i|^2 \Big) < +\infty. \tag{0.4}$$

The second equation is a viscous scalar conservation law in the one dimensional space case and presents a local nonlinearity :

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\partial A(u)}{\partial x}, \; (t,x) \in [0,+\infty) \times \mathbb{R}, \quad u(0,x) = m((-\infty, x]) = H * m(x) \tag{0.5}$$

where $A : \mathbb{R} \to \mathbb{R}$ is a C^2 function, $H(y) = 1_{\{y\geq 0\}}$ denotes the Heaviside function and $m \in \mathcal{M}(\mathbb{R})$ is a bounded signed measure on $\mathbb{R}$.
This equation has also been studied from a probabilistic point of view in the particular case $m \in \mathcal{P}(\mathbb{R})$. When $A(u) = u^2/2$ (viscous Burgers equation), Bossy and Talay [1] construct a probability measure $P \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ with time marginals $(P_t)_{t\geq 0}$ such that $u(t,x) = H * P_t(x)$ solves (0.5). This probability measure is characterized as the unique solution of the nonlinear martingale problem :

$$\begin{cases} P_0 = m \\ \forall \phi \in C_b^2(\mathbb{R}), \; \phi(X_t) - \phi(X_0) - \int_0^t \Big(\frac{1}{2}\phi''(X_s) + A'(H * P_s(X_s))\phi'(X_s) \Big) ds \text{ is a } P\text{-martingale.} \end{cases} \tag{0.6}$$

They also prove a propagation of chaos result to P for the particle systems :

$$X_t^{i,n} = \xi^i + B_t^i + \int_0^t A'(H * \mu_s^n(X_s^{i,n}))ds, \; 1 \leq i \leq n, \qquad \mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}} \tag{0.7}$$

where $(B^i)_{i\in\mathbb{N}^*}$ are independent $\mathbb{R}$-valued Brownian motions and $(\xi^i)_{i\in\mathbb{N}^*}$ are random variables I.I.D. according to m. Indeed for any $k \in \mathbb{N}^*$, they show that the laws of $(X^{1,n}, \ldots, X^{k,n})$ converge weakly to $P^{\otimes k}$ as $n \to +\infty$. In [5], we adapt these results for $A(u) = |u|^q/q, \; q \geq 2$.

For both equations (0.1) and (0.5), our aim is to generalize the probabilistic interpretation to take into account any bounded signed measure $m \neq 0$. In [2], Bossy and Talay adapt to the viscous Burgers equation the approach developped by Marchioro and Pulvirenti [6] who study the 2-dimensional Navier-Stokes equation for an uncompressible fluid. For any $m \in \mathcal{M}(\mathbb{R})$ they construct a function $t \to m_t \in \mathcal{M}(\mathbb{R})$ with $m_0 = m$ such that $u(t,x) = m_t((-\infty, x]) = H * m_t(x)$ solves (0.5) for $A(u) = u^2/2$.
Our approach is similar to theirs in its principle but presents one main innovation : we construct a probability measure P on the sample-paths space such that the knowledge of P entails the knowledge of a function $t \to m_t$ such that $m_0 = m$ and

- in the case of the McKean-Vlasov model, $t \to m_t$ solves (0.1)
- in the case of the second model, $u(t,x) = m_t((-\infty, x]) = H * m_t(x)$ solves (0.5)

This enables us to take advantage of the classical framework of propagation of chaos (see Sznitman [11] and the references cited in it). Let us introduce a few notations in order to precise the link between P and the function $t \to m_t$.
For $m \neq 0$ a bounded signed measure on $\mathbb{R}^d$, let $|m|$, $\|m\|$ and h denote respectively the absolute value of m, the total variation of m and a density of m with respect to the probability measure $|m|/\|m\|$ with values in $\{-\|m\|, \|m\|\}$. For Q a probability measure on $C([0,+\infty), \mathbb{R}^d)$, we define $\tilde{Q}_t \in \mathcal{M}(\mathbb{R}^d)$ by

$$\forall B \text{ Borel subset of } \mathbb{R}^d, \; \tilde{Q}_t(B) = \mathbb{E}^Q(1_B(X_t)h(X_0))$$

where X denotes the canonical process on $C([0,+\infty), \mathbb{R}^d)$. Each sample-path is given a signed weight depending on the initial position.
The probability measure P that we construct has initial marginal $P_0 = |m|/\|m\|$ and the function $t \to m_t$ is defined as follows : $\forall t \geq 0, \; m_t = \tilde{P}_t$. For both models we generalize the results obtained when m a is probability measure by replacing the time marginals P_s and μ_s^n respectively by $\tilde{P}_s$ and $\tilde{\mu}_s^n$.

In the case of the McKean-Vlasov model, we obtain P as the law of the unique solution of the nonlinear stochastic differential equation defined like (0.2) with $\sigma[Y_s, \tilde{P}_s]$ (resp. $b[Y_s, \tilde{P}_s]$) replacing $\sigma[Y_s, P_s]$ (resp. $b[Y_s, P_s]$) and Y_0 distributed according to $|m|/\|m\|$. We also generalize estimate (0.4) for the particle system $(X^{1,n}, \ldots, X^{n,n})$ defined like (0.3) with $\sigma[X_s^{i,n}, \tilde{\mu}_s^n]$ (resp. $b[X_s^{i,n}, \tilde{\mu}_s^n]$) replacing $\sigma[X_s^{i,n}, \mu_s^n]$ (resp. $b[X_s^{i,n}, \mu_s^n]$) and $(Y_0^i)_{i \in \mathbb{N}^*}$ random variables I.I.D. with law $|m|/\|m\|$. Note that, as for $s > 0$ the measure

$$\tilde{\mu}_s^n = \frac{1}{n} \sum_{j=1}^{n} h(Y_0^j) \; \delta_{X_s^{j,n}},$$

depends on the initial positions, the particle system is no longer markovian. The main difficulty is to prove existence and uniqueness for the nonlinear stochastic differential equation. To adapt the fixed-point method developped by Sznitman [11], we have to replace the Vaserstein metric by a stronger metric for which the mapping $Q \to \tilde{Q}_s$ is continuous. In return, the trajectorial estimate (0.4) can be proved in the same way as in the case $m \in \mathcal{P}(\mathbb{R}^d)$.

In the second model, P is obtained as the unique solution of the martingale problem defined like (0.6) with the condition $P_0 = m$ replaced by $P_0 = |m|/\|m\|$ and the drift coefficient $A'(H * P_s(x))$ replaced by $A'(H * \tilde{P}_s(x))$. We also prove propagation of chaos to P for the non-markovian particle systems defined like (0.7) with $A'(H * \mu_s^n(X_s^{i,n}))$ replaced by $A'(H * \tilde{\mu}_s^n(X_s^{i,n}))$. This time, the proof of existence and uniqueness for the martingale problem is very similar to the one given in [5] in the particular case $m \in \mathcal{P}(\mathbb{R})$ and $A(u) = |u|^q/q$, $q \geq 2$. However, the lack of regularity of the density h is a new difficulty in the proof of the propagation of chaos result which is a weak convergence result.

Several authors have studied another probabilistic interpretation of the viscous Burgers equation ((0.5) with $A(u) = u^2/2$) in terms of nonlinear diffusion processes when the initial data is not a distribution function but a probability measure on $\mathbb{R}$. Calderoni and Pulvirenti [3], Oelschläger [8], Sznitman [10] have constructed probability measures $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ such that $\forall t > 0$, $P_t = u(t,x)dx$ and the function u is the solution of Burgers equation for initial data P_0. In [9] Roynette and Vallois deal with the more general nonlinearity $A(u) = |u|^q/q$ with $q > 1$. Here, we are not interested in this approach.

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1 The McKean-Vlasov model with a signed measure as initial data

Let $m \neq 0 \in \mathcal{M}(\mathbb{R}^d)$, Y_0 be a random variable with distribution $|m|/\|m\|$ and B be a k-dimensional brownian motion independent of Y_0. We are going to study the nonlinear stochastic differential equation

$$\begin{cases} Y_t = Y_0 + \int_0^t \sigma[Y_s, \tilde{P}_s] dB_s + \int_0^t b[Y_s, \tilde{P}_s] ds \\ P \text{ is the distribution of } Y. \end{cases} \tag{1.1}$$

If Y solves this problem, then for any $\phi \in C_b^{1,2}([0,+\infty) \times \mathbb{R}^d)$, by Itô's formula,

$$\phi(t, Y_t) = \phi(0, Y_0) + \sum_{l=1}^{k} \int_0^t \sum_{i=1}^{d} \left(\sigma_{il}(s, Y_s) \frac{\partial \phi}{\partial x_i}(s, Y_s) \right) dB_s^l$$
$$+ \int_0^t \left(\frac{\partial \phi}{\partial s}(s, Y_s) + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}[Y_s, \tilde{P}_s] \frac{\partial^2 \phi}{\partial x_i \partial x_j}(s, Y_s) + \sum_{i=1}^{d} b_i[Y_s, \tilde{P}_s] \frac{\partial \phi}{\partial x_i}(s, Y_s) \right) ds$$

Multiplying this equality by $h(Y_0)$ and taking expectations, we deduce that $s \to \tilde{P}_s$ is a weak solution of the McKean-Vlasov equation (0.1).

1.1 Existence and uniqueness for the stochastic differential equation (1.1)

This section is dedicated to the proof of the following result

Theorem 1.1 *There is existence and uniqueness (trajectorial and in law) for the stochastic differential equation (1.1).*

Let $\mathcal{P}_m(C([0,+\infty),\mathbb{R}^d)) = \{Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}^d)),\ Q_0 = |m|/\|m\|\}$. Taking up the approach of Sznitman [11], we introduce the map Θ which associates with $Q \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d))$ the distribution of the unique solution of the stochastic differential equation :

$$Y_t = Y_0 + \int_0^t \sigma[Y_s, \tilde{Q}_s]dB_s + \int_0^t b[Y_s, \tilde{Q}_s]ds. \tag{1.2}$$

If the stochastic process Y solves (1.1) then its distribution is a fixed-point of Θ. Conversely, if Q is a fixed-point of Θ, the solution Y of (1.2) solves (1.1). To deal with the fixed-point problem for Θ, we need continuity of the map $Q \in \mathcal{P}(C([0,+\infty),\mathbb{R}^d)) \to (\sigma[x,\tilde{Q}_s], b[x,\tilde{Q}_s])$. That is why for any $T > 0$, we endow the space $\mathcal{P}_m(C([0,T],\mathbb{R}^d)) = \{Q \in \mathcal{P}(C([0,T],\mathbb{R}^d)),\ Q_0 = |m|/\|m\|\}$ with the metric

$$D_T(Q^1,Q^2) = \inf\left\{\mathbb{E}^R\left(\sup_{s\le T}|X^1_s - X^2_s| \wedge 1\right),\ R \in \mathcal{P}(C([0,T],\mathbb{R}^d)^2)\right.$$
$$\left. R\circ {X^1}^{-1} = Q^1,\ R\circ {X^2}^{-1} = Q^2,\ R(\{X^1_0 = X^2_0\}) = 1\right\}$$

where (X^1,X^2) denotes the canonical process on $C([0,T],\mathbb{R}^d)^2$. If $Q^1,Q^2 \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d))$, $D_T(Q^1,Q^2)$ will denote the metric between the images of Q^1 and Q^2 by the canonical restriction from $C([0,+\infty),\mathbb{R}^d)$ to $C([0,T],\mathbb{R}^d)$.
In [11], Sznitman works with the Vaserstein metric which is defined like D_T but without the condition $R(\{X^1_0 = X^2_0\}) = 1$ and is therefore smaller than D_T.

Let $Q^1,Q^2 \in \mathcal{P}_m(C([0,T],\mathbb{R}^d))$, $x,y\in\mathbb{R}^d$ and $s \le T$ and $R \in \mathcal{P}(C([0,T],\mathbb{R}^d)^2)$ be such that $R\circ {X^1}^{-1} = Q^1$, $R\circ {X^2}^{-1} = Q^2$ and $R(\{X^1_0 = X^2_0\}) = 1$. Using the Lipschitz continuity and the boundedness of σ, we obtain

$$\begin{aligned}|\sigma[x,\tilde{Q}^1_s] - \sigma[y,\tilde{Q}^2_s]| &= |\mathbb{E}^R(h(X^1_0)\sigma(x,X^1_s) - h(X^2_0)\sigma(y,X^2_s))|\\ &= |\mathbb{E}^R(h(X^1_0)(\sigma(x,X^1_s) - \sigma(y,X^2_s)))| \le \|m\|\mathbb{E}^R|\sigma(x,X^1_s) - \sigma(y,X^2_s)|\\ &\le K(\mathbb{E}^R(|X^1_s - X^2_s|\wedge 1) + |x-y|)\end{aligned}$$

And a similar upper-bound holds for $|b[x,\tilde{Q}^1_s] - b[y,\tilde{Q}^2_s]|$. Taking the infimum for R with marginals Q^1 and Q^2 and stisfying $R(\{X^1_0 = X^2_0\}) = 1$, we deduce

$$\forall s\le T,\ \forall x,y\in\mathbb{R}^d,\ |\sigma[x,\tilde{Q}^1_s] - \sigma[y,\tilde{Q}^2_s]| + |b[x,\tilde{Q}^1_s] - b[y,\tilde{Q}^2_s]| \le K(D_T(Q^1,Q^2) + |x-y|) \tag{1.3}$$

This Lipschitz property which does not hold for the Vaserstein metric enables us to prove the following contraction lemma :

Lemma 1.2

$$\forall t\le T,\ \forall Q^1,Q^2 \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d)),\quad D_t^2(\Theta(Q^1),\Theta(Q^2)) \le K_T\int_0^t D_s^2(Q^1,Q^2)ds. \tag{1.4}$$

Proof of Lemma 1.2 : Let $Q^1,Q^2 \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d))$ and $Y^i,\ i=1,2$ be the solution of

$$Y^i_t = Y_0 + \int_0^t \sigma[Y^i_s,\tilde{Q}^i_s]dB_s + \int_0^t b[Y^i_s,\tilde{Q}^i_s]ds.$$

Using Burckholder inequality, the Lipschitz property (1.3) and Gronwall's lemma, it is quite easy to prove

$$\forall t \leq T,\ \mathbb{E}\Big(\sup_{s\leq t}|Y^1_s - Y^2_s|^2\Big) \leq K_T \int_0^t D^2_s(Q^1,Q^2)ds.$$

As $\Theta(Q^i)$ is the law of Y^i and a.s., $Y^1_0 = Y^2_0$,

$$D^2_t(\Theta(Q^1),\Theta(Q^2)) \leq \mathbb{E}\Big(\sup_{s\leq t}|Y^1_s - Y^2_s|^2\Big).$$

Hence (1.4) holds. ∎

Uniqueness for (1.1):

If Y^1 and Y^2 solve the nonlinear stochastic differential equation (1.1), their distributions P^1 and P^2 are fixed-points for Θ. By (1.4) and Gronwall's lemma, we get $\forall T > 0,\ D_T(P^1,P^2) = 0$. Hence $P^1 = P^2$ and uniqueness in law holds for (1.1). As for any $Q \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d))$ there is trajectorial uniqueness for the linear equation (1.2), trajectorial uniqueness holds for (1.1).

Existence for (1.1):

Let $Q \in \mathcal{P}_m(C([0,+\infty),\mathbb{R}^d))$. Iterating (1.4), we obtain

$$\forall n \in \mathbb{N},\ D^2_T(\Theta^{n+1}(Q),\Theta^n(Q)) \leq \frac{(K_T T)^n}{n!} D^2_T(\Theta(Q),Q).$$

By (1.3), we deduce that $\forall s \leq T$, $\forall x \in \mathbb{R}^d$,

$$|\sigma[x,\widetilde{\Theta^{n+1}(Q)}_s] - \sigma[x,\widetilde{\Theta^n(Q)}_s]| + |b[x,\widetilde{\Theta^{n+1}(Q)}_s] - b[x,\widetilde{\Theta^n(Q)}_s]| \leq K\sqrt{\frac{(K_T T)^n}{n!}} D_T(\Theta(Q),Q).$$

Hence $\forall x \in \mathbb{R}^d$, $\forall s \geq 0$, $\sigma[x,\widetilde{\Theta^n(Q)}_s]$ and $b[x,\widetilde{\Theta^n(Q)}_s]$ converge respectively to $\sigma_\infty(s,x)$ and $b_\infty(s,x)$ when n goes to $+\infty$. Moreover for any $T > 0$, the convergence is uniform for $(s,x) \in [0,T] \times \mathbb{R}^d$. Clearly the maps $x \to \sigma_\infty(s,x)$ and $x \to b_\infty(s,x)$ are bounded and Lipschitz continuous uniformly for $s \geq 0$. Let Y be the solution of the equation

$$Y_t = Y_0 + \int_0^t \sigma_\infty(s,Y_s)dB_s + \int_0^t b_\infty(s,Y_s)ds.$$

and Y^n be the solution of

$$Y^n_t = Y_0 + \int_0^t \sigma[Y^n_s,\widetilde{\Theta^n(Q)}_s]dB_s + \int_0^t b[Y^n_s,\widetilde{\Theta^n(Q)}_s]ds.$$

The uniform convergence on $[0,T] \times \mathbb{R}^d$ of the coefficients of the stochastic equation satisfied by Y^n to the coefficients of the equation satisfied by Y and the Lipschitz continuity of the coefficients of the last equation imply

$$\lim_{n\to+\infty} \mathbb{E}\Big(\sup_{s\leq T}|Y^n_s - Y_s|^2\Big) = 0.$$

Hence if P denotes the distribution of Y, $\forall T > 0$, $\lim_{n\to+\infty} D_T(P,\Theta^n(Q)) = 0$. By (1.3) we deduce that $\sigma_\infty(s,x) = \sigma[x,\tilde{P}_s]$ and $b_\infty(s,x) = b[x,\tilde{P}_s]$. We conclude that Y solves the nonlinear stochastic differential equation (1.1).

Remark 1.3 *It is also possible to adapt the proof given by Dobrushin [4] for the Vaserstein metric to prove that $(\mathcal{P}_m(C([0,T],\mathbb{R}^d)),D_T)$ is a complete metric space. The contraction estimate (1.4) then implies that the map which associates with $Q \in \mathcal{P}_m(C([0,T],\mathbb{R}^d))$ the distribution of the solution of (1.2) for $t \leq T$ admits a unique fixed-point P_T. It is easy to check that the family $P_T, T \geq 0$ is consistent and that the corresponding probability measure on $C([0,+\infty),\mathbb{R}^d)$ is a fixed-point of Θ.*

1.2 Propagation of chaos

Let $(B^i)_{i\in\mathbb{N}^*}$ be a sequence of k-dimensional independent Brownian motions and $(Y_0^i)_{i\in\mathbb{N}^*}$ a sequence of independent random variables distributed according to the probability measure $|m|/\|m\|$ (and independent of the Brownian motions).
Let $n \in \mathbb{N}^*$. We define a system of n interacting particles by the following stochastic differential equation :

$$X_t^{i,n} = Y_0^i + \int_0^t \sigma[X_s^{i,n},\tilde{\mu}_s^n]dB_s^i + \int_0^t b[X_s^{i,n},\tilde{\mu}_s^n]ds, \;\; 1 \leq i \leq n \tag{1.5}$$

where $\mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}} \in \mathcal{P}(C([0,+\infty),\mathbb{R}^d))$ is the empirical measure of the system.

Lemma 1.4 *There is existence and trajectorial uniqueness for the particle system (1.5).*

Proof : Writing the definition of $\tilde{\mu}_s^n$, we get

$$X_t^{i,n} = Y_0^i + \int_0^t \left(\frac{1}{n}\sum_{j=1}^n \sigma(X_s^{i,n},X_s^{j,n})h(X_0^{j,n})\right)dB_s^i + \int_0^t \left(\frac{1}{n}\sum_{j=1}^n b(X_s^{i,n},X_s^{j,n})h(X_0^{j,n})\right)ds$$

Without the term $h(X_0^{j,n})$, this equation would enter the classical Itô's framework.
If $X^n = (X^{1,n},\ldots,X^{n,n})$ and $\tilde{X}_n = (\hat{X}^{1,n},\ldots,\hat{X}^{n,n})$ are two solutions, $h(X_0^{j,n}) = h(\hat{X}_0^{j,n}) = h(Y_0^j)$. Hence, by the boundedness of h and the Lipschitz continuity assumptions made on σ, we get

$$\left|\frac{1}{n}\sum_{j=1}^n \sigma(X_s^{i,n},X_s^{j,n})h(X_0^{j,n}) - \frac{1}{n}\sum_{j=1}^n \sigma(\hat{X}_s^{i,n},\hat{X}_s^{j,n})h(\hat{X}_0^{j,n})\right| \leq K|X_s^n - \hat{X}_s^n|.$$

The same is true for b. We easily deduce

$$\mathbb{E}\left(\sup_{s\leq T}|X_s^n - \hat{X}_s^n|^2\right) \leq K\int_0^T \mathbb{E}(|X_s^n - \hat{X}_s^n|^2)ds.$$

By Gronwall's lemma we conclude that trajectorial uniqueness holds for (1.5).
Similar computations enable us to apply the classical iteration scheme to prove existence. ∎

For $i \in \mathbb{N}^*$, let Y^i be the solution of the nonlinear equation (1.1) for the initial condition Y_0^i and the Brownian motion B^i. By the coupling between the independent processes Y^i and the particle systems (1.5), we generalize the trajectorial estimate obtained in the case $m \in \mathcal{P}(\mathbb{R}^d)$ (see for instance Sznitman [11]) :

Theorem 1.5

$$\forall T > 0,\ \sup_n n\, \mathbb{E}\left(\sup_{s \le T} |X_s^{i,n} - Y_s^i|^2 \right) < +\infty \tag{1.6}$$

Remark 1.6 *This estimate obviously implies that the laws of the particle systems* $(X^{1,n}, \dots, X^{n,n})$ *are* P*-chaotic where* P *denotes the common distribution of the nonlinear processes* Y^i *i.e.*

$$\forall k \in \mathbb{N}^*,\ \mathcal{L}((X^{1,n}, \dots, X^{k,n})) \to_{n \to +\infty} P^{\otimes k} \quad \textit{weakly.}$$

Proof : By Burckholder inequality, for any $t \le T$

$$\begin{aligned}
\mathbb{E}\left(\sup_{s \le t} |X_s^{i,n} - Y_s^i|^2 \right) \le K \mathbb{E}\Bigg(& \int_0^t \left| \frac{1}{n} \sum_{j=1}^n h(Y_0^j)(\sigma(X_s^{i,n}, X_s^{j,n}) - \sigma(Y_s^i, Y_s^j)) \right|^2 ds \\
& + \int_0^t \left| \frac{1}{n} \sum_{j=1}^n h(Y_0^j)\sigma(Y_s^i, Y_s^j) - \sigma[Y_s^i, \tilde{P}_s] \right|^2 ds \\
& + \ \text{similar terms corresponding to } b \Bigg)
\end{aligned} \tag{1.7}$$

For notational simplicity we also denote by σ a fixed coefficient of the matrix σ. If $l \le n$ satisfies $l \ne i$ and $l \ne j$, by independence of Y^l and (Y^i, Y^j),

$$\begin{aligned}
& \mathbb{E}\left((h(Y_0^j)\sigma(Y_s^i, Y_s^j) - \sigma[Y_s^i, \tilde{P}_s])(h(Y_0^l)\sigma(Y_s^i, Y_s^l) - \sigma[Y_s^i, \tilde{P}_s]) \right) \\
& \quad = \mathbb{E}\left((h(Y_0^j)\sigma(Y_s^i, Y_s^j) - \sigma[Y_s^i, \tilde{P}_s]) \mathbb{E}\left(h(Y_0^l)\sigma(Y_s^i, Y_s^l) - \sigma[Y_s^i, \tilde{P}_s] \,\middle|\, (Y^i, Y^j) \right) \right) = 0
\end{aligned}$$

Hence

$$\mathbb{E}\left(\left| \frac{1}{n} \sum_{j=1}^n h(Y_0^j)\sigma(Y_s^i, Y_s^j) - \sigma[Y_s^i, \tilde{P}_s] \right|^2 \right) \le \frac{K}{n}.$$

This inequality and the Lipschitz properties of σ enable us to upper-bound respectively the second term and the first term of the right-hand-side of (1.7) and obtain

$$\mathbb{E}\left(\sup_{s \le t} |X_s^{i,n} - Y_s^i|^2 \right) \le K \left(\frac{1}{n} + \mathbb{E}(|X_s^{i,n} - Y_s^i|^2) + \frac{1}{n} \sum_{j=1}^n \mathbb{E}(|X_s^{j,n} - Y_s^j|^2) \right)$$

Then summing over i, using the exchangeability of the variables $(Y^i, X^{i,n}),\ 1 \le i \le n$ and applying Gronwall's lemma, we conclude that (1.6) holds. ∎

2 A viscous scalar conservation law

This section is dedicated to the viscous scalar conservation law :

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{\partial A(u)}{\partial x},\ (t, x) \in [0, +\infty) \times \mathbb{R}, \quad u(0, x) = u_0(x). \tag{2.1}$$

where $A : \mathbb{R} \to \mathbb{R}$ is a C^2 function. To give a probabilistic interpretation of this equation when the initial function $u_0(x)$ is equal to $H * m(x)$ where $m \neq 0 \in \mathcal{M}(\mathbb{R})$, we study the following martingale problem.

Définition 2.1 *We say that $P \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ solves the martingale problem (MP_A) starting at m if $P_0 = |m|/\|m\|$ and*

$$\forall \phi \in C_b^2(\mathbb{R}),\ \phi(X_t) - \phi(X_0) - \int_0^t \left(\frac{1}{2}\phi''(X_s) + A'(H * \tilde{P}_s(X_s))\phi'(X_s) \right) ds \quad \textit{is a P-martingale.} \tag{2.2}$$

2.1 Existence and uniqueness for the martingale problem (MP_A)

Proposition 2.2 *For any $m \neq 0 \in \mathcal{M}(\mathbb{R})$, the martingale problem (MP_A) starting at m admits a unique solution P.*

Proof : The main idea is the following : P solves problem (PM_A) starting at m if and only if $s \to \tilde{P}_s$ is a fixed-point of the mapping which associates with $s \to \nu_s \in \mathcal{M}(\mathbb{R})$ the function $s \to \tilde{P}_s^\nu$ where P^ν solves the linear martingale problem defined like (PM_A) with $H * \nu_s$ replacing $H * \tilde{P}_s$ in (2.2). We are going to apply Picard fixed-point theorem locally in time.

We endow $\mathcal{M}(\mathbb{R})$ with the total variation norm $\| \ \|$ and define for $T > 0$

$$L_T = \{\nu \in C((0,T], \mathcal{M}(\mathbb{R})),\ \sup_{t \in [0,T]} \|\nu(t)\| \leq \|m\|\}$$

which is complete for the metric $D_T(\nu, \nu') = \sup_{t \in (0,T]} \|\nu(t) - \nu'(t)\|$.
For $\nu \in L_T$, the function $(s,x) \in (0,+\infty) \times \mathbb{R} \to A'(H * \nu_{s \wedge T}(x))$ is measurable and bounded by $M_m = \sup_{|x| \leq \|m\|} |A'(x)|$. By Girsanov theorem, there exists a unique $P^\nu \in \mathcal{P}(C([0,+\infty), \mathbb{R}))$ such that $P_0^\nu = |m|/\|m\|$ and $\forall \phi \in C_b^2(\mathbb{R})$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \frac{1}{2}\phi''(X_s) + A'(H * \nu_{s \wedge T}(X_s))\phi'(X_s) ds \quad \text{is } P^\nu\text{-martingale.}$$

Moreover $\forall s > 0$, P_s^ν is absolutely continuous w.r.t. Lebesgue measure. We set $\psi_T(\nu)(s) = \tilde{P}_s^\nu$, $s \in (0,T]$. The absolute continuity of P_s^ν entails that $\psi_T(\nu)(s)$ has a density $p_\nu(s,.)$. We are going to prove that ψ_T is a contraction for T small enough by writing an evolution equation satisfied by $p_\nu(s,.)$.
By Lévy characterization, $X_t - X_0 - \int_0^t A'(H * \nu_{s \wedge T}(X_s)) ds$ is a P^ν Brownian motion. Let $t \in (0,T]$. If $\phi \in C_b^{1,2}([0,t] \times \mathbb{R})$ and

$$M_s^\phi = \phi(s, X_s) - \phi(0, X_0) - \int_0^s \left(\frac{\partial \phi}{\partial r}(r, X_r) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(r, X_r) + A'(H * \nu_{r \wedge T}(X_r))\frac{\partial \phi}{\partial x}(r, X_r) \right) dr,$$

then $(M_s^\phi)_{s\in[0,t]}$ is a P^ν-martingale. So is $(h(X_0)M_s^\phi)_{s\in[0,t]}$. As the expectation of this martingale is constant,

$$\int_{\mathbb{R}} \phi(t,x)p_\nu(t,x)dx = \int_{\mathbb{R}} \phi(0,x)m(dx) + \int_{(0,t]\times\mathbb{R}} \left(\frac{\partial\phi}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(s,x) + A'(H*\nu_{s\wedge T}(x))\frac{\partial\phi}{\partial x}(s,x)\right)p_\nu(s,x)dxds$$

The choice $\phi(s,x) = N_{t-s}*f(x)$ where $f:\mathbb{R}\to\mathbb{R}$ is a C^2 function with compact support and N_s denotes the heat kernel on $\mathbb{R}$ ($N_s(x) = \exp(-x^2/2s)/\sqrt{2\pi s}$) enables us to get rid of $\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}$ and obtain

$$dx \text{ a.e.,} \quad p_\nu(t,x) = N_t*m(x) - \int_0^t \frac{\partial N_{t-s}}{\partial x} * \left(p_\nu(s,.)A'(H*\nu_{s\wedge T}(.))\right)(x)ds.$$

Combining the upper-bound $\|p_\nu(s,.)\|_{L^1} \le \mathbb{E}^{P^\nu}(|h|(X_0)) = \|m\|$, the continuity of $t\to N_t*m$ in $L^1(\mathbb{R})$ for $t>0$, and the computation

$$\left\|\frac{\partial N_s(.)}{\partial x}\right\|_{L^1(\mathbb{R})} = \sqrt{\frac{2}{\pi s}}, \tag{2.3}$$

we easily prove that $t\to p_\nu(t,.)\in C((0,T],L^1(\mathbb{R}))$ and deduce that $\psi_T(\nu)\in L_T$.
Let ν' be another element of L_T and $K_m = \sup_{|x|\le\|m\|}|A''(x)|$. Using (2.3), we obtain for $t\in(0,T]$,

$$\begin{aligned}
\|p_\nu(t,.) - p_{\nu'}(t,.)\|_{L^1} &\le \int_0^t \left\|\frac{\partial N_{t-s}}{\partial x}\right\|_{L^1} \|p_\nu(s,.)A'(H*\nu_s(.)) - p_{\nu'}(s,.)A'(H*\nu'_s(.))\|_{L^1}ds \\
&\le \sqrt{\frac{2}{\pi}}\int_0^t \Big(\|p_\nu(s,.)\|_{L^1}\|\|A'(H*\nu_s(.)) - A'(H*\nu'_s(.))\|_{L^\infty} \\
&\qquad + \|p_\nu(s,.) - p_{\nu'}(s,.)\|_{L^1}\|A'(H*\nu'_s(.))\|_{L^\infty}\Big)\frac{ds}{\sqrt{t-s}} \\
&\le 2\sqrt{\frac{2T}{\pi}}\Big(\|m\|K_mD_T(\nu,\nu') + M_mD_T(\psi_T(\nu),\psi_T(\nu'))\Big)
\end{aligned}$$

Hence

$$(1-2\sqrt{2T/\pi}\ M_m)D_T(\psi_T(\nu),\psi_T(\nu')) \le 2\sqrt{2T/\pi}\ \|m\|K_mD_T(\nu,\nu').$$

From now on, we set $\tau = \pi/(8(M_m+2\|m\|K_m)^2)$. Then for $\nu,\nu'\in L_\tau$, $D_\tau(\psi_\tau(\nu),\psi_\tau(\nu')) \le D_\tau(\nu,\nu')/2$. Hence ψ admits a unique fixed-point $\nu^1\in L_\tau$.

In order to iterate the fixed-point technique, we need a few more notations. For $\bar\nu\in L_T$, we define

$$L_{\bar\nu,\tau} = \{\nu\in L_{T+\tau} \text{ such that } \forall t\in(0,T],\ \nu(t)=\bar\nu(t)\}$$

which is complete for the metric $\sup_{t\in[T,T+\tau]}\|\nu(t)-\nu'(t)\|$.
When $\nu\in L_{\bar\nu,\tau}$, we set $\psi_{\bar\nu}(\nu)(s) = \tilde P_s^\nu$, $s\in(0,T+\tau]$.

If $\nu\in L_{\nu^1,\tau}$, P^{ν^1} and P^ν coincide on the σ-field $\mathcal{F}_\tau = \sigma(X_s,\ s\in[0,\tau])$. We deduce that $\psi_{\nu^1}(L_{\nu^1,\tau})\subset L_{\nu^1,\tau}$. By computations similar to the ones that we have made for ψ_τ, we prove that ψ_{ν^1} is a contraction on $L_{\nu^1,\tau}$. This mapping admits a unique fixed-point ν^2. By induction, we construct for $n \geq 2$, $\nu^n \in L_{\nu^{n-1},\tau}$ fixed-point of $\psi_{\nu^{n-1}}$. When $i, j \in \mathbb{N}^*$, the restrictions of ν^i and ν^j to $(0, (i \wedge j)\tau]$ are equal. This property enables us to check that the probability measures P^{ν^n} converge weakly to a solution P of problem (PM_A) starting at m when $n \to +\infty$.

We still have to prove uniqueness for this problem. Let P be a solution. By the reasoning made to prove that $\psi_T(L_T) \subset L_T$, we obtain that for any $T > 0$, $s \in (0, T] \to \tilde{P}_s$ belongs to L_T. Therefore $t \in (0, \tau] \to \tilde{P}_t$ is a fixed-point of ψ_τ in L_τ and $\forall t \in (0, \tau]$, $\tilde{P}_t = \nu^1(t)$. By induction, we obtain $\forall n \in \mathbb{N}^*$, $\forall t \in (0, n\tau]$, $\tilde{P}_t = \nu^n(t)$. Hence P solves the linear martingale problem with drift coefficient $(t, x) \to \sum_{n\in\mathbb{N}^*} 1_{((n-1)\tau, n\tau]}(t) A'(H * \nu^n_t(x))$. By Girsanov theorem uniqueness holds for this problem, which puts an end to the proof. ∎

2.2 Link with equation (2.1)

We say that $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a weak solution of the Cauchy problem (2.1) if for any C^∞ with compact support function $\psi : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$:

$$\int_{(0,+\infty)\times\mathbb{R}} \left(u\frac{\partial\psi}{\partial s} + \frac{u}{2}\frac{\partial^2\psi}{\partial x^2} + A(u)\frac{\partial\psi}{\partial x} \right)(s, x)dsdx = -\int_{\mathbb{R}} \psi(0, x)u_0(x)dx. \tag{2.4}$$

Proposition 2.3 *Let $m \neq 0 \in \mathcal{M}(\mathbb{R})$ and P be the solution of problem (PM_A) starting at m. The function $u(t, x) = H * \tilde{P}_t(x)$ is the unique weak solution of (2.1) for the initial condition $u_0(x) = H * m(x)$.*

Remark 2.4 *So far, our approach enables us to give a probabilistic interpretation of (2.1) for any initial condition u_0 with bounded variation satisfying $lim_{x\to-\infty} u_0(x) = 0$. It is not difficult to get rid of the last condition. Let u_0 be any non-constant function on $\mathbb{R}$ with bounded variation. Then $\exists c \in \mathbb{R}$, $\lim_{x\to-\infty} u_0(x) = c$. If $A_c(.) = A(c + .)$ and P denotes the solution of the martingale problem (PM_{A_c}) starting at the measure du_0, then an easy adaptation of the proof of Proposition 2.3 ensures that the function $u(t, x) = c + H * \tilde{P}_t(x)$ is the unique weak solution of (2.1) for the initial condition $u_0(x)$.*

Proof : We first check that $u(t, x) = H * \tilde{P}_t(x)$ is a weak solution of (2.1). Let $\psi : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be a C^∞ with compact support function. We set $\phi(t, x) = \int_{-\infty}^x \psi(t, y)dy$. The function ϕ is C^∞, bounded together with its derivatives and equal to 0 for t big enough.
By Lévy characterization, $X_t - X_0 - \int_0^t A'(H * \tilde{P}_s(X_s))ds$ is a P-Brownian motion. Hence

$$M^\phi_t = \phi(t, X_t) - \phi(0, X_0) - \int_0^t \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2} \right)(s, X_s) + A'(H * \tilde{P}_s(X_s))\frac{\partial\phi}{\partial x}(s, X_s)ds$$

is a P-martingale. So is $h(X_0)M^\phi_t$. Hence

$$\int_0^{+\infty} \int_{\mathbb{R}} \left[\left(\frac{\partial\phi}{\partial s} + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2} \right)(s, x) + A'(H * \tilde{P}_s(x))\frac{\partial\phi}{\partial x}(s, x) \right] \tilde{P}_s(dx)ds = -\int_{\mathbb{R}} \phi(0, x)m(dx) \tag{2.5}$$

Let $s>0$. By Girsanov theorem, P_s is absolutely continuous with respect to Lebesgue measure, which implies that $\tilde{P}_s$ does not weight points. Thus the distribution function of the measure $A'(H*\tilde{P}_s(.))\tilde{P}_s$ is $A(H*\tilde{P}_s(.))-A(0)=A(u(s,.))-A(0)$. Applying Stieljes integration by parts formula in the spatial integrals of equality (2.5), we get

$$\int_0^{+\infty}\left(\tilde{P}_s(\mathbb{R})\int_{\mathbb{R}}\frac{\partial\psi}{\partial s}(s,y)dy\right)ds-\int_{(0,+\infty)\times\mathbb{R}}\left(u\frac{\partial\psi}{\partial s}+\frac{u}{2}\frac{\partial^2\psi}{\partial x^2}+A(u)\frac{\partial\psi}{\partial x}\right)(s,x)dsdx$$
$$=-m(\mathbb{R})\int_{\mathbb{R}}\psi(0,y)dy+\int_{\mathbb{R}}\psi(0,x)H*m(x)dx$$

As $\forall s\geq 0,\ \tilde{P}_s(\mathbb{R})=\mathbb{E}^P\left(h(X_0)\right)=m(\mathbb{R})$,

$$m(\mathbb{R})\int_{\mathbb{R}}\psi(0,y)dy+\int_0^{+\infty}\left(\tilde{P}_s(\mathbb{R})\int_{\mathbb{R}}\frac{\partial\psi}{\partial s}(s,y)dy\right)ds=0.$$

Hence u is a weak solution of (2.1) for the initial condition $u_0(x)=H*m(x)$. The boundedness of u on $[0,+\infty)\times\mathbb{R}$ is clear. To prove that u is continuous on $(0,+\infty)\times\mathbb{R}$, we set $t>0$, $s\in[\frac{t}{2},\frac{3t}{2}]$ and $x,y\in\mathbb{R}$.

$$|u(t,x)-u(s,y)|\leq|\mathbb{E}^P\left((H(x-X_t)-H(x-X_s))h(X_0)\right)|+\|m\|\mathbb{E}^P\left(|H(x-X_s)-H(y-X_s)|\right)$$

Girsanov theorem implies that the L^2 norm of the density of P_s w.r.t. Lebesgue measure is bounded by $K(t)<+\infty$ uniformly for $s\in[\frac{t}{2},\frac{3t}{2}]$. By Cauchy-Schwarz inequality, we deduce

$$|u(t,x)-u(s,y)|\leq|\mathbb{E}^P\left((H(x-X_t)-H(x-X_s))h(X_0)\right)|+\|m\|K(t)\sqrt{x-y}.$$

As P_t does not weight points, the first term of the right hand side converges to 0 when $s\to t$. We conclude that u is continuous on $(0,+\infty)\times\mathbb{R}$.

Let v be a weak solution of (2.1) for the initial condition $u_0=H*m$, bounded by M_v on $[0,+\infty)\times\mathbb{R}$ and continuous on $(0,+\infty)\times\mathbb{R}$. For $t>0$ and $f:\mathbb{R}\to\mathbb{R}$ a C^2 with compact support function, we set $\psi(s,x)=1_{[0,t]}(s)N_{t-s}*f(x)$ (we recall that $N_s(x)=\exp(-x^2/2s)/\sqrt{2\pi s}$.). It is possible to approximate ψ, its first order time derivative and its first and second order spatial derivatives in $L^1([0,t]\times\mathbb{R})$ by C^∞ functions ψ^n with compact support on $[0,t)\times\mathbb{R}$ and their derivatives. Writing (2.4) for ψ^n and taking the limit $n\to+\infty$, we get

$$\int_{\mathbb{R}}v(t,x)\psi(t,x)dx=\int_{\mathbb{R}}\psi(0,x)u_0(x)dx+\int_{(0,t)\times\mathbb{R}}\left(v\frac{\partial\psi}{\partial s}+\frac{v}{2}\frac{\partial^2\psi}{\partial x^2}+A(v)\frac{\partial\psi}{\partial x}\right)(s,x)dxds$$

As $\frac{\partial\psi}{\partial s}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}$ on $[0,t]\times\mathbb{R}$, we deduce

$$\int_{\mathbb{R}}f(x)v(t,x)dx=\int_{\mathbb{R}}N_t*f(x)u_0(x)dx+\int_0^t\int_{\mathbb{R}}A(v(s,x))\frac{\partial N_{t-s}}{\partial x}*f(x)dxds.$$

Hence

$$\forall(t,x)\in(0,+\infty)\times\mathbb{R},\ v(t,x)=N_t*u_0(x)-\int_0^t\left(\frac{\partial N_{t-s}}{\partial x}*A(v(s,.))\right)(x)ds.$$

This equation is also satisfied by $u(t,x)=H*\tilde{P}_t(x)$. Let $K=\sup_{\{|x|\leq\|m\|\vee M_v\}}|A'(x)|$. Writing the equation satisfied by $v-u$ and taking (2.3) into account, we obtain

$$\|v(t,.)-u(t,.)\|_{L^\infty(\mathbb{R})}\leq\sqrt{\frac{2}{\pi}}\int_0^t\frac{K\|v(s,.)-u(s,.)\|_{L^\infty(\mathbb{R})}}{\sqrt{t-s}}ds.$$

We iterate this inequality and apply Gronwall lemma to conclude that $u=v$ on $[0,+\infty)\times\mathbb{R}$. ∎

2.3 Propagation of chaos

Let $(\epsilon_n)_n$ be a sequence of strictly positive numbers converging to 0. We approximate the Heaviside function by the Lipschitz continuous functions

$$H_n(x) = \frac{x+\epsilon_n}{\epsilon} 1_{\{-\epsilon \le x \le 0\}} + 1_{\{x>0\}}.$$

Let $(B^i)_{i\in\mathbb{N}^*}$ be a sequence of independent one-dimensional Brownian motions and $(\xi^i)_i$ be a sequence of I.I.D. random variables with law $|m|/\|m\|$ (independent of the Brownian motions). We define a system of n interacting particles by the following stochastic differential equation

$$X_t^{i,n} = \xi^i + B_t^i + \int_0^t A'(H_n * \tilde{\mu}_s^n(X_s^{i,n}))ds, \; 1 \le i \le n \tag{2.6}$$

where $\mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}}$.
By a proof similar to the one given for Lemma 1.4, we obtain existence and trajectorial uniqueness for this stochastic differential equation.

Proposition 2.5 *The particle systems $(X^{1,n}, \ldots, X^{n,n})$ are P-chaotic where P denotes the unique solution of the problem (MP_A) starting at m.*

Remark 2.6 *By Girsanov theorem, the stochastic differential equation (2.6) where H_n is replaced by H admits a unique weak solution. Let Q^n be the law of the solution. It is possible to prove that the sequence $(Q^n)_n$ is P-chaotic by computations similar to the ones made in the proof of Proposition 2.5. We have introduced H_n in the stochastic differential equation (2.6) only to obtain a strong solution.*

The propagation of chaos result that we prove is a convergence in distribution result. To deal with such a convergence, we need continuous functions. But the density h may be irregular. The following lemma ensures that it is possible to approximate h by Lipschitz continuous functions.

Lemma 2.7 *Let B be a Borel subset of $\mathbb{R}^d$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$. For any $\alpha > 0$, there is a Lipschitz continuous function $f_{B,\alpha}$ with $0 \le f_{B,\alpha} \le 1$ such that $\nu(\{1_B \ne f_{B,\alpha}\}) \le \alpha$.*

Proof : Since the probability measure ν is regular, there exits a closed set $F \subset B$ such that $\nu(B \setminus F) \le \alpha/2$.
Let $f^k(x) = (1 - kd(F,x)) \vee 0$ where $d(F,x) = \inf_{y\in F}|x-y|$. The function f^k is Lipschitz continuous and satisfies $0 \le f^k \le 1$. Moreover as $k \to +\infty$, $\{f^k \ne 1_F\} \searrow \emptyset$. Hence there is k_0 such that $\nu(\{f^{k_0} \ne 1_F\}) \le \alpha/2$.

$$\nu(\{f^{k_0} \ne 1_B\}) \le \nu(\{f^{k_0} \ne 1_F\}) + \nu(\{1_F \ne 1_B\}) \le \alpha/2 + \nu(B \setminus F) \le \alpha.$$

We set $f_{B,\alpha} = f^{k_0}$. ∎

Applying this lemma with $d=1$, $\nu = |m|/\|m\|$, $B = \{h = \|m\|\}$, we define $h_n = 2\|m\|\left(f_{B,\frac{1}{n}} - \frac{1}{2}\right)$.
Then h_n is a Lipschitz continuous function with values in $[-\|m\|, \|m\|]$. Moreover,

$$\frac{|m|}{\|m\|}(\{h_n \ne h\}) \le \frac{1}{n} \tag{2.7}$$

Proof : The particles are exchangeable. Hence the propagation of chaos result is equivalent to the weak convergence of the laws π^n of the empirical measures μ^n considered as $\mathcal{P}(C([0,+\infty),\mathbb{R}))$-valued random variables to δ_P (see for instance [11] and the references cited in it). We are going to prove this convergence. Again by exchangeability, the tightness of the sequence $(\pi^n)_n$ is equivalent to the tightness of the laws of the random variables $(X^{1,n})_n$ which is an easy consequence of the bound uniform in n

$$\forall(x^1,\ldots,x^n)\in\mathbb{R}^n,\ \forall(y^1,\ldots,y^n)\in\mathbb{R}^n,\ \left|A'\left(\frac{1}{n}\sum_{j=1}^n H_n(x^1-x^j)h(y^j)\right)\right|\leq\sup_{\{|x|\leq\|m\|\}}|A'(x)|.$$

Let π^∞ be the limit of a convergent subsequence that we still index by n for notational simplicity. In order to check that $\pi^\infty=\delta_P$, we set $p\in\mathbb{N}^*$, $\phi\in C_b^2(\mathbb{R})$, $g\in C_b(\mathbb{R}^p)$ and $t\geq s\geq s_1\geq\ldots\geq s_p\geq 0$ and define a mapping F on $\mathcal{P}(C([0,+\infty),\mathbb{R}))$ by

$$F(Q)=<Q,\left(\phi(X_t)-\phi(X_s)-\int_s^t\frac{1}{2}\phi''(X_r)+A'(H*\tilde{Q}_r(X_r))\phi'(X_r)dr\right)g(X_{s_1},\ldots,X_{s_p})>.$$

For $k\in\mathbb{N}^*$, we define F_k and G_k like F replacing $H*\tilde{Q}_r$ respectively by $H_k*\tilde{Q}_r$ and $H_k*\tilde{Q}_r^k$ where the measure $\tilde{Q}_r^k$ is defined by :

$$\forall B\ \text{ Borel subset of }\mathbb{R},\ \tilde{Q}_r^k(B)=\mathbb{E}^Q(1_B(X_r)h_k(X_0)).$$

The functions $Q\to H_k*\tilde{Q}_r^k(x)=<Q,H_k(x-X_r)h_k(X_0)>$ are equicontinuous and uniformly bounded by $\|m\|$ for $(r,x)\in[s,t]\times\mathbb{R}$. As A' is Lipschitz continuous on $[-\|m\|,\|m\|]$, we deduce that the functions G_k are continuous. Hence

$$\begin{aligned}\mathbb{E}^{\pi^\infty}((F(Q))^2)&\leq 2\limsup_k\mathbb{E}^{\pi^\infty}((F(Q)-G_k(Q))^2)+4\limsup_n\mathbb{E}((F_n(\mu^n))^2)\\&\quad+4\limsup_k\limsup_n\mathbb{E}((F_n(\mu^n)-G_k(\mu^n))^2)\end{aligned}\tag{2.8}$$

We are going to prove that each term of the right hand side of (2.8) is 0.
Taking into account the Lipschitz continuity of A' on $[-\|m\|,\|m\|]$, we get

$$\mathbb{E}^{\pi^\infty}((F(Q)-G_k(Q))^2)\leq K\mathbb{E}^{\pi^\infty}\left(<Q,\int_s^t|H*\tilde{Q}_r(X_r)-H_k*\tilde{Q}_r^k(X_r)|dr>\right)$$

$$\begin{aligned}\text{Now,}\quad|H*\tilde{Q}_r(x)-H_k*\tilde{Q}_r^k(x)|&=|<Q,H(x-X_r)h(X_0)-H_k(x-X_r)h_k(X_0)>|\\&\leq\|m\||H-H_k|*Q_r(x)+<Q,|h(X_0)-h_k(X_0)|>\end{aligned}$$

Therefore $\mathbb{E}^{\pi^\infty}((F(Q)-G_k(Q))^2)$ is bounded by

$$K\left(\mathbb{E}^{\pi^\infty}\left(<Q,\int_s^t|H-H_k|*Q_r(X_r)dr>\right)+\mathbb{E}^{\pi^\infty}(<Q,|h(X_0)-h_k(X_0)|>)\right)$$

Since $|H-H_k|$ goes pointwise to 0 as $k\to+\infty$, Lebesgue theorem implies the first term of the right hand side converges to 0. Moreover, as π^∞ a.s., $Q_0=|m|/\|m\|$, by (2.7), we obtain that the second term also goes to 0. Hence the first term of the right hand side of (2.8) is 0.

By Itô's formula, $F_n(\mu^n)=\frac{1}{n}\sum_{i=1}^n g(X_{s_1}^{i,n},\ldots,X_{s_p}^{i,n})\int_s^t\phi'(X_r^{i,n})dB_r^i$.
Hence $\mathbb{E}((F_n(\mu^n))^2)\leq K/n$ and the second term of the right hand side of (2.8) is zero.

We still have to prove that the third term is zero. By computations similar to the ones made for the first term, we get

$$\mathbb{E}((G_k - F_n)^2(\mu^n)) \leq K\mathbb{E}(< \mu^n, \int_s^t |H_n - H_k| * \mu_r^n(X_r)dr >) + K\mathbb{E}(< \mu^n, |h(X_0) - h_k(X_0)| >) \tag{2.9}$$

$$\leq K\Bigg(\mathbb{E}\Bigg(< \mu^n \otimes \mu^n, \int_s^t 1_{\{|X_r - Y_r| \leq \epsilon_n \vee \epsilon_k\}} dr > \Bigg) + \mathbb{E}(|h(\xi^1) - h_k(\xi^1)|)\Bigg) \tag{2.10}$$

where (X, Y) denotes the canonical process on $C([0, +\infty), \mathbb{R})^2$.
Since the distribution of ξ^1 is $|m|/\|m\|$, according to (2.7), the second term of the right hand side of (2.10) goes to 0 when $k \to +\infty$.
The variables $X^{i,n}$ are exchangeable. Moreover, the convergence of $(\epsilon_n)_n$ to 0 ensures that $\epsilon_n \leq \epsilon_k$ for n big enough. Hence

$$\begin{aligned}\limsup_{n\to+\infty} \mathbb{E}\Bigg(< \mu^n \otimes \mu^n, \int_s^t 1_{\{|X_r - Y_r| \leq \epsilon_n \vee \epsilon_k\}} dr > \Bigg) &= \limsup_{n\to+\infty} \mathbb{E}\Bigg(\int_s^t 1_{\{|X_r^{1,n} - X_r^{2,n}| \leq \epsilon_k\}} dr\Bigg) \\ &\leq \limsup_{n\to+\infty} \mathbb{E}\Bigg(\int_s^t 1_{\{|X_r^{1,n} - X_r^{2,n}| \leq \epsilon_k\}} 1_{\{|X_r^{1,n}| \leq \frac{1}{\sqrt{\epsilon_k}}\}} dr\Bigg) + \limsup_{n\to+\infty} \int_s^t \mathbb{P}\Bigg(|X_r^{1,n}| \geq \frac{1}{\sqrt{\epsilon_k}}\Bigg) dr\end{aligned} \tag{2.11}$$

The inequality

$$\mathbb{P}\Bigg(|X_r^{1,n}| \geq \frac{1}{\sqrt{\epsilon_k}}\Bigg) \leq \mathbb{P}\Bigg(|B_r^1| \geq \frac{1}{2\sqrt{\epsilon_k}} - \frac{rM_m}{2}\Bigg) + \mathbb{P}\Bigg(|\xi^1| \geq \frac{1}{2\sqrt{\epsilon_k}} - \frac{rM_m}{2}\Bigg)$$

where $M_m = \sup_{\{|x| \leq \|m\|\}} |A'(x)|$ implies that the second term of the right hand side of (2.11) goes to 0 when $k \to +\infty$.
We easily obtain by Girsanov theorem the following bound for the couples $(X^{1,n}, X^{2,n})_n$:

$$\forall f \in L^2(\mathbb{R}^2),\ \forall n \geq 2,\ \forall r > 0,\ |\mathbb{E}(f(X_r^{1,n}, X_r^{2,n}))| \leq \frac{1}{\sqrt{2\pi r}} \exp(M_m^2 r) \|f\|_{L^2(\mathbb{R}^2)}$$

Hence $\forall n \geq 2$, $\mathbb{E}\Bigg(\int_s^t 1_{\{|X_r^{1,n} - X_r^{2,n}| \leq \epsilon_k\}} 1_{\{|X_r^{1,n}| \leq \frac{1}{\sqrt{\epsilon_k}}\}} dr\Bigg) \leq K\epsilon_k^{\frac{1}{4}}$. We deduce that the first term of the right hand side of (2.11) goes to 0 when $k \to +\infty$. Inequalities (2.10) and (2.11) enable us to conclude that $\limsup_k \limsup_n \mathbb{E}((F_n(\mu_n) - G_k(\mu_n))^2) = 0$.

Since each term of the right hand side of (2.8) is 0, $\mathbb{E}^{\pi^\infty}((F(Q))^2) = 0$. We deduce that π^∞ a.s., Q solves the martingale problem (MP_A) starting at m. Hence $\pi^\infty = \delta_P$. ∎

Remark 2.8 *The propagation of chaos result implies of course that it is possible to approximate the function $u(t, x) = H * \tilde{P}_t(x)$ thanks to the particle systems. For instance, when $(t, x) \in (0, +\infty) \times \mathbb{R}$, $\frac{1}{n} \sum_{j=1}^n H(x - X_s^{j,n}) h(\xi^j)$ converges in L^1 to $u(t, x)$. Indeed,*

$$\begin{aligned}\mathbb{E}\Bigg| u(t, x) - \frac{1}{n} \sum_{j=1}^n H(x - X_t^{j,n}) h(\xi^j) \Bigg| &\leq |u(t, x) - < P, H_k(x - X_t) h_k(X_0) > | \\ &+ \mathbb{E}\Bigg| < P - \mu^n, H_k(x - X_t) h_k(X_0) > \Bigg| \\ &+ \|m\| \mathbb{E}(|H_k - H|(x - X_t^{1,n})) + \mathbb{E}|h_k(\xi^1) - h(\xi^1)|\end{aligned}$$

The first and the fourth term of the right hand side converge to 0 *when* k *goes to* $+\infty$. *By Proposition (2.5), the random variables* $\mu^n \circ (X_0, X_t)^{-1}$ *converge in probability to the constant* $P \circ (X_0, X_t)^{-1}$. *As* $(y, z) \in \mathbb{R}^2 \to H_k(x-y)h_k(z)$ *is a Lipschitz continuous function, for fixed* k, *the second term goes to* 0 *as* $n \to +\infty$. *Lastly, the estimate* $\forall f \in L^2(\mathbb{R}), \forall n \in \mathbb{N}^*,\ |\mathbb{E}(f(X_t^{1,n})| \leq K_t \|f\|_{L^2}$ *which is a consequence of Girsanov theorem, implies that the third term converges to* 0 *uniformly in* n *when* $k \to +\infty$.

Remark 2.9 *Let* $(\nu_n)_n$ *be a sequence of probability measures on* $\mathbb{R}$ *converging weakly to* $|m|/\|m\|$. *The study of the asymptotic behaviour for* $n \to +\infty$ *of the particle systems*

$$X_t^{i,n} = \xi^{i,n} + B_t^i + \int_0^t A'(H_n * \tilde{\mu}_s^n(X_s^{i,n}))ds,\ 1 \leq i \leq n \tag{2.12}$$

where the initial positions $(\xi^{i,n})_{1\leq i \leq n}$ *are I.I.D. according to* ν_n *(and independent of the Brownian motions) is a natural question. Unlike the solution of (2.6), the solution of (2.12) may depend on the choice of the density* h. *We are going to give a sufficient condition on* m *ensuring that, for a good choice of* h, *the propagation of chaos result still holds.*

If m *admits a continuous density* f *w.r.t. Lebesgue measure (condition equivalent to* $u_0 = H * m \in C^1(\mathbb{R})$*), we set* $h = \|m\| \left(1_{\{f \geq 0\}} - 1_{\{f<0\}}\right)$. *For any sequence* $(\nu_n)_n$ *converging weakly to* $|m|/\|m\|$, *the particles* $(X^{1,n}, \dots, X^{n,n})$ *solving (2.12) are* P *chaotic. The only significant difference with the proof of Proposition 2.5 is the treatment of the second term of the right hand side of (2.9) :* $\mathbb{E}(< \mu^n, |h(X_0) - h_k(X_0)| >)$.
Since $B = \{f \geq 0\}$ *is a closed subset of* $\mathbb{R}$, *it is possible to ensure* $h_k \geq h$ *by choosing*

$$h_k(x) = 2\|m\| \left(((1 - n_k d(B, x)) \vee 0) - 1/2\right)$$

for n_k *big enough. Then*

$$\mathbb{E}(< \mu^n, |h(X_0) - h_k(X_0)| >) = < \nu_n, h_k > - \|m\|(\nu_n(\{f \geq 0\}) - \nu_n(\{f < 0\})).$$

The continuity of f *ensures that the boundaries* $\delta\{f \geq 0\}$ *and* $\delta\{f < 0\}$ *are included in* $\{f = 0\}$. *Hence* $|m|/\|m\|(\delta\{f \geq 0\}) = |m|/\|m\|(\delta\{f < 0\}) = 0$. *We conclude that*

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \mathbb{E}(< \mu^n, |h(X_0) - h_k(X_0)| >) = \lim_{k\to+\infty} < \frac{|m|}{\|m\|}, h_k - h >= 0.$$

More generally, if there exists B *a closed subset of* $\mathbb{R}$ *such that* $|m|(\delta B) = 0$ *and* $\|m\| (1_B - 1_{B^c})$ *is a density of* m *w.r.t.* $|m|/\|m\|$, *then, for the choice* $h = \|m\| (1_B - 1_{B^c})$, *the previous proof can be adapted and the propagation of chaos result holds.*

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