RATE OF CONVERGENCE OF A PARTICLE METHOD FOR THE SOLUTION OF A 1D VISCOUS SCALAR CONSERVATION LAW IN A BOUNDED INTERVAL

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Abstract

In this paper, we give a probabilistic interpretation of a viscous scalar conservation law in a bounded interval thanks to a nonlinear martingale problem. The underlying nonlinear stochastic process is reflected at the boundary to take into account the Dirichlet conditions. After proving uniqueness for the martingale problem, we show existence thanks to a propagation of chaos result. Indeed we exhibit a system of N interacting particles, the empirical measure of which converges to the unique solution of the martingale problem as $N \to +\infty$. As a consequence, the solution of the viscous conservation law can be approximated thanks to a numerical algorithm based on the simulation of the particle system. When this system is discretized in time thanks to the Euler-Lépingle scheme [10], we show that the rate of convergence of the error is in $\mathcal{O}(\Delta t + 1/\sqrt{N})$ where Δt denotes the time step. Finally, we give numerical results which confirm this theoretical rate.

1 Introduction

We are interested in the following viscous scalar conservation law with non homogeneous Dirichlet boundary conditions on the interval $[0,1]$:

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} v(t,x) - \frac{\partial}{\partial x} A(v(t,x)), \forall (t,x) \in (0,+\infty)\times(0,1) \\ \forall x \in [0,1],\ v(0,x) = v_0(x), \\ \forall t > 0,\ v(t,0) = 0 \text{ and } v(t,1) = 1, \end{cases} \tag{1.1}$$

We suppose that $A : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function and that the initial data v_0 is the cumulative distribution function of a probability measure U_0 on $[0,1]$, which writes $\forall x \in [0,1],\ v_0(x) = U_0([0,x]) = H * U_0(x)$ where $H(y) = 1_{\{y \geq 0\}}$ denotes the Heaviside function.
After giving a probabilistic interpretation of the solution of this equation thanks to a non-linear martingale problem, we want to derive and study a particle approximation of this solution. Our main motivation is that the spatial domain in equation (1.1) is bounded. To our knowledge, the only paper about a probabilistic particle interpretation for the solution of a partial differential equations posed in a bounded spatial domain is [1], that is dedicated to the 2d inviscid Navier Stokes equation. In [1], the authors do not prove the convergence

of the proposed particle method. By considering the much simpler equation (1.1), we are able not only to prove the convergence but also to bound the associated rate.

When the viscous scalar conservation law is posed in the spatial domain $\mathbb{R}$ instead of $[0,1]$, one can show that its unique weak solution is equal to $H * P_t(x)$ where $(P_t)_{t\geq 0}$ denote the time-marginals of the probability measure P on $C([0,+\infty),\mathbb{R})$ characterized the following martingale problem nonlinear in the sense of McKean [4] [8] :

$$\begin{cases} P_0 = U_0 \\ \forall \varphi \in C_b^2(\mathbb{R}),\ \varphi(X_t) - \varphi(X_0) - \int_0^t \left[\frac{\sigma^2}{2}\varphi''(X_s) - A'(H * P_s(X_s))\varphi'(X_s)\right] ds \text{ is a } P\text{-martingale} \end{cases}$$

where X denotes the canonical process on $C([0,\infty),\mathbb{R})$.

Here, we follow a similar approach. To take into account the Dirichlet boundary conditions, we work with a diffusion process with reflection. That is why we introduce (X, K) the canonical process on the sample path space $\mathcal{C} = C([0,+\infty),[0,1]) \times C([0,+\infty),\mathbb{R})$ (endowed with the topology of uniform convergence on compact sets). For P in $\mathcal{P}(\mathcal{C})$ the set of probability measures on $\mathcal{C}$, $(\bar{P}_t)_{t\geq 0}$ is the set of time-marginals of the probability measure $\bar{P}$ on $C([0,+\infty),[0,1])$ defined by $\bar{P} = P \circ X^{-1}$. We associate the following nonlinear problem with (1.1)

Definition 1.1 *A probability measure $P \in \mathcal{P}(\mathcal{C})$ solves the martingale problem (MP) starting at $U_0 \otimes \delta_0 \in \mathcal{P}([0,1]\times\mathbb{R})$, if*

i) $P \circ (X_0, K_0)^{-1} = U_0 \otimes \delta_0$

ii) $\forall \varphi \in C_b^2(\mathbb{R}),\ \varphi(X_t - K_t) - \varphi(X_0 - K_0) - \int_0^t \frac{\sigma^2}{2}\varphi''(X_s - K_s) + A'(H * \bar{P}_s(X_s))\varphi'(X_s - K_s) ds$ *is a P martingale*

iii) *P a.s., $\forall t \geq 0,\ \int_0^t d|K|_s < +\infty,\ \ |K|_t = \int_0^t 1_{\{0,1\}}(X_s) d|K|_s$ and $K_t = \int_0^t (1 - 2X_s) d|K|_s$.*

In section 2, we prove that if P solves problem (MP), then $(t,x) \to H * \bar{P}_t(x)$ is the unique weak solution of (1.1). We deduce uniqueness for the martingale problem. Existence is obtained thanks to a propagation of chaos result for a system of weakly interacting diffusion processes.

In section 3, we discretize this system in time thanks to the version of the Euler scheme introduced by Lépingle [10]. This way, we derive a numerical method to approximate the solution of (1.1). We prove a theoretical rate of convergence in $\mathcal{O}(\Delta t + 1/\sqrt{N})$ where Δt and N denote respectively the time-step and the number of particles. This rate is the same as the one obtained by Bossy [3] when the spatial domain is $\mathbb{R}$. As an important step in the proof, we show that in case the diffusion coefficient is a constant, the weak error of the Euler Lépingle scheme is in $\mathcal{O}(\Delta t)$. To our knowledge, this is the first result concerning the weak error of this scheme.

The last section is devoted to numerical experiments which confirm the theoretical rate of convergence of our particle method. The treatment of the reflection by the Euler Lépingle scheme does not alter the convergence whereas we exhibit a sublinear numerical dependence on the time step Δt when the particle system is discretized thanks to the cruder Euler projection scheme.

To conclude the introduction, we should mention that using signed weights like in [8] and [3], we could extend our approach to deal with the following more general boundary conditions in (1.1) : $\forall t > 0,\ v(t,0) = a$ and $v(t,1) = b$, and $\forall x \in [0,1],\ v_0(x) = U_0([0,x])$ where U_0 is a bounded signed measure on $[0,1]$ satisfying the compatibility condition $U_0([0,1]) = b - a$. But we restrict ourselves to a simple case without weights to avoid further complication of the already technical developments.

2 Probabilistic interpretation of the viscous scalar conservation law equation

For $T > 0$ let $\mathcal{Q}_T = (0,T) \times (0,1)$ and $W_2^{0,1}(\mathcal{Q}_T)$, $W_2^{1,1}(\mathcal{Q}_T)$ denote the Hilbert spaces with respective scalar products (cf. [9])

$$\begin{aligned}(u,v)_{W_2^{0,1}(\mathcal{Q}_T)} &= \int_{\mathcal{Q}_T} (uv + \partial_x u \partial_x v) dx dt,\\ (u,v)_{W_2^{1,1}(\mathcal{Q}_T)} &= \int_{\mathcal{Q}_T} (uv + \partial_x u \partial_x v + \partial_t u \partial_t v) dx dt.\end{aligned}$$

We introduce the Banach space $V_2^{0,1}(\mathcal{Q}_T) = \{u \in W_2^{0,1}(\mathcal{Q}_T) \cap C((0,T), L^2(0,1))$ such that $\|u\|_{V_2^{0,1}(\mathcal{Q}_T)} = \sup_{0 \le t \le T} \|u(t,x)\|_{L^2(0,1)} + \|\partial_x u\|_{L^2(\mathcal{Q}_T)} < +\infty\}$. The corresponding subspaces consisting in elements which vanish on $[0,T] \times \{0,1\}$ are respectively denoted by $\overset{\circ}{W}{}_2^{0,1}(\mathcal{Q}_T)$, $\overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$, $\overset{\circ}{V}{}_2^{0,1}(\mathcal{Q}_T)$.

We first prove uniqueness of weak solutions of problem (1.1) defined in the following way :

Definition 2.1 *A weak solution of (1.1) is a function $v : [0,+\infty) \times [0,1] \to \mathbb{R}$ satisfying the boundary conditions and such that for any $T > 0$, $v \in V_2^{0,1}(\mathcal{Q}_T) \cap L^\infty(\mathcal{Q}_T)$ and for all ϕ in $\overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$ and all $t \in [0,T]$,*

$$\begin{aligned}\int_0^1 v(t,x)\phi(t,x)dx &= \int_0^1 v_0(x)\phi(0,x)dx + \int_0^t \int_0^1 \frac{\partial}{\partial s}\phi(s,x)v(s,x)dxds\\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial x}\phi(s,x)A(v(s,x))dxds \qquad (2.1)\\ &- \int_0^t \int_0^1 \frac{\sigma^2}{2}\frac{\partial}{\partial x}\phi(s,x)\frac{\partial}{\partial x}v(s,x)dxds.\end{aligned}$$

Then we check that when P solves the martingale problem (MP), $V(t,x) = H * \bar{P}_t(x)$ is a weak solution of (1.1). Uniqueness for the martingale problem is derived from uniqueness for this equation. The probabilistic interpretation is completed by a propagation of chaos result which ensures existence for problem (MP).

2.1 Uniqueness result for equation (1.1)

Lemma 2.2 *Equation (1.1) has no more than one weak solution in the sense of Definition 2.1.*

Proof : Let v^1 and v^2 be two weak solutions of (1.1) and $T > 0$. We set $w = v^1 - v^2$. Then w is in $\overset{\circ}{V}{}_2^{0,1}(\mathcal{Q}_T)$ and $w(0,x) = 0$ for all $x \in [0,1]$. Moreover, for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$,

$$\begin{aligned}\int_0^1 w(t,x)\phi(t,x)dx &= \int_0^t \int_0^1 \frac{\partial}{\partial s}\phi(s,x)w(s,x)dxds\\ &+ \int_0^t \int_0^1 \frac{\partial}{\partial x}\phi(s,x)\left\{A(v^1(s,x)) - A(v^2(s,x))\right\}dxds \qquad (2.2)\\ &- \int_0^t \int_0^1 \frac{\sigma^2}{2}\frac{\partial}{\partial x}\phi(s,x)\frac{\partial}{\partial x}w(s,x)dxds.\end{aligned}$$

Thus, w is a generalized solution in the sense of Ladyzenskaja, Solonnikov and Ural'ceva (cf. [9]) of a linear equation with uniformly bounded coefficients. We can apply results of chapter 3 of [9]. In particular the identity ([9],2.13) of section 2, used to establish the energy inequality holds and becomes in our case

$$\begin{aligned}&\frac{1}{2}\int_0^1 w^2(x,t)dx + \int_0^t \int_0^1 \frac{\sigma^2}{2}\left(\frac{\partial w}{\partial x}\right)^2(s,x)dxds\\ &= \int_0^t \int_0^1 \frac{\partial w}{\partial x}(s,x)\left\{A(v^1(s,x)) - A(v^2(s,x))\right\}dxds.\end{aligned} \qquad (2.3)$$

Formally, this identity is obtained by taking $\phi = w$ in (2.2) and integrating by part the first term of the right hand side. As w is not sufficiently smooth to do so, the proof of (2.3) relies on two steps. The first one consists in working with Steklov averagings in time of functions ϕ and showing that is it possible to integrate by parts the first term of the right hand side of (2.2). The second one consists in proving that it is possible to replace ϕ by w in the obtained identity.

Following techniques from chapter 3 of [9], we deduce from (2.3) that

$$\begin{aligned}\min(\frac{1}{2}, \frac{\sigma^2}{2}) &\left[\|w(t)\|^2_{L^2([0,1])} + \int_{Q_t} \left(\frac{\partial w}{\partial x}\right)^2 (s,x) dx ds \right] \\ &\le \int_{Q_t} \frac{\partial w}{\partial x}(s,x) \left\{ A(v^1(s,x)) - A(v^2(s,x)) \right\} dx ds.\end{aligned}$$

Now we observe that for $M = \|v^1\|_{L^\infty(Q_T)} \vee \|v^2\|_{L^\infty(Q_T)}$

$$\begin{aligned}&\int_{Q_t} \frac{\partial w}{\partial x}(s,x) \left\{ A(v^1(s,x)) - A(v^2(s,x)) \right\} dx ds \\ &\le \sup_{|x|\le M} |A'(x)| \int_{Q_t} |w|(s,x) \left| \frac{\partial w}{\partial x} \right| (s,x) dx ds \\ &\le \sup_{|x|\le M} |A'(x)| \sup_{s\le t} \|w(s)\|_{L^2[0,1]} \int_0^t \|\frac{\partial w}{\partial x}(s)\|_{L^2[0,1]} ds \\ &\le \sup_{|x|\le M} |A'(x)| \frac{\sqrt{t}}{2} \left[\sup_{s\le t} \|w(s)\|^2_{L^2(0,1)} + \|\frac{\partial w}{\partial x}\|^2_{L^2(Q_t)} \right].\end{aligned}$$

by using Cauchy-Schwarz inequality and the upper-bound $2ab \le a^2 + b^2$. Thus

$$\min(1, \sigma^2) \left[\sup_{s\le t} \|w(s)\|^2_{L^2(0,1)} + \int_{Q_t} \left(\frac{\partial w}{\partial x}\right)^2 (s,x) dx ds \right] \le C\sqrt{t} \|w\|^2_{V_2^{0,1}(Q_t)}$$

and hence,

$$\min(1, \sigma^2) \|w\|^2_{V_2^{0,1}(Q_t)} \le C\sqrt{t} \|w\|^2_{V_2^{0,1}(Q_t)}.$$

Choose t_1 such that

$$t_1 < \frac{\min(1, \sigma^4)}{C^2} \wedge T$$

then $\|w\|_{V_2^{0,1}(Q_{t_1})} = 0$. Now, for $t \ge t_1$, Equality (2.3) gives

$$\begin{aligned}&\frac{1}{2} \int_0^1 w^2(x,t) dx + \int_{t_1}^t \int_0^1 \frac{\sigma^2}{2} \left(\frac{\partial w}{\partial x}\right)^2 (s,x) dx ds \\ &= \int_{t_1}^t \int_0^1 \frac{\partial w}{\partial x}(s,x) \left\{ A(v^1(s,x)) - A(v^2(s,x)) \right\} dx ds\end{aligned}$$

and the previous computation shows that for any $t_2 \le T$ such that

$$t_2 - t_1 < \frac{\min(1, \sigma^4)}{C^2}$$

$\|w\|_{V_2^{0,1}(Q_{t_1,t_2})} = 0$. Finally, we can iterate this procedure to obtain that $\|w\|_{V_2^{0,1}(Q_T)} = 0$. Since T is arbitrary $v^1 = v^2$. ∎

2.2 Uniqueness for the martingale problem (MP) and link with equation (1.1)

Proposition 2.3 *If P solves the martingale problem (MP) starting at $U_0 \otimes \delta_0$, then $V(t,x) = H * \bar{P}_t(x)$ is a weak solution of (1.1). Moreover, uniqueness holds for the martingale problem (MP).*

Proof : Clearly the function $V(t,x) = H * \bar{P}_t(x)$ is bounded by 1. Let $T > 0$. We check that the function V belongs to $V_2^{0,1}(\mathcal{Q}_T)$ and satisfies the non homogeneous Dirichlet boundary conditions in (1.1) thanks to the following Lemma the proof of which is postponed.

Lemma 2.4 *If P solves the martingale problem (MP) then for any $t > 0$, $\bar{P}_t$ has a density $\bar{p}_t$ which belongs to $L^2([0,1])$ and it holds that*

$$\|\bar{p}_t\|_{L^2([0,1])} \leq C(1 + t^{-1/4}) \exp(Ct).$$

We still have to check that V satisfies the identity (2.1). Let ϕ be a C^∞ function on $[0,T] \times [0,1]$ with $\phi(t,0) = \phi(t,1) = 0$ for all $t \in [0,T]$. We set $\psi(t,x) = \int_0^x \phi(t,y)dy$. Then ψ is C^∞ with $\frac{\partial \psi}{\partial x}(t,0) = \frac{\partial \psi}{\partial x}(t,1) = 0$. According to Definition 1.1 ii), under the probability measure P, $\frac{1}{\sigma}(X_t - K_t - \int_0^t A'(V(s,X_s))ds)$ is a local martingale with quadratic variation t i.e. a Brownian motion. Thus, by Itô's formula

$$\mathbb{E}\psi(t,X_t) = \mathbb{E}\psi(0,X_0) + \mathbb{E}\int_0^t \left[\frac{\partial \psi}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 \psi}{\partial x^2}\right](s,X_s)ds + \mathbb{E}\int_0^t \frac{\partial \psi}{\partial x}(s,X_s)A'(V(s,X_s))ds$$

As $\bar{p}_s = \frac{\partial V}{\partial x}(s,.)$, we deduce that

$$\begin{aligned}
\int_0^1 \psi(t,x)\frac{\partial V}{\partial x}(t,x)dx &= \int_0^1 \psi(0,x)\frac{\partial V_0}{\partial x}(x)dx + \int_0^t \int_0^1 \frac{\partial \psi}{\partial s}(s,x)\frac{\partial V}{\partial x}(s,x)dxds \\
&\quad + \int_0^t \int_0^1 \frac{\sigma^2}{2}\frac{\partial \phi}{\partial x}(s,x)\frac{\partial V}{\partial x}(s,x)dxds \\
&\quad + \int_0^t \int_0^1 \phi(s,x)A'(V(s,x))\frac{\partial V}{\partial x}(s,x)dxds.
\end{aligned}$$

Applying Stieljes integration by parts formula in the spatial integrals in the first and last lines of the previous equality, we get identity (2.1) for ϕ a C^∞ function vanishing for $x = 0$ and $x = 1$. As V is in $V_2^{0,1}(\mathcal{Q}_T)$ we can extend the identity easily by density for any function ϕ in $\overset{\circ}{W}{}_2^{1,1}(\mathcal{Q}_T)$.
Hence $V(t,x) = H * \bar{P}_t(x)$ is a weak solution in the sense of Definition 2.1.

Uniqueness for the martingale problem (MP) is derived from the uniqueness result for the problem (1.1): if P and Q solve (MP), then for any $(t,x) \in [0,+\infty) \times \mathbb{R}$, $H * \bar{P}_t(x) = H * \bar{Q}_t(x)$. Hence P and Q solve a linear martingale problem with bounded drift term $A'(H * \bar{P}_t(x))$ and by Girsanov theorem, $P = Q$. ∎

Proof of Lemma 2.4 : We just have to adapt to the case of reflected diffusion processes the proof of Proposition 1.1 of Méléard and Roelly [12]. According to Definition 1.1 ii), under the probability measure P, $\frac{1}{\sigma}(X_t - K_t - \int_0^t A'(V(s,X_s))ds)$ a Brownian motion. As $\sup_{[0,1]} |A'(x)| < +\infty$, by Girsanov theorem, under the probability measure $Q \in \mathcal{P}(\mathcal{C})$ such that

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \frac{1}{Z_t} \quad \text{where} \quad Z_t = \exp\left(\int_0^t \frac{1}{\sigma^2}A'(H * \bar{P}_s(X_s))d(X_s - K_s) - \frac{1}{2\sigma^2}A'^2(H * \bar{P}_s(X_s))ds\right),$$

$\beta_t = \frac{1}{\sigma}(X_t - K_t)$ is a Brownian motion starting at $\frac{1}{\sigma}X_0$ and $(X_t)_{t \geq 0}$ is the doubly reflected process associated with $(\sigma\beta_t)_{t \geq 0}$.

For ψ bounded and measurable, since $\mathbb{E}^P(\psi(X_t)) = \mathbb{E}^Q(\psi(X_t)Z_t)$, by Cauchy-Schwarz inequality

$$\mathbb{E}^P(\psi(X_t)) \leq \left(\int_0^1 \psi^2(x)u_t(x)dx\right)^{\frac{1}{2}} \exp\left(\frac{t}{2\sigma^2}\sup_{[0,1]}|A'^2(x)|\right)$$

where $u_t(x) = \int_0^1 p_{\sigma^2 t}(z,x)U_0(dz)$ and

$$p_t(z,x) = \frac{1}{\sqrt{2\pi t}}\sum_{n\in\mathbb{Z}}\left(e^{-\frac{(x-z-2n)^2}{2t}} + e^{-\frac{(x+z+2n)^2}{2t}}\right)$$

denotes the transition density of the doubly reflected Brownian motion in $[0,1]$. For any $(z,x) \in \mathbb{R}^2$ we easily check that $p_t(z,x) \leq \frac{2}{\sqrt{2\pi t}} + 1$. Thus,

$$\mathbb{E}^P(\psi(X_t)) \leq C(1+t^{-1/4})\exp(Ct)\|\psi\|_{L^2([0,1])}$$

which gives the lemma. ∎

2.3 The propagation of chaos result

The system of weakly interacting diffusion processes with normal reflecting boundary conditions is given by the stochastic differential equation :

$$\begin{cases} X_t^{i,N} = X_0^{i,N} + \sigma W_t^i + \int_0^t A'(H * \bar{\mu}_s^N(X_s^{i,N}))ds + K_t^{i,N} \\ |K^{i,N}|_t = \int_0^t 1_{\{0,1\}}(X_s^{i,N})d|K^{i,N}|_s, \quad K_t^{i,N} = \int_0^t (1-2X_s^{i,N})d|K^{i,N}|_s, \quad i \leq N \end{cases} \tag{2.4}$$

where $\bar{\mu}_s^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}}$ and $(W^1,\ldots,W^N)$ is a N-dimensional Brownian motion independent of the initial variables $(X_0^{1,N},\ldots,X_0^{N,N})$ which are I.I.D. with law U_0.
As $\sup_{[0,1]}|A'(x)|$ is bounded, by Girsanov theorem, this equation admits a unique weak solution. Existence for problem (MP) is ensured by the following propagation of chaos result:

Theorem 2.5 *The particle systems $((X^{1,N},K^{1,N}),\ldots,(X^{N,N},K^{N,N}))$ are P-chaotic where P denotes the unique solution of the martingale problem (MP) starting at $U_0\otimes\delta_0$ i.e. for fixed $j \in \mathbb{N}^*$ the law of $((X^{1,N},K^{1,N}),\ldots,(X^{j,N},K^{j,N}))$ converges weakly to $P^{\otimes j}$ as $N \to +\infty$.*

Proof : Except in the treatment of the discontinuity of the Heaviside function, we follow the proof given by Sznitman [14] Theorem 1.4. When possible, we take advantage of the particular form of our diffusion domain (the interval $[0,1]$) to simplify the arguments.
By Proposition 2.3, uniqueness holds for problem (MP). As the particles $(X^{i,N},K^{i,N})_{i\leq N}$ are exchangeable, the propagation of chaos result is equivalent to the weak convergence of the law π^N of the empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X^{i,N},K^{i,N})}$ to a probability measure concentrated on solutions of problem (MP) when $N \to +\infty$ (see [13] and the references cited in it).
Again by exchangeability the tightness of the sequence $(\pi^N)_N$ is equivalent to the tightness of the laws of the couples $(X^{1,N},K^{1,N})$. As $\sup_{[0,1]}|A'(x)| < +\infty$, the laws of the processes $Y^{1,N} = X^{1,N} - K^{1,N}$ are tight. Since the map sending $y \in C([0,+\infty),\mathbb{R})$ to the solution $(x,k) \in \mathcal{C}$ of the Skorokhod problem is continuous (see [11]), we deduce that the laws of the couples $(X^{1,N},K^{1,N})$ are tight. Hence $(\pi^N)_N$ is a tight sequence.
Let π^∞ be the limit of a converging subsequence that we still index by N for simplicity and Q denote the canonical variable on $\mathcal{P}(\mathcal{C})$. We are going to prove that π^∞ a.s., Q solves problem (MP). Clearly, π^∞ a.s., $Q\circ(X_0,K_0)^{-1} = u_0(x)dx\otimes\delta_0$ i.e. π^∞ a.s., condition i) in Definition

1.1 is satisfied. To deal with condition ii), we set $\varphi \in C_b^2(\mathbb{R})$, $p \geq 1$, $t \geq s \geq s_1 \geq \ldots \geq s_p \geq 0$, $g \in C_b(\mathbb{R}^{2p})$ and define a mapping F on the set $\mathcal{P}(\mathcal{C})$ of probability measures on $\mathcal{C}$ by

$$F(Q) = < Q, \left(\varphi(X_t - K_t) - \varphi(X_s - K_s) - \int_s^t \frac{\sigma^2}{2}\varphi''(X_r - K_r) + A'(H * \bar{Q}_r(X_r))\varphi'(X_r - K_r)dr\right)$$
$$g(X_{s_1}, K_{s_1}, \ldots, X_{s_p}, K_{s_p}) > .$$

The mapping F_k defined like F with the Heaviside function H replaced by the Lipschitz continuous approximation $H_k(x) = k(x + \frac{1}{k})1\!\!1_{\{-\frac{1}{k} < x < 0\}} + 1\!\!1_{\{x \geq 0\}}$ is continuous and bounded. Hence the weak convergence of π^N to π^∞ implies

$$\mathbb{E}^{\pi^\infty}|F(Q)| \leq \limsup_k \mathbb{E}^{\pi^\infty}|F - F_k(Q)| + \limsup_k \limsup_N \mathbb{E}^{\pi^N}|F - F_k(Q)| + \limsup_N \mathbb{E}^{\pi^N}|F(Q)| \tag{2.5}$$

As the mappings F_k converge pointwise to F and are bounded uniformly in k, the first term of the right-hand-side is equal to 0. Applying Itô's formula, we check that the third term is also nil. By the Lipschitz continuity of A' on $[0, 1]$,

$$\mathbb{E}^{\pi^N}|F - F_k(Q)| \leq C\mathbb{E}\left(< \bar{\mu}^N, \int_s^t (H_k - H) * \bar{\mu}_r^N(X_r)dr > \right).$$

Using the exchangeability of the particles $X^{i,N}$, $i \leq N$, we deduce that

$$\limsup_N \mathbb{E}^{\pi^N}|F - F_k(Q)| \leq C \limsup_N \mathbb{E}\left(\int_s^t (H_k - H)(X_r^{1,N} - X_r^{2,N})dr\right). \tag{2.6}$$

Using Girsanov theorem like in the proof of Lemma 2.4, we obtain that $\forall N \geq 2$, the couple $(X_r^{1,N}, X_r^{2,N})$ has a density that belongs to $L^2([0,1] \times [0,1])$ with a norm smaller than $C\left(1 + r^{-1/2}\right)\exp(Ct)$. Hence $\forall r \in [0,t]$, $\mathbb{E}((H_k - H)(X_r^{1,N} - X_r^{2,N})) \leq C(t)\left(1 + r^{-1/2}\right)k^{-\frac{1}{4}}$. By (2.6), we deduce that $\limsup_k \limsup_N \mathbb{E}^{\pi^N}|F - F_k(Q)| = 0$. Hence each term of the right-hand-side of (2.5) is nil and $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$. As a consequence, π^∞ a.s., Q satisfies condition ii) in Definition 1.1.

Let us check that π^∞ a.s., Q satisfies condition iii). The closeness of the following subset of $\mathcal{C}$

$$F^{T,M} = \left\{(x,k) : \ |k|_T \leq M, \ |k|_T = \int_0^T 1_{\{0,1\}}(x_s)d|k|_s, \ \forall t \leq T, \ k_t = \int_0^t (1 - 2x_s)d|k|_s\right\}$$

which is stated in Lemma 2.6 implies that $\{Q \in \mathcal{P}(\mathcal{C}) : \ Q(F^{T,M}) \geq 1 - \epsilon\}$ is also closed. By the weak convergence of π^N to π^∞, we deduce

$$\begin{aligned}
\pi^\infty(\{Q : Q(F^{T,M}) \geq 1 - \epsilon\}) &\geq \limsup_N \pi^N(\{Q : Q(F^{T,M}) \geq 1 - \epsilon\}) \\
&= 1 - \liminf_N \pi^N(\{Q : Q(\{|k|_T > M\}) > \epsilon\}) \\
&\geq 1 - \liminf_N \mathbb{E}^{\pi^N}\left(\frac{< Q, |k|_T >}{M\epsilon}\right) = 1 - \frac{\liminf_N \mathbb{E}(|K^{1,N}|_T)}{M\epsilon}.
\end{aligned} \tag{2.7}$$

As $|K^{1,N}|_T = \int_0^T (1 - 2X_s^{1,N})dK_s^{1,N}$, by Itô's formula, we obtain that $|K^{1,N}|_T$ is equal to

$$\left(X_0^{1,N} - \frac{1}{2}\right)^2 - \left(X_T^{1,N} - \frac{1}{2}\right)^2 + \int_0^T (2X_s^{1,N} - 1)A'(H * \bar{\mu}_s^N(X_s^{1,N}))ds + \int_0^T \sigma(2X_s^{1,N} - 1)dW_s^1 + \sigma^2 T.$$

Hence $\sup_N \mathbb{E}(|K^{1,N}|_T) < +\infty$. With (2.7), we deduce that $\pi^\infty(\{Q : Q(\cup_{M>0} F^{M,T}) \geq 1 - \epsilon\}) = 1$. As ϵ is arbitrary, we conclude that

$$\pi^\infty\left(\left\{Q : Q\left(\bigcap_{T>0}\bigcup_{M>0} F^{M,T}\right) = 1\right\}\right) = 1$$

i.e. π^∞ a.s. Q satisfies Definition 1.1 iii) which puts an end to the proof. ∎

Lemma 2.6 *The subset $F^{T,M}$ of $\mathcal{C}$ which consists in the couples (x,k) such that $|k|_T \leq M$, $|k|_T = \int_0^T 1_{\{0,1\}}(x_s)d|k|_s$ and $\forall t \leq T, k_t = \int_0^t (1-2x_s)d|k|_s$ is closed.*

Proof : Let $(x^n, k^n) \in F^{T,M}$ converge to (x,k) in $\mathcal{C}$. As $\forall n \geq 0$, $|k^n|_T \leq M$, by extraction of a subsequence, we can suppose that the measures $d|k|^n$ (resp. dk^n) on the compact set $[0,T]$ converge weakly to da, a positive measure with mass smaller than M (resp. db a signed measure). As k^n converges uniformly to k on $[0,T]$, $t \in [0,T] \to k_t$ is the cumulative distribution function of the measure db.
If $f : [0,T] \to \mathbb{R}$ is continuous, as x^n converges uniformly to x on $[0,T]$,

$$\int_0^T f(s)db_s = \lim_n \int_0^T f(s)dk_s^n = \lim_n \int_0^T f(s)(1-2x_s^n)d|k^n|_s = \int_0^T f(s)(1-2x_s)da_s.$$

Hence $(1-2x_s)$ is a density of db w.r.t. da and

$$\forall t \in [0,T],\ k_t = \int_0^t (1-2x_s)da_s. \tag{2.8}$$

As $(x^n,k^n) \in F^{T,M}$, $\int_0^T x_s^n(1-x_s^n)d|k^n|_s = 0$. Letting $n \to +\infty$, we get $\int_0^T x_s(1-x_s)da_s = 0$ i.e. da_s a.e. $x_s \in \{0,1\}$ and $|1-2x_s| = 1$. With (2.8), we deduce that da is the total variation of dk and conclude that $(x,k) \in F^{T,M}$. ∎

Corollary 2.7 *It is possible to approximate the weak solution $V(t,x) = H * \bar{P}_t(x)$ of (1.1) thanks to the empirical cumulative distribution function $H * \bar{\mu}_t^N(x)$ of the particle system. More precisely $\forall (t,x) \in [0,+\infty) \times [0,1]$, $\lim_{N\to+\infty} \mathbb{E}|V(t,x) - H * \bar{\mu}_t^N(x)| = 0$.*

Proof : For $t > 0$ and $x \in [0,1]$, according to Lemma 2.4, the function $Q \in \mathcal{P}(\mathcal{C}) \to |H * \bar{P}_t(x) - H * \bar{Q}_t(x)| \in \mathbb{R}$ is continuous at P. The weak convergence of the sequence $(\pi^N)_N$ to $\pi^\infty = \delta_P$ implies

$$\lim_{N\to+\infty} \mathbb{E}|H * \bar{P}_t(x) - H * \bar{\mu}_t^N(x)| = \mathbb{E}^{\pi^\infty}|H * \bar{P}_t(x) - H * \bar{Q}_t(x)| = 0.$$

In case $t = 0$, we conclude by the strong law of large numbers. ∎

3 Particle method

In this section we describe a numerical particle method to approximate the solution V of equation (1.1) on $[0,T] \times [0,1]$ (where T is a positive constant) and analyse its rate of convergence. According to Corollary 2.7, it is possible to approximate $V(t,x)$ by the empirical cumulative distribution function $H * \bar{\mu}_t^N(x) = \frac{1}{N}\sum_{j=1}^N H(x - X_t^{j,N})$ of the particle system (2.4). To transform this convergence result into a numerical approximation procedure, we need to discretize in time the N-dimensional stochastic differential equation (2.4). To do so we use the version of the Euler scheme introduce by Lépingle [10] which mimics the reflection at the boundary. We choose $\Delta t > 0$ and $L \in \mathbb{N}$ such that $T = L\Delta t$ and denote by $Y_{t_l}^i$ the position of the i-th particle $(1 \leq i \leq N)$ at the discretization time $t_l = l\Delta t$ $(0 \leq l \leq L)$. The Euler-Lépingle scheme consists in setting $0 < \alpha_0 < \alpha_1 < 1$ and in generating exact reflexion on the lower-boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \leq \alpha_0$ and exact reflexion on the upper-boundary on $[t_l, t_{l+1}]$ when $Y_{t_l}^i \geq \alpha_1$. The other cases of reflexion are treated by projection onto $[0,1]$. We will actually let α_0 and α_1 depend on Δt in order to reduce the computational effort but to simplify notations we do not emphasize this dependence unless necessary. Taking advantage of the one-dimensional space domain, we invert the initial cumulative distribution function $V_0(x) = H * U_0(x)$ to construct the set of initial positions of the numerical particles :

$$y_0^i = \inf\left\{z : H * U_0(z) \geq \frac{i}{N}\right\} \quad \text{for} \quad 1 \leq i \leq N. \tag{3.1}$$

At time t_l, the function $V(t_l, x)$ is approximated thanks to the empirical cumulative distribution function

$$\overline{V}(t_l, x) = \frac{1}{N}\sum_{i=1}^{N} H(x - Y^i_{t_l})$$

and the positions of the ith particle are given inductively by

$$\begin{cases} Y^i_{t_0} = y^i_0 \\ \forall t \in [t_l, t_{l+1}], Y^i_t = 0 \vee \left(Y^i_{t_l} + \sigma(W^i_t - W^i_{t_l}) + (t - t_l)A'(\overline{V}(t_l, Y^i_{t_l})) + C^i_t\right) \wedge 1 \\ C^i_t = \mathbb{1}_{\{Y^i_{t_l} \leq \alpha_0\}} \sup\limits_{s\in[t_l,t]} \left(Y^i_{t_l} + \sigma(W^i_s - W^i_{t_l}) + (s - t_l)A'(\overline{V}(t_l, Y^i_{t_l}))\right)^- \\ \qquad - \mathbb{1}_{\{Y^i_{t_l} \geq \alpha_1\}} \sup\limits_{s\in[t_l,t]} \left(Y^i_{t_l} - 1 + \sigma(W^i_s - W^i_{t_l}) + (s - t_l)A'(\overline{V}(t_l, Y^i_{t_l}))\right)^+ . \end{cases} \quad (3.2)$$

Since it is possible to simulate jointly the Brownian increment $(W^i_{t_{l+1}} - W^i_{t_l})$ and the corresponding $\sup_{s\in[t_l,t_{l+1}]} \left(W^i_s - W^i_{t_l} + (s - t_l)\alpha\right)$, this discretization scheme is feasible.

To obtain the optimal rate of convergence $\mathcal{O}(1/\sqrt{N} + \Delta t)$ we are going to make rather strong assumptions on the initial condition $v_0(x) = H * U_0(x)$ ensuring that the weak solution of (1.1) is in fact a classical solution. These hypotheses are possibly too restrictive but they avoid further complications of the already technical proof. For the solution of (1.1) to be classical i.e. $C^{1,2}$ (C^1 in the time variable t and C^2 in the space variable x), it is necessary that v_0 is C^2. Moreover, since the Dirichlet boundary conditions are constant in time, for $x = 0$ or $x = 1$, $\frac{\sigma^2}{2}\partial_{xx}v(t,x) - A'(v(t,x))\partial_x v(t,x) = \partial_t v(t,x) = 0$. At time $t = 0$, we obtain the necessary compatibility conditions $\sigma^2 v_0''(0) = 2A'(0)v_0'(0)$ and $\sigma^2 v_0''(1) = 2A'(1)v_0'(1)$. Our hypothesis

$$(H) \begin{cases} v_0 \in C^{2+\beta}([0,1]) \ (C^2 \text{ with } v_0'' \text{ Hölder continuous with exponent } \beta) \text{ where } \beta \in]0,1[, \\ \sigma^2 v_0''(0) = 2A'(0)v_0'(0) \quad \text{and} \quad \sigma^2 v_0''(1) = 2A'(1)v_0'(1), \\ A \text{ is a } C^3 \text{ function} \end{cases}$$

is slightly stronger than these necessary conditions. Combining Theorem 6.1 pp.452-453 [9] which gives existence of a classical solution on $[0,T]\times[0,1]$ for (1.1) and the proof of Lemma 2.2 which gives uniqueness of weak solutions on $[0,T]\times[0,1]$, we obtain

Lemma 3.1 *Under hypothesis (H), the solution $V(t,x) = H * \bar{P}_t(x)$ of (1.1) belongs to $C^{1,2}([0,T]\times[0,1])$ and $\partial_x V(t,x)$ is Hölder continuous with exponent $(1+\beta)/2$ in the time variable t on $[0,T]\times[0,1]$.*

In order to reduce the effort needed to compute the correction terms C^i_t in (3.2), it is interesting to let α_0 and α_1 depend on the time-step Δt and converge respectively to 0 and 1 as $\Delta t \to 0$. Supposing that these convergence are not too quick, we obtain the following estimate for the convergence rate of the particle method :

Theorem 3.2 *Under hypothesis (H), if we assume that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a\Delta t^\gamma$ for $0 \leq \gamma < 1/2$ and $a > 0$, then there exists a strictly positive constant C depending on A, U_0, T, σ, a and γ such that*

$$\forall 0 \leq l \leq L, \ \sup_{x\in[0,1]} \mathbb{E}|V(t_l, x) - \overline{V}(t_l, x)| \leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right).$$

The proof follows the main ideas of Bossy [3] who deals with the convergence rate of a particle approximation for the solution of the scalar conservation law with spatial domain $\mathbb{R}$ similar to (1.1) even if some new difficulties arise in the present framework because of the reflexion. Let W_t denote a standard Brownian motion. To analyse the convergence rate, for $y \in [0,1]$, we introduce the stochastic differential equation with normal reflexion, constant diffusion coefficient and drift coefficient $A'(V(s,x))$:

$$\begin{cases} X^y_t = y + \sigma W_t + \int_0^t A'(V(s, X^y_s))ds + K^y_t, \\ |K^y|_t = \int_0^t \mathbb{1}_{\{0,1\}}(X^y_s)d|K^y|_s, \quad K^y_t = \int_0^t (1 - 2X^y_s)d|K^y|_s. \end{cases} \quad (3.3)$$

Under hypothesis (H), according to Lemma 3.1, the function $b(s,x) \stackrel{\text{def}}{=} A'(V(s,x))$ is Lipschitz continuous in the space variable x uniformly for $s \in [0,T]$ and bounded. As a consequence, for any $y \in [0,1]$ the above stochastic differential equation has a unique solution (see for instance [11] Remark 3.3 p.525).
We are interested in the upper bound of

$$\begin{aligned}\text{Error}(t_l) &= \sup_{x\in[0,1]} \mathbb{E}|V(t_l,x) - \overline{V}(t_l,x)| \\ &\leq \sup_{x\in[0,1]} \left| V(t_l,x) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left(H(x - X_{t_l}^{y_0^i})\right)\right| \\ &+ \sup_{x\in[0,1]} \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left(H(x - X_{t_l}^{y_0^i})\right) - \overline{V}(t_l,x)\right|. \end{aligned} \tag{3.4}$$

The first term of the right-hand-side is an initialization error, that we upper-bound in the next paragraph. Since the dynamics of the particle system (3.2) on $[t_l, t_{l+1}]$ depends on the approximate solution $\overline{V}(t_l,.)$ of (1.1) at time t_l, the analysis of the second term of the right-hand-side is more complicated than the analysis of the weak error of the discretization by the Euler-Lépingle scheme of the Stochastic Differential Equation (3.3) where the drift coefficient $b(s,x) = A'(V(s,x))$ is supposed to be known. We are going to deal with the latter problem as a model problem : the obtained results are useful to solve the former problem. They are also interesting by themselves because although limited to the case of a constant diffusion coefficient, they form the first study of the weak rate of the Euler-Lépingle scheme to our knowledge.

3.1 Initialization error

Lemma 3.3 *Under hypothesis (H), the solution (X_t^y) of (3.3) can be chosen continuous in $(t,y) \in [0,T]\times[0,1]$ and nondecreasing in y for fixed $t \in [0,T]$. Moreover, $\forall (t,x) \in [0,T]\times[0,1]$, $V(t,x) = \mathbb{E}(\int_0^1 H(x - X_t^y) U_0(dy))$.*

Proof of Lemma 3.3 : As $(X_t^x - X_t^y)_{t\in[0,T]}$ is a continuous process with bounded variation,

$$(X_t^x - X_t^y)^+ = (x-y)^+ + \int_0^t \mathbb{1}_{\{X_s^x > X_s^y\}}(b(s,X_s^x) - b(s,X_s^y))ds + \int_0^t \mathbb{1}_{\{X_s^x > X_s^y\}}(dK_s^x - dK_s^y)$$

The third term of the right-hand-side is nonpositive. By the Lipschitz continuity of $x \to b(s,x)$ and Gronwall's lemma, we deduce that for some real constant C_T,

$$a.s., \sup_{t\in[0,T]} (X_t^x - X_t^y)^+ \leq C_T(x-y)^+. \tag{3.5}$$

Using the symmetric inequality for $(X_t^y - X_t^x)^+$, we obtain that a.s. $\sup_{t\in[0,T]}|X_t^x - X_t^y| \leq C_T|x-y|$. According to Kolmogorov continuity theorem, the $C([0,T],[0,1])$-valued process $(t \to X_t^y)$ indexed by $y \in [0,1]$ admits a continuous version that we still denote by X_t^y to simplify notations. By (3.5), a.s., $\forall q \leq q' \in [0,1]\cap\mathbb{Q}$, $\forall t \in [0,T]$, $X_t^q \leq X_t^{q'}$. With the a.s. continuity of $y \in [0,1] \to X_t^y$ in the variable y, we conclude that this function is a.s. nondecreasing.
Let now X_0 be an initial variable with law U_0 independent of the Brownian motion W. We easily check that the law of $(X_t^{X_0}, K_t^{X_0})_{t\in[0,T]}$ on $C([0,T],[0,1]\times\mathbb{R})$ solves the martingale problem

i) $Q_0 = U_0 \otimes \delta_0$

ii) $\forall \varphi \in C_b^2(\mathbb{R})$, $\varphi(X_t - K_t) - \varphi(X_0 - K_0) - \int_0^t \frac{\sigma^2}{2}\varphi''(X_s - K_s) + A'(V(s,X_s))\varphi'(X_s - K_s)ds$ is a Q martingale

iii) Q a.s., $\forall t \geq 0$, $\int_0^t d|K|_s < +\infty$, $|K|_t = \int_0^t \mathbb{1}_{\{0,1\}}(X_s)d|K|_s$ and $K_t = \int_0^t (1-2X_s)d|K|_s$.

By Girsanov theorem, uniqueness holds for this martingale problem. Since the image of the solution P of problem (MP) starting at $U_0 \otimes \delta_0$ by the canonical restriction from $C([0,+\infty),[0,1]\times\mathbb{R})$ to $C([0,T],[0,1]\times\mathbb{R})$ solves this martingale problem, we deduce that $\forall (t,x)\in[0,T]\times[0,1]$, $V(t,x) = H * \bar{P}_t(x) = \mathbb{E}\left(H(x - X_t^{X_0})\right)$. By independence of X_0 and W, we conclude that $V(t,x) = \mathbb{E}(\int_0^1 H(x - X_t^y)U_0(dy))$. ■

We easily deduce that the initialization error is smaller than $1/N$.

Lemma 3.4

$$\forall t \geq 0, \sup_{x\in[0,1]} \left| V(t,x) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}(H(x - X_t^{y_0^i})) \right| \leq \frac{1}{N}.$$

Proof : Let $\overline{U}_0 = \frac{1}{N}\sum_{i=1}^N \delta_{y_0^i}$. Clearly,

$$a.s., \int_{[0,1]} H(x - X_t^y)U_0(dy) - \frac{1}{N}\sum_{i=1}^N H(x - X_t^{y_0^i}) = (U_0 - \overline{U}_0)(\{y \in [0,1] : X_t^y \leq x\}).$$

Since by Lemma 3.3, $y \to X_t^y$ is a.s. continuous and increasing, if nonempty, the set $\{y : X_t^y \leq x\}$ is equal to $[0,\phi_t(x)]$ where $\phi_t(x) = \inf\{y : X_t^y > x\}$. By definition of the initial positions y_0^i (see (3.1)), $\forall y \in [0,1]$, $0 \leq (U_0 - \overline{U}_0)([0,y]) \leq 1/N$. Hence

$$0 \leq \int_{[0,1]} H(x - X_t^y)U_0(dy) - \frac{1}{N}\sum_{i=1}^N H(x - X_t^{y_0^i}) \leq \frac{1}{N}.$$

We conclude by taking expectations. ■

3.2 Weak error of the Euler-Lépingle scheme

We recall that $T = L\Delta t$ ($\Delta t > 0$, $L \in \mathbb{N}$) and $t_l = l\Delta t$ for $0 \leq l \leq L$. The Euler-Lépingle discretization of the Stochastic Differential Equation

$$\begin{cases} X_t^y = y + \sigma W_t + \int_0^t b(s, X_s^y)ds + K_t^y \\ |K^y|_t = \int_0^t \mathbb{1}_{\{0,1\}}(X_s^y)d|K^y|_s, \quad K_t^y = \int_0^t (1 - 2X_s^y)d|K^y|_s \end{cases} \tag{3.6}$$

is given by

$$\begin{cases} \widetilde{X}_{t_0}^y = y \\ \forall t \in [t_l, t_{l+1}],\ \widetilde{X}_t^y = 0 \vee \left(\widetilde{X}_{t_l}^y + \sigma(W_t - W_{t_l}) + b(t_l, \widetilde{X}_{t_l}^y)(t - t_l) + C_t\right) \wedge 1 \\ C_t = \mathbb{1}_{\{\widetilde{X}_{t_l}^y \leq \alpha_0\}} \sup_{s\in[t_l,t]} \left(\widetilde{X}_{t_l}^y + \sigma(W_s - W_{t_l}) + b(t_l, \widetilde{X}_{t_l}^y)(s - t_l)\right)^- \\ \qquad - \mathbb{1}_{\{\widetilde{X}_{t_l}^y \geq \alpha_1\}} \sup_{s\in[t_l,t]} \left(\widetilde{X}_{t_l}^y - 1 + \sigma(W_s - W_{t_l}) + b(t_l, \widetilde{X}_{t_l}^y)(s - t_l)\right)^+ \end{cases}$$

In the next Proposition, assuming a regularity condition on the drift coefficient $b(s,x)$ which is satisfied by $A'(V(s,x))$ under hypothesis (H) (see Lemma 3.1), we upper-bound the weak convergence rate of this scheme :

Proposition 3.5 *Assume that b is $C^{1,2}$ on $[0,T]\times[0,1]$, that for some $\alpha > 0$, $\partial_x b(t,x)$ is Hölder continuous with exponent α in t and that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a\Delta t^\gamma$ for $\gamma \in [0,1/2)$ and $a > 0$. Then there is a constant C depending on σ, T, b, a, γ but not on y and Δt such that when $f : [0,1] \to \mathbb{R}$ is a function with bounded variation and m denotes its distribution derivative,*

$$\forall l \leq L, \left|\mathbb{E}\left(f(X_{t_l}^y) - f(\widetilde{X}_{t_l}^y)\right)\right| \leq C\Delta t \int_0^1 |m|(dx).$$

The error proceeds from two sources. The first one is the usual Euler discretization of the drift coefficient. The second contribution is the inexact treatment of the reflexion on the lower boundary (resp. the upper boundary, resp. both boundaries) between t_l and t_{l+1} when $\widetilde{X}^y_{t_l} \geq \alpha_1$ (resp. $\widetilde{X}^y_{t_l} \leq \alpha_0$, resp. $\widetilde{X}^y_{t_l} \in (\alpha_0, \alpha_1)$) which will turn out to be neglectable. To get rid of it, we introduce the Euler-Peano discretization of (3.6). The Euler-Peano is a theoretical discretization scheme which consists in freezing the drift coefficient on each interval $[t_l, t_{l+1}]$ whereas the normal reflexion remains exact :

$$\begin{cases} \hat{X}^y_t = y + \sigma W_t + \int_0^t b(\tau_s, \hat{X}^y_{\tau_s}) ds + \hat{K}^y_t \\ |\hat{K}^y|_t = \int_0^t \mathbb{1}_{\{0,1\}}(\hat{X}^y_s) d|\hat{K}^y|_s, \quad \hat{K}^y_t = \int_0^t (1 - 2\hat{X}^y_s) d|\hat{K}^y|_s \end{cases} \tag{3.7}$$

where $\tau_s = \Delta t \left[\frac{s}{\Delta t}\right]$ and $[x]$ denotes the integral part of x.

Lemma 3.6 *Assume that $b(\cdot,\cdot)$ is bounded and that $0 < \alpha_0(\Delta t) \leq \alpha_1(\Delta t) < 1$ satisfy $\alpha_0(\Delta t) \wedge (1 - \alpha_1(\Delta t)) \geq a\Delta t^\gamma$ for $\gamma \in [0, 1/2)$ and $a > 0$. Then for some positive constants c and C independent of Δt and y*

$$\forall l \leq L,\ \mathbb{P}(\exists k \leq l,\ \hat{X}^y_{t_k} \neq \widetilde{X}^y_{t_k}) \leq C\Delta t^{-\gamma - \frac{1}{2}} e^{-c\Delta t^{2\gamma - 1}}. \tag{3.8}$$

Proof : The proof follows the ideas of [10] even if this upper-bound is not stated. To simplify notations, we do not emphasize the dependence of α_0 and α_1 on Δt.

$$\mathbb{P}(\exists k \leq l,\ \hat{X}^y_{t_k} \neq \widetilde{X}^y_{t_k}) \leq \sum_{k=0}^{l-1} \mathbb{P}(\hat{X}^y_{t_k} = \widetilde{X}^y_{t_k},\ \hat{X}^y_{t_{k+1}} \neq \widetilde{X}^y_{t_{k+1}}). \tag{3.9}$$

When $\alpha_0 < \hat{X}^y_{t_k} = \widetilde{X}^y_{t_k} < \alpha_1$, we remark that $\forall t \in [t_k, t_{k+1}]$, $\hat{X}^y_t = \widetilde{X}^y_t$ unless both processes reach 0 or 1 before t_{k+1}. As a consequence,

$$\begin{aligned} &\mathbb{P}(\alpha_0 < \hat{X}^y_{t_k} = \widetilde{X}^y_{t_k} < \alpha_1,\ \hat{X}^y_{t_{k+1}} \neq \widetilde{X}^y_{t_{k+1}}) \\ &\qquad \leq \mathbb{P}\left(\sup_{t \in [t_k, t_{k+1}]} (\sigma(W_t - W_{t_k}) + b(t_k, \widetilde{X}^y_{t_k})(t - t_k)) > 1 - \alpha_1\right) \\ &\qquad + \mathbb{P}\left(\inf_{t \in [t_k, t_{k+1}]} (\sigma(W_t - W_{t_k}) + b(t_k, \widetilde{X}^y_{t_k})(t - t_k)) < -\alpha_0\right) \\ &\qquad \leq \mathbb{P}\left(\sup_{t \in [t_k, t_{k+1}]} |W_t - W_{t_k}| > (a\Delta t^\gamma - \sup|b(.,.)|\Delta t)/\sigma\right) \end{aligned}$$

Since $\gamma < 1$, for Δt small enough, $a\Delta t^\gamma - \sup|b(.,.)|\Delta t \geq a\Delta t^\gamma/2$. Then

$$\begin{aligned} &\mathbb{P}(\alpha_0 < \hat{X}^y_{t_k} = \widetilde{X}^y_{t_k} < \alpha_1,\ \hat{X}^y_{t_{k+1}} \neq \widetilde{X}^y_{t_{k+1}}) \leq \mathbb{P}\left(\sup_{s \in [t_k, t_{k+1}]} |W_s - W_{t_k}| \geq a\Delta t^\gamma/(2\sigma)\right) \\ &\leq 2\sqrt{\frac{2}{\pi\Delta t}} \int_{a\Delta t^\gamma/(2\sigma)}^{+\infty} e^{-z^2/2\Delta t} dz \leq \frac{4\sigma}{a}\sqrt{\frac{2}{\pi}} \Delta t^{\frac{1}{2} - \gamma} e^{-a^2\Delta t^{2\gamma - 1}/8\sigma^2}. \end{aligned}$$

Since $\alpha_0 \leq \alpha_1$, $\alpha_1 \wedge (1 - \alpha_0) \geq a\Delta t^\gamma$ and remarking that $\forall t \in [t_k, t_{k+1}]$, $C_t \leq \sigma \sup_{s \in [t_k, t_{k+1}]} |W_s - W_{t_k}| + \sup|b(.,.)|\Delta t$, we easily obtain similar bounds for $\mathbb{P}(\hat{X}^y_{t_k} = \widetilde{X}^y_{t_k} \geq \alpha_1,\ \hat{X}^y_{t_{k+1}} \neq \widetilde{X}^y_{t_{k+1}})$ and $\mathbb{P}(\hat{X}^y_{t_k} = \widetilde{X}^y_{t_k} \leq \alpha_0,\ \hat{X}^y_{t_{k+1}} \neq \widetilde{X}^y_{t_{k+1}})$ and we conclude by (3.9) since $l \leq L = T/\Delta t$. ∎

Let $l \geq 1$, $f : [0,1] \to \mathbb{R}$ be a function with bounded variation and m denote its distribution derivative. According to the previous Lemma,

$$\begin{aligned} \left|\mathbb{E}\left(f(X^y_{t_l}) - f(\widetilde{X}^y_{t_l})\right)\right| &\leq \left|\mathbb{E}\left(f(X^y_{t_l}) - f(\hat{X}^y_{t_l})\right)\right| + \sup_{z, z' \in [0,1]} |f(z) - f(z')| C\Delta t^{-\gamma - \frac{1}{2}} e^{-c\Delta t^{2\gamma - 1}} \\ &\leq \left|\mathbb{E}\left(f(X^y_{t_l}) - f(\hat{X}^y_{t_l})\right)\right| + C\Delta t^{-\gamma - \frac{1}{2}} e^{-c\Delta t^{2\gamma - 1}} \int_0^1 |m|(dy). \end{aligned}$$

The set D_f of discontinuity points of f is denumerate. For $y \leq z \in [0,1] \setminus D_f$, $f(z) = f(y) + \int_y^z m(dx) = f(y) + \int_0^1 H(x-y) - H(x-z)m(dx)$. As the function $b(.,.)$ is bounded, by Girsanov theorem both variables $X^y_{t_l}$ and $\hat{X}^y_{t_l}$ have densities w.r.t. Lebesgue measure. Hence $\mathbb{P}(X^y_{t_l} \in D_f) + \mathbb{P}(\hat{X}^y_{t_l} \in D_f) = 0$ and

$$\left|\mathbb{E}\left(f(X^y_{t_l}) - f(\hat{X}^y_{t_l})\right)\right| = \left|\int_0^1 \mathbb{E}\left(H(x - X^y_{t_l}) - H(x - \hat{X}^y_{t_l})\right) m(dx)\right|.$$

Therefore the proof of Proposition 3.5 is completed as soon as we obtain the following weak convergence rate for the Euler Peano scheme :

Proposition 3.7 *Under the assumptions of Proposition 3.5, there is a constant C depending on σ, T, b but not on x, y and Δt such that*

$$\forall l \leq L,\ \forall x, y \in [0,1],\ \left|\mathbb{E}\left(H(x - X^y_{t_l}) - H(x - \hat{X}^y_{t_l})\right)\right| \leq C\Delta t.$$

In case $l = 0$, the conclusion is clear. As for all other values of l the proof is the same, we are only going to deal with the case $l = L$ i.e. $t_l = T$. By Girsanov theorem both variables X^y_T and $\hat{X}^y_T$ admit densities with respect to Lebesgue measure. Hence

$$\begin{aligned}
\mathbb{E}\left(H(0 - X^y_T) - H(0 - \hat{X}^y_T)\right) &= \mathbb{P}(X^y_T = 0) - \mathbb{P}(\hat{X}^y_T = 0) = 0 - 0 = 0\\
\mathbb{E}\left(H(1 - X^y_T) - H(1 - \hat{X}^y_T)\right) &= \mathbb{P}(X^y_T \leq 1) - \mathbb{P}(\hat{X}^y_T \leq 1) = 1 - 1 = 0
\end{aligned}$$

and the conclusion holds for $x \in \{0,1\}$.
$>$From now on, we assume that $x \in (0,1)$. We follow the idea first introduced by Talay and Tubaro : if the function v solves the parabolic problem

$$\begin{cases}\partial_t v + \frac{\sigma^2}{2}\partial_z^2 v + b(t,z)\partial_z v = 0,\ (t,z) \in [0,T) \times [0,1]\\ \forall z \in [0,1],\ v(T,z) = H(x-z),\ \forall t \in [0,T], \partial_z v(t,0) = \partial_z v(t,1) = 0\end{cases} \tag{3.10}$$

computing formally $H(x - \hat{X}^y_T) - H(x - X^y_T) = v(T, \hat{X}^y_T) - v(T, X^y_T)$ by Itô's formula and taking expectations we obtain

$$\mathbb{E}(H(x - \hat{X}^y_T) - H(x - X^y_T)) = \mathbb{E}\left(\int_0^T (b(s, \hat{X}^y_s) - b(\tau_s, \hat{X}^y_{\tau_s}))(-\partial_z v(s, \hat{X}^y_s))ds\right).$$

The function v appears only through the opposite of its spatial derivative, which justifies our interest in the parabolic problem satisfied by $w = -\partial_z v$

$$\begin{cases}\partial_t w + \frac{\sigma^2}{2}\partial_z^2 w + b(t,z)\partial_z w + \partial_z b(t,z) w = 0,\ (t,z) \in [0,T) \times [0,1],\\ \forall t \in [0,T), w(t,0) = w(t,1) = 0,\ w(T,.) = \delta_x(.).\end{cases} \tag{3.11}$$

According to section IV.16 [9] and section 3.7 [7] which are dedicated to Green's functions, the following holds :

Lemma 3.8 *Under the assumptions of Proposition 3.5, there is a continuous function $(x,t,z) \in (0,1) \times [0,T) \times [0,1] \to w(x,t,z) \in \mathbb{R}$ such that :*

- *for fixed $x \in (0,1)$, $(t,z) \to w(x,t,z) \in C^{1,2}([0,T) \times [0,1])$, solves*

$$\begin{cases}\partial_t w + \frac{\sigma^2}{2}\partial_z^2 w + b(t,z)\partial_z w + \partial_z b(t,z) w = 0,\ (t,z) \in [0,T) \times [0,1]\\ \forall t \in [0,T), w(x,t,0) = w(x,t,1) = 0\end{cases} \tag{3.12}$$

  and takes the terminal value $w(x,T,.) = \delta_x(.)$ in the distribution sense.
- *For any integers r and s such that $2r + s \leq 2$,*

$$\forall t \in [0,T),\ \forall x, z,\ |\partial_t^r \partial_z^s w(x,t,z)| \leq C(T-t)^{-\frac{1+2r+s}{2}} \exp\left(-c\frac{(z-x)^2}{T-t}\right).$$

- *for any function φ continuous on $[0,1]$, the function*

$$w^{\varphi}(t,z) = \begin{cases} \int_0^1 w(x',t,z)\varphi(x')dx' & \text{if } t < T \\ \varphi(z) & \text{if } t = T \end{cases}$$

  is continuous on $[0,T]\times[0,1]$ and satisfies (3.12).

Thanks to these results, we express rigorously $\mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T))$ in terms of w :

Lemma 3.9

$$\mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T)) = \mathbb{E}\left(\int_0^T (b(s,\hat{X}^y_s) - b(\tau_s,\hat{X}^y_{\tau_s}))w(x,s,\hat{X}^y_s)ds\right). \qquad (3.13)$$

Proof : For $\epsilon > 0$, we set $\varphi^\epsilon(x') = e^{-(x'-x)^2/(2\epsilon)}/\sqrt{2\pi\epsilon}$ and $v^\epsilon(t,z) = \int_z^1 w^{\varphi^\epsilon}(t,z')dz'$. By Lemma 3.8, the function v^ϵ is continuous on $[0,T]\times[0,1]$ and satisfies

$$\begin{cases} \forall (t,z) \in [0,T)\times[0,1],\ \partial_t v^\epsilon + \frac{\sigma^2}{2}\partial_z^2 v^\epsilon + b(t,z)\partial_z v^\epsilon + \frac{\sigma^2}{2}\partial_z w^{\varphi^\epsilon}(t,1) = 0 \\ \forall t \in [0,T),\ \partial_z v^\epsilon(t,0) = \partial_z v^\epsilon(t,1) = 0. \end{cases}$$

Hence for $t < T$, by Itô's formula,

$$\begin{aligned} v^\epsilon(t,\hat{X}^y_t) - v^\epsilon(t,X^y_t) &= \int_0^t \sigma(\partial_z v^\epsilon(s,\hat{X}^y_s) - \partial_z v^\epsilon(s,X^y_s))dW_s \\ &+ \int_0^t (b(s,\hat{X}^y_s) - b(\tau_s,\hat{X}^y_{\tau_s}))(-\partial_z v^\epsilon(s,\hat{X}^y_s))ds. \end{aligned}$$

This equation still holds for $t = T$ by continuity of both sides, since for $s < T$, by the upperbound of w given in Lemma 3.8 and the convolution property of Gaussian kernels

$$|\partial_z v^\epsilon(s,z)| \le \int_0^1 \varphi^\epsilon(x')|w(x',s,z)|dx' \le C/\sqrt{\epsilon + (T-s)}. \qquad (3.14)$$

Taking expectations, we deduce that

$$\mathbb{E}(v^\epsilon(T,\hat{X}^y_T)) - v^\epsilon(T,X^y_T)) = \mathbb{E}\left(\int_0^T (b(s,\hat{X}^y_s) - b(\tau_s,\hat{X}^y_{\tau_s}))(-\partial_z v^\epsilon(s,\hat{X}^y_s))ds\right). \qquad (3.15)$$

The function $z \to v^\epsilon(T,z) = \int_z^1 \varphi^\epsilon(z')dz'$ is bounded by 1 uniformly in ϵ and converges pointwise to $\mathbb{1}_{\{z=x\}}/2 + \mathbb{1}_{\{z<x\}}$ as $\epsilon \to 0$. By Lebesgue theorem, the left-hand-side of (3.15) converges to

$$\mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T)) + \frac{1}{2}(\mathbb{P}(X^y_T = x) - \mathbb{P}(\hat{X}^y_T = x)) = \mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T))$$

since by Girsanov theorem both variables X^y_T and $\hat{X}^y_T$ admit densities w.r.t. Lebesgue measure.

By continuity of the function w, $\forall (s,z) \in [0,T)\times[0,1]$, $-\partial_z v^\epsilon(s,z) = \int_0^1 w(x',s,z)\varphi^\epsilon(x')dx'$ converges to $w(x,s,z)$ as $\epsilon \to 0$. Using (3.14), we obtain by Lebesgue theorem that the right-hand-side of (3.15) converges to $\mathbb{E}\left(\int_0^T (b(s,\hat{X}^y_s) - b(\tau_s,\hat{X}^y_{\tau_s}))w(x,s,\hat{X}^y_s)ds\right)$. Hence (3.13) holds. ∎

In the sequel $x \in (0,1)$ is fixed and we denote $w(t,z)$ instead of $w(x,t,z)$.

Applying Itô's formula to the function $g_l(t,z) = (b(t,z) - b(t_l,\hat{X}^x_{t_l}))w(t,z)$ we get that for $l \le L-1$ and $s \in [t_t, t_{l+1})$.

$$\begin{aligned} (b(s,\hat{X}^y_s) - b(t_l,\hat{X}^y_{t_l}))w(s,\hat{X}^y_s) &= \int_{t_l}^s (\partial_t + \frac{\sigma^2}{2}\partial_z^2 + b(t_l,\hat{X}^y_{t_l})\partial_z)g_l(\theta,\hat{X}^y_\theta)d\theta \\ &+ \int_{t_l}^s \partial_z g_l(\theta,\hat{X}^y_\theta)d\hat{K}^y_\theta + \sigma\int_{t_l}^s \partial_z g_l(\theta,\hat{X}^y_\theta)dW_\theta \qquad (3.16) \\ &\stackrel{\text{def}}{\equiv} T^1_s + T^2_s + T^3_s \end{aligned}$$

We divide the integral on $[0,T]$ in the right-hand-side of (3.13) into integrals on $[t_l, t_{l+1}]$, $0 \leq l \leq L-1$ and treat separetely the first and last term,

$$\begin{aligned}\left|\mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T))\right| \leq & \left|\mathbb{E}\left(\int_0^{t_1} (b(s,\hat{X}^y_s) - b(0,y))w(s,X^y_s)ds\right)\right| \\ & + \left|\mathbb{E}\left(\sum_{l=1}^{L-2}\int_{t_l}^{t_{l+1}} T^1_s + T^2_s + T^3_s ds\right)\right| \\ & + \left|\mathbb{E}\left(\int_{t_{L-1}}^{T} (b(s,\hat{X}^y_s) - b(t_{L-1},\hat{X}^y_{t_{L-1}}))w(s,X^y_s)ds\right)\right|.\end{aligned} \tag{3.17}$$

To upper-bound the last term we need the following Lemma the proof of which is postponed :

Lemma 3.10

$$\forall 0 \leq t \leq s \leq T,\ \mathbb{E}\left(\sup_{\theta\in[t,s]} (\hat{X}^y_\theta - \hat{X}^y_s)^2 + (|\hat{K}^y|_s - |\hat{K}^y|_t)^2\right) \leq C(s-t).$$

Combining this result, the regularity assumptions on b and the upperbound $|w(s,.)| \leq C(T-s)^{-1/2}$ given in Lemma 3.8, we get

$$\begin{aligned}&\left|\mathbb{E}\Big(\int_{t_{L-1}}^{T} (b(s,\hat{X}^y_s) - b(t_{L-1},\hat{X}^y_{T_{L-1}}))w(s,\hat{X}^y_s)ds\Big)\right| \\ &\qquad \leq \mathbb{E}\sup_{[t_{L-1},T]}\left|b(s,\hat{X}^y_s) - b(t_{L-1},\hat{X}^y_{T_{L-1}})\right|\int_{t_{L-1}}^{T} C(T-s)^{-1/2}ds \leq C\Delta t\end{aligned}$$

The same bound is valid for $\left|\mathbb{E}\left(\int_0^{t_1}(b(s,\hat{X}^y_s) - b(0,y))w(s,\hat{X}^y_s)ds\right)\right|$. Once we check that

Lemma 3.11

$$\forall s\in[0,T],\ \mathbb{E}|T^1_s| \leq C\int_{\tau_s}^{s}\left((T-\theta)^{-1/2} + (T-\theta)^{-2/3}\theta^{-1/3}\right)d\theta \tag{3.18}$$

$$\forall \eta\in(0,1/2),\ \forall 1\leq l\leq L-1,\ \forall s\in[t_l,t_{l+1}[,\ \mathbb{E}|T^2_s| \leq Ct_l^{-(1-2\eta)/4(1-\eta)}\Delta t^{(5-4\eta)/4}(T-s)^{-1} \tag{3.19}$$

$$\mathbb{E}\left(\int_{t_1}^{t_{L-1}} T^3_s ds\right) = 0. \tag{3.20}$$

we deduce from (3.17) that

$$\begin{aligned}\left|\mathbb{E}(H(x-\hat{X}^y_T) - H(x-X^y_T))\right| \leq & C\Delta T + C\int_{t_1}^{t_{L-1}}\int_{\tau_s}^{s}\left((T-\theta)^{-1/2} + (T-\theta)^{-2/3}\theta^{-1/3}\right)d\theta \\ & + C\Delta t^{(5-4\eta)/4}\sum_{l=1}^{L-2} t_l^{-(1-2\eta)/4(1-\eta)}(T-t_{l+1})^{-1}\Delta t \\ & \leq C\left(\Delta t + \Delta t^{(5-4\eta)/4}|\ln\Delta t|\right) \leq C\Delta t \ \text{ by choosing } \ \eta < 1/4.\end{aligned}$$

i.e. Proposition 3.7 holds.

Proof of Lemma 3.10 : Let $\phi : x\in\mathbb{R} \to 1 - |1 - x + 2[x/2]| \in [0,1]$ and $0 \leq t \leq T$. By Girsanov theorem, since b is bounded, the stochastic differential equation

$$Y_\theta = \hat{X}^y_t + \sigma(W_\theta - W_t) + \int_t^\theta (-1)^{[Y_r]}b(\tau_r,\phi(Y_{\tau_r}))dr$$

has a unique weak solution. Moreover the processes $(\phi(Y_\theta))_{\theta\geq t}$ and $(\hat{X}^y_\theta)_{\theta\geq t}$ have the same law. Since ϕ is Lipschitz continuous with constant 1, we deduce that

$$\mathbb{E}\left(\sup_{\theta\in[t,s]}(\hat{X}^y_\theta - \hat{X}^y_t)^2\right) \leq \mathbb{E}\left(\sup_{\theta\in[t,s]}(Y_\theta - \hat{X}^y_t)^2\right) \leq C(s-t).$$

Now applying Itô's formula to compute $(\hat X^y_s-\frac12)^2-(\hat X^y_t-\frac12)^2$, we get

$$|\hat K^y|_s-|\hat K^y|_t=\int_t^s(2\hat X^y_r-1)(\sigma dW_r+b(\tau_r,\hat X^y_{\tau_r})dr)+(s-t)+(\hat X^y_t-\hat X^y_s)(\hat X^y_t+\hat X^y_s-1).$$

Using the previous upper-bound, we conclude that $\mathbb{E}\left((|\hat K^y|_s-|\hat K^y|_t)^2\right)\le C(s-t)$. ∎

Proof of Lemma 3.11 : Using (3.11) we check that $\forall 0\le l\le L-1$, $\forall s\in[t_l,t_{l+1})$,

$$\begin{aligned}T^1_s=&\int_{t_l}^s w(\theta,\hat X^y_\theta)\left(\partial_t+\frac{\sigma^2}{2}\partial_z^2+(2b(t_l,\hat X^y_{t_l})-b(\theta,\hat X^y_\theta))\partial_z\right)b(\theta,\hat X^y_\theta)d\theta\\&+\int_{t_l}^s\left(\sigma^2\partial_z b(\theta,\hat X^y_\theta)-(b(\theta,\hat X^y_\theta)-b(t_l,\hat X^y_{t_l}))^2\right)\partial_z w(\theta,\hat X^y_\theta)d\theta\end{aligned}$$

By the regularity assumptions on the function $b(.,.)$ and the upper-bound $|w(\theta,.)|\le C(T-\theta)^{-1/2}$ given in Lemma 3.8, we deduce that $\mathbb{E}|T^1_s|\le C\int_{t_l}^s(T-\theta)^{-1/2}+\mathbb{E}|\partial_z w(\theta,\hat X^y_\theta)|d\theta$. Since the rough upper-bound $C/(T-\theta)$ of $\partial_z w(\theta,.)$ is not integrable for $\theta\in[0,T]$ we are going to make use of Girsanov theorem : for a well-chosen exponential martingale $\hat Z_t$, if β^y_t denotes the doubly reflected process associated with the Brownian motion $y+\sigma W_t$, we have

$$\mathbb{E}|\partial_z w(\theta,\hat X^y_\theta)|=\mathbb{E}\left(|\partial_z w(\theta,\beta^y_\theta)|\hat Z_T\right)\le\left(\mathbb{E}(\hat Z^3_T)\right)^{1/3}\left(\mathbb{E}|\partial_z w(\theta,\beta^y_\theta)|^{3/2}\right)^{2/3}.$$

Since $\forall\theta\in]0,T]$, $\forall y\in[0,1]$, the density of β^y_θ is smaller than $C\theta^{-1/2}$, by Lemma 3.8,

$$\mathbb{E}|\partial_z w(\theta,\beta^y_\theta)|^{3/2}\le C\int_0^1(T-\theta)^{-3/2}\exp\left(-c\frac{(z-x)^2}{T-\theta}\right)\theta^{-1/2}dz\le C(T-\theta)^{-1}\theta^{-1/2}$$

and we deduce that $\mathbb{E}|\partial_z w(\theta,\hat X^y_\theta)|\le C(T-\theta)^{-2/3}\theta^{-1/3}$. Hence (3.18) holds.

We turn to the proof of (3.19). Let $0\le l\le L-1$ and $s\in[t_l,t_{l+1})$. As $d|\hat K^y|_\theta=\mathbb{1}_{\{0,1\}}(\hat X^y_\theta)d|\hat K^y|_\theta$ and $w(t,0)=w(t,1)=0$,

$$T^2_s=\int_{t_l}^s(b(\theta,\hat X^y_\theta)-b(t_l,\hat X^y_{t_l}))\partial_z w(\theta,\hat X^y_\theta)\mathbb{1}_{\{0,1\}}(\hat X^y_\theta)d\hat K^y_\theta.$$

We deduce that

$$\begin{aligned}\mathbb{E}|T^2_s|\le&\sup_{[t_l,s]}|\partial_z w(\theta,0)|\vee|\partial_z w(\theta,1)|\times\left(\mathbb{E}\Big(\sup_{[t_l,t_{l+1}]}\left((b(\theta,\hat X^y_\theta)-b(t_l,\hat X^y_{t_l}))^2\mathbb{1}_{\{0,1\}}(\hat X^y_\theta)\right)\Big)\right)^{1/2}\\&\times\left(\mathbb{E}\left(|\hat K^y|_{t_{l+1}}-|\hat K^y|_{t_l}\right)^2\right)^{1/2}.\end{aligned}\tag{3.21}$$

Let us upper-bound the three terms of the right-hand-side. By Lemma 3.8, $\sup_{[t_l,s]}|\partial_z w(\theta,0)|\vee|\partial_z w(\theta,1)|\le C(T-s)^{-1}$. Let $\eta\in(0,1/2)$,

$$\begin{aligned}\mathbb{E}\Big(\sup_{[t_l,t_{l+1}]}((b(\theta,\hat X^y_\theta)-b(t_l,\hat X^y_{t_l}))^2\mathbb{1}_{\{\hat X^y_\theta=0\}}\Big)\le&4\sup|b(.,.)|^2\mathbb{P}\left(\hat X^y_{t_l}\ge\Delta t^{(1-\eta)/2},\ \inf_{[t_l,t_{l+1}]}\hat X^y_\theta=0\right)\\&+\mathbb{E}\left(\mathbb{1}_{\{\hat X^y_{t_l}\le\Delta t^{(1-\eta)/2}\}}\sup_{[t_l,t_{l+1}]}(b(\theta,0)-b(t_l,\hat X^y_{t_l}))^2\right)\end{aligned}$$

Following the same approach as in the proof of (3.8), we upper-bound the first term of the right-hand-side by $C\Delta t^{\eta/2}e^{-c/\Delta t^{2\eta}}$. The second term is smaller than $C\Delta t^{1-\eta}\mathbb{P}(\hat X^y_{t_l}\le\Delta t^{(1-\eta)/2})$ and by using Girsanov Theorem like in the derivation of (3.18), when $l\ge1$, we get

$$\begin{aligned}\mathbb{P}(\hat X^y_{t_l}\le\Delta t^{(1-\eta)/2})&\le\mathbb{E}(\hat Z_T^{(1-\eta)/\eta})^{\eta/1-\eta}\mathbb{P}(\beta^y_{t_l}\le\Delta t^{(1-\eta)/2})^{(1-2\eta)/(1-\eta)}\\&\le Ct_l^{-(1-2\eta)/2(1-\eta)}\Delta t^{(1-2\eta)/2}.\end{aligned}$$

Treating in a symmetric way $\mathbb{E}(\sup_{[t_l,t_{l+1}]}((b(\theta,\hat{X}^y_\theta)-b(t_l,\hat{X}^y_{t_l}))^2 1\!\!1_{\{\hat{X}^y_\theta=1\}})$, we deduce that when $l\geq 1$,

$$\mathbb{E}\bigg(\sup_{[t_l,t_{l+1}]}((b(\theta,\hat{X}^y_\theta)-b(t_l,\hat{X}^y_{t_l}))^2 1\!\!1_{\{0,1\}}(\hat{X}^y_\theta)\bigg)\leq Ct_l^{-(1-2\eta)/2(1-\eta)}\Delta t^{(3-4\eta)/2}.$$

Since according to Lemma 3.10, the third term of right-hand-side of (3.21) is smaller than $C\Delta t^{1/2}$ we conclude that (3.19) holds.

Let us finally check (3.20). By the integration by parts formula, for $l\leq L-2$

$$\int_{t_l}^{t_{l+1}} T^3_s ds=\sigma\int_{t_l}^{t_{l+1}}(t_{l+1}-\theta)\left((b(\theta,\hat{X}^y_\theta)-b(t_l,\hat{X}^y_{t_l}))\partial_z w(\theta,\hat{X}^y_\theta)+\partial_z b(\theta,\hat{X}^y_\theta)w(\theta,\hat{X}^y_\theta)\right)dW_\theta.$$

Since according to Lemma 3.8, $|w(\theta,.)|\leq C(T-\theta)^{-1/2}$, $|\partial_z w(\theta,.)|\leq C(T-\theta)^{-1}$ and the functions b and $\partial_z b$ are bounded, we deduce that $\forall 0\leq l\leq L-2$, $\mathbb{E}\left(\int_{t_l}^{t_{l+1}} T^3_s ds\right)=0$. Hence (3.20) holds. ■

Remark 3.12 *Our proof only works in case the diffusion coefficient is constant because otherwise the analysis of the error would involve higher order derivatives of the Green's function w.*

3.3 Proof of Theorem 3.2

We come back to the analysis of the stochastic particle method and the estimation of $\sup_{x\in[0,1]}\mathbb{E}|V(t_l,x)-\overline{V}(t_l,x)|=\text{Error}(t_l)$, for $0\leq l\leq L$. Now, we set

$$b(t,x)=A'(V(t,x)).$$

By Lemma 3.1, this drift function satisfies the regularity assumptions made in the study of the weak error of the Euler-Lépingle scheme. By (3.4) and Lemma 3.4,

$$\text{Error}(t_l)\leq\frac{1}{N}+\sup_{x\in[0,1]}\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left(H(x-X^{y^i_0}_{t_l})\right)-\overline{V}(t_l,x)\right|$$

To deal with the inexact treatment of the reflexion by the Euler-Lépingle scheme, we introduce the system of processes $(Z^i, i=1,\dots,N)$ evolving according to the Euler-Peano scheme on $[t_l,t_{l+1})$ and reinitialized at the positions $(Y^i_{t_{l+1}},\ 1\leq i\leq N)$ at time t_{l+1} (for $0\leq l\leq L-1$) :

$$\begin{cases}\forall 0\leq l\leq L-1,\ \forall t\in[t_l,t_{l+1}), Z^i_t=Y^i_{t_l}+\sigma(W^i_t-W^i_{t_l})+(t-t_l)A'(\overline{V}(t_l,Y^i_{t_l}))+\hat{K}^i_t-\hat{K}^i_{t_l},\\ |\hat{K}^i|_t=\int_0^t 1\!\!1_{\{0,1\}}(Z^i_s)d|\hat{K}^i|_s,\ \text{and}\ \hat{K}^i_t=\int_0^t(1-2Z^i_t)d|\hat{K}^i|_s.\end{cases} \tag{3.22}$$

Since we assume that $\gamma<1/2$, according to Lemma 3.6, $\forall 1\leq i\leq N$, $\mathbb{P}(\exists k\leq l:\ Z^i_{t_k^-}\neq Y^i_{t_k})\leq C\Delta t$. Hence

$$\text{Error}(t_l)\leq\frac{1}{N}+C\Delta t+\sup_{x\in[0,1]}\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N 1\!\!1_{\{\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}\}}\left(\mathbb{E}(H(x-X^{y^i_0}_{t_l}))-H(x-Z^i_{t_l^-})\right)\right|. \tag{3.23}$$

We introduce the solution $w(x,t,z)$ of the parabolic problem

$$\begin{cases}\partial_t w+\frac{\sigma^2}{2}\partial^2_z w+b(t,z)\partial_z w+\partial_z b(t,z)w=0,\ (t,z)\in[0,t_l)\times[0,1],\\ \forall t\in[0,t_l), w(t,0)=w(t,1)=0,\ w(t_l,.)=\delta_x(.).\end{cases}$$

Lemma 3.8 remains valid with t_l replacing T.

Lemma 3.13

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}\}}\left(\mathbb{E}(H(x-X^{y_0^i}_{t_l}))-H(x-Z^i_{t_l^-})\right.\right.$$
$$\left.\left.-\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}}w(x,s,Z^i_s)(A'(\overline{V}(t_k,Y^i_{t_k}))-b(s,Z^i_s))ds\right)\right|\leq C\left(\frac{1}{\sqrt{N}}+\Delta t\right),$$

where the constant $C\geq 0$ does not depend on $x\in[0,1]$.

Proof : For $\epsilon>0$, we set $\varphi^\epsilon(x')=e^{-(x'-x)^2/(2\epsilon)}/\sqrt{2\pi\epsilon}$ and $v^\epsilon(t,z)=\int_z^1 w^{\varphi^\epsilon}(t,z')dz'=\int_z^1\int_0^1 w(x',t,z')\varphi^\epsilon(x')dx'dz'$. Applying Itô's formula like in the proof of Lemma 3.9, we obtain that if $\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}$

$$\mathbb{E}v^\epsilon(t_l,X^{y_0^i}_{t_l})-v^\epsilon(t_l,Z^i_{t_l})-\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}}(b(s,Z^i_s)-A'(\overline{V}(t_k,Y^i_{t_k})))\partial_z v^\epsilon(s,Z^i_s)ds$$
$$=-\sigma\int_0^{t_l}\partial_z v^\epsilon(s,Z^i_s)dW^i_s=-\sigma\int_0^{t_l}\partial_z v^\epsilon(s,\hat{Z}^i_s)dW^i_s \tag{3.24}$$

where $\hat{Z}^i$ is the continuous process satisfying

$$\begin{cases}\hat{Z}^i_0=y^i_0\\ \forall t\in[t_k,t_{k+1}],\hat{Z}^i_t=\hat{Z}^i_{t_k}+\sigma(W^i_t-W^i_{t_k})+(t-t_k)A'(\overline{V}(t_k,Y^i_{t_k}))+\hat{K}^i_t\\ |\hat{K}^i|_t=\int_0^t\mathbb{1}_{\{0,1\}}(\hat{Z}^i_s)d|\hat{K}^i|_s,\ \text{and}\ \hat{K}^i_t=\int_0^t(1-2\hat{Z}^i_t)d|\hat{K}^i|_s.\end{cases}$$

Since according to Lemma 3.8, $(\partial_z v^\epsilon(s,z))^2\leq\frac{C}{t_l-s}\exp\left(-c\frac{(z-x)^2}{t_l-s}\right)$, the inequality

$$\mathbb{E}(\partial_z v^\epsilon(s,\hat{Z}^i_s))^2\leq C(t_l-s)^{-2/3}s^{-1/3}$$

where the constant C does not depend on ϵ and x, is obtained like the upperbound of $\mathbb{E}|\partial_z w(\theta,\hat{X}^y_\theta)|$ in the proof of Lemma 3.11. Hence $\mathbb{E}\left(\int_0^{t_l}(\partial_z v^\epsilon(s,\hat{Z}^i_s))^2ds\right)\leq C$ and

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}\}}\int_0^{t_l}\partial_z v^\epsilon(s,\hat{Z}^i_s)dW^i_s\right|$$
$$\leq\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\int_0^{t_l}\partial_z v^\epsilon(s,\hat{Z}^i_s)dW^i_s\right|+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\mathbb{1}_{\{\exists 1\leq k\leq l,\ Z^i_{t_k^-}\neq Y^i_{t_k}\}}\int_0^{t_l}\partial_z v^\epsilon(s,\hat{Z}^i_s)dW^i_s\right|$$
$$\leq\left(\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\int_0^{t_l}(\partial_z v^\epsilon(s,\hat{Z}^i_s))^2ds\right)^{1/2}$$
$$+\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{P}(\exists 1\leq k\leq l,\ Z^i_{t_k^-}\neq Y^i_{t_k})\right)^{1/2}\left(\mathbb{E}\int_0^{t_l}(\partial_z v^\epsilon(s,\hat{Z}^i_s))^2ds\right)^{1/2}$$
$$\leq C\left(\frac{1}{\sqrt{N}}+\Delta t\right)\quad\text{according to Lemma 3.6.}$$

By (3.24), we deduce that

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}\}}\left(\mathbb{E}(v^\epsilon(t_l,X^{y_0^i}_{t_l}))-v^\epsilon(t_l,Z^i_{t_l^-})\right.\right.$$
$$\left.\left.-\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}}(A'(\overline{V}(t_k,Y^i_{t_k}))-b(s,Z^i_s))(-\partial_z v^\epsilon(s,Z^i_s)ds\right)\right|\leq C\left(\frac{1}{\sqrt{N}}+\Delta t\right)$$

and we conclude by taking the limit $\epsilon \to 0$ like in the proof of Lemma 3.9. ∎

By (3.23) and the previous Lemma, we obtain that

$$\begin{aligned}
\mathrm{Error}(t_l) &\leq C\left(\tfrac{1}{\sqrt{N}} + \Delta t\right) \\
&+ \sup_{x\in[0,1]} \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{\forall 1\leq k\leq l,\ Z^i_{t_k^-}=Y^i_{t_k}\}} \sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(A'(\overline{V}(t_k,Y^i_{t_k})) - b(s,Z^i_s)\right) ds \right|.
\end{aligned}$$

Since according to Lemma 3.8, $w(x,s,z) \leq C/\sqrt{t_l - s}$, using once more Lemma 3.6, we get

$$\begin{aligned}
\mathrm{Error}(t_l) &\leq C\left(\tfrac{1}{\sqrt{N}} + \Delta t\right) \\
&+ \sup_{x\in[0,1]} \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(A'(\overline{V}(t_k,Y^i_{t_k})) - b(s,Z^i_s)\right) ds \right|.
\end{aligned}$$

We consider now the last term in the upperbound of $\mathrm{Error}(t_l)$. We split it in two parts, in order to introduce the difference between the drift function $b(t_k,\cdot) = A'(V(t_k,.))$ and its approximation $A'(\overline{V}(t_k,\cdot))$ at the same point $Y^i_{t_k}$. As $\forall 0 \leq k \leq L-1,\ Z^i_{t_k} = Y^i_{t_k}$, we get

$$\begin{aligned}
&\mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(A'(\overline{V}(t_k,Y^i_{t_k})) - b(s,Z^i_s)\right) ds \right| \\
&\leq \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(b(s,Z^i_s) - b(t_k,Z^i_{t_k})\right) ds \right| \qquad (3.25)\\
&+ \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(b(t_k,Y^i_{t_k}) - A'(\overline{V}(t_k,Y^i_{t_k}))\right) ds \right|.
\end{aligned}$$

The first term in the right-hand-side of (3.25) is a time discretization error. In order to obtain an error bound of order $\mathcal{O}(\Delta t)$, we need an expectation inside the absolute value. If for $k \in \{0,\ldots,L\}$, we denote $\mathcal{F}_{t_k} \stackrel{\text{def}}{=} \sigma(W^i_s; 0 \leq s \leq t_k, i = 1,\ldots,N)$, then for all $s \in [t_k,t_{k+1})$, the variables $(R^i_{t_k,s} \stackrel{\text{def}}{=} w(x,s,Z^i_s)\left[b(s,Z^i_s) - b(t_k,Z^i_{t_k})\right], i = 1,\ldots,N)$ are $\mathcal{F}_{t_k}$-independent. Hence,

$$\mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N} R^i_{t_k,s} - \mathbb{E}^{\mathcal{F}_{t_k}}\left(R^i_{t_k,s}\right)\right| \leq \frac{1}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(R^i_{t_k,s}\right)^2} \leq \frac{C}{\sqrt{N}}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(w(x,s,Z^i_s)\right)^2}$$

Using once more that $w(x,s,z) \leq C/\sqrt{t_l - s}$, we easily obtain that

$$\begin{aligned}
&\mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(b(s,Z^i_s) - b(t_k,Z^i_{t_k})\right) ds \right| \\
&\leq \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} \mathbb{E}^{\mathcal{F}_{t_k}}\left\{w(x,s,Z^i_s)\left[b(s,Z^i_s) - b(t_k,Z^i_{t_k})\right]\right\} ds \right| + \frac{C}{\sqrt{N}}.
\end{aligned}$$

To obtain an upper-bound of order $\mathcal{O}(\Delta t)$ for the first term in the right-hand-side of the previous inequality, we just have to remark that we are now in the same context as in the proof of Proposition 3.7 : Equality (3.16) and Lemma 3.11 are valid replacing $\hat{X}$ by Z^i and T by t_l. Following the proof of Proposition 3.7 we conclude that

$$\sup_{x\in[0,1]} \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z^i_s)\left(b(s,Z^i_s) - b(t_k,Z^i_{t_k})\right) ds \right| \leq C\Delta t + \frac{C}{\sqrt{N}}. \qquad (3.26)$$

For the second term in (3.25), by the upper-bound $C/\sqrt{t_l - s}$ for $w(x,s,z)$ and since by definition $b(t,.) = A'(V(t,.))$, we get

$$\begin{aligned}
&\sup_{x\in[0,1]} \mathbb{E}\left| \frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{l-1}\int_{t_k}^{t_{k+1}} w(x,s,Z_s^i)\left(b(t_k,Y_{t_k}^i) - A'(\overline{V}(t_k,Y_{t_k}^i))\right) ds\right| \\
&\leq C \sup_{v\in[0,1]} |A'(v)| \sum_{k=0}^{l-1}\left(\int_{t_k}^{t_{k+1}} \frac{1}{\sqrt{t_l - s}} ds\right) \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_k,Y_{t_k}^i) - \overline{V}(t_k,Y_{t_k}^i)\right|.
\end{aligned} \tag{3.27}$$

Lemma 3.14

$$\forall 0 \leq l \leq L, \quad \frac{1}{N}\sum_{i=1}^{N}\left|V(t_l,Y_{t_l}^i) - \overline{V}(t_l,Y_{t_l}^i)\right| \leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right).$$

Using Lemma 3.14 in (3.27), with (3.26) we come back to (3.25) and deduce that

$$\text{Error}(t_l) \leq C\left(\frac{1}{\sqrt{N}} + \Delta t\right),$$

which ends the proof of Theorem 3.2.

We are now concentrated on the upper bound of

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_l,Y_{t_l}^i) - \overline{V}(t_l,Y_{t_l}^i)\right|$$

given in Lemma 3.14. Because of the complex form of $\overline{V}(t_l,Y_{t_l}^i)$, we need to introduce another auxiliary family of discrete time processes : let $(\overline{X}_t^i, t\in[0,T], i=1,\dots,N)$ denote the solution of the following Euler-Peano equations,

$$\begin{cases}
\overline{X}_0^i = y_0^i \\
\forall t\in[t_l,t_{l+1}], \overline{X}_t^i = \overline{X}_{t_l}^i + \sigma(W_t^i - W_{t_l}^i) + (t-t_l)b(t_l,\overline{X}_{t_l}^i) + \overline{K}_t^i \\
|\overline{K}^i|_t = \int_0^t \mathbb{1}_{\{0,1\}}(\overline{X}_s^i) d|\overline{K}^i|_s, \text{ and } \overline{K}_t^i = \int_0^t (1 - 2\overline{X}_t^i) d|\overline{K}^i|_s.
\end{cases} \tag{3.28}$$

We will compare $\overline{V}(t_l,Y_{t_l}^i) = \frac{1}{N}\sum_{j=1}^{N} H(Y_{t_l}^i - Y_{t_l}^j)$ with the same expression written with the system of independent particles $\frac{1}{N}\sum_{j=1}^{N} H(\overline{X}_{t_l}^i - \overline{X}_{t_l}^j)$.

First, we note that

$$\begin{aligned}
\mathbb{E}\left|V(t_l,Y_{t_l}^i) - V(t_l,\overline{X}_{t_l}^i)\right| &= \mathbb{E}\left|V(t_l,Z_{t_l}^i) - V(t_l,\overline{X}_{t_l}^i)\right| \\
&\leq \mathbb{E}\left|V(t_l,Z_{t_l^-}^i) - V(t_l,\overline{X}_{t_l}^i)\right| + C\Delta t^2 \\
&\leq C|Z_{t_l^-}^i - \overline{X}_{t_l}^i| + C\Delta t^2.
\end{aligned}$$

The first inequality is obtained thanks to Lemma 3.6 which compares Peano and Lépingle schemes. The second one uses the Lipschitz property of V stated in Lemma 3.1. Now, using arguments similar to those given at the beginning of the proof of Lemma 3.3, one can easily check that

$$\begin{aligned}
|Z_{t_l^-}^i - \overline{X}_{t_l}^i| &\leq |Y_{t_{l-1}}^i - \overline{X}_{t_{l-1}}^i| + \Delta t C|\overline{V}(t_{l-1},Y_{t_{l-1}}^i) - V(t_{l-1},\overline{X}_{t_{l-1}}^i)| \\
&\leq |Z_{t_{l-1}^-}^i - \overline{X}_{t_{l-1}}^i| + C\Delta t^2 + \Delta t C|\overline{V}(t_{l-1},Y_{t_{l-1}}^i) - V(t_{l-1},\overline{X}_{t_{l-1}}^i)|.
\end{aligned}$$

By induction, we deduce that

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|V(t_l,Y_{t_l}^i) - V(t_l,\overline{X}_{t_l}^i)\right| \leq C\Delta t \sum_{m=0}^{l-1} \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left|\overline{V}(t_m,Y_{t_m}^i) - V(t_m,\overline{X}_{t_m}^i)\right| + C\Delta t. \tag{3.29}$$

For all $k \in \{0,\dots,L\}$ we set

$$\bar{E}(t_k) \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\overline{V}(t_k, Y^i_{t_k}) - V(t_k, \overline{X}^i_{t_k})\right|,$$

so that, by (3.29),

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_l, Y^i_{t_l}) - \overline{V}(t_l, Y^i_{t_l})\right| \quad\leq\quad \bar{E}(t_l) + C\Delta t\sum_{m=0}^{l-1}\bar{E}(t_m) + C\Delta t. \tag{3.30}$$

We have transformed the estimation of $\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_l, Y^i_{t_l}) - \overline{V}(t_l, Y^i_{t_l})\right|$ into the estimation of each $\bar{E}(t_m)$ for $0 \leq m \leq L$. For $m = 0$, $\bar{E}(0) = \frac{1}{N}\sum_{i=1}^{N}|\overline{V}(0, y^i_0) - V(0, y^i_0)| \leq \frac{1}{N}$, by Lemma 3.4. For $m \geq 1$, we insert the term $\frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_l} - \overline{X}^j_{t_l})$ in the expression of $\bar{E}(t_m)$ to split it in two part :

$$\begin{aligned}
\bar{E}(t_m) \quad &\leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\overline{V}(t_m, Y^i_{t_m}) - \frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m})\right| \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m}) - V(t_m, \overline{X}^i_{t_m})\right| \\
&\leq \frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left|H(Y^i_{t_m} - Y^j_{t_m}) - H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m})\right| \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m}) - V(t_m, \overline{X}^i_{t_m})\right|.
\end{aligned} \tag{3.31}$$

To deal with the first term in the right-hand-side, we introduce the errors $\bar{E}(t_k)$ for $k \leq m-1$. The second term is very similar to error terms we have already treated. The upper-bound of these terms are respectively given in the following Lemmas the proofs of which are postponed :

Lemma 3.15

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\mathbb{E}\left|H(Y^i_{t_m} - Y^j_{t_m}) - H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m})\right| \leq C\Delta t + C\Delta t\sum_{k=0}^{m-1}\frac{\bar{E}(t_k)}{\sqrt{t_m - t_k}}.$$

Lemma 3.16

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m}) - V(t_m, \overline{X}^i_{t_m})\right| \leq \frac{C}{\sqrt{N}} + C\Delta t.$$

Coming back to (3.31), we have obtained that

$$\bar{E}(t_m) \leq C\Delta t + \frac{C}{\sqrt{N}} + C\Delta t\sum_{k=0}^{m-1}\frac{\bar{E}(t_k)}{\sqrt{t_m - t_k}}.$$

Using a discrete time version of Gronwall's Lemma, we obtain that $\forall m \leq L$, $\bar{E}(t_m) \leq C\Delta t + \frac{C}{\sqrt{N}}$. By (3.30), we conclude that Lemma 3.14 holds.

Proof of Lemma 3.15 : The main difficulty is to deal with the non Lipschitz Heaviside function H. To overcome this difficulty, the idea consists in smoothing H thanks to the probability transition density of the Euler-Peano scheme. First, we note that

$$\begin{aligned}
&H(Y^i_{t_m} - Y^j_{t_m}) - H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m}) \\
&= \sum_{k=0}^{m-1} H\left(\overline{X}^{i,t_{m-k},Y^i_{t_{m-k}}}_{t_m} - \overline{X}^{j,t_{m-k},Y^j_{t_{m-k}}}_{t_m}\right) - H\left(\overline{X}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}}_{t_m} - \overline{X}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}_{t_m}\right).
\end{aligned}$$

where for $0 \leq k \leq L$, $y \in [0,1]$ and $1 \leq i \leq N$, $(\overline{X}_t^{i,t_k,y})_{t\in[t_k,T]}$ denote the Euler Peano process starting from $\overline{X}_{t_k}^{i,t_k,y} = y$ at time t_k and with posterior evolution given by (3.28). By Lemma 3.6, replacing $Y^i_{t_{m-k}}$ by $Z^i_{t^-_{m-k}}$ in the expression above has a cost of order $\mathcal{O}(\Delta t^2)$. Hence, using the inequality

$$\mathbb{E}|H(A) - H(B)| = \mathbb{P}(A \geq 0, B < 0) + \mathbb{P}(A < 0, B \geq 0) \leq \mathbb{P}\left(|B| \leq |B - A|\right),$$

$$\begin{aligned}
&\mathbb{E}\left|H\left(\overline{X}_{t_m}^{i,t_{m-k},Y^i_{t_{m-k}}} - \overline{X}_{t_m}^{j,t_{m-k},Y^j_{t_{m-k}}}\right) - H\left(\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_l}^{j,t_{l-m-1},Y^j_{t_{l-m-1}}}\right)\right| \\
&\leq C\Delta t^2 + \mathbb{E}\left|H\left(\overline{X}_{t_m}^{i,t_{m-k},Z^i_{t^-_{m-k}}} - \overline{X}_{t_m}^{j,t_{m-k},Z^j_{t^-_{m-k}}}\right) - H\left(\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}\right)\right| \\
&\leq C\Delta t^2 + \mathbb{P}\left(\left|\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}\right|\right. \\
&\qquad\qquad \left.\leq \left|\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}} - \overline{X}_{t_m}^{i,t_{m-k},Z^i_{t^-_{m-k}}} + \overline{X}_{t_m}^{j,t_{m-k},Z^j_{t^-_{m-k}}}\right|\right) \\
&\leq C\Delta t^2 + \mathbb{P}\Bigg(\left|\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}\right| \\
&\qquad\qquad \leq C\Delta t\left(\left|V - \overline{V}\right|(t_{m-k-1}, Y^i_{t_{m-k-1}}) + \left|V - \overline{V}\right|(t_{m-k-1}, Y^j_{t_{m-k-1}})\right)\Bigg),
\end{aligned}$$

as for $i = 1, \ldots, N$,

$$\begin{aligned}
\left|\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{i,t_{m-k},Z^i_{t^-_{m-k}}}\right| &\leq C\left|\overline{X}_{t_{m-k}}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - Z^i_{t^-_{m-k}}\right| \\
&\leq C\Delta t\left|V - \overline{V}\right|(t_{m-k-1}, Y^i_{t_{m-k-1}}).
\end{aligned}$$

The variable $\left|V - \overline{V}\right|(t_{m-k-1}, Y^i_{t_{m-k-1}}) + \left|V - \overline{V}\right|(t_{m-k-1}, Y^j_{t_{m-k-1}})$ is $\mathcal{F}_{t_{m-k-1}}$ measurable. Moreover, for $i \neq j$, conditionally on $\mathcal{F}_{t_{m-k-1}}$, the variables $\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}}$ and $\overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}$ are independent and admit densities with respect to Lebesgue measure with a L^2 norm smaller than $C/(t_{k+1})^{1/4}$ (by Girsanov Theorem, as in the proof of Lemma 2.4). Hence, conditionally on $\mathcal{F}_{t_{m-k-1}}$, the variable $\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}$ admits a density with respect to Lebesgue measure with a L^∞ norm smaller than $C/\sqrt{t_{k+1}}$ and we deduce that

$$\begin{aligned}
&\mathbb{P}\Bigg(\left|\overline{X}_{t_m}^{i,t_{m-k-1},Y^i_{t_{m-k-1}}} - \overline{X}_{t_m}^{j,t_{m-k-1},Y^j_{t_{m-k-1}}}\right| \\
&\qquad\qquad \leq C\Delta t\left(\left|V - \overline{V}\right|(t_{m-k-1}, Y^i_{t_{m-k-1}})\left|V - \overline{V}\right|(t_{m-k-1}, Y^j_{t_{m-k-1}})\right)\Bigg) \\
&\leq C\frac{\Delta t}{\sqrt{t_{k+1}}}\mathbb{E}\left(\left|V - \overline{V}\right|(t_{m-k-1}, Y^i_{t_{m-k-1}}) + \left|V - \overline{V}\right|(t_{m-k-1}, Y^j_{t_{m-k-1}})\right).
\end{aligned}$$

We conclude that

$$\frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\left|H(Y^i_{t_m} - Y^j_{t_m}) - H(\overline{X}^i_{t_m} - \overline{X}^j_{t_m})\right| \leq C\Delta t + C\Delta t\sum_{k=0}^{m-1}\frac{\bar{E}(t_{m-k-1})}{\sqrt{t_{k+1}}}.$$

∎

Proof of Lemma 3.16 : We are going to decompose the expression of interest in order to introduce error terms that we have already bounded.

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|V(t_m,\overline{X}^i_{t_m})-\frac{1}{N}\sum_{j=1}^{N}H(\overline{X}^i_{t_m}-\overline{X}^j_{t_m})\right| \\
&\leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big(|V(t_m,x)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}H(x-X^{y_0^j}_{t_m})|\Big|_{x=\overline{X}^i_{t_m}}\Big) \\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{N}|\mathbb{E}H(x-X^{y_0^j}_{t_m})-\mathbb{E}H(x-\overline{X}^j_{t_m})|\Big|_{x=\overline{X}^i_{t_m}}\right) \\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(|\frac{1}{N}\sum_{j=1}^{N}(\mathbb{E}H(x-\overline{X}^j_{t_m})|_{x=\overline{X}^i_{t_m}}-H(\overline{X}^i_{t_m}-\overline{X}^j_{t_m}))|\right) \\
&\leq \sup_{x\in[0,1]}\left|V(t_m,x)-\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}H(x-X^{y_0^j}_{t_m})\right| \\
&\quad+\sup_{x\in[0,1],j\leq N}\left|\mathbb{E}H(x-X^{y_0^j}_{t_m})-\mathbb{E}H(x-\overline{X}^j_{t_m})\right| \\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\Big(\frac{1}{N^2}\sum_{j,k=1}^{N}(\mathbb{E}H(x-\overline{X}^j_{t_m})|_{x=\overline{X}^i_{t_m}}-H(\overline{X}^i_{t_m}-\overline{X}^j_{t_m}))\right. \\
&\qquad\qquad\left.(\mathbb{E}H(x-\overline{X}^k_{t_m})|_{x=\overline{X}^i_{t_m}}-H(\overline{X}^i_{t_m}-\overline{X}^k_{t_m}))\Big)\right)^{\frac{1}{2}}.
\end{aligned}
$$

The first term in the right-hand-side is the initialisation error bounded in Lemma 3.4 by $\frac{1}{N}$. The second term is the weak time discretization error for the Euler-Peano scheme bounded in Proposition 3.7 by $C\Delta t$. The last term is a statistical error : it is smaller than $1/\sqrt{N}$ since by independence of the variables $(\overline{X}^j_{t_m}, 1\leq j\leq N)$, each term of the summation $\sum_{j,k=1}^{N}$ with $j\neq k$ is nil. ∎

4 Numerical experiments

As a numerical benchmark, we consider the following Dirichlet problem for the viscous Burgers equation which corresponds to the choice $A(x)=x^2/2$:

$$
\begin{cases}
\frac{\partial}{\partial t}v(t,x)=\dfrac{\partial^2 v}{\partial x^2}(t,x)-v(t,x)\dfrac{\partial v}{\partial x}(t,x), t>0, x\in[0,2\pi] \\
v(0,x)=\dfrac{2\sin(x)}{\cos(x)+\mathrm{e}}, x\in[0,2\pi] \text{ and } \forall t\geq 0,\ v(t,0)=0, v(t,2\pi)=0,
\end{cases}
\tag{4.1}
$$

The exact solution is (see [2]) $V(t,x)=2\sin(x)/(\cos(x)+\mathrm{e}^{(1+t)})$.
The spatial domain $[0,2\pi]$ is different from the one considered so far but our results remain true for any bounded interval replacing $[0,1]$. The fact that the distribution derivative $m_0(x)dx$ of the initial data $v(0,x)$ given by $m_0(x)=(2+2\mathrm{e}\cos(x))/(\cos(x)+\mathrm{e})^2$ is not a probability measure but a bounded signed measure represents a more significant modification. In fact, we could not find any explicit solution when $v(0,.)$ is the cumulative distribution function of a probability measure.
To take into account this modification, we use weighted particles $(Y^i_{t_l},w^i)_{1\leq i\leq N}$ (see for instance [8] which deals with a spatial domain equal to $\mathbb{R}$). The N initial locations $y_0^i = \inf\{y; H*|m_0|/\|m_0\|_{L^1([0,2\pi])}(y)=\frac{i}{N}\}$ are chosen in order to approximate the cumulative distribution function of the probability measure $|m_0|(x)dx/\|m_0\|_{L^1([0,2\pi])}$ and the

corresponding weights are $w_i = \|m_0\|_{L^1([0,2\pi])}\text{sign}(m_0(y_0^i))$. The approximate solution is given by the weighted cumulative distribution function of the particle system $\overline{V}(t_l,x) = \frac{1}{N}\sum_{i=1}^N w^i H(x - Y_{t_l}^i)$ where the successive positions are defined inductively by (3.2) but with $\wedge 1$ (resp. -1) replaced by $\wedge 2\pi$ (resp. -2π) in the second (resp. last) line.
The parameters of Lépingle scheme are $\alpha_0 = 0.25$ and $\alpha_1 = 2\pi - 0.25$. We have plotted on Figure 1 the numerical solution at time $t = 1$. As the dependence of the error on the number

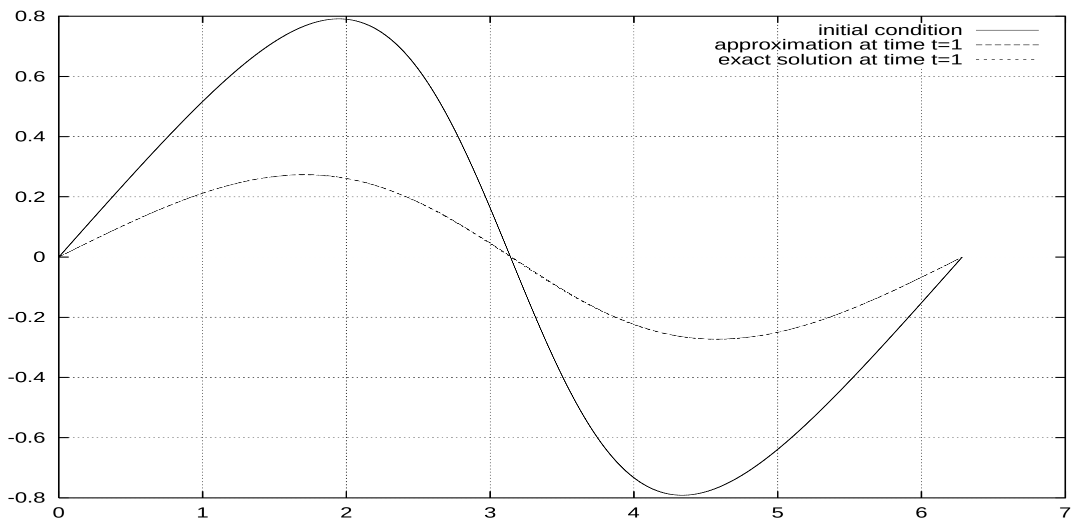

Figure 1: *Exact and numerical solutions of (4.1) obtained at time $t = 1$, for 10^6 particles and $\Delta t = 10^{-2}$ with the Lépingle scheme.*

of particles is standard and corresponds to the usual central limit theorem rate (see [4][5][8] for numerical results in case the spatial domain is $\mathbb{R}$), we concentrate our numerical study on the dependence on the time step. That is why we take a large number of particles $N = 10^6$. According to Theorem 3.2, $\mathbb{E}\|V(1,.)-\overline{V}(1,.)\|_{L^1([0,2\pi])} \leq 2\pi \sup_{x\in[0,2\pi]} \mathbb{E}|V(1,x)-\overline{V}(1,x)| \leq C(\Delta t + N^{-1/2})$. Since it is not possible to compute the last quantity, we compute the first one by averaging $\|V(1,.) - \overline{V}(1,.)\|_{L^1([0,2\pi])}$ over 20 runs of the particle method and give the dependence of the result on Δt in Table 1 and Figure 2.

Δt	Lépingle scheme	Confidence interval at 95%	Projection scheme	Confidence interval at 95%
2^{-1}	0.0940	[0.0933,0.0946]	0.2510	[0.2501,0.2519]
2^{-2}	0.0585	[0.0579,0.0591]	0.2320	[0.2309,0.2329]
2^{-3}	0.0329	[0.0322,0.0336]	0.1964	[0.1953,0.1975]
2^{-4}	0.0173	[0.0166,0.0180]	0.1568	[0.1557,0.1578]
2^{-5}	0.0083	[0.0076,0.0090]	0.1241	[0.1227,0.1254]
2^{-6}	0.0053	[0.0045,0.0060]	0.0982	[0.0969,0.0995]
2^{-7}	0.0049	[0.0043,0.0055]	0.0779	[0.0765,0.0793]
2^{-8}	0.0050	[0.0042,0.0058]	0.0635	[0.0627,0.0643]

Table 1: *Expectation of L^1 norm of error at $t = 1$ for $N = 10^6$ particles ($\|V(\cdot,1)\|_{L^1([0,2\pi])} = 1.09$)*

We need to check that our test case (4.1) produces a significant rate of effective reflections. If this rate is too small, we only observe the effect of the classical Euler scheme (without reflection) with weak convergence also in Δt, and we cannot conclude on the convergence of the Lépingle scheme. The rate of effective reflections is around 10% for this test case : more precisely there are about 10% of the particles in $[0,\alpha_0]\cup[\alpha_1,2\pi]$ at each time-step. For these particles, we compute the correction term C in (3.2). When we discretize the particle system according to the projected Euler scheme, which treats the reflection simply by projection onto $[0,1]$, we clearly observe a sublinear convergence in Δt (see Table 1 and Figure 2). The projected Euler scheme does not use the correction term C whatever the position of the particle and its weak convergence rate is in $\mathcal{O}(\Delta t^{1/2})$, (see [6]). Therefore we can conclude that the quasi-linear decreasing of the error for the Lépingle scheme confirms our theoretical analysis.

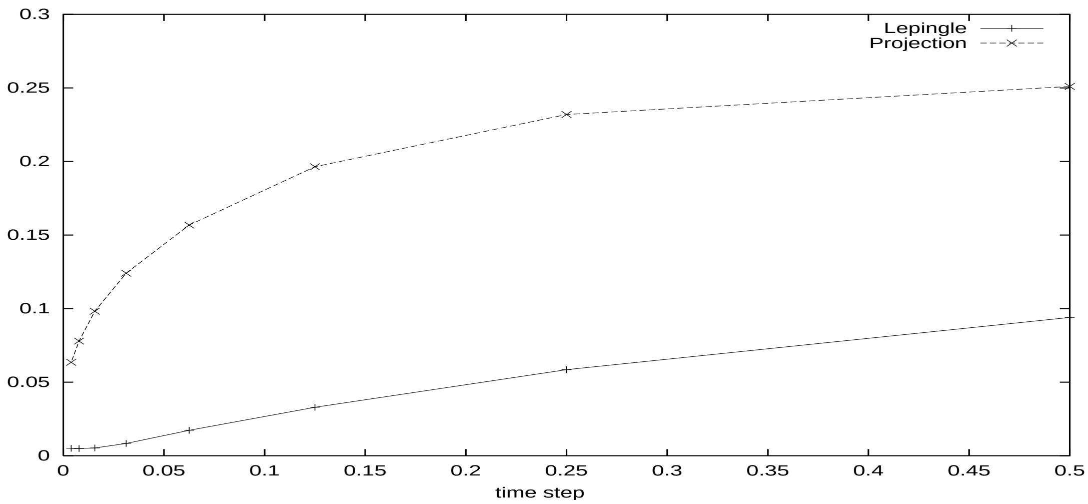

Figure 2: $\mathbb{E}\|V(\cdot,1)-\overline{V}(\cdot,1)\|_{L^1(\mathbb{R})}$ *in terms of* Δt *(*$N=10^6$*).*

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