Probabilistic gradient approximation for a viscous scalar conservation law in space dimension $d \geq 2$

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Abstract

In this paper, we are interested in the solution of a viscous scalar conservation law. We remark that its first order spatial derivatives solve a system of partial differential equations presenting a nonlocal nonlinearity. We associate a nonlinear martingale problem with this system. After proving existence and uniqueness for the martingale problem, we obtain a propagation of chaos result for a system of interacting diffusion processes. We deduce that it is possible to approximate the solution of the viscous scalar conservation law thanks to the interacting diffusions.

Let $\nu > 0$ and $A = (A_1, \ldots, A_d)$ be a C^2 $\mathbb{R}^d$-valued function on $\mathbb{R}$ satisfying $A(0) = 0$. We are interested in the parabolic equation obtained by adding a second order diffusion term to the scalar conservation law $\partial_t u + \nabla.A(u) = 0$:

$$\begin{cases} \partial_t u = \nu \Delta u - \nabla.A(u), \ (t,x) \in [0,+\infty) \times \mathbb{R}^d \\ u(0,x) = u_0(x). \end{cases} \tag{0.1}$$

where $\nabla.$ stands for the divergence with respect to the space variables. This equation presents a local nonlinearity. Formally, it is possible to associate a nonlinear martingale problem with it when $u_0(x)$ is a probability density on $\mathbb{R}^d$. If a probability measure Q on $C([0,+\infty),\mathbb{R}^d)$ with time-marginals $Q_t, t \geq 0$ is such that $Q_0 = u_0(x)dx$, $\forall t > 0$, $Q_t = q(t,x)dx$ and $\forall \phi \in C_b^2(\mathbb{R}^d)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t \nu \Delta \phi(X_s) + \alpha(q(s,X_s)).\nabla \phi(X_s) ds \quad \text{is a } Q \text{ martingale} \tag{0.2}$$

where $\alpha(u) = A(u)/u$, then $q(t,x)$ is a weak solution of (0.1). One could try to prove existence of a unique solution Q to this problem and to show propagation of chaos to Q for a sequence of moderately interacting diffusion processes (see Oelschläger [11] who first introduced moderate interaction).

We are not interested in this point of view. We want to generalize for $d \geq 2$ the one-dimensional approach developped by Bossy and Talay [2] in the case of the viscous Burgers equation ($A(u) = u^2/2$). Differentiating (0.1) with respect to the $i-th$ space variable, we obtain that $\forall i \leq d$, $v_i = \partial_i u$ satisfies

$$\begin{cases} \partial_t v_i = \nu \Delta v_i - \nabla.(A'(u) v_i), \\ v_i(0,.) = \partial_i u_0. \end{cases}$$

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To obtain a closed system for $(v_1, \dots, v_d)$, we need to express u in terms of its gradient.
When $d = 1$, the function u is easily recovered from its spatial derivative v by convolution with the Heaviside function $H(y) = 1_{\{y \geq 0\}}$ i.e. $u(t,x) = c + (H * v(t,.))(x)$. The equation satisfied by v, $\partial_t v = \nu \partial_{xx} v - \partial_x(A'(c + H * v)v)$, presents a nonlocal nonlinearity. It is therefore possible to associate with it a nonlinear martingale problem with a drift coefficient which depends globally on the time-marginals of its solution and not locally on their densities like in (0.2). Moreover a propagation of chaos result to the solution of this martingale problem can be proved for a system of weakly interacting diffusion processes (see [2], [7]). Hence differenciation of equation (0.1) simplifies its probabilistic interpretation.

From now on, we suppose that $d \geq 2$. The way the scalar field $u(t,x)$ can be recovered from its gradient is not as obvious as in the one-dimensional case. Anderson [1] first suggested to use the fundamental solution of the Laplacian in $\mathbb{R}^d$ for this purpose in a particle-method framework (see also [4], [12]). If $x \to \gamma(x)$ denotes this fundamental solution and $f = c + \phi$ where ϕ is a C^∞ function with compact support on $\mathbb{R}^d$, then $f = c + \gamma * \Delta f$ and by integration by parts, $f = c + \sum_{i=1}^d \partial_i \gamma * \partial_i f$. As mentionned in [12], although this method appears to be very different from the one-dimensional method, it is actually a natural generalization. Indeed the fundamental solution of the Laplacian on $\mathbb{R}$, $x \to \frac{|x|}{2}$, has a derivative equal to $H(x) - \frac{1}{2}$.
In the last part of this introduction we remark that the equality $f = c + \sum_{i=1}^d \partial_i \gamma * \partial_i f$ still holds under less restrictive assumptions on f i.e. if $f \in W^{1,1+\infty}(\mathbb{R}^d)$ where $W^{1,1+\infty}(\mathbb{R}^d)$ denotes the subspace of $L^\infty(\mathbb{R}^d)$ consisting in functions with first order distribution derivatives in $L^1 \cap L^\infty(\mathbb{R}^d)$. In the first section of the paper, we recall that for u_0 continuous and bounded, the scalar conservation law (0.1) admits a unique classical solution bounded by $\|u_0\|_\infty$. Replacing A by a C^2 function $\bar{A}$ equal to A on $[-\|u_0\|_\infty, \|u_0\|_\infty]$ and admitting bounded first and second order derivatives, we obtain that when $u_0 \in W^{1,1+\infty}(\mathbb{R}^d)$, $\forall t \geq 0, u(t,.) \in W^{1,1+\infty}(\mathbb{R}^d)$ and the constant c in the equality $u(t,.) = c + \sum_{i=1}^d \partial_i \gamma * \partial_i u(t,.)$ does not depend on t. As a consequence, the derivatives $(\partial_1 u, \dots, \partial_d u)$ solve the following system with a nonlocal nonlinearity.

$$\begin{cases} \partial_t v_i = \nu \Delta v_i - \nabla.(v_i \bar{A}'(c + \sum_{j=1}^d \partial_j \gamma * v_j(t,.))), \\ v_i(0,.) = \partial_i u_0, \quad\quad i \leq d. \end{cases} \tag{0.3}$$

In the second section of the paper we associate a nonlinear martingale problem with this system and we prove existence of a unique solution for this problem.

The last section is dedicated to a propagation of chaos result. The main difficulty encountered is the singularity of the kernel $\nabla\gamma$ at the origin. To overcome this problem, we follow the approach of Marchioro Pulvirenti [9] and Méléard [10] who are interested in the two-dimensional incompressible Navier-Stokes equation. The vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ of the velocity field $u = (u_1, u_2)$ solution of the Navier-Stokes equation satisfies

$$\partial_t \omega = \nu \Delta \omega - \nabla.(u\omega). \tag{0.4}$$

The velocity field is recovered from its vorticity thanks to the Biot and Savart kernel :

$$u(t,x) = (-\partial_2 \gamma * w(t,.)(x), \partial_1 \gamma * w(t,.)(x)).$$

In the framework first introduced by Marchioro and Pulvirenti, Méléard proves a propagation of chaos result to the solution of a nonlinear martingale problem associated with (0.4). As the kernel $\nabla\gamma$ and the Biot and Savart kernel are very closely connected, we adapt their ideas and define the n-particles system with a cutoffed kernel replacing $\nabla\gamma$. We first prove that when the cutoffed kernel converges to $\nabla\gamma$ as n tends to $+\infty$, the empirical measure of the particle system converges in probability to the unique solution of the nonlinear martingale problem. As

an easy consequence, it is possible to approximate the solution of (0.1) thanks to the empirical measure. Supposing moreover that the cutoff depends on n in a precise asymptotics given by the computations like in Méléard [10], we obtain a trajectorial rate of convergence.
To our knowledge, these are the first convergence results for a method based on gradient particles when the space dimension is strictly greater than 1.
In the numerical simulation of the n-particles system, the complexity of the naïve computation of the interaction between particles is $\mathcal{O}(n^2)$. One advantage of the gradient approach in space dimension one is that the drift coefficient of a particle is obtained by calculating the convolution of the Heaviside function with the empirical measure at the position of the particle i.e. by counting the number of particles under this position [2]. The numerical complexity can be reduced drastically by sorting the particles. In space dimension $d \geq 2$, computation of the drift coefficient involves convolution of the empirical measure with the cutoffed kernel replacing $\nabla\gamma$. This corresponds to the computation of a Coulombic-like interaction. Some fast approximation algorithms have been developped for the Coulombic interaction [5] [6]. We also point out that for simulation, the convergence of the cutoffed kernels to $\nabla\gamma$ is far easier to handle than the convergence of the approximations of identity which appear in the definition of the moderately interacting diffusion systems associated with the martingale problem (0.2).

Notations

- C denotes a real constant which can change from line to line.

- The Euclidian norm of $x \in \mathbb{R}^d$ is denoted by $|x|$.

- Let $\hat{P}(C([0,+\infty),\mathbb{R}^d))$ denote the set of probability measures Q on $C([0,+\infty),\mathbb{R}^d)$ with time-marginals $Q_t,\ t \geq 0$ admitting densities with respect to Lebesgue measure on $\mathbb{R}^d$ that belong to $L^\infty(\mathbb{R}^d)$.

- The space of bounded signed measures on $\mathbb{R}^d$ is denoted by $\mathcal{M}(\mathbb{R}^d)$.

- For a function $f(x)$ defined on $\mathbb{R}^d$, $\partial_i f$ $(i \leq d)$ and ∇f denote respectively the partial derivative of f with respect to the i-th variable and its gradient. For a function $u(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}^d$, $\partial_t u$ and $\partial_i u$ denote respectively the derivatives of u with respect to its first (time) variable and its $(i+1)$-th (i-th space) variable.

- The heat kernel on $\mathbb{R}^d$ is denoted by $G_t^\nu(x) = \frac{1}{(4\pi\nu t)^{d/2}} e^{-|x|^2/4\nu t}$. By an easy computation, we get

$$\forall t > 0,\ \forall i \leq d,\ \|\partial_i G_t^\nu\|_1 = 1/\sqrt{\pi\nu t}. \tag{0.5}$$

- Let $C_b^2(\mathbb{R}^d)$ be the space of C^2 functions on $\mathbb{R}^d$ bounded together with their first and second order derivatives.

- $L^{1+\infty}(\mathbb{R}^d) = L^1 \cap L^\infty(\mathbb{R}^d)$ endowed with the norm $|||f||| = ||f||_1 \vee \|f\|_\infty$ is a complete space.

- Let $W^{1,1+\infty}(\mathbb{R}^d)$ denote the Sobolev space of functions belonging to $L^\infty(\mathbb{R}^d)$ with first order distribution derivatives belonging to $L^{1+\infty}(\mathbb{R}^d)$ endowed with the norm $\|f\|_{1,1+\infty} = \|f\|_\infty + \sum_{i=1}^d |||\partial_i f|||$. Note that if $f \in W^{1,1+\infty}(\mathbb{R}^d)$, then f admits a globally Lipschitz continuous representative still denoted by f and

$$\forall x,y \in \mathbb{R}^d,\ |f(x)-f(y)| \leq C\|f\|_{1,1+\infty}|x-y| \quad \text{where } C \text{ does not depend on } f. \tag{0.6}$$

- For $g(r)=\begin{cases}\ln(r)/S_2 & \text{if } d=2\\ -1/(S_d\, r^{d-2}) & \text{if } d\geq 3\end{cases}$ where S_d denotes the area of the unit sphere in $\mathbb{R}^d$,

the function $x\in\mathbb{R}^d\to g(|x|)$ is the fundamental solution of the Laplacian in $\mathbb{R}^d$. Let $K(x)=(K_1(x),\ldots,K_d(x))=g'(|x|)x/|x|$ denotes its gradient. The next Lemma groups some useful properties of the kernel K. It is proved in the Appendix.

Lemma 0.1 *1. The function $|K|(.)$ is bounded in the complementary set of the unit ball $B(0,1)$ and belongs to $L^p(B(0,1))$ for any $1\leq p<d/(d-1)$.*
Moreover, for $v=(v_1,\ldots,v_d)\in(L^1\cap L^\infty(\mathbb{R}^d))^d$, the function $K*v:\mathbb{R}^d\to\mathbb{R}$ defined by $K*v(x)=\sum_{i=1}^d K_i*v_i(x),\ x\in\mathbb{R}^d$ is continuous and bounded and satisfies

$$\|K*v\|_\infty\leq C\sum_{i=1}^d |||v_i|||. \tag{0.7}$$

Last, $\Delta(K*v)=\sum_{i=1}^d\partial_i v_i$ in the distribution sense.

2. For $f\in W^{1,1+\infty}(\mathbb{R}^d)$, $\exists c\in\mathbb{R},\ f=K*\nabla f+c$.

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1 Existence and uniqueness for the viscous scalar conservation law (0.1)

Let u_0 be a continuous and bounded function on $\mathbb{R}^d$. We say that $u:[0,+\infty)\times\mathbb{R}^d\to\mathbb{R}$ is a classical solution of (0.1) if u is continuous on $[0,+\infty)\times\mathbb{R}^d$, continuously differentiable with respect to the time variable and twice continuously differentiable with respect to the space variables on $(0,+\infty)\times\mathbb{R}^d$ and satisfies

$$\forall x\in\mathbb{R}^d,\ u(0,x)=u_0(x)\quad\text{and}\quad\forall(t,x)\in(0,+\infty)\times\mathbb{R}^d,\ \partial_t u=\nu\Delta u-\nabla.A(u).$$

From now on, $\bar{A}$ denotes a C^2 function equal to A on $[-\|u_0\|_\infty,\|u_0\|_\infty]$ and admitting bounded first and second order derivatives.

Proposition 1.1 *Let u_0 be a continuous and bounded function on $\mathbb{R}^d$. Then the viscous scalar conservation law (0.1) admits a unique classical solution u bounded by $\|u_0\|_\infty$. Moreover, the function u has the following integral representation*

$$u(t,x)=G_t^\nu*u_0-\int_0^t\nabla G_{t-s}^\nu*\bar{A}(u(s,.))(x)ds \tag{1.1}$$

where $\nabla G_{t-s}^\nu*\bar{A}(u(s,.))$ stands for $\sum_{i=1}^d\partial_i G_{t-s}^\nu*\bar{A}_i(u(s,.))$.

Proof : Existence of a classical solution is a consequence of results concerning quasilinear parabolic equations given by Ladyzenskaja, Solonnikov and Ural'ceva [8] (Theorem 8.1 p.495

and Remark 8.1 p. 495). By the maximum principle ([8] Theorem 2.5 p.18), any such solution is bounded by $\|u_0\|_\infty$.

As the initial condition u_0 is only continuous, it is not possible to apply the maximum principle like in [8] p.494-495 to prove uniqueness. That is why we are going to show that any classical solution u bounded by $\|u_0\|_\infty$ has the integral representation (1.1). Using the integration by parts formula, we easily check that u is such that for any C^∞ with compact support function $\psi : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$, $\forall t > 0$,

$$\int_{\mathbb{R}^d} \psi(t,x)u(t,x)dx = \int_{\mathbb{R}^d} \psi(0,x)u_0(x)dx + \int_{(0,t]\times\mathbb{R}^d} \left(u\frac{\partial \psi}{\partial s} + \nu u \Delta\psi + \bar{A}(u).\nabla\psi \right)(s,x)dsdx \tag{1.2}$$

For $t > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ a C^2 with compact support function, let $\phi(s,x) = 1_{[0,t]}(s)G^\nu_{t-s} * f(x)$. It is possible to approximate the function ϕ, its first order time derivative and its first and second order spatial derivatives in $L^1([0,t] \times \mathbb{R}^d)$ by C^∞ functions ψ^n with compact support and their corresponding derivatives such that $\psi^n(t,.)$ and $\psi^n(0,.)$ converge to $\phi(t,.)$ and $\phi(0,.)$ in $L^1(\mathbb{R}^d)$. Writing (1.2) for ψ^n and taking the limit $n \to +\infty$, we get that (1.2) holds for ψ replaced by ϕ. As $\partial_s\phi + \nu\Delta\phi = 0$ on $[0,t] \times \mathbb{R}^d$, we deduce

$$\int_{\mathbb{R}^d} f(x)u(t,x)dx = \int_{\mathbb{R}^d} G^\nu_t * f(x)u_0(x)dx + \int_0^t \int_{\mathbb{R}^d} \bar{A}(u(s,x)).\nabla G^\nu_{t-s} * f(x)dxds.$$

This equality is satisfied for any C^2 with compact support function f on $\mathbb{R}^d$. Hence (1.1) holds. Let now u and v be two classical solutions of (0.1) bounded by $\|u_0\|_\infty$.

$$\begin{aligned}\text{By (1.1) and (0.5),}\quad \|u(t,.) - v(t,.)\|_\infty &\leq \int_0^t \sum_{i=1}^d \|\partial_i G^\nu_{t-s}\|_1 \|\bar{A}'_i\|_\infty \|u(s,.) - v(s,.)\|_\infty ds \\ &\leq \frac{\sum_{i=1}^d \|\bar{A}'_i\|_\infty}{\sqrt{\pi\nu}} \int_0^t \frac{\|u(s,.) - v(s,.)\|_\infty}{\sqrt{t-s}} ds \qquad (1.3)\end{aligned}$$

Iterating this inequality, we get $\|u(t,.) - v(t,.)\|_\infty \leq C \int_0^t \|u(r,.) - v(r,.)\|_\infty dr$ and we conclude that $u = v$ by Gronwall's lemma. ■

Remark 1.2 *If u_0 is constant and equal to c, then $u \equiv c$ is the solution of (0.1) given by Proposition 1.1.*
Suppose now that u_0 does not depend on the variables $x_{i_1}, \ldots, x_{i_k}$, $1 \leq i_1 < \ldots < i_k \leq d$ where $1 \leq k \leq d-1$ i.e. $u_0(x_{i_1}, \ldots, x_{i_k}, y) = w_0(y)$ where $y \in \mathbb{R}^{d-k}$ groups the variables x_j, $j \neq i_1, \ldots, i_k$. Let $w(t,y)$ denote the solution of the $d-k$-dimensional viscous scalar conservation law with initial condition $w_0(y)$ where $A = (A_i)_{i\leq d}$ is replaced by $(A_j)_{j\neq i_1,\ldots,i_k}$. We easily check that $u(t,x) = w(t,y)$ is the solution of (0.1) for the initial condition u_0.

We are now going to prove that in the particular case $u_0 \in W^{1,1+\infty}(\mathbb{R}^d)$, for $i \leq d$ the derivative $\partial_i u(t,.)$ satisfy an integral equation similar to (1.1).

Proposition 1.3 *If $u_0 \in W^{1,1+\infty}(\mathbb{R}^d)$ then u, the classical solution of (0.1) given by Proposition 1.1, is such that $\forall t \geq 0$, $u(t,.) \in W^{1,1+\infty}(\mathbb{R}^d)$ and $\sup_{s\in[0,t]} \|u(s,.)\|_{1,1+\infty} < +\infty$. For $i \leq d$ the partial derivative with respect to the $i-th$ space variable satisfies the integral equation*

$$\partial_i u(t,x) = G^\nu_t * \partial_i u_0(x) - \int_0^t \nabla G^\nu_{t-s} * (\bar{A}'(u(s,.))\partial_i u(s,.))(x)ds \tag{1.4}$$

Last, $\forall (t,x)\in[0,+\infty)\times\mathbb{R}^d,\ u(t,x)=c+K*\nabla u(t,.)(x)$ *where c is a real constant which does not depend on t.*

Proof : Let $u_0\in W^{1,1+\infty}(\mathbb{R}^d)$, $T>0$ and $\mathcal{C}_T=C([0,T],L^\infty(\mathbb{R}^d))$ endowed with the norm $\|v\|_{\mathcal{C}_T}=\sup_{t\in[0,T]}\|v(t)\|$. For $v\in\mathcal{C}_T$, we define

$$\Theta(v)(t)=G_t^\nu*u_0-\int_0^t\nabla G_{t-s}^\nu*\bar{A}(v(s))ds,\ 0\le t\le T.$$

As u_0 is Lipschitz continuous (see (0.6)) and bounded and $\|\bar{A}_i(v(s))\|_\infty\le\|\bar{A}_i'\|_\infty\|v(s)\|_\infty$, we easily check that $\Theta(v)\in\mathcal{C}_T$. By computations similar to (1.3), we obtain that for $T_0=\pi\nu/(4\sum_{i=1}^d\|\bar{A}_i'\|_\infty)^2$, the mapping Θ is a contraction on $\mathcal{C}_{T_0}$. Moreover, its fixed-point is equal to $t\in[0,T_0]\to u(t,.)$ and when $\|v\|_{\mathcal{C}_{T_0}}\le 2\|u_0\|_\infty$, then $\|\Theta(v)\|_{\mathcal{C}_{T_0}}\le 2\|u_0\|_\infty$.
Let $v\in\mathcal{C}_{T_0}$ be such that $\forall s\in[0,T_0],\ v(s)\in W^{1,1+\infty}(\mathbb{R}^d)$ and $\forall i\le d$, $\sup_{s\in[0,T_0]}|||\partial_i v(s)|||\le 2|||\partial_i u_0|||$. As $\bar{A}_j(0)=A_j(0)=0$ and $\bar{A}_j$ is C^1 with a bounded derivative, by Brezis [3] Proposition IX.5 p.155, $\bar{A}_j(v(s))\in W^{1,1+\infty}(\mathbb{R}^d)$ with gradient $\bar{A}_j'(v(s))\nabla v(s)$. Differentiating $\Theta(v)(t)$, we get

$$\begin{aligned}\forall t\le T_0,\ |||\partial_i\Theta(v)(t)|||&\le|||\partial_i u_0|||+\frac{\sum_{j=1}^d\|\bar{A}_j'\|_\infty}{\sqrt{\nu\pi}}\int_0^t\frac{|||\partial_i v(s)|||}{\sqrt{t-s}}ds\\&\le|||\partial_i u_0|||+\frac{4\sum_{j=1}^d\|\bar{A}_j'\|_\infty\sqrt{T_0}|||\partial_i u_0|||}{\sqrt{\nu\pi}}=2|||\partial_i u_0|||\end{aligned}$$

Hence the sequence of fixed-point iterations defined in $\mathcal{C}_{T_0}$ by $\forall t\in[0,T_0],\ v^0(t)=u_0$ and $v^{k+1}=\Theta(v^k)$ for $k\ge 0$ is such that

$$\forall k\ge 0,\ \forall t\in[0,T_0],\ \|v^k(t)\|_\infty\le 2\|u_0\|_\infty\ \text{ and }\ \forall i\le d, |||\partial_i v^k(t)|||\le 2|||\partial_i u_0|||.$$

As $\forall t\in[0,T_0],\ v^k(t)\to u(t,.)$ in the distribution sense, we deduce that $\forall t\in[0,T_0],\ u(t,.)\in W^{1,1+\infty}(\mathbb{R}^d)$, $\|u(t,.)\|_\infty\le 2\|u_0\|_\infty$ and $\forall i\le d, |||\partial_i u(t,.)|||\le 2|||\partial_i u_0|||$.
By induction on n, using the mapping defined like Θ with u_0 replaced by $u(nT_0,.)$, we conclude

$$\begin{aligned}&\forall n\in\mathbb{N},\ \forall t\in[nT_0,(n+1)T_0],\ u(t,.)\in W^{1,1+\infty}(\mathbb{R}^d),\\&\qquad\|u(t,.)\|_\infty\le 2^{n+1}\|u_0\|_\infty\ \text{ and }\ \forall i\le d, |||\partial_i u(t,.)|||\le 2^{n+1}|||\partial_i u_0|||.\end{aligned}\tag{1.5}$$

The integral equation (1.4) is obtained from (1.1) by differentiation with respect to x_i.

Since $u(t,.)\in W^{1,1+\infty}(\mathbb{R}^d)$, by Lemma 0.1, there exists a constant $c(t)\in\mathbb{R}$ such that $u(t,.)=c(t)+K*\nabla u(t,.)$. Similarly, since $\bar{A}_j(u(t,.))\in W^{1,1+\infty}(\mathbb{R}^d)$ with gradient $\bar{A}_j'(u(t,.))\nabla u(t,.)$, there exists a constant $b(t)\in\mathbb{R}^d$ such that $\forall j\le d,\ \bar{A}_j(u(t,.))=b_j(t)+K*(\bar{A}_j'(u(t,.))\nabla u(t,.))$. Using (1.4) to compute $K*\nabla u(t,.)$, we get

$$K*\nabla u(t,x)=G_t^\nu*K*\nabla u_0(x)-\int_0^t\sum_{j=1}^d\partial_j G_{t-s}^\nu*(K*(\bar{A}_j'(u(s,.))\nabla u(s,.)))(x)ds.$$

Hence $u(t,x)-c(t)=G_t^\nu*u_0(x)-G_t^\nu*c(0)-\int_0^t\nabla G_{t-s}^\nu*(\bar{A}(u(s,.))-b(s))(x)ds$. As $G_t^\nu*c(0)=c(0)$ and $\nabla G_{t-s}^\nu*b(s)=0$, using (1.1) we conclude that $t\to c(t)$ is constant. ∎

2 Existence and uniqueness for the nonlinear martingale problem

From now on, let $u_0 \in W^{1,1+\infty}(\mathbb{R}^d)$. We are interested in giving a probabilistic representation of the solution of (0.1) given by Proposition 1.1. As u_0 does not depend on the space variables x_i for which $\|\partial_i u_0\|_1 = 0$, by Remark 1.2, the solution of (0.1) is easily derived from the solution of the similar problem obtained by removing these space variables. Hence we can suppose that $\forall i \leq d$, $\|\partial_i u_0\|_1 > 0$ which ensures that $|\partial_i u_0|/\|\partial_i u_0\|_1$ is a probability density.
For $i \leq d$ let $h_i(x) = \|\partial_i u_0\|_1 \partial_i u_0(x)/|\partial_i u_0(x)|$ (with convention $\frac{0}{0} = 1$). Using the functions $h_i, i \leq d$ as signed weights, we associate with $P = (P^1, \ldots, P^d)$ in $(P(C([0,+\infty), \mathbb{R}^d)))^d$ the signed measures $\tilde{P}_s = (\tilde{P}_s^1, \ldots, \tilde{P}_s^d) \in (\mathcal{M}(\mathbb{R}^d))^d$ defined by

$$\forall i \leq d,\ \forall B \in \mathcal{B}(\mathbb{R}^d),\ \tilde{P}_s^i(B) = \mathbb{E}^{P^i}(1_B(X_s) h_i(X_0))$$

where X denotes the canonical process on $C([0,+\infty), \mathbb{R}^d)$.

Lemma 2.1 *Let $P = (P^1, \ldots, P^d)$ in $(\hat{P}(C([0,+\infty), \mathbb{R}^d)))^d$. For $i \leq d$ and $s \geq 0$, the measure $\tilde{P}_s^i$ admits a density $\tilde{p}_i(s,.)$ with respect to Lebesgue measure on $\mathbb{R}^d$ which satisfies $\|\tilde{p}_i(s,.)\|_1 \leq \|\partial_i u_0\|_1$ and belongs to $L^\infty(\mathbb{R}^d)$ with a norm smaller than $\|\partial_i u_0\|_1$ times the L^∞ norm of the density of P_s^i.*

Proof : Let λ denote the Lebesgue measure on $\mathbb{R}^d$.

$$\forall B \in \mathcal{B}(\mathbb{R}^d),\ |\tilde{P}_s^i(B)| = |\mathbb{E}^{P^i}(1_B(X_s) h_i(X_0))| \leq \|\partial_i u_0\|_1 P_s^i(B) \tag{2.1}$$

As P_s^i admits a density with respect to λ, $\lambda(B) = 0$ implies that $\tilde{P}_s^i(B) = P_s^i(B) = 0$. Hence $\tilde{P}_s^i$ admits a density $\tilde{p}_i(s,.)$ with respect to λ. Summing (2.1) for $B^+ = \{x : \tilde{p}_i(s,x) > 0\}$ and $B^- = \{x : \tilde{p}_i(s,x) < 0\}$, we get $\|\tilde{p}_i(s,.)\|_1 \leq \|\partial_i u_0\|_1$.
Let $c_i(s)$ denote the L^∞ norm of the density of P_s^i. By (2.1), $|\tilde{P}_s^i(B)| \leq c_i(s)\|\partial_i u_0\|_1 \lambda(B)$. Therefore $\lambda(\{x : \tilde{p}_i(s,x) > c_i(s)\|\partial_i u_0\|_1\}) = \lambda(\{x : \tilde{p}_i(s,x) < -c_i(s)\|\partial_i u_0\|_1\}) = 0$ and $\|\tilde{p}_i(s,.)\|_\infty \leq c_i(s)\|\partial_i u_0\|_1$. ∎

Combining this result and Lemma 0.1, we obtain that for $P \in (\hat{P}(C([0,+\infty), \mathbb{R}^d)))^d$, $K * \tilde{P}_s(x) = \sum_{i=1}^d (K_i * \tilde{p}_i(s,.))(x)$ makes sense.
To simplify notations, we set

$$\bar{\bar{A}}(w) = \bar{A}(c+w) \quad \text{where } c \in \mathbb{R} \text{ is such that } u_0 = c + K * \nabla u_0 \text{ (see Lemma 0.1).} \tag{2.2}$$

Definition 2.2 *We say that $P \in (\hat{P}(C([0,+\infty), \mathbb{R}^d)))^d$ solves the nonlinear martingale problem (MP) starting at u_0 if $\forall i \leq d$, P_0^i has density $|\partial_i u_0(x)|/\|\partial_i u_0\|_{L^1}$ with respect to Lebesgue measure and $\forall \phi \in C_b^2(\mathbb{R}^d)$,*

$$M_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \nu \Delta\phi(X_s) + \bar{\bar{A}}'(K * \tilde{P}_s(X_s)).\nabla\phi(X_s)ds \ \text{ is a } P^i \text{ martingale.}$$

where X denotes the canonical process on $C([0,+\infty), \mathbb{R}^d)$.

Remark 2.3 *The martingale problem (MP) is linked to the scalar conservation law (0.1) through the system (0.3) obtained from (0.1) by spatial derivation. Indeed, if P solves the martingale problem, the constancy of the expectation of the P^i martingale $h_i(X_0)M_t^\phi$ implies that the densities*

$\tilde{p}_i(s,.)$ *of the measures* $\tilde{P}^i_s$ *satisfy*

$$\int_{\mathbb{R}^d} \phi(x)\tilde{p}_i(t,x)dx = \int_{\mathbb{R}^d} \phi(x)\partial_i u_0(x)dx$$
$$+ \int_0^t \int_{\mathbb{R}^d} \nu\Delta\phi(x)\tilde{p}_i(s,x) + \bar{A}'\Big(c + \sum_{j=1}^d (K_j * \tilde{p}_j(s,.))(x)\Big).\nabla\phi(x)\tilde{p}_i(s,x)\ dxds.$$

And $(\tilde{p}_1,\ldots,\tilde{p}_d)$ *is a weak solution of (0.3).*

Theorem 2.4 *The nonlinear martingale problem* (MP) *starting at* u_0 *admits a unique solution* $P \in (\hat{P}(C([0,+\infty),\mathbb{R}^d)))^d$. *Moreover,* $(s,x) \to c + K * \tilde{P}_s(x)$ *is equal to the solution of (0.1) studied in Proposition 1.1.*

To prove this result, we need the following lemma obtained by a reasoning similar to the one made in the proof of Proposition 1.3 (see (1.5)).

Lemma 2.5 *Let* $w : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ *denote a measurable function and* $v_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. *If* $t \to v(t) \in L^1(\mathbb{R}^d)$ *is a bounded mapping such that*

$$\forall t \geq 0,\ v(t) = G^\nu_t * v_0 - \int_0^t \nabla G^\nu_{t-s} * (\bar{A}'(w(s))v(s))ds \tag{2.3}$$

then $\forall t \geq 0,\ \|v(t)\|_\infty \leq 2^{1+\frac{t}{T_0}}\|v_0\|_\infty$ *where* $T_0 = \pi\nu/(4\sum_{i=1}^d \|\bar{A}'_i\|_\infty)^2$.

Proof of Theorem 2.4 : Existence

Let $(B_t)_{t\geq 0}$ be a d-dimensional Brownian motion and $X(0) = (X_1(0),\ldots,X_d(0))$ an independent $\mathbb{R}^{d\times d}$-valued random variable such that $\forall i \leq d$, $X_i(0)$ has the density $|\partial_i u_0|/\|\partial_i u_0\|_1$ with respect to Lebesgue measure. Let $u(t,x)$ denote the solution of (0.1) studied in Proposition 1.1. Combining the estimation $\sup_{s\in[0,t]}\|u(s,.)\|_{1,1+\infty}$ given by Proposition 1.3 and (0.6), we obtain that for any $t > 0$, $x \in \mathbb{R}^d \to u(s,x)$ is Lipschitz continuous uniformly for $s \in [0,t]$. Hence $\forall i \leq d$, existence and pathwise uniqueness hold for the stochastic differential equation

$$X_i(t) = X_i(0) + \sqrt{2\nu}B_t + \int_0^t \bar{A}'(u(s,X_i(s)))ds$$

Let P^i denote the distribution of X^i. As $\bar{A}'$ is bounded, by Girsanov theorem, $\forall s \geq 0$, P^i_s admits a density $p_i(s,.)$ with respect to Lebesgue measure. As in Lemma 2.1, we deduce that $\tilde{P}^i_s$ admits a density $\tilde{p}_i(s,.)$ with $\|\tilde{p}_i(s,.)\|_1 \leq \|\partial_i u_0\|_1$.
Let $t > 0$, f be a C^2 function with compact support on $\mathbb{R}^d$ and $\psi(s,x) = G^\nu_{t-s} * f(x)$. As $(\partial_s + \nu\Delta)\psi = 0$ on $[0,t]\times\mathbb{R}$, Itô's formula yields

$$f(X_i(t)) = \psi(0,X_i(0)) + \int_0^t \bar{A}'(u(s,X_i(s))).\nabla\psi(s,X_i(s))ds + \sqrt{2\nu}\int_0^t \nabla\psi(s,X_i(s)).dB_s \tag{2.4}$$

Multiplying by $h_i(X^i_0)$ and taking expectations, we deduce that

$$\int_{\mathbb{R}^d} f(x)\tilde{p}_i(t,x)dx = \int_{\mathbb{R}^d} G^\nu_t * f(x)\partial_i u_0(x)dx + \int_0^t \int_{\mathbb{R}^d} \bar{A}'(u(s,x)).\nabla(G^\nu_{t-s} * f)(x)\tilde{p}_i(s,x)dxds.$$

Hence $\forall t \geq 0$, dx a.e.,

$$\tilde{p}_i(t,x) = G_t^\nu * \partial_i u_0(x) - \int_0^t \nabla G_{t-s}^\nu * (\bar{A}'(u(s,.))\tilde{p}_i(s,.))(x)ds$$

Combining this equation with (1.4), we conclude that $\forall t \geq 0$, $\|\tilde{p}_i(t,.) - \partial_i u(t,.)\|_1 = 0$ by Gronwall's lemma.
By Proposition 1.3, $\forall t \geq 0$, $u(t,.) = c + K * \nabla u(t,.)$. Hence $\forall t \geq 0$, $\forall x \in \mathbb{R}^d$, $\bar{A}'(u(t,x)) = \bar{A}'(c + K * \tilde{P}_t(x)) = \bar{\bar{A}}'(K * \tilde{P}_t(x))$ where $P = (P^1, \ldots, P^d)$. Applying Itô's formula and replacing $\bar{A}'(u(s,.))$ by $\bar{\bar{A}}'(K * \tilde{P}_s(.))$, we check that $\forall \phi \in C_b^2(\mathbb{R}^d)$, $M_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t (\nu \Delta \phi(X_s) + \bar{\bar{A}}'(K * \tilde{P}_s(X_s)).\nabla\phi(X_s))ds$ is a P^i martingale.
Taking expectations in (2.4), we obtain that the densities $p_i(s,.)$ of the time-marginals P_s^i satisfy

$$\forall t \geq 0,\ dx\ a.e.,\ p_i(t,x) = G_t^\nu * \frac{|\partial_i u_0|}{\|\partial_i u_0\|_1}(x) - \int_0^t \nabla G_{t-s}^\nu * (\bar{A}'(u(s,.))p_i(s,.))(x)ds.$$

As $|\partial_i u_0| \in L^\infty(\mathbb{R}^d)$, Lemma 2.5 implies that $\forall t \geq 0$, $p_i(t,.) \in L^\infty(\mathbb{R}^d)$. Hence $\forall i \leq d$, $P^i \in \hat{P}(C([0,+\infty),\mathbb{R}^d))$ and $P = (P^1, \ldots, P^d)$ solves problem (MP) starting at u_0.

<u>Uniqueness</u>
Let $P = (P^1, \ldots, P^d)$ and $Q = (Q^1, \ldots, Q^d)$ be two solutions.
By Paul Levy's characterization, $X_t - X_0 - \int_0^t \bar{\bar{A}}'(K * \tilde{P}_s(X_s))ds$ is a P^i Brownian motion $\forall i \leq d$. Hence by a reasoning similar to the one made in the proof for existence, we get that the densities $\tilde{p}_i(s,.)$ of the measures $\tilde{P}_s^i$ satisfy

$$\forall t \geq 0,\ dx\ a.e.,\ \tilde{p}_i(t,x) = G_t^\nu * \partial_i u_0(x) - \int_0^t \nabla G_{t-s}^\nu * (\bar{\bar{A}}'(K * \tilde{P}_s(.))\tilde{p}_i(s,.))(x)ds.$$

For $t_0 > 0$, as $\partial_i u_0 \in L^\infty(\mathbb{R}^d)$, Lemma 2.5 implies that $\|\tilde{p}_i(t,.)\|_\infty$ is bounded on $[0,t_0]$. The densities $\tilde{q}_i(s,.)$ of the measures $\tilde{Q}_s^i$ satisfy similar properties. As according to Lemma 0.1, $\|K * \tilde{P}_s - K * \tilde{Q}_s\|_\infty \leq C \sum_{j=1}^d |||\tilde{p}_j(s) - \tilde{q}_j(s)|||$, using the boundedness of $\bar{\bar{A}}'_j$ and $\bar{\bar{A}}''_j$ $j \leq d$, we obtain that for $t \leq t_0$,

$$\begin{aligned}
|||\tilde{p}_i(t) - \tilde{q}_i(t)||| &\leq \frac{1}{\sqrt{\pi\nu}} \int_0^t \sum_{j=1}^d |||\bar{\bar{A}}'_j(K * \tilde{P}_s)\tilde{p}_i(s) - \bar{\bar{A}}'_j(K * \tilde{Q}_s)\tilde{q}_i(s)||| \frac{ds}{\sqrt{t-s}} \\
&\leq \frac{1}{\sqrt{\pi\nu}} \int_0^t \sum_{j=1}^d \left(\|\bar{\bar{A}}'_j\|_\infty |||\tilde{p}_i(s) - \tilde{q}_i(s)||| + \|\bar{\bar{A}}''_j\|_\infty \|K * \tilde{P}_s - K * \tilde{Q}_s\|_\infty |||\tilde{p}_i(s)||| \right) \frac{ds}{\sqrt{t-s}} \\
&\leq C(i,t_0) \int_0^t \sum_{j=1}^d |||\tilde{p}_j(s) - \tilde{q}_j(s)||| \frac{ds}{\sqrt{t-s}}
\end{aligned}$$

Summing this inequality for $i \leq d$, and iterating the result, we conclude by Gronwall's Lemma that $\forall i \leq d$, $\forall t \leq t_0$, $|||\tilde{p}_i(t) - \tilde{q}_i(t)||| = 0$. Hence $\forall i \leq d$, both P^i and Q^i solve the classical martingale problem with diffusion matrix equal to $2\nu \times I_d$ (where I_d is the identity $d \times d$ matrix), bounded drift coefficient equal to $\bar{\bar{A}}'(K * \tilde{P}_s(x))$ and initial marginal $\frac{|\partial_i u_0(x)|}{\|\partial_i u_0\|_1}dx$. By Girsanov theorem, we conclude that $\forall i \leq d$, $P^i = Q^i$. ∎

3 Probabilistic approximation of the solution of (0.1)

3.1 Approximation of the kernel K

Because of the explosion of K at the origin, we are going to replace this kernel by Lipschitz continuous and bounded ones in the definition of the interacting particle systems. Following Marchioro and Pulvirenti [9] who deal with the Biot-Savart kernel ($d = 2$), for $\epsilon > 0$, we set

$$\text{for } \ d = 2, \ g_\epsilon(r) = \begin{cases} \frac{\ln(r)}{S_2} & \text{if } \ r \geq \epsilon \\ \frac{r^2}{2S_2\epsilon^2} + \frac{1}{S_2}(\ln(\epsilon) - \frac{1}{2}) & \text{if } \ 0 < r \leq \epsilon \end{cases}$$

$$\text{for } \ d \geq 3, \ g_\epsilon(r) = \begin{cases} \frac{-1}{S_d\, r^{d-2}} & \text{if } \ r \geq \epsilon \\ \frac{(d-2)r^2}{2S_d\epsilon^d} - \frac{d}{2S_d\epsilon^{d-2}} & \text{if } \ 0 < r \leq \epsilon \end{cases}$$

and $K^\epsilon(x) = \nabla(g^\epsilon(|x|)) = g'_\epsilon(|x|)x/|x|$. The next lemma groups the properties of these kernels that will be needed in the sequel.

Lemma 3.1 *The function K^ϵ is bounded by M_ϵ and Lipschitz continuous with constant L_ϵ where*

$$\textit{for} \quad d = 2, \ M_\epsilon = \frac{1}{S_2\epsilon} \quad \textit{and} \quad L_\epsilon = \frac{3}{S_2\epsilon^2}$$
$$\textit{for} \quad d \geq 3, \ M_\epsilon = \frac{d-2}{S_d\epsilon^{d-1}} \quad \textit{and} \quad L_\epsilon = \frac{3(d-2)(d-1)}{S_d\epsilon^d}$$

Last, $\forall i \leq d$, $\|K_i^\epsilon - K_i\|_1 \leq C\epsilon$ where C does not depend on ϵ.

Proof : It is easy to check that g_ϵ is C^1 on $[0, +\infty)$ and g'_ϵ is C^1 on $[0, \epsilon) \cup (\epsilon, +\infty)$. Moreover,

$$\text{for } d = 2, \ \forall r > 0, \ 0 \leq g'_\epsilon(r) \leq \min\left(g'(r), \frac{1}{S_2\epsilon}\right) \ \text{ and } \ |g''_\epsilon(r)| \leq \min\left(\frac{1}{S_2\epsilon^2}, |g''(r)|\right)$$
$$\text{for } d \geq 3, \ \forall r > 0, \ 0 \leq g'_\epsilon(r) \leq \min\left(g'(r), \frac{d-2}{S_d\epsilon^{d-1}}\right) \ \text{ and } \ |g''_\epsilon(r)| \leq \min\left(\frac{(d-2)(d-1)}{S_d\epsilon^d}, |g''(r)|\right).$$

As $|K^\epsilon(x)| = |g'_\epsilon(|x|)|$, we deduce that K^ϵ is bounded by M_ϵ. The Lipschitz continuity property is obtained by the following computation.

$$\begin{aligned} \text{As } g'_\epsilon(0) = 0, \ |K^\epsilon(x) - K^\epsilon(y)| &\leq |g'_\epsilon(|x|) - g'_\epsilon(|y|)| + |g'_\epsilon(|y|)|\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \\ &\leq \|g''_\epsilon\|_\infty\left(|x-y| + |y|\left|\frac{x}{|x|} - \frac{y}{|y|}\right|\right) \leq 3\|g''_\epsilon\|_\infty|x-y| \end{aligned}$$

For $|x| \geq \epsilon$, $g'_\epsilon(|x|) = g'(|x|)$ and $K^\epsilon(x) = K(x)$. Moreover, as $0 \leq g'_\epsilon \leq g'$, $|K_i^\epsilon - K_i| \leq |K_i|$. Hence,

$$\|K_i - K_i^\epsilon\|_1 \leq \int_{B(0,\epsilon)} |K_i(x)|dx \leq C(d)\int_{B(0,\epsilon)} \frac{1}{|x|^{d-1}}dx \leq C(d)\epsilon.$$

$\blacksquare$

3.2 The weak propagation of chaos result

Let $(Z^k(0) = (Z_1^k(0), \dots, Z_d^k(0)))_{k\in\mathbb{N}^*}$ be a sequence of initial variables independent and identically distributed according to a probability measure on $\mathbb{R}^{d\times d}$ with $i-th$ marginal $\frac{|\partial_i u_0(x)|}{\|\partial_i u_0\|_1}dx$ for $i \le d$ and $(B_t^k)_{k\in\mathbb{N}^*}$ a sequence of independent $\mathbb{R}^d$-valued Brownian motions independent of the initial variables. For $(\epsilon_n)_n$ a sequence of strictly positive numbers, the n-particles system is defined as the unique solution of the stochastic differential equation

$$Z_i^{k,n}(t) = Z_i^k(0) + \sqrt{2\nu}B_t^k + \int_0^t \bar{\bar{A}}'\left(\sum_{j=1}^d \frac{1}{n-1}\sum_{l\neq k} K_j^{\epsilon_n}(Z_i^{k,n}(s) - Z_j^{l,n}(s))h_j(Z_j^l(0))\right) ds \quad (3.1)$$

where $1 \le i \le d$ and $1 \le k \le n$.
Computation of the interaction between particles involves Coulombic field calculations. Indeed, apart from the cutoff, the drift coefficient of $Z_i^{k,n}$ is obtained by composition of $\bar{\bar{A}}'$ with the summation over $j \le d$ of the j-th component of the electrostatic field generated at $Z_i^{k,n}(s)$ by the charges $h_j(Z_j^l(0))/(n-1)$ at positions $Z_j^{l,n}(s)$, $l \neq k$.
The empirical measure $\mu^n = (\mu_1^n, \dots, \mu_d^n) \in (P(C([0,+\infty), \mathbb{R}^d)))^d$ is defined by

$$\mu^n = \left(\frac{1}{n}\sum_{k=1}^n \delta_{Z_1^{k,n}}, \dots, \frac{1}{n}\sum_{k=1}^n \delta_{Z_d^{k,n}}\right) \quad \text{and for } t \ge 0, \text{ we have}$$

$$\tilde{\mu}^n(t) = (\tilde{\mu}_1^n(t), \dots, \tilde{\mu}_d^n(t)) = \left(\frac{1}{n}\sum_{k=1}^n h_1(Z_1^k(0))\delta_{Z_1^{k,n}(t)}, \dots, \frac{1}{n}\sum_{k=1}^n h_d(Z_d^k(0))\delta_{Z_d^{k,n}(t)}\right).$$

Theorem 3.2 *If ϵ_n converges to 0 as $n \to +\infty$, the empirical measures $\mu^n = (\mu_1^n, \dots, \mu_d^n)$ converge in probability to P, the unique solution of problem (MP) starting at u_0.*

To prove this result, we have to deal with two difficulties. The first one is the singularity of the kernel K at the origin. We overcome it thanks to the following lemma which is an easy consequence of Girsanov theorem :

Lemma 3.3 *Let $1 < q < \frac{d}{d-1}$.*

$$\forall 1 \le k < l \le n,\ \forall 1 \le i,j \le d,\ \forall \alpha \le 1,\ \forall t \ge 0,$$
$$\mathbb{E}(|K|1_{B(0,\alpha)}(Z_i^{k,n}(t) - Z_j^{l,n}(t)))$$
$$\le \||K|(.)\|_{L^q(B(0,\alpha))}\left(\frac{\|\partial_i u_0\|_\infty}{\|\partial_i u_0\|_1} \wedge \frac{\|\partial_j u_0\|_\infty}{\|\partial_j u_0\|_1}\right)^{1/q} \exp\left(\frac{\sum_{m=1}^d \|\bar{\bar{A}}'_m\|_\infty^2}{2\nu(q-1)}t\right) \quad (3.2)$$

$$\forall x \in \mathbb{R}^d,\ \mathbb{E}(|K|1_{B(0,\alpha)}(x - Z_i^{k,n}(t))) \le \||K|(.)\|_{L^q(B(0,\alpha))}\left(\frac{\|\partial_i u_0\|_\infty}{\|\partial_i u_0\|_1}\right)^{1/q} \exp\left(\frac{\sum_{m=1}^d \|\bar{\bar{A}}'_m\|_\infty^2}{4\nu(q-1)}t\right) \quad (3.3)$$

where by convention $|K|(0) = +\infty$.

The second difficulty is the possible lack of continuity of the density h_i. Approximating h_i by functions of the form $\|\partial_i u_0\|_1((1 - kd(x,F)) \vee -1)$ where F is a closed set included in $\{h_i(x) = \|\partial_i u_0\|_1\}$ and using the regularity of the probability measure $|\partial_i u_0|(x)dx/\|\partial_i u_0\|_1$, we obtain that

Lemma 3.4 *For any $1 \leq i \leq d$, $\forall \epsilon > 0$, there exists a Lipchitz continuous function h_i^ϵ bounded by $\|\partial_i u_0\|_1$ such that*

$$\int_{\mathbb{R}^d} 1_{\{h_i^\epsilon(x) \neq h_i(x)\}} \frac{|\partial_i u_0|(x)dx}{\|\partial_i u_0\|_1} \leq 1/M_\epsilon^2. \tag{3.4}$$

We are now ready to prove the Theorem.

Proof : Let π^n denote the distribution of μ^n and $i \leq d$. Since the processes $Z_i^{1,n}, \ldots, Z_i^{n,n}$ are exchangeable, according to [13], the tightness of the distribution of the variables $(\mu_i^n)_n$ is equivalent to the tightness of the distributions of the processes $(Z_i^{1,n})_n$. Because of the boundedness of $\bar{\bar{A}}'$, both sequences are tight. By Prokhorov theorem, we deduce that $(\pi^n)_n$ is tight. Let π^∞ denote the limit of a converging subsequence that we still denote by n for notational simplicity. We are going to check that $\pi^\infty = \delta_P$.

To do so, we introduce $Q = (Q^1, \ldots, Q^d)$ the canonical variable on $(P(C[0,+\infty), \mathbb{R}^d))^d$ and $(X, X^1, \ldots, X^n)$ the canonical process on $(C([0,+\infty), \mathbb{R}^d))^{d+1}$. We define $F_i^\epsilon(Q)$ to be equal to

$$< Q^i \otimes Q^1 \otimes \ldots \otimes Q^d, \Bigg(\phi(X_t) - \phi(X_s) - \int_s^t \nu \Delta \phi(X_r) dr \\ - \int_s^t \bar{\bar{A}}' \Bigg(\sum_{j=1}^d K_j^\epsilon (X_r - X_r^j) h_j^\epsilon (X_0^j) \Bigg) . \nabla \phi(X_r) dr \Bigg) g(X_{s_1}, \ldots, X_{s_p}) > .$$

where $0 \leq s \leq t$, $\;s_k \leq s$ for any $k \leq p$, $\phi \in C_b^2(\mathbb{R}^d)$ and $g \in C_b(\mathbb{R}^{p \times d})$. The function G_i^ϵ (resp. G_i) is defined like F_i^ϵ but with h_j replacing h_j^ϵ for $j \leq d$ (resp K_j and h_j replacing K_j^ϵ and h_j^ϵ). As F_i^ϵ is continuous and bounded,

$$\mathbb{E}^{\pi^\infty} |F_i^\epsilon(Q)| = \lim_{n \to +\infty} \mathbb{E}^{\pi^n} |F_i^\epsilon(Q)| \leq \limsup_{n \to +\infty} \mathbb{E}^{\pi^n} (|F_i^\epsilon - G_i^\epsilon|(Q)) + \limsup_{n \to +\infty} \mathbb{E}^{\pi^n} |G_i^\epsilon(Q)| \tag{3.5}$$

Using the Lipschitz continuity of $\bar{\bar{A}}'$ and the boundedness of $\nabla \phi$ and g, we get

$$|F_i^\epsilon - G_i^\epsilon|(\mu^n) \leq \frac{C}{n^2} \sum_{k,l=1}^n \sum_{j=1}^d \int_s^t |K_j^\epsilon (Z_i^{k,n}(r) - Z_j^{l,n}(r)).(h_j^\epsilon - h_j)(Z_j^l(0))| dr.$$

As $Z_j^l(0)$ has the density $|\partial_j u_0(x)|/\|\partial_j u_0\|_1$, using (3.4), we deduce

$$\mathbb{E}^{\pi^n} (|F_i^\epsilon - G_i^\epsilon|(Q)) \leq C M_\epsilon (t-s) \sum_{j=1}^d \mathbb{E}(|h_j^\epsilon - h_j|(Z_j^1(0))) \leq \frac{C}{M_\epsilon}. \tag{3.6}$$

To upper-bound the second term of the r.h.s. of (3.5), we compute $\phi(Z_i^{k,n}(t))$ by Itô's formula :

$$\phi(Z_i^{k,n}(t)) - \phi(Z_i^{k,n}(s)) - \int_s^t \bar{\bar{A}}' \Bigg(\sum_{j=1}^d \frac{1}{n-1} \sum_{l \neq k} K_j^{\epsilon_n} (Z_i^{k,n}(r) - Z_j^{l,n}(r)) h_j(Z_j^l(0)) \Bigg) . \nabla \phi(Z_i^{k,n}(r)) dr \\ - \int_s^t \nu \Delta \phi(Z_i^{k,n}(r)) dr = \sqrt{2\nu} \int_s^t \nabla \phi(Z_i^{k,n}(r)).dB_r^k.$$

Hence

$$|G_i^\epsilon(\mu^n)| \leq \left| \frac{\sqrt{2\nu}}{n} \sum_{k=1}^n \left(g(Z_i^{k,n}(s_1), \ldots, Z_i^{k,n}(s_p)) \int_s^t \nabla\phi(Z_i^{k,n}(r)).dB_r^k \right) \right|$$

$$+ \frac{C}{n} \sum_{k=1}^n \int_s^t \sum_{j=1}^d \frac{1}{n-1} \sum_{l\neq k} |(K_j^{\epsilon_n} - K_j^\epsilon)(Z_i^{k,n}(r) - Z_j^{l,n}(r)).h_j(Z_j^l(0))| dr$$

$$+ \frac{C}{n} \sum_{k=1}^n \int_s^t \sum_{j=1}^d \left| \sum_{l=1}^n \left(\frac{1_{\{l\neq k\}}}{n-1} - \frac{1}{n} \right) K_j^\epsilon(Z_i^{k,n}(r) - Z_j^{l,n}(r)).h_j(Z_j^l(0)) \right| dr$$

As $|K_j^{\epsilon_n} - K_j^\epsilon| \leq |K| 1_{B(0,\epsilon\vee\epsilon_n)}$, taking expectations and using (3.2), we get for $1 < q < d/(d-1)$

$$\mathbb{E}^{\pi^n} |G_i^\epsilon(Q)| \leq C\left(\frac{1}{\sqrt{n}} + \| |K|(.)\|_{L^q(B(0,\epsilon\vee\epsilon_n))} + \frac{M_\epsilon}{n} \right).$$

We combine this estimation with (3.5) and (3.6) to conclude that

$$\forall i \leq d, \ \lim_{\epsilon\to 0} \mathbb{E}^{\pi^\infty} |F_i^\epsilon(Q)| = 0. \tag{3.7}$$

Since $|K| \wedge M_\epsilon$ is a continuous and bounded function we get that $\forall 1 \leq i,j \leq d$, $\forall r \geq 0$,

$$\begin{aligned}
&\mathbb{E}^{\pi^\infty}(< Q^i \otimes Q^1 \otimes \ldots \otimes Q^d, |K|(X_r - X_r^j) >) \\
&\quad = \lim_{\epsilon\to 0} \lim_{n\to+\infty} \mathbb{E}\left(\frac{1}{n^2} \sum_{k,l=1}^n (|K| \wedge M_\epsilon)(Z_i^{k,n}(r) - Z_j^{l,n}(r)) \right) \\
&\quad \leq \limsup_{\epsilon\to 0} \limsup_{n\to+\infty} \left(\frac{M_\epsilon}{n} + \frac{n-1}{n} (\| |K|(.)\|_{L^\infty(B(0,1)^c)} + C \| |K|(.)\|_{L^q(B(0,1))}) \right) < +\infty,
\end{aligned}$$

by using (3.2) for $1 < q < d/(d-1)$. We deduce that π^∞ a.s., Q^i a.s., dr a.e. $K * \tilde{Q}_r(X_r)$ makes sense i.e. π^∞ a.s., $G_i(Q)$ makes sense. Using the previous upper-bound and remarking that π^∞ a.s. $Q_0^i = \frac{|\partial_i u_0|(x)dx}{\|\partial_i u_0\|_1}$, we check that $\lim_{\epsilon\to 0} \mathbb{E}^{\pi^\infty}(|G_i - F_i^\epsilon|(Q)) = 0$.
With (3.7), we obtain that $\forall i \leq d$, $\mathbb{E}^{\pi^\infty} |G_i(Q)| = 0$. Paul Levy's characterization implies that π^∞ a.s. $\forall i \leq d$, $X_t - X_0 - \int_0^t \bar{\bar{A}}'(K * \tilde{Q}_s(X_s))ds$ is a Q^i Brownian which implies that $Q^i \in \hat{P}(C([0,+\infty), \mathbb{R}^d))$ by arguments given in the proof of theorem 2.4. Hence π^∞ a.s., Q solves problem (MP) starting at u_0 which puts an end to the proof. ∎

As a consequence, it is possible to approximate the solution $u(t,x)$ of (0.1) thanks to the empirical measure of the particle system :

Corollary 3.5 *For any $t_0 > 0$,*

$$\lim_{n\to+\infty} \sup_{(t,x)\in[0,t_0]\times\mathbb{R}^d} \mathbb{E}|u(t,x) - c - (K^{\epsilon_n} * \tilde{\mu}^n(t))(x)| = 0.$$

Proof : Let $\epsilon > 0$ and $(X^1, \ldots, X^d)$ be the canonical process on $(C([0,+\infty), \mathbb{R}^d))^d$. Since

$u(t,x) = c + K * \tilde{P}_t(x)$,

$$\mathbb{E}|u(t,x) - c - (K^{\epsilon_n} * \tilde{\mu}^n(t))(x)| \leq$$

$$|< P^1 \otimes \ldots \otimes P^d, \sum_{j=1}^{d} (K_j(x - X_t^j) h_j(X_0^j) - K_j^\epsilon(x - X_t^j) h_j^\epsilon(X_0^j)) > |$$

$$+ \mathbb{E}| < P^1 \otimes \ldots \otimes P^d - \mu_1^n \otimes \ldots \otimes \mu_d^n, \sum_{j=1}^{d} K_j^\epsilon(x - X_t^j) h_j^\epsilon(X_0^j) > |$$

$$+ \mathbb{E}| < \mu_1^n \otimes \ldots \otimes \mu_d^n, \sum_{j=1}^{d} (K_j^\epsilon(x - X_t^j) h_j^\epsilon(X_0^j) - K_j^{\epsilon_n}(x - X_t^j) h_j(X_0^j)) > |$$

According to Theorem 3.2, for fixed ϵ the second term of the right-hand-side converges to 0 uniformly on $[0,t_0] \times \mathbb{R}^d$ as $n \to +\infty$. Replacing (3.2) by (3.3) to adapt the computations made in the proof of this Theorem, we obtain that the third term is smaller than $C(\||K|(.)\|_{L^q(B(0,\epsilon \vee \epsilon_n))} + 1/M_\epsilon)$ where $1 < q < d/(d-1)$ and the constant C does not depend on ϵ, n, $(t,x) \in [0,t_0] \times \mathbb{R}^d$. Last, since $\forall i \leq d$, $P_0^i = |\partial_i u_0|(x)dx/\|\partial_i u_0\|_1$ and P_t^i has a density bounded uniformly for $t \in [0,t_0]$, we check that the first term of the right-hand-side converges to 0 uniformly on $[0,t_0] \times \mathbb{R}^d$ as $\epsilon \to 0$. ∎

Remark 3.6 *Because of the boundedness of $\bar{\bar{A}}'$ and the polarity of the Brownian path in space dimension $d \geq 2$, by Girsanov theorem, we obtain weak existence for the stochastic differential equation without cutoff :*

$$Z_i^{k,n}(t) = Z_i^k(0) + \sqrt{2\nu}\beta_i^k(t) + \int_0^t \bar{\bar{A}}' \Big(\sum_{j=1}^{d} \frac{1}{n-1} \sum_{l \neq k} K_j(Z_i^{k,n}(s) - Z_j^{l,n}(s)) h_j(Z_j^l(0)) \Big) ds$$

where $(\beta_i^k)_{k \leq n, i \leq d}$ are independent $\mathbb{R}^d$-valued Brownian motions independent of the initial variables $(Z_i^k(0), \ldots, Z_d^k(0))_{k \leq n}$ which are IID according to a probability measure with $i-th$ $(i \leq d)$ marginal $|\partial_i u_0(x)|dx/\|\partial_i u_0\|_1$. Note that unlike in (3.1), we have to assume independence for the Brownian motions governing the evolution of the different coordinates of the $k-th$ particle. The previous propagation of chaos result can be adapted for this particle system.

3.3 Trajectorial estimates

To obtain a trajectorial result, we define a sequence of independent limit processes indexed by $k \geq 1$: $X^k = (X_1^k, \ldots, X_d^k)$ is the solution of the nonlinear stochastic differential equation

$$\begin{cases} X_i^k(t) = Z_i^k(0) + \sqrt{2\nu}B_t^k + \int_0^t \bar{\bar{A}}'(K * \tilde{P}_s(X_i^k(s))ds, \ i \leq d \\ P = (P^1, \ldots, P^d) \in (\hat{P}(C([0,+\infty), \mathbb{R}^d)))^d \ \text{ is such that } \forall i \leq d, \ P^i \text{ is the law of } X_i^k. \end{cases} \tag{3.8}$$

By the existence part of the proof of Theorem 2.4, existence holds for this equation. Moreover, any solution is such that P solves problem (MP) starting at u_0. Therefore $\bar{\bar{A}}'(K * \tilde{P}_s(x)) = \bar{A}'(u(s,x))$ where $u(s,x)$, the solution of (0.1) is Lipschitz continuous in x uniformly for $s \in [0,t]$ $(\forall t > 0)$ according to (0.6) and Proposition 1.3. Hence trajectorial uniqueness holds. The main result of this section is the following one :

Theorem 3.7 *Let $t_0 > 0$.*

$$\mathbb{E}\Big(\sum_{i=1}^{d}\sup_{t\le t_0}|Z_i^{k,n}(t)-X_i^k(t)|\Big) \le C(t_0,u_0,\bar{A})\epsilon_n + \frac{dM_{\epsilon_n}}{2L_{\epsilon_n}\sqrt{n-1}}\exp\Big(2d\||\bar{A}''|(.)\|_\infty \sup_{j\le d}\|\partial_j u_0\|_1 L_{\epsilon_n} t_0\Big).$$

where $C(t_0,u_0,\bar{A})$ is a constant depending on t_0, u_0 and $\bar{A}$ but not on n.

Remark 3.8 *If the sequence ϵ_n converges to 0 in such a way that*

$$\lim_{n\to+\infty}\frac{M_{\epsilon_n}}{L_{\epsilon_n}\sqrt{n-1}}\exp\Big(2d\||\bar{A}''|(.)\|_\infty \sup_{j\le d}\|\partial_j u_0\|_1 L_{\epsilon_n} t_0\Big) = 0,$$

then $\lim_{n\to+\infty}\mathbb{E}\Big(\sum_{i=1}^{d}\sup_{t\le t_0}|Z_i^{k,n}(t)-X_i^k(t)|\Big) = 0$ i.e. trajectorial propagation of chaos holds.

To prove this result we need to introduce the processes $Y^{k,n}$ solution of the nonlinear stochastic differential equation defined like (3.8) with K^{ϵ_n} replacing K :

$$\begin{cases} Y_i^{k,n}(t) = Z_i^k(0) + \sqrt{2\nu}B_t^k + \int_0^t \bar{A}'(K^{\epsilon_n} * \tilde{P}_s^{\epsilon_n}(Y_i^{k,n}(s))ds, \ i\le d \\ P^{\epsilon_n} = (P^{\epsilon_n,1},\dots,P^{\epsilon_n,d}) \in (P(C([0,+\infty),\mathbb{R}^d)))^d \text{ is such that } \forall i\le d,\ P^{\epsilon_n,i} \text{ is the law of } Y_i^{k,n}. \end{cases} \tag{3.9}$$

As the kernel K^{ϵ_n} is Lipschitz continuous and bounded, without the signed weights h_i that appear in the definition of the measures $\tilde{P}_s^{\epsilon_n,i}$, this stochastic equation would enter in the classical McKean-Vlasov framework. Existence and uniqueness can be proved by an adaptation of the arguments of Sznitman [13] Theorem 1.1 p.172 (see [7]). The first step in the proof of Theorem 3.7 consists in the following estimation :

Proposition 3.9 *Let $t_0 > 0$.*

$$\mathbb{E}\Big(\sum_{i=1}^{d}\sup_{t\le t_0}|Y_i^{k,n}(t)-Z_i^{k,n}(t)|\Big) \le \frac{dM_{\epsilon_n}}{2L_{\epsilon_n}\sqrt{n-1}}\exp\Big(2d\||\bar{A}''|(.)\|_\infty \sup_{j\le d}\|\partial_j u_0\|_1 L_{\epsilon_n} t_0\Big).$$

Proof :

$$\begin{aligned}
&\sup_{s\le t}|Y_i^{k,n}(s)-Z_i^{k,n}(s)| \\
&\le \||\bar{A}''|(.)\|_\infty \int_0^t \sum_{j=1}^d \Big| K_j^{\epsilon_n} * \tilde{P}_s^{\epsilon_n,j}(Y_i^{k,n}(s)) - \frac{1}{n-1}\sum_{l\ne k} K_j^{\epsilon_n}(Z_i^{k,n}(s)-Z_j^{l,n}(s))h_j(Z_j^l(0))\Big| ds \\
&\le \||\bar{A}''|(.)\|_\infty \int_0^t \Bigg(\sum_{j=1}^d \Big|\frac{1}{n-1}\sum_{l\ne k}(K_j^{\epsilon_n} * \tilde{P}_s^{\epsilon_n,j}(Y_i^{k,n}(s)) - K_j^{\epsilon_n}(Y_i^{k,n}(s)-Y_j^{l,n}(s))h_j(Z_j^l(0)))\Big| \\
&\qquad + \frac{L_{\epsilon_n}}{n-1}\sum_{j=1}^d(\|\partial_j u_0\|_1 \sum_{l\ne k}(|Y_i^{k,n}(s)-Z_i^{k,n}(s)| + |Y_j^{l,n}(s)-Z_j^{l,n}(s)|))\Bigg) ds
\end{aligned} \tag{3.10}$$

As the processes $(Y^{l,n})_{l\geq 1}$ are independent and the common law of the processes $(Y_j^{l,n})_{l\geq 1}$ is $P^{\epsilon_n,j}$,

$$\mathbb{E}\left|\frac{1}{n-1}\sum_{l\neq k}(K_j^{\epsilon_n}*\tilde{P}_s^{\epsilon_n,j}(Y_i^{k,n}(s))-K_j^{\epsilon_n}(Y_i^{k,n}(s)-Y_j^{l,n}(s))h_j(Z_j^l(0)))\right|$$

$$\leq\Bigg(\frac{1}{(n-1)^2}\sum_{l,m\neq k}\mathbb{E}\Big((K_j^{\epsilon_n}*\tilde{P}_s^{\epsilon_n,j}(Y_i^{k,n}(s))-K_j^{\epsilon_n}(Y_i^{k,n}(s)-Y_j^{l,n}(s))h_j(Z_j^l(0)))$$

$$(K_j^{\epsilon_n}*\tilde{P}_s^{\epsilon_n,j}(Y_i^{k,n}(s))-K_j^{\epsilon_n}(Y_i^{k,n}(s)-Y_j^{m,n}(s))h_j(Z_j^m(0)))\Big)\Bigg)^{\frac{1}{2}}$$

$$\leq\frac{M_{\epsilon_n}\|\partial_j u_0\|_1}{\sqrt{n-1}}\qquad\text{since the above expectation equals 0 for } m\neq l.$$

Summing inequality (3.10) over $i\leq d$, taking expectations and using exchangeability of the processes $(Y^{l,n},Z^{l,n})_{1\leq l\leq n}$, we get

$$\mathbb{E}\Big(\sum_{i=1}^d\sup_{s\leq t}|Y_i^{k,n}(s)-Z_i^{k,n}(s)|\Big)$$

$$\leq\||\bar{A}''|(.)\|_\infty\Big(\frac{d^2M_{\epsilon_n}\sup_j\|\partial_j u_0\|_1 t}{\sqrt{n-1}}+2dL_{\epsilon_n}\sup_j\|\partial_j u_0\|_1\int_0^t\mathbb{E}\Big(\sum_{i=1}^d|Y_n^{i,k}(s)-Z_n^{i,k}(s)|\Big)ds\Big)$$

We conclude by Gronwall's lemma. ∎

The second step in the proof of Theorem 3.9 consists in showing that the solution $Y^{k,n}$ of the nonlinear stochastic differential equation with kernel K^{ϵ_n} converges to the solution X^k of the nonlinear stochastic differential equation with kernel K as $\epsilon_n\to 0$.

Proposition 3.10 *Let $t_0>0$.*

$$\mathbb{E}\Big(\sum_{i=1}^d\sup_{t\leq t_0}|Y_i^{k,n}(t)-X_i^k(t)|\Big)\leq C(t_0,u_0,\bar{A})\epsilon_n$$

where $C(t_0,u_0,\bar{A})$ is a constant depending on t_0, u_0 and $\bar{A}$ but not on n.

Combining both the last Propositions, we obtain that Theorem 3.7 holds.

3.3.1 Proof of Proposition 3.10

As the result does not depend on k, to simplify notations, we replace B^k, X^k and $Y^{k,n}$ by B, X and Y^n. Since the drift coefficients of the stochastic differential equation satisfied by X_i and Y_i are respectively $\bar{\bar{A}}'(K*\tilde{P}_s(x))$ and $\bar{\bar{A}}'(K^{\epsilon_n}*\tilde{P}_s^{\epsilon_n}(x))$, we need to compare $K*\tilde{P}_s(x)$ and $K^{\epsilon_n}*\tilde{P}_s^{\epsilon_n}(x)$. By the existence part of the proof of Theorem 2.4, the densities $\tilde{p}_i(s,.)$ of the measures $\tilde{P}_s^i$ satisfy

$$\forall t\geq 0,\ dx\text{ a.e., }\tilde{p}_i(t,x)=G_t^\nu*\partial_i u_0(x)-\int_0^t\nabla G_{t-s}^\nu*(\bar{\bar{A}}'(K*\tilde{P}_s(.))\tilde{p}_i(s,.))(x)ds\qquad(3.11)$$

Similarly, for any $n \geq 0$, $\tilde{P}_s^{\epsilon_n,i}$ admits a density $\tilde{p}_i^n(s,.)$ with respect to Lebesgue measure,

$$\forall t \geq 0,\ dx \text{ a.e., } \tilde{p}_i^n(t,x) = G_t^\nu * \partial_i u_0(x) - \int_0^t \nabla G_{t-s}^\nu * (\bar{\bar{A}}'(K^{\epsilon_n} * \tilde{P}_s^{\epsilon_n}(.))\tilde{p}_i^n(s,.))(x)ds. \quad (3.12)$$

By Lemmas 2.5 and 2.1,

$$\forall i \leq d,\ \forall t \geq 0,\ |||\tilde{p}_i(t,.)||| \vee \sup_n |||\tilde{p}_i^n(t,.)||| \leq 2^{1+t/T}|||\partial_i u_0|||. \quad (3.13)$$

Comparing the integral equations (3.11) and (3.12) like in Méléard [10] (Theorem 3.4, Lemma 3.5 and Corollary 3.6), we are going to show :

Lemma 3.11 *Let $t_0 > 0$.*

$$\sup_{[0,t_0]\times\mathbb{R}^d} |K * \tilde{P}_t(x) - K^{\epsilon_n} * \tilde{P}_t^{\epsilon_n}(x)| \leq C(t_0, u_0, \bar{A})\epsilon_n$$

where $C(t_0, u_0, \bar{A})$ is a constant depending on t_0, u_0 and $\bar{A}$ but not on n.

Proof : Let $t_0 > 0$, $t \in [0, t_0]$ and $x \in \mathbb{R}^d$. By (3.13) and Lemmas 0.1 and 3.1,

$$\begin{aligned}
|K * \tilde{P}_t(x) - K^{\epsilon_n} * \tilde{P}_t^{\epsilon_n}(x)| &\leq |K * (\tilde{P}_t - \tilde{P}_t^{\epsilon_n})(x)| + \sum_{i=1}^d |(K_i - K_i^{\epsilon_n}) * \tilde{p}_i^n(t,.)(x)| \\
&\leq C \sum_{i=1}^d (|||\tilde{p}_i(t) - \tilde{p}_i^n(t)||| + \|K_i - K_i^{\epsilon_n}\|_1 \|\tilde{p}_i^n(t)\|_\infty) \\
&\leq C \sum_{i=1}^d (|||\tilde{p}_i(t) - \tilde{p}_i^n(t)||| + 2^{1+t_0/T}|||\partial_i u_0|||\epsilon_n) \quad (3.14)
\end{aligned}$$

By (3.11) and (3.12), we have

$$\begin{aligned}
&|||\tilde{p}_i(t) - \tilde{p}_i^n(t)||| \leq \frac{1}{\sqrt{\pi\nu}} \int_0^t \sum_{j=1}^d |||\bar{\bar{A}}_j'(K * \tilde{P}_s(.))\tilde{p}_i(s) - \bar{\bar{A}}_j'(K^{\epsilon_n} * \tilde{P}_s^{\epsilon_n}(.))\tilde{p}_i^n(s)||| \frac{ds}{\sqrt{t-s}} \\
&\leq \frac{1}{\sqrt{\pi\nu}} \int_0^t \sum_{j=1}^d \left(\|\bar{A}_j''\|_\infty \|K * \tilde{P}_s - K^{\epsilon_n} * \tilde{P}_s^{\epsilon_n}\|_\infty |||\tilde{p}_i(s)||| + \|\bar{A}_j'\|_\infty |||\tilde{p}_i(s) - \tilde{p}_i^n(s)||| \right) \frac{ds}{\sqrt{t-s}}
\end{aligned}$$

Using (3.13) and (3.14), we deduce that there exists a constant C depending on t_0, u_0 and $\bar{A}$ but not on n such that

$$\forall i \leq d,\ \forall t \leq t_0,\ |||\tilde{p}_i(t) - \tilde{p}_i^n(t)||| \leq C \int_0^t \left(\epsilon_n + \sum_{j=1}^d |||\tilde{p}_j(s) - \tilde{p}_j^n(s)||| \right) \frac{ds}{\sqrt{t-s}}.$$

Summing this inequality for $i \leq d$ and iterating the result, we obtain by Gronwall's lemma that

$$\sup_{t\in[0,t_0]} \sum_{i=1}^d |||\tilde{p}_i(t) - \tilde{p}_i^n(t)||| \leq C\epsilon_n.$$

Combining this inequality with (3.14), we conclude the proof. ∎

Let $t \leq t_0$ and $i \leq d$.

$$\sup_{s\leq t}|Y_i^n(s)-X_i(s)| \leq \||\bar{A}''|(.)\|_\infty \int_0^t \Big(|K^{\epsilon_n} * \tilde{P}_s^{\epsilon_n}(Y_i^n(s)) - K * \tilde{P}_s(Y_i^n(s))| + |K * \tilde{P}_s(Y_i^n(s)) - K * \tilde{P}_s(X_i(s))| \Big) ds$$

By Theorem 2.4, $K * \tilde{P}_s(x) = u(s,x) - c$ where $u(s,x)$, the solution of (0.1) given by Proposition 1.1 satisfies $\sup_{[0,t_0]} \|u(s,.)\|_{1,1+\infty} < +\infty$ according to Proposition 1.3. Using (0.6), we deduce that $x \to K * \tilde{P}_s(x)$ is Lipschitz continuous uniformly for $s \leq t_0$. By this Lipschitz property and Lemma 3.11, we get

$$\forall t \leq t_0,\ \mathbb{E}\Big(\sup_{s\leq t}|Y_i^n(s)-X_i(s)|\Big) \leq C\left(\epsilon_n + \int_0^t \mathbb{E}\Big(\sup_{r\leq s}|Y_i^n(r)-X_i(r)|\Big)ds\right).$$

where the constant C depends on t_0, u_0 and $\bar{A}$ but not on n. Gronwall's lemma implies the desired inequality.

3.3.2 Approximation of the solution of 0.1

Thanks to the previous trajectorial estimates, it is possible to bound the rate of the convergence stated in Corollary 3.5.

Proposition 3.12 *Let $t_0 > 0$.*

$$\sup_{[0,t_0]\times\mathbb{R}^d} \mathbb{E}|u(t,x) - c - (K^{\epsilon_n} * \tilde{\mu}^n(t))(x)| \leq C(t_0,u_0,\bar{A})\epsilon_n + \frac{dM_{\epsilon_n}\sup_{j\leq d}\|\partial_j u_0\|_1}{2\sqrt{n-1}}\left(2 + \exp\left(2d\||\bar{A}''|(.)\|_\infty \sup_{j\leq d}\|\partial_j u_0\|_1 L_{\epsilon_n} t_0\right)\right)$$

Proof : Let $t \leq t_0$ and $x \in \mathbb{R}^d$. As, by Theorem 2.4, $u(t,x) = c + K * \tilde{P}_t(x)$,

$$\begin{aligned}
\mathbb{E}|u(t,x) - c - (K^{\epsilon_n} * \tilde{\mu}^n(t))(x)| &\leq \sup_{[0,t_0]\times\mathbb{R}^d}|K * \tilde{P}_t(x) - K^{\epsilon_n} * \tilde{P}_t^{\epsilon_n}(x)| \\
&+ \sum_{j=1}^d \mathbb{E}\left| K_j^{\epsilon_n} * \tilde{P}_t^{\epsilon_n,j}(x) - \frac{1}{n}\sum_{k=1}^n K_j^{\epsilon_n}(x - Y_j^{k,n}(t))h_j(Z_j^k(0))\right| \\
&+ \sum_{j=1}^d \mathbb{E}\left|\frac{1}{n}\sum_{k=1}^n K_j^{\epsilon_n}(x - Y_j^{k,n}(t))h_j(Z_j^k(0)) - \frac{1}{n}\sum_{k=1}^n K_j^{\epsilon_n}(x - Z_j^{k,n}(t))h_j(Z_j^k(0))\right| \\
&\leq \sup_{[0,t_0]\times\mathbb{R}^d}|K * \tilde{P}_t(x) - K^{\epsilon_n} * \tilde{P}_t^{\epsilon_n}(x)| + \frac{dM_{\epsilon_n}\sup_j\|\partial_j u_0\|_1}{\sqrt{n}} \\
&+ L_{\epsilon_n}\sup_j\|\partial_j u_0\|_1 \sup_{[0,t_0]}\mathbb{E}\Big(\sum_{i=1}^d |Y_i^{1,n}(t) - Z_i^{1,n}(t)|\Big)
\end{aligned}$$

Combining Lemma 3.11 and Proposition 3.9, we easily conclude the proof. ∎

Appendix : proof of Lemma 0.1

1. Let $B(0,1)$ denote the unit ball in $\mathbb{R}^d$. We easily check that for $1 \leq i \leq d$, K_i belongs to $L^p(B(0,1))$ for $1 \leq p < d/(d-1)$ and to $L^q(B(0,1)^c)$ for $d/(d-1) < q \leq +\infty$. Hence for $v = (v_1, \ldots, v_d) \in (L^1 \cap L^\infty(\mathbb{R}^d))^d$,

$$|K * v(x)| \leq \sum_{i=1}^{d} (\|K_i\|_{L^1(B(0,1))} \|v_i\|_\infty + \|K_i\|_{L^\infty(B(0,1)^c)} \|v_i\|_1)$$
$$\leq (\|K_1\|_{L^1(B(0,1))} + \|K_1\|_{L^\infty(B(0,1)^c)}) \sum_{i=1}^{d} |||v_i|||.$$

To prove the continuity of $K * v$, we set $i \leq d$, $\alpha > 0$ and suppose that $|x - y| \leq \frac{\alpha}{2}$.

$$|K_i * v_i(x) - K_i * v_i(y)| \leq \int_{B(x,\alpha)} (|K_i(x-z)| + |K_i(y-z)|)|v_i(z)|dz$$
$$+ \int_{B(x,\alpha)^c} |K_i(x-z) - K_i(y-z)||v_i(z)|dz$$

The first term of the right-hand-side is smaller that $2\|v_i\|_\infty \|K_i\|_{L^1(B(0,\frac{3\alpha}{2}))}$ and converges to 0 as $\alpha \to 0$. For fixed α, by Lebesgue theorem, the second term converges to 0 as $y \to x$, since the integrand is smaller than $2\|K_i\|_{L^\infty(B(0,\frac{\alpha}{2})^c)}|v_i(z)|$. We deduce that $K_i * v_i$ is continuous. Hence $K * v$ is continuous.

Let ϕ be a C^∞ function with compact support on $\mathbb{R}^d$. By Fubini's theorem and the integration by parts formula,

$$\int_{\mathbb{R}^d} \Delta\phi(x) K_i * v_i(x) dx = \int_{\mathbb{R}^d} v_i(y) \left(\int_{\mathbb{R}^d} K_i(x-y) \Delta\phi(x) dx \right) dy$$
$$= -\int_{\mathbb{R}^d} v_i(y) \left(\int_{\mathbb{R}^d} g(|x-y|) \Delta \partial_i \phi(x) dx \right) dy$$

As $g(|x|)$ is the fundamental solution of the Laplacian in $\mathbb{R}^d$, we obtain

$$\int_{\mathbb{R}^d} \Delta\phi(x) K_i * v_i(x) dx = -\int_{\mathbb{R}^d} v_i(y) \partial_i \phi(y) dy$$

i.e. $\Delta(K_i * v_i) = \partial_i v_i$ in the distribution sense.

2. When $f \in W^{1,1+\infty}(\mathbb{R}^d)$, applying **1.** to $v = \nabla f$, we get that $K * \nabla f$ is bounded and satisfies $\Delta(K * \nabla f) = \Delta f$ in the distribution sense. Hence $f - K * \nabla f$ is a bounded harmonic function and therefore a constant.

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