Nonlinear processes associated with the discrete Smoluchowski coagulation fragmentation equation

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Abstract

This paper is dedicated to the probabilistic interpretation of the mass-flow equation which is associated with the discrete Smoluchowski coagulation fragmentation equation. The mass-flow equation describes the evolution in time of the distribution of the mass with respect to the size of the clusters when the expected numbers of clusters follow Smoluchowski's equation. Under various assumptions on the coagulation and the fragmentation kernels, we construct nonlinear processes linked with the mass-flow equation : the time-marginals of their law solve this equation. When possible, we approximate these processes thanks to simulable interacting particle systems. We deduce some existence and uniqueness results concerning the discrete Smoluchowski coagulation fragmentation equation which seem to be new.

The discrete Smoluchowski coagulation fragmentation equation describes the evolution of the expected number $c_i(t)$ of clusters with mass $i \in \mathbb{N}^*$ when two clusters with respective masses j and k coagulate at rate $K_{j,k}$ to form a cluster with mass $j+k$ whereas a cluster with mass $j+k$ breaks up at rate $F_{j,k}$ into two clusters with masses j and k :

$$\begin{cases} \partial_t c_t(i) = \frac{1}{2}\sum_{j=1}^{i-1}\left(K_{i-j,j}c_t(i-j)c_t(j) - F_{i-j,j}c_t(i)\right) - \sum_{j\in\mathbb{N}^*}\left(K_{i,j}c_t(j)c_t(i) - F_{i,j}c_t(i+j)\right) \\ c_0(i) = \gamma(i). \end{cases} \tag{0.1}$$

We assume that the initial distribution $\gamma \in \mathbb{R}_+^{\mathbb{N}^*}$ has finite mass i.e $\sum_{i\in\mathbb{N}^*} i\gamma(i) < +\infty$. The kernels $K_{j,k}$ and $F_{j,k}$ are supposed to be non-negative and symmetric : $K_{j,k} = K_{k,j}$ and $F_{j,k} = F_{k,j}$. Since both in the coagulation phenomenom $(j,k) \to j+k$ and the reverse fragmentation reaction $j+k \to (j,k)$, the mass is conserved, one would expect a solution of (0.1) to satisfy $\forall t \geq 0,\ \sum_{i\in\mathbb{N}^*} ic_t(i) = \sum_{i\in\mathbb{N}^*} i\gamma(i)$. In the pure fragmentation case ($K_{j,k} \equiv 0$), it is possible to construct solutions with increasing mass (see [3]). These solutions have to be rejected for obvious physical reasons. In the pure coagulation case ($F_{j,k} = 0$), it may happen that the mass decreases after a finite time. Intuitively, this phenomenom called gelation corresponds to the formation of an infinite cluster. That is why we consider solutions of Smoluchowski's equation in the following sense : for $T \in (0,+\infty]$, we say $t \in [0,T) \to c_t \in \{c : \mathbb{N}^* \to \mathbb{R}_+,\ \sum_{i\in\mathbb{N}^*} ic(i) \leq \sum_{i\in\mathbb{N}^*} i\gamma(i)\}$ solves this equation on $[0,T)$ if $\forall i \in \mathbb{N}^*$, $\forall t \in [0,T)$, $s \to \sum_{j\in\mathbb{N}^*} K_{i,j}c_s(j)$ and

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$s \to \sum_{j\in\mathbb{N}^*} F_{i,j} c_s(i+j)$ are integrable on $(0,t)$ and

$$c_t(i) = \gamma(i) + \int_0^t \frac{1}{2}\sum_{j=1}^{i-1} (K_{i-j,j} c_s(i-j)c_s(j) - F_{i-j,j}c_s(i)) - \sum_{j\in\mathbb{N}^*} (K_{i,j}c_s(j)c_s(i) - F_{i,j}c_s(i+j))\, ds.$$

Since c_t solves (0.1) if and only if $c_t / \sum_{i\in\mathbb{N}^*} i\gamma(i)$ solves the same equation but with coagulation kernel $K_{j,k}$ multiplied by $\sum_{i\in\mathbb{N}^*} i\gamma(i)$ and initial data $\gamma/\sum_{i\in\mathbb{N}^*} i\gamma(i)$, we can suppose without restriction that $\sum_{i\in\mathbb{N}^*} i\gamma(i) = 1$ i.e. that $(\rho(i) = i\gamma(i))_{i\in\mathbb{N}^*}$ is a probability distribution on $\mathbb{N}^*$. By symmetry of the kernels K and F,

$$\begin{aligned}\frac{i}{2}\sum_{j=1}^{i-1} K_{i-j,j}c_s(i-j)c_s(j) &= \frac{1}{2}\sum_{j=1}^{i-1}((i-j)+j)K_{i-j,j}c_s(i-j)c_s(j)\\ &= \sum_{j=1}^{i-1} \frac{K_{i-j,j}}{j}(jc_s(j))((i-j)c_s(i-j)),\end{aligned}$$

$$\text{and}\quad \frac{1}{2}\sum_{j=1}^{i-1} F_{i-j,j} = \frac{1}{2}\sum_{j=1}^{i-1} \frac{(i-j)+j}{i} F_{i-j,j} = \sum_{j=1}^{i-1} \frac{i-j}{i} F_{i-j,j}.$$

Hence setting

$$\tilde{K}_{k,j} = \frac{K_{k,j}}{j} \quad\text{and}\quad \tilde{F}_{k,j} = \frac{kF_{k,j}}{j+k},$$

we obtain that $(c_t(i))$ solves (0.1) if and only if $(p_t(i) = ic_t(i))$ solves the mass-flow equation (see [2] [10] where this link is made respectively for the discrete and the general Smoluchowski equation without fragmentation)

$$\begin{cases} p_0(i) = \rho(i) \\ \partial_t p_t(i) = \sum_{j=1}^{i-1}\left(\tilde{K}_{i-j,j}p_t(i-j)p_t(j) - \tilde{F}_{i-j,j}p_t(i)\right) - \sum_{j\in\mathbb{N}^*}\left(\tilde{K}_{i,j}p_t(i)p_t(j) - \tilde{F}_{i,j}p_t(i+j)\right) \\ \qquad = \sum_{j=1}^{i-1} W_{i-j,j}(p_t) - \sum_{j\in\mathbb{N}^*} W_{i,j}(p_t) \text{ for } W_{k,j}(p_t) = \tilde{K}_{k,j}p_t(k)p_t(j) - \tilde{F}_{k,j}p_t(k+j) \end{cases} \tag{0.2}$$

in the following sense :

Definition 0.1 *Let $T \in (0,+\infty]$. We say that $t \to p_t$ solves (0.2) on $[0,T)$ if $\forall t \in (0,T)$,*

(i) $p_t \in \{q : \mathbb{N}^* \to \mathbb{R}_+,\ \sum_{i\in\mathbb{N}^*} q(i) \le 1\}$

(ii) $\forall i \in \mathbb{N}^*$, *$s \to \sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j}p_s(j)$ and $s \to \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j}p_s(i+j)$ are integrable on $(0,t)$ and*

$$p_t(i) = \rho(i) + \int_0^t \sum_{j=1}^{i-1}\left(\tilde{K}_{i-j,j}p_s(i-j)p_s(j) - \tilde{F}_{i-j,j}p_s(i)\right) - \sum_{j\in\mathbb{N}^*}\left(\tilde{K}_{i,j}p_s(j)p_s(i) - \tilde{F}_{i,j}p_s(i+j)\right) ds.$$

In this paper, we are going to study (0.2) under various kind of assumptions on the coagulation and fragmentation kernels. Following [7], we say that

$(K\alpha)$ holds for some $\alpha \in (1/2, 1]$ if $\exists \kappa > 0$ s.t. $\forall i,j \ge 1,\ K_{i,j} \le \kappa i^\alpha j^\alpha$ i.e. $\tilde{K}_{i,j} \le \kappa i^\alpha j^{\alpha-1}$.

$(F\gamma)$ holds for some $\gamma > 0$ if $\forall \mu \ge 0,\ \exists C(\mu) > 0,\ \forall i \ge 3,\ \sum_{j=1}^{[(i-1)/2]} j^\mu F_{j,i-j} \ge C(\mu) i^{\gamma+\mu}$ where $[x]$ denotes the integer part of x.

Hypothesis $(F\gamma)$ is the so-called strong fragmentation condition and is satisfied for the kernel $F_{j,k} = (jk)^\beta$ with $\beta = (\gamma-1)/2$ and for the kernel $F_{j,k} = (j+k)^\beta$ with $\beta = \gamma - 1$. Note that when $(K\alpha)$ holds for $\alpha \in (1/2, 1]$ then for $q : \mathbb{N}^* \to \mathbb{R}_+$ such that $\sum_{j\in\mathbb{N}^*} q(j) \le 1$, $\sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j} q(j) \le \kappa i^\alpha \sum_{j\in\mathbb{N}^*} j^{\alpha-1} q(j) \le \kappa i^\alpha$ and in definition 0.1, the integrability condition on $\sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j} p_s(j)$ in (ii) is a consequence of (i).

Many mathematical studies have been devoted to the Smoluchowski coagulation equation particularly in the absence of fragmentation $(F_{i,j} \equiv 0)$: see for instance the survey of Aldous [1] and the references cited therein. In the presence of fragmentation, less is known. In [17], assuming $K_{i,j} \le \varphi(i)\varphi(j)$ with $\varphi(i)/i \to 0$ as $i \to +\infty$ and boundedness of the total fragmentation rate $\frac{1}{2}\sum_{j=1}^{i-1} F_{i-j,j}$, Spouge proved existence of a global non-negative solution to (0.1). Ball and Carr [3] proved existence of a global mass-conserving solution in case $K_{i,j} \le \kappa(i+j)$ but without any assumption on the fragmentation kernel. To obtain uniqueness, further assumptions were needed, in particular, some restrictions on the growth of the fragmentation coefficients. In the so-called strong fragmentation case, assuming that for some $\alpha \in (1/2, 1]$ and $\gamma > \alpha$ hypotheses $(K\alpha)$ and $(F\gamma)$ hold, Da Costa [7] obtained existence of a unique global solution : see Proposition 2.1 below. Guias [12] and Jeon [13] studied probabilistic approximations of (0.1) based on Markov jump processes. Assuming boundedness of the coagulation kernel and of the total fragmentation rate sequence $(\frac{1}{2}\sum_{j=1}^{i-1} F_{i-j,j})_i$, Guias obtained existence of a unique mass-conserving solution of (0.1) and convergence of the probabilistic approximations. Among other studies of the approximate Markov jumps processes, Jeon proved convergence to a solution of (0.1) in case $\lim_{i+j\to+\infty} \frac{K_{i,j}}{ij} + F_{i,j} = 0$.
More recently, in the absence of fragmentation, Babovski [2], Eibeck and Wagner [10] and Deaconu Fournier Tanré [8] [9] have worked on the probabilistic interpretation of the mass-flow equation (0.2). Papers [10], [8] and [9] are devoted to the general mass-flow equation corresponding to the non-necessarily discrete Smoluchowski coagulation equation but we are only going to present their results in the discrete case. In [8], Deaconu, Fournier and Tanré prove existence of a nonlinear process linked with (0.2) in case $\sum_{i\in\mathbb{N}^*} i^2\rho(i) < +\infty$ and $K_{i,j} \le \kappa ij$ (resp. $K_{i,j} \le \kappa(i+j)$) : the time-marginals of the law of this process provide a local (resp. global) solution of (0.2). In case $K_{i,j} \le \varphi(i)\varphi(j)$ with $\varphi(i)/i$ non-increasing and $\lim_{i+j\to+\infty} \frac{K_{i,j}}{\varphi(i)\varphi(j)} = 0$, Eibeck and Wagner [10] prove convergence to a solution of (0.2) of approximations based on Markov jump processes. In [9], the authors prove convergence of similar stochastic approximations in case $K_{i,j} \le \kappa(i+j)$ and $\sum_{i\in\mathbb{N}^*} i^2\rho(i) < +\infty$.

In the first section of this paper, we introduce a class of Markov jump processes which enables us to take into account the possible formation of infinite clusters called gelation in the probabilistic interpretation and approximation of (0.2).
The second section is devoted to the strong fragmentation case introduced by Da Costa [7]. The regularizing effect of the fragmentation prevents gelation. We introduce a nonlinear martingale problem such that the time-marginals of any solution provide a solution of (0.2). After checking existence and uniqueness for this martingale problem thanks to the results given in [7], we prove propagation of chaos to its solution for a sequence of simulable interacting particle systems.
In the third section, we do not make any assumption on the fragmentation kernel. In balance, we suppose that $(K1)$ holds and that the initial data is small in the following sense : $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$. We obtain a local (in time) existence and uniqueness result for (0.2). Moreover we construct an associated nonlinear process. In case the coagulation satisfies the stronger upper-bound $K_{i,j} \le \kappa(i+j)$, the existence and uniqueness results turn out to be global and the propagation of chaos result introduced in the strong framentation case still holds. Translated in terms of the Smoluchowski equation (0.1), our existence and uniqueness results seem to be new.
In the last section, we suppose that $\forall i \in \mathbb{N}^*$, $\lim_{j\to+\infty}(K_{i,j} + F_{i,j})/j = 0$. We obtain a global

existence result for (0.2) and consequently for (0.1) by considering the limit behaviour of the particle system introduced in the second section as the total number of particles goes to $+\infty$. Our hypothesis on the fragmentation (resp. coagulation) kernel is far (resp. slightly) less restrictive than the ones made by Jeon [13] who assumes that $\lim_{i+j\to+\infty} \frac{K_{i,j}}{ij} + F_{i,j} = 0$ to obtain existence for (0.1). Moreover we can deal with coagulation kernels such as $K_{i,j} = (ij)^\beta$ with $1/2 < \beta < 1$, for which the existence result of Ball and Carr [3] does not apply.

1 A Class of jump processes

Let $\mathcal{E} = \mathbb{N}^* \cup \{+\infty\}$. In order to be able to take into account the gelification phenomenom, we introduce for $N \in \mathbb{N}^*$ a class of Markov jump processes on $\mathcal{E}^N$ such that some coordinates become infinite when jumps accumulate. We prove existence and weak uniqueness for processes among this class.
More precisely we endow $\mathcal{E}^N$ with the metric $d((x^1,\dots,x^N),(y^1,\dots,y^N)) = \sum_{n=1}^N \left|\frac{1}{x^n} - \frac{1}{y^n}\right|$ (convention : $\frac{1}{+\infty} = 0$) and set

$$\mathcal{D}_N = \Big\{X : t \in \mathbb{R}_+ \to X_t = (X_t^1,\dots,X_t^N) \in \mathcal{E}^N \text{ càdlàg such that for } 1 \le n \le N,\ X \text{ is continuous} \\ \text{at } \sigma_n = \inf\{s \ge 0,\ X^n_{s-} \vee X^n_s = +\infty\} \text{ and satisfies } \forall s \in [\sigma_n, +\infty),\ X^n_s = +\infty\Big\}. \tag{1.1}$$

The space $\mathcal{D}_N$ is endowed with the trace of the Skorokhod topology on the space $D([0,+\infty),\mathcal{E}^N)$ of càdlàg functions from $\mathbb{R}_+$ to $\mathcal{E}^N$ and with the corresponding Borel sigma field. We have $D([0,+\infty),{\mathbb{N}^*}^N) \subset \mathcal{D}_N \subset D([0,+\infty),\mathcal{E}^N)$.

Definition 1.1 *A function $\lambda : (s,x,y) \in \mathbb{R}_+ \times \mathcal{E}^N \times \mathcal{E}^N \to \lambda(s,x,y) \in \mathbb{R}_+$ is called a transition function on $\mathcal{E}^N$ if*

(i) $\forall x \in \mathcal{E}^N,\ \sup_{s\ge 0} \sum_{y\in\mathcal{E}^N} \lambda(s,x,y) = \Lambda(x) < +\infty$

(ii) $\forall x, y \in \mathcal{E}^N$ *with* $x^n = +\infty$ *and* $y^n < +\infty$ *for some* $1 \le n \le N$,

$$\forall s \ge 0,\ \lambda(s,x,y) = \lambda(s,y,x) = 0.$$

(iii) $\forall 1 \le n \le N,\ \forall i \in \mathbb{N}^*,\ \sup_{s\ge0} \sup_{x\in\mathcal{E}^N : x^n = i} \sum_{y\in\mathcal{E}^N : y^n \ne i} \lambda(s,x,y) < +\infty.$

Definition 1.2 *For a probability measure ν on ${\mathbb{N}^*}^N$ and a transition function λ on $\mathcal{E}^N$, we say that the $\mathcal{D}$-valued process $(X_t)_{t\ge0}$ is a jump process with transition function λ starting from ν if*

1. *X_0 is distributed according to ν,*
2. *$\forall \varphi : \mathcal{E}^N \to \mathbb{R}$ bounded and s.t. for some $m \in \mathbb{N}^*$, $\forall x \in \mathcal{E}^N$, $\varphi(x) = \varphi((x^1 \wedge m, \dots, x^N \wedge m))$,*

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t \sum_{y\in\mathcal{E}} (\varphi(y) - \varphi(X_s))\lambda(s, X_s, y)ds \text{ is a martingale.}$$

Proposition 1.3 *For any probability measure ν on ${\mathbb{N}^*}^N$ and any transition function λ on $\mathcal{E}^N$, there exists a jump process with transition function λ starting from ν on a well-chosen probability space. Moreover two jump processes with transition function λ starting from ν have the same law.*

1.1 Proof of existence

Let X_0 be a random variable with law ν independent of a sequence of independent Poisson processes with marks $(T^x_k, U^x_k)_{k\geq 1}$ indexed by $x \in \mathcal{E}^N$. More precisely for fixed $x \in \mathcal{E}^N$, we suppose that $(T^x_k)_k$ is the sequence of successive jump times of a Poisson process with rate $\Lambda(x)$ given by Definition 1.1 (ii) independent of the marks $(U^x_k)_k$ which are i.i.d. according to the uniform distribution on $[0,1]$.
The process X_t is constructed by induction. We set $\tau_0 = 0$. Supposing that the process is constructed up to time τ_l, we define $L = \inf\{k : T^{X_{\tau_l}}_k > \tau_l\}$. We set $\tau_{l+1} = T^{X_{\tau_l}}_L$, fix $X_t = X_{\tau_l}$ for all $t \in [\tau_l, \tau_{l+1})$ and $X_{\tau_{l+1}} = \psi(\tau_{l+1}, X_{\tau_l}, U^{X_{\tau_l}}_L)$ with $\psi : \mathbb{R}_+ \times \mathcal{E}^N \times [0,1] \to \mathcal{E}^N$ defined by

$$\psi(s,x,u) = \begin{cases} y \text{ if } \sum_{z<y} \lambda(s,x,z) \leq \Lambda(x)u < \sum_{y\leq z} \lambda(s,x,z) \\ x \text{ if } \sum_{z\in\mathcal{E}^N} \lambda(s,x,z) \leq \Lambda(x)u \end{cases}$$

where $\mathcal{E}^N$ is endowed with the lexicographical order. This way the process X_t is constructed on the time interval $[0, \lim_l \tau_l)$. We have to deal with the case $\lim_l \tau_l < +\infty$. For $x \in \mathcal{E}^N$, we introduce the Poisson random measure $N(x, ds, du) = \sum_{k\geq 1} \delta_{(T^x_k, U^x_k)}$ on $\mathbb{R}_+ \times [0,1]$. Let $1 \leq n \leq N$ and $i \in \mathbb{N}^*$. The number of jumps leading from $X^n_{s^-} = i$ to $X^n_s \neq i$ on $[0, t\wedge \tau_l]$ is equal to

$$\sum_{x:x^n=i} \sum_{y:y^n\neq i} \int_{\mathbb{R}_+\times[0,1]} 1_{\{s\leq \tau_l\wedge t\}} 1_x(X_{s^-}) 1_y(\psi(s,x,u)) N(x,ds,du).$$

By compensation of the Poisson random measures, its expectation is equal to

$$\mathbb{E}\left(\int_{\mathbb{R}_+\times[0,1]} 1_{\{s\leq\tau_l\wedge t\}} 1_i(X^n_s) \sum_{y\in\mathcal{E}^N:y^n\neq i} \lambda(s, X_s, y) ds\right) \leq t \sup_{s\geq 0} \sup_{x\in\mathcal{E}^N:x^n=i} \sum_{y\in\mathcal{E}^N:y^n\neq i} \lambda(s,x,y).$$

With assumption **(iii)** concerning the transition function λ, we easily deduce that $\forall t > 0$, a.s. there are at most finitely many jumps leading from $X^n_{s^-} = i$ to $X^n_s \neq i$ on $[0, \lim_l \tau_l \wedge t)$. Hence a.s. on $\{\lim_l \tau_l < +\infty\}$, $\forall 1 \leq n \leq N$, $\forall i \in \mathbb{N}^*$, there are at most finitely many jumps leading from $X^n_{s^-} = i$ to $X^n_s \neq i$ for $s \in [0, \lim_l \tau_l)$. As a consequence a.s. on $\{\lim_l \tau_l < +\infty\}$, $\lim_l X_{\tau_l}$ exists in $\mathcal{E}^N$. We set $X_{\lim_l \tau_l} = \lim_l X_{\tau_l}$ and carry on the construction : the next jump time is given by $T^{X_{\lim_l \tau_l}}_L$ where $L = \inf\{k : T^{X_{\lim_l \tau_l}}_k > \lim_l \tau_l\}$ and so on. By assumptions **(i)** and **(ii)** concerning the transition function λ, a.s. on $\lim_l \tau_l < +\infty$,

$$\exists 1 \leq n \leq N,\ \forall s \in [0, \lim_l \tau_l),\ X^n_s < +\infty \text{ and } \lim_l X^n_{\tau_l} = +\infty. \tag{1.2}$$

Because of assumption **(ii)** on λ, the coordinates which become infinite at time $\lim_l \tau_l$ remain so afterwards. More generally, in the construction of the process X_t, a.s. at each finite accumulation point of jump times, at least one of the coordinates which was finite so far becomes infinite and remains so afterwards. As a consequence, there are at most N such finite accumulation points and the process is constructed for $t \in [0, +\infty)$.
Because of assumption **(ii)** on λ, up to time $\lim_l \tau_l$, the process X_t only depends on the variable

X_0 and the Poisson processes with indexes in $\mathbb{N}^{*N}$. Hence using (1.2) and the independence assumptions on the initial variable X_0 and Poisson processes, we have

$$\mathbb{P}\left(\lim_l \tau_l < +\infty,\ \exists k :\ T_k^{X_{\lim_l \tau_l}} = \lim_l \tau_l\right) \leq \sum_{n=1}^{N} \sum_{x:x^n=+\infty} \mathbb{P}\left(\lim_l \tau_l < +\infty,\ \exists k :\ T_k^x = \lim_l \tau_l\right) = 0.$$

Since the same property holds for all the finite accumulation points of jump times, we easily check that a.s.

$$\forall t \geq 0,\ X_t = X_0 + \sum_{x \in \mathcal{E}^N} \int_{[0,t]\times[0,1]} 1_x(X_{s^-})(\psi(s, X_{s^-}, u) - X_{s^-})N(x, ds, du). \tag{1.3}$$

We are now going to check that condition 2. in Definition 1.2 is satisfied by compensation of the Poisson measures. Let $\varphi : \mathcal{E}^N \to \mathbb{R}$ bounded and $m \in \mathbb{N}^*$ be such that $\forall x \in \mathcal{E}^N,\ \varphi(x) = \varphi((x^1 \wedge m, \ldots, x^n \wedge m))$. For $t > 0$, in the computation of $\varphi(X_t)$ from (1.3), only the jumps such that for some $1 \leq n \leq N$, $X^n_{s^-} < m$ and $X^n_s \neq X^n_{s^-}$ or $X^n_{s^-} \geq m$ and $X^n_s < m$ contribute. For fixed n, the total number of such jumps on $[0,t]$ is necessarily smaller than one plus twice the number of jumps of the first category (those leading from $X^n_{s^-} < m$ to $X^n_s \neq X^n_{s^-}$), the expectation of which is smaller than $t\sum_{i=1}^{m-1} \sup_{s\geq 0} \sup_{x\in\mathcal{E}^N : x^n = i} \sum_{y\in\mathcal{E}^N : y^n \neq i} \lambda(s,x,y) < +\infty$. Hence the expectation of the number of jumps on $[0,t]$ contributing to $\varphi(X_t)$ is finite. As a consequence, a.s.,

$$\forall t \geq 0,\ \varphi(X_t) = \varphi(X_0) + \sum_{x \in \mathcal{E}^N} \int_{[0,t]\times[0,1]} 1_x(X_{s^-})(\varphi(\psi(s, X_{s^-}, u)) - \varphi(X_{s^-}))N(x, ds, du)$$

and M_t^φ is a martingale by compensation of the Poisson random measures. Hence X_t is a jump process with transition function λ starting from ν.

1.2 Proof of weak uniqueness

Let P and Q denote the respective laws of two jump processes with transition function λ starting from ν. We denote by $(X_t)_{t\geq 0}$ the canonical process on $\mathcal{D}$ and by $\mathcal{F}_t = \sigma(X_s, s \leq t)$ ($\mathcal{F}_\infty = \sigma(X_s, s \geq 0)$) its natural filtration. For a stopping time τ relative to $(\mathcal{F}_t)$, we define $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty :\ \forall t \geq 0,\ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$. According to [16] Exercise (4.21) p.45, $\mathcal{F}_\tau = \sigma(X_{s\wedge\tau},\ s \geq 0)$.
We need to introduce the successive times when some of the coordinates of the process $(X_t)_{t\geq 0}$ become infinite. Let $T_1 = \inf\{s \geq 0 :\ \exists 1 \leq n \leq N,\ X^n_{s^-} \vee X^n_s = +\infty\}$ (convention $\inf \emptyset = +\infty$). On $\{T_1 < +\infty\}$, we set $\mathcal{N}_1 = \{1 \leq n \leq N :\ X^n_{T_1^-} \vee X^n_{T_1} < +\infty\}$ and $T_2 = \inf\{s \geq T_1 :\ \exists n \in \mathcal{N}_1,\ X^n_{s^-} \vee X^n_s = +\infty\}$. On the contrary event, $\mathcal{N}_1 = \emptyset$, $T_2 = +\infty$. Inductively we obtain stopping times $T_1 \leq T_2 \leq \ldots \leq T_{N+1} = +\infty$ and sets of indexes $\emptyset = \mathcal{N}_N \subset \ldots \subset \mathcal{N}_1 \subset \{1, \ldots, N\}$ with for $1 \leq k \leq N$, $\mathcal{N}_k = \emptyset$, $T_{k+1} = +\infty$ if $T_k = +\infty$ and $\mathcal{N}_k = \{n \in \mathcal{N}_{k-1} : X^n_{T_k^-} \vee X^n_{T_k} < +\infty\}$, $T_{k+1} = \inf\{s \geq T_k :\ \exists n \in \mathcal{N}_k,\ X^n_{s^-} \vee X^n_s = +\infty\}$ otherwise.
We also introduce another increasing sequence of localizing stopping times. For $m \in \mathbb{N}^*$, let $\tau_m = \inf\{s \geq 0 :\ \exists 1 \leq n \leq N,\ X^n_s \geq m\}$. Clearly $\lim_m \tau_m = T_1$. By Definition 1.2, the images of P and Q by the mapping $(X_t)_t \in \mathcal{D} \to (X_t \wedge \tau_m)_t$ both solve a martingale problem with jump rates bounded because of assumption **(i)** on the transition function λ. According to [11] Theorem 7.3 p.223, uniqueness holds for this problem. Hence P and Q coincide on $\sigma(X_{s\wedge\tau_m}, s \geq 0) = \mathcal{F}_{\tau_m} \subset \mathcal{F}_{T_1}$ for any m and therefore on the sigma algebra $\vee_m \sigma(X_{s\wedge\tau_m}, s \geq 0) \subset \mathcal{F}_{T_1}$. On $\{T_1 < +\infty\}$, by definition of $\mathcal{D}$ (see (1.1)), $t \to X_t$ is continuous at T_1. As a consequence $\forall s \geq 0$, $X_{s\wedge T_1} = \lim_m X_{s\wedge\tau_m}$ is mesurable w.r.t. the sigma algebra $\vee_m \sigma(X_{s\wedge\tau_m}, s \geq 0)$ which therefore contains $\mathcal{F}_{T_1} = \sigma(X_{s\wedge T_1}, s \geq 0)$. Hence P and Q coincide on $\mathcal{F}_{T_1}$. Again by (1.1), on $T_1 < +\infty$,

$\forall n \in \{1, \dots, N\} \setminus \mathcal{N}_1$, $\forall s \geq T_1$, $X^n_s = +\infty$. Thanks to property **(ii)** of the transition function λ, we check that on $\{T_1 < +\infty\}$, P and Q a.s., conditionally on $\mathcal{F}_{T_1}$, $(X^n_{T_1+s},\ n \in \mathcal{N}_1)_{s\geq 0}$ is a jump process on $(\mathbb{N}^* \cup \{+\infty\})^{\mathrm{card}(\mathcal{N}_1)}$ starting from the Dirac mass at $(X^n_{T_1},\ n \in \mathcal{N}_1)$ and with modified transition function $\lambda_1(s, \varkappa, \xi) = \lambda(T_1 + s, x, y)$ where $x, y \in \mathcal{E}^N$ are obtained from $\varkappa, \xi \in (\mathbb{N}^* \cup \{+\infty\})^{\mathrm{card}(\mathcal{N}_1)}$ by setting the coordinates in $\{1, \dots, N\} \setminus \mathcal{N}_1$ equal to $+\infty$. Moreover, $T_2 - T_1$ is the first time when a coordinate of this process becomes infinite. Using the partial uniqueness result already obtained, we deduce that P and Q coincide on $\mathcal{F}_{T_2}$. By induction, we conclude that P and Q coincide on $\mathcal{F}_{T_{N+1}} = \mathcal{F}_\infty$.

2 The strong fragmentation case : for $\alpha \in (1/2, 1]$ and $\gamma > \alpha$ $(K\alpha)$ and $(F\gamma)$ hold.

Because of the link between (0.1) and (0.2), Theorem 5.1 [7] yields existence for (0.2). Moreover, since we assume in our definition of solutions that the mass at time t is smaller than the initial mass 1, by an easy adaptation of the proof of Theorem 6.1 [7], uniqueness also holds.

Proposition 2.1 *There is a unique solution p_t of (0.2) on $[0, +\infty)$. This solution is mass conserving (i.e. $\forall t \geq 0$, $\sum_{i\in\mathbb{N}^*} p_t(i) = 1$) and such that $\forall \epsilon > 0$,*

$$\forall t,\ \sum_{i\in\mathbb{N}^*} i^{\gamma-\epsilon} \int_0^t p_s(i)ds < +\infty.$$

Because of the mass-conserving property of the solution p_t of (0.2), the paths of the process that we are going to associate with it belong to the space $D([0, +\infty), \mathbb{N}^*)$ of càdlàg functions from $[0, +\infty)$ to $\mathbb{N}^*$ which is strictly included in $\mathcal{D}_1$. We endow $D([0, +\infty), \mathbb{N}^*)$ with the Skorokhod topology. Let $\mathcal{P}(D([0, +\infty), \mathbb{N}^*))$, $(X_t)_{t\geq 0}$ denote respectively the set of probability measures and the canonical process on this space. We associate the following nonlinear martingale problem with (0.2) :

Definition 2.2 *A probability measure P on $D([0, +\infty), \mathbb{N}^*)$ with time-marginals $(P_t)_{t\geq 0}$ solves the nonlinear martingale problem (MP) if*
(i)$P_0 = \rho$ *i.e.* $\forall i \in \mathbb{N}^*,\ P_0(i) = \rho(i)$
(ii)$\forall \varphi : \mathbb{N}^* \to \mathbb{R}$ *s.t.* $\exists m \in \mathbb{N}^*$, $\forall l \geq m,\ \varphi(l) = \varphi(m)$,

$$M^\varphi_t = \varphi(X_t) - \varphi(X_0) - \int_0^t \Bigg(\sum_{j\in\mathbb{N}^*} \tilde{K}_{X_s,j}(\varphi(X_s + j) - \varphi(X_s))P_s(j) + \sum_{j=1}^{X_s-1} \tilde{F}_{X_s-j,j}(\varphi(X_s - j) - \varphi(X_s))\Bigg) ds \text{ is a } P\text{-martingale.}$$

If P is a probability measure on $D([0,+\infty),\mathbb{N}^*)$, then $\forall t \geq 0$, P_t satisfies condition (i) in Definition 0.1. For $i \in \mathbb{N}^*$, let $\varphi(l) = 1_i(l)$. By $(K\alpha)$,

$$\left| \sum_{j\in\mathbb{N}^*} \tilde{K}_{X_s,j}(1_i(X_s+j) - 1_i(X_s))P_s(j) - \sum_{j=1}^{X_s-1} \tilde{F}_{X_s-j,j} 1_i(X_s) \right|$$
$$\leq \max\left(\max_{1\leq j\leq i-1} \tilde{K}_{i-j,j}P_s(j), \sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j}P_s(j) + \sum_{j=1}^{i-1} \tilde{F}_{i-j,j} \right) \leq \kappa i^\alpha + \sum_{j=1}^{i-1} \tilde{F}_{i-j,j}.$$

Hence if P solves problem (MP), the integrability of M_t^φ yields that $\mathbb{E}(\int_0^t \sum_{j=1}^{X_s-1} \tilde{F}_{X_s-j,j} 1_i(X_s - j)ds) = \int_0^t \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j}P_s(i+j)ds < +\infty$. Moreover, by the constancy of the expectation of the P-martingale M_t^φ, we get

$$P_t(i) = P_0(i) + \int_0^t \sum_{j=1}^{i-1} \tilde{K}_{i-j,j}P_s(i-j)P_s(j) - \sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j}P_s(i)P_s(j)ds$$
$$+ \int_0^t \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j}P_s(i+j) - \sum_{j=1}^{i-1} \tilde{F}_{i-j,j}P_s(i)ds$$

Hence we have established the following link between problem (MP) and equation (0.2):

Lemma 2.3 *If P solves problem (MP) then $t \to P_t$ solves* (0.2) *on* $[0,+\infty)$.

2.1 Existence and uniqueness for problem (MP)

Theorem 2.4 *The martingale problem (MP) has a unique solution P. Moreover, P_t is the unique solution of* (0.2) *on* $[0,+\infty)$.

Proof of uniqueness : If P and Q both solve (MP), then according to Lemma 2.3, P_t and Q_t both solve (0.2) on $[0,+\infty)$ and we deduce from Proposition 2.1 that $\forall t \geq 0$, $P_t = Q_t = p_t$. Hence under both P and Q, the canonical process is a jump process starting from ρ and with transition function

$$\forall (s,i,j) \in \mathbb{R}_+ \times \mathcal{E} \times \mathcal{E},\ \lambda(s,i,j) = 1_{\{i<+\infty\}} \left(\tilde{F}_{j,i-j} 1_{\{1\leq j<i\}} + \tilde{K}_{i,j-i} p_s(j-i) 1_{\{i<j<+\infty\}} \right). \tag{2.1}$$

According to the weak uniqueness result in Proposition 1.3, $P = Q$. ∎

We still have to prove existence. According to Proposition 2.1, (0.2) has a solution $t \to p_t$. Let X be a jump process with transition function given by (2.1) starting from ρ and q_s denote the law of X_s on $\mathbb{N}^* \cup \{+\infty\}$. The fact that the law of the process X solves problem (MP) is a consequence of the following Proposition :

Proposition 2.5 $\forall t \geq 0$, $\forall i \geq 1$, $q_t(i) = p_t(i)$.

Since by Proposition 2.1, $\forall t\geq 0$, $\sum_{i\in\mathbb{N}^*}p_t(i)=1$, this result implies in particular that $\forall t\geq 0$, a.s. $X_t<+\infty$. By definition of $\mathcal{D}_1$, we deduce that $\forall t\geq 0$, a.s. $\forall s\in[0,t]$, $X_{s^-}\vee X_s<+\infty$ i.e. $\sup_{s\in[0,t]}X_s<+\infty$. Therefore a.s. $X\in D([0,+\infty),\mathbb{N}^*)$.

Proof of Proposition 2.5 : According to 2. in Definition 1.2, for $i\in\mathbb{N}^*$, $1_i(X_t)-1_i(X_0)-\int_0^t\sum_{j\in\mathbb{N}^*}(1_i(j)-1_i(X_s))\lambda(s,X_s,j)ds$ is a martingale. Following the same line of reasoning as in the proof of Lemma 2.3, we deduce from the constancy of its expectation that $t\to q_t$ solves the following linear equation

$$\begin{cases}\partial_t q_t(i)=\sum_{j=1}^{i-1}\left(\tilde{K}_{i-j,j}q_t(i-j)p_t(j)-\tilde{F}_{i-j,j}q_t(i)\right)-\sum_{j\in\mathbb{N}^*}\left(\tilde{K}_{i,j}q_t(i)p_t(j)-\tilde{F}_{i,j}q_t(i+j)\right)\\ q_0(i)=\rho(i).\end{cases} \tag{2.2}$$

So does the solution p_t of (0.2). Hence it is enough to prove uniqueness for this equation to conclude. Without the fragmentation terms, we could prove that $t\to p_t(i)$ and $t\to q_t(i)$ are equal by induction on i. Here, we take advantage of the strong fragmentation hypotheses $(F\gamma)$ and $(K\alpha)$ with $\gamma>\alpha\in(\frac{1}{2},1]$ and adapt ideas developped by Da Costa [7] in order to prove uniqueness for (0.1) in the same framework. Let $sg_s(i)$ denote the sign of $p_s(i)-q_s(i)$. Since when $s\to f(s)$ is absolutely continuous with derivative $g(s)$, $|f(s)|$ is absolutely continuous with derivative $\mathrm{sign}(f(s))g(s)$, combining (0.2) and (2.2), we have

$$\begin{aligned}\sum_{i=1}^n|p_t(i)-q_t(i)|=\int_0^t\sum_{i=1}^n sg_s(i)\Bigg(&\sum_{j=1}^{i-1}(\tilde{K}_{i-j,j}p_s(j)(p_s(i-j)-q_s(i-j))-\tilde{F}_{i-j,j}(p_s(i)-q_s(i)))\\ &-\sum_{j\in\mathbb{N}^*}(\tilde{K}_{i,j}p_s(j)(p_s(i)-q_s(i))-\tilde{F}_{i,j}(p_s(i+j)-q_s(i+j)))\Bigg)ds\end{aligned}$$

Exchanging summations over i and j in $\sum_{i=1}^n sg_s(i)\sum_{j=1}^{i-1}(\tilde{K}_{i-j,j}p_s(j)(p_s(i-j)-q_s(i-j))-\tilde{F}_{i-j,j}(p_s(i)-q_s(i)))$, then setting $k=i-j$ and exchanging summations over j and k we get that this term writes $\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}sg_s(k+j)(\tilde{K}_{k,j}p_s(j)(p_s(k)-q_s(k))-\tilde{F}_{k,j}(p_s(k+j)-q_s(k+j)))$. Hence $\sum_{i=1}^n|p_t(i)-q_t(i)|=\int_0^t U_n(s)+V_n(s)ds$ where

$$U_n(s)=\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}(sg_s(i+j)-sg_s(i))(\tilde{K}_{i,j}p_s(j)(p_s(i)-q_s(i))-\tilde{F}_{i,j}(p_s(i+j)-q_s(i+j)))\leq 0$$

since $(sg_s(i+j)-sg_s(i))(p_s(i)-q_s(i))\leq 0$ and $(sg_s(i+j)-sg_s(i))(p_s(i+j)-q_s(i+j))\geq 0$,

$$\begin{aligned}\text{and}\ \ V_n(s)&=-\sum_{i=1}^n\sum_{j\geq n+1-i}sg_s(i)(\tilde{K}_{i,j}p_s(j)(p_s(i)-q_s(i))-\tilde{F}_{i,j}(p_s(i+j)-q_s(i+j)))\\ &\leq\sum_{i=1}^n\sum_{j\geq n+1-i}\tilde{F}_{i,j}(p_s(i+j)+q_s(i+j)).\end{aligned}$$

Integrating (0.2) on $[0,t]$ and summing the obtained result for $1\leq i\leq n$, we get

$$\int_0^t\sum_{i=1}^n\sum_{j\geq n+1-i}\tilde{F}_{i,j}p_s(i+j)ds=\sum_{i=1}^n p_t(i)-\sum_{i=1}^n\rho(i)+\int_0^t\sum_{i=1}^n\sum_{j\geq n+1-i}\tilde{K}_{i,j}p_s(j)p_s(i)ds.$$

Since $\alpha<\gamma$, combining $(K\alpha)$, the conservation of mass for p_t ($\forall s\geq 0$, $\sum_{j\in\mathbb{N}^*}p_s(j)=1$) and the estimation given in Proposition 2.1, we get

$$\int_0^t\sum_{i,j\in\mathbb{N}^*}\tilde{K}_{i,j}p_s(j)p_s(i)ds\leq\kappa\int_0^t\sum_{i\in\mathbb{N}^*}i^\alpha p_s(i)ds<+\infty.$$

We deduce that $\int_0^t \sum_{i=1}^n \sum_{j\geq n+1-i} \tilde{K}_{i,j} p_s(j) p_s(i) ds$ and $\int_0^t \sum_{i=1}^n \sum_{j\geq n+1-i} \tilde{F}_{i,j} p_s(i+j) ds$ converge to 0 as $n \to +\infty$. By the following estimation the proof of which is postponed,

Lemma 2.6 $\forall \epsilon > 0$, $\forall t \geq 0$, $\int_0^t \sum_{i\in\mathbb{N}^*} i^{\gamma-\epsilon} q_s(i) ds < +\infty$

we obtain similarly that $\int_0^t \sum_{i=1}^n \sum_{j\geq n+1-i} \tilde{F}_{i,j} q_s(i+j) ds$ converges to $\sum_{i\in\mathbb{N}^*} q_t(i) - 1 \leq 0$. Hence $\limsup_{n\to+\infty} \int_0^t V_n(s) ds \leq 0$. We conclude that $\forall t \geq 0$, $\sum_{i\in\mathbb{N}^*} |p_t(i) - q_t(i)| = 0$. ∎

Proof of Lemma 2.6 : Let $\lambda \in [-1, 0)$. Integrating (2.2) on $[0,t]$ summing the obtained result multiplied by i^λ for $1 \leq i \leq n$ and removing one of the two coagulations terms and the term involving the initial condition ρ, we get

$$\int_0^t \left(\sum_{i=1}^n i^\lambda \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j} q_s(i+j) - \sum_{i=1}^n i^\lambda q_s(i) \sum_{j=1}^{i-1} \tilde{F}_{i-j,j} \right) ds \leq \int_0^t \sum_{i=1}^n i^\lambda \sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j} q_s(i) p_s(j) ds + \sum_{i=1}^n i^\lambda q_t(i) \tag{2.3}$$

Setting $l = i + j$, exchanging summations then replacing indexes (l, i) by (i, j) we get

$$\sum_{i=1}^n i^\lambda \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j} q_s(i+j) = \sum_{l\geq 2} q_s(l) \sum_{i=1}^{n\wedge(l-1)} i^\lambda \tilde{F}_{i,l-i} \geq \sum_{i=2}^n q_s(i) \sum_{j=1}^{i-1} j^\lambda \tilde{F}_{j,i-j}.$$

Inserting this bound, $(K\alpha)$ and $\sum_{i\in\mathbb{N}^*} i^\lambda q_t(i) \leq \sum_{i\in\mathbb{N}^*} q_t(i) \leq 1$ in (2.3) and using moreover $\sum_{j=1}^{i-1} \tilde{F}_{i-j,j} = \sum_{j=1}^{i-1} \tilde{F}_{j,i-j}$ we obtain that

$$\int_0^t \sum_{i=1}^n q_s(i) \sum_{j=1}^{i-1} (j^\lambda - i^\lambda) \tilde{F}_{j,i-j} ds \leq 1 + \kappa \int_0^t \sum_{i\in\mathbb{N}^*} i^{\lambda+\alpha} q_s(i) ds.$$

By the strong fragmentation hypothesis $(F\gamma)$, for $C_\lambda > 0$, $\forall i \geq 3$,

$$\sum_{j=1}^{i-1} (j^\lambda - i^\lambda) \tilde{F}_{j,i-j} = \frac{1}{i} \sum_{j=1}^{i-1} j^{1+\lambda} \left(1 - \left(\frac{i}{j}\right)^\lambda \right) F_{j,i-j} \geq \frac{1-2^\lambda}{i} \sum_{j=1}^{[(i-1)/2]} j^{1+\lambda} F_{j,i-j} \geq C_\lambda i^{\lambda+\gamma}.$$

Hence for $\beta = \lambda + \alpha \in [-1+\alpha, \alpha)$,

$$C_{\beta-\alpha} \int_0^t \sum_{3\leq i<+\infty} i^{\beta+\gamma-\alpha} q_s(i) ds \leq 1 + \kappa 2^\alpha + \kappa \int_0^t \sum_{3\leq i<+\infty} i^\beta q_s(i) ds. \tag{2.4}$$

We have $\gamma - \alpha > 0$. Let $\beta_l = l(\gamma - \alpha)$ and $L = \inf\{l : \beta_l \geq \alpha\}$. Since $\forall s \geq 0$, $\sum_{i\in\mathbb{N}^*} q_s(i) \leq 1$, $\int_0^t \sum_{3\leq i<+\infty} i^{\beta_0} q_s(i) ds < t$. Using (2.4), we deduce inductively that for any l smaller than L, $\int_0^t \sum_{3\leq i<+\infty} i^{\beta_l} q_s(i) ds < +\infty$. Hence $\int_0^t \sum_{3\leq i<+\infty} i^\alpha q_s(i) ds < +\infty$. We complete the proof by choosing $\beta = \alpha - \epsilon$ in (2.4). ∎

2.2 Propagation of chaos

2.2.1 The system of N particles

The particle system $(Y_t^{1,N},\dots,Y_t^{N,N})$ that we consider is a jump process on $\mathcal{E}^N$ starting from $\rho^{\otimes N}$ and with time-homogeneous transition function equal to zero for transitions modifying more than one coordinate and for transitions involving an infinite coordinate. If $(e_1,\dots,e_N)$ denotes the canonical basis on $\mathbb{R}^N$, the transitions involving the n-th coordinate when finite are given by

$$\forall y\in\mathcal{E}^N \text{ with } y^n<+\infty,\ \forall 1\le j\le y^n-1,\ \lambda(y,y-je_n)=\tilde{F}_{y^n-j,j}$$
$$\text{and } \forall 1\le m\le N \text{ with } y^m<+\infty,\ \lambda(y,y+y^m e_n)=\frac{1}{N}\tilde{K}_{y^n,y^m}. \quad (2.5)$$

Without fragmentation and with coagulation kernel $K_{i,j}=\kappa i^\alpha j^\alpha$ for $\alpha\in(1/2,1]$, particles may become infinite in finite time. For $N=1$ and $Y_0^{1,1}=1$, at each jump the size of the particle doubles and consequently the times between the successive jumps are independent exponential variables with successive expectations $(\frac{1}{\kappa}2^{-(2\alpha-1)n})_{n\ge 0}$. Since this sequence is summable, a.s. the particle becomes infinite in finite time.

2.2.2 Tightness

We are first going to prove that the hypotheses on the coagulation and fragmentation kernels imply that a.s. no particle becomes infinite in finite time. As a consequence, the empirical measure $\mu^N=\frac{1}{N}\sum_{n=1}^N\delta_{Y^{n,N}}$ of the particle system is a r.v. with values in $\mathcal{P}(D([0,+\infty),\mathbb{N}^*))$. Then, we are going to prove tightness of the sequence of the laws of these variables μ^N.

Since the initial measure $\rho^{\otimes N}$ and the transition function (2.5) are symmetric, by the weak uniqueness result in Proposition 1.3, the particles are exchangeable. Hence for fixed $N\ge 2$, $\mathbb{P}(Y_t^{n,N}=i)$ (resp $\mathbb{P}(Y_t^{n,N}=i,Y_t^{m,N}=j)$) is independent of $n\in[1,N]$ (resp independent of (n,m) with $n\neq m\in[1,N]$). Let $p_t^{1,N}(i)$ and $p_t^{2,N}(i,j)$ denote respectively this one-particle (resp. two particles) measure. By a reasoning analogous to the one made to obtain (2.2), we check that for $i\in\mathbb{N}^*$,

$$\partial_t p_t^{1,N}(i)=\sum_{j=1}^{i-1}\left(\frac{1}{N}\tilde{K}_{i-j,j}\left((N-1)p_t^{2,N}(i-j,j)+1_{\{i-j=j\}}p_t^{1,N}(j)\right)-\tilde{F}_{i-j,j}p_t^{1,N}(i)\right)$$
$$-\sum_{j\in\mathbb{N}^*}\left(\frac{1}{N}\tilde{K}_{i,j}\left((N-1)p_t^{2,N}(i,j)+1_{\{i=j\}}p_t^{1,N}(i)\right)-\tilde{F}_{i,j}p_t^{1,N}(i+j)\right). \quad (2.6)$$

Since because of the possibility for particles to become infinite, $\sum_{j\in\mathbb{N}^*}p_t^{2,N}(i,j)\le p_t^{1,N}(i)$,

$$\text{by } (K\alpha),\ \sum_{j\in\mathbb{N}^*}\frac{1}{N}\tilde{K}_{i,j}\left((N-1)p_t^{2,N}(i,j)+1_{\{i=j\}}p_t^{1,N}(i)\right)\le\kappa i^\alpha p_t^{1,N}(i).$$

Hence by an easy adaptation of the proof of Lemma 2.6, we obtain that (2.4) still holds with q_s replaced by $p_s^{1,N}$ and deduce the first assertion in the following Lemma :

Lemma 2.7 $\forall\epsilon>0,\ \forall t\ge 0,\ \sup_{N\ge1}\int_0^t\sum_{i\in\mathbb{N}^*}i^{\gamma-\epsilon}p_s^{1,N}(i)ds<+\infty.$
Moreover, $\forall t\ge 0,\ \forall 1\le n\le N,\ \mathbb{P}(Y_t^{n,N}<+\infty)=1.$

$$\textit{Lastly, }\begin{cases}\textit{if } \alpha\in(\frac{1}{2},1),\ \forall t\ge 0,\ \sup_N\sup_{1\le n\le N}\mathbb{E}\left(\sup_{s\le t}(Y_s^{n,N})^{1-\alpha}-(Y_0^{n,N})^{1-\alpha}\right)<+\infty\\ \textit{if } \alpha=1,\ \forall t\ge 0,\ \sup_N\sup_{1\le n\le N}\mathbb{E}\left(\ln\left(\sup_{s\le t}Y_s^{n,N}/Y_0^{n,N}\right)\right)<+\infty\end{cases}$$

Proof : We only have to prove the second and last assertions. Integrating (2.6) on $[0,t]$, summing the obtained result for $1 \leq i \leq I$ and removing the fragmentation terms, we get

$$\sum_{i=1}^{I} p_t^{1,N}(i) + \int_0^t \sum_{i=1}^{I} \sum_{j \geq I-i+1} \frac{1}{N} \tilde{K}_{i,j} \left((N-1) p_s^{2,N}(i,j) + 1_{\{i=j\}} p_s^{1,N}(i) \right) ds \geq \sum_{i=1}^{I} \rho(i)$$

Since by $(K\alpha)$, $\sum_{i \in \mathbb{N}^*} \sum_{j \in \mathbb{N}^*} \frac{1}{N} \tilde{K}_{i,j} \left((N-1) p_s^{2,N}(i,j) + 1_{\{i=j\}} p_s^{1,N}(i) \right) ds \leq \kappa \sum_{i \in \mathbb{N}^*} i^\alpha p_s^{1,N}(i)$ which is integrable on $[0,t]$ by the first assertion, the second term of the left-hand-side converges to 0 as $I \to +\infty$. Hence $\sum_{i \in \mathbb{N}^*} p_t^{1,N}(i) \geq \sum_{i \in \mathbb{N}^*} \rho(i) = 1$ and the second assertion holds.
Let us now suppose that $\alpha \in (\frac{1}{2}, 1)$. Then $1 - \alpha \in (0, \frac{1}{2})$. By exchangeability of the particles, we only need to check the upper-bound for $n = 1$. The variable $\sup_{s \leq t} (Y_s^{1,N})^{1-\alpha}$ is smaller than the sum of $(Y_0^{1,N})^{1-\alpha}$ and of the contributions of the a.s. finite (otherwise $Y_t^{1,N}$ would be equal to $+\infty$) number of jumps of $s \in [0,t] \to Y_s^{1,N}$ with $Y_s^{1,N} > Y_{s^-}^{1,N}$ i.e.

$$\sup_{s \leq t} (Y_s^{1,N})^{1-\alpha} - (Y_0^{1,N})^{1-\alpha} \leq \sum_{s \leq t} 1_{\{Y_s^{1,N} > Y_{s^-}^{1,N}\}} \left((Y_s^{1,N})^{1-\alpha} - (Y_{s^-}^{1,N})^{1-\alpha} \right).$$

Taking expectations, using the inequality $(y+y')^{1-\alpha} - y^{1-\alpha} \leq (y')^{1-\alpha}$, hypothesis $(K\alpha)$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\sup_{s \leq t} (Y_s^{1,N})^{1-\alpha} - (Y_0^{1,N})^{1-\alpha} \right) &\leq \mathbb{E}\left(\int_0^t \frac{1}{N} \sum_{m=1}^{N} \tilde{K}_{Y_s^{1,N}, Y_s^{m,N}} \left((Y_s^{1,N} + Y_s^{m,N})^{1-\alpha} - (Y_s^{1,N})^{1-\alpha} \right) ds \right) \\
&\leq \frac{\kappa}{N} \int_0^t \sum_{m=1}^{N} \mathbb{E}((Y_s^{1,N})^\alpha (Y_s^{m,N})^{\alpha-1} (Y_s^{m,N})^{1-\alpha}) ds \\
&\leq \kappa \int_0^t \sum_{i \in \mathbb{N}^*} i^\alpha p_s^{1,N}(i) ds < +\infty \text{ since } \alpha < \gamma.
\end{aligned}$$

In case $\alpha = 1$, the conclusion is obtained in the same way by using the inequality $\forall y, y' \in \mathbb{N}^*$, $\ln(y+y') - \ln(y) \leq y'/y$. ∎

Proposition 2.8 *The sequence of the laws of the empirical measures μ^N considered as random variables with values in $\mathcal{P}(D([0,+\infty), \mathbb{N}^*))$ is tight.*

Proof : By exchangeability of the particles, according to [18] and the references therein, the Proposition is equivalent to the tightness of the laws of the variables $(Y^{1,N})_N$ in $D([0,+\infty), \mathbb{N}^*)$. Since $D([0,+\infty), \mathbb{N}^*)$ is a closed subset of $D([0,+\infty), \mathbb{R})$ endowed with the Skorokhod topology, it is enough to prove the tightness of the laws of the variables $(Y^{1,N})_N$ in $D([0,+\infty), \mathbb{R})$. Indeed by the closed sets characterization of weak convergence ([4] Theorem 2.1 (iii)), when probability measures on $D([0,+\infty), \mathbb{R})$ giving full weight to $D([0,+\infty), \mathbb{N}^*)$ converge weakly, their restrictions to $D([0,+\infty), \mathbb{N}^*)$ also converge weakly. We are going to do so by checking that Aldous tightness criterion (see for instance [14] p.35) is satisfied.

Let $t \geq 0$ and $M \in \mathbb{N}^*$. Supposing that $\alpha < 1$, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leq t} Y_s^{1,N} > M\right) &= \mathbb{P}\left(\sup_{s\leq t}(Y_s^{1,N})^{1-\alpha} > M^{1-\alpha}\right) \\
&\leq \mathbb{P}\left((Y_0^{1,N})^{1-\alpha} > \frac{M^{1-\alpha}}{2}\right) + \mathbb{P}\left(\sup_{s\leq t}(Y_s^{1,N})^{1-\alpha} - (Y_0^{1,N})^{1-\alpha} > \frac{M^{1-\alpha}}{2}\right) \\
&\leq \mathbb{P}\left(Y_0^{1,N} > \frac{M}{2^{1/(1-\alpha)}}\right) + \frac{2}{M^{1-\alpha}}\mathbb{E}\left(\sup_{s\leq t}(Y_s^{1,N})^{1-\alpha} - (Y_0^{1,N})^{1-\alpha}\right)
\end{aligned}$$

By the third assertion in Lemma 2.7, we deduce that when $\alpha < 1$,

$$\forall t \geq 0, \lim_{M\to+\infty} \sup_N \mathbb{P}\left(\sup_{s\leq t} Y_s^{1,N} \geq M\right) = 0. \tag{2.7}$$

We check this property for $\alpha = 1$ by replacing $y \to y^{1-\alpha}$ by $y \to \ln(y)$ in the above computation. As a consequence $\forall s \geq 0$ the laws of the real variables $(Y_s^{1,N})_N$ are tight.
Let $T > 0$ and for $N \geq 1$, τ_N be a stopping time of the filtration $\mathcal{F}_t^N = \sigma((Y_s^{1,N}, \ldots, Y_s^{N,N}),\ s \leq t)$ smaller than T. For $\delta, \eta > 0$,

$$\begin{aligned}
\sup_{\theta\in[0,\delta]} \mathbb{P}(|Y_{\tau_N+\theta}^{1,N} - Y_{\tau_N}^{1,N}| > \eta) &\leq \mathbb{P}\left(\sup_{\theta\in[0,\delta]} |Y_{\tau_N+\theta}^{1,N} - Y_{\tau_N}^{1,N}| > \eta\right) \\
&\leq \mathbb{P}\left(\sup_{s\leq T} Y_s^{1,N} \geq M\right) + \mathbb{P}\left(Y_{\tau_N}^{1,N} \leq M \text{ and } \exists\theta \in [0,\delta] \text{ s.t. } Y_{\tau_N+\theta}^{1,N} \neq Y_{\tau_N}^{1,N}\right)
\end{aligned} \tag{2.8}$$

By (2.7), the first term of the right-hand-side is arbitrarily small uniformly in N for M big enough. Therefore it is enough to check that for fixed M the second term is arbitrarily small uniformly in (N, τ_N) for δ small to conclude that Aldous tightness criterion holds. Let $1 \leq i \leq M$, $\sigma_i = \inf\{s \geq \tau_N : Y_s^{1,N} \neq i\}$ and $\varphi(y^1, \ldots, y^N) = 1_i(y^1)$. For the jump process $(Y^{1,N}, \ldots, Y^{N,N})$ with transition function defined by (2.5), the martingale M_t^φ given by Definition 1.2 is such that a.s. on $\{Y_{\tau_N}^{1,N} = i\}$,

$$M^\varphi_{\sigma_i\wedge(\tau_N+\delta)} - M^\varphi_{\tau_N} = -1_{\{\sigma_i\leq\tau_N+\delta\}} + \int_{\tau_N}^{\sigma_i\wedge(\tau_N+\delta)} \frac{1}{N}\sum_{n=1}^N \tilde{K}_{i,Y_s^{n,N}} + \sum_{j=1}^{i-1} \tilde{F}_{i-j,j} ds.$$

As $\mathbb{E}\left(1_{\{Y_{\tau_N}^{1,N}=i\}}(M^\varphi_{\sigma_i\wedge(\tau_N+\delta)} - M^\varphi_{\tau_N})\right) = 0$, we deduce that

$$\mathbb{P}\left(Y_{\tau_N}^{1,N} = i \text{ and } \exists\theta \in [0,\delta] \text{ s.t. } Y_{\tau_N+\theta}^{1,N} \neq Y_{\tau_N}^{1,N}\right) \leq \delta\left(\kappa i^\alpha + \sum_{j=1}^{i-1} \tilde{F}_{i-j,j}\right)\mathbb{P}(Y_{\tau_N}^{1,N} = i).$$

By summation over $i \in [1, N]$, we deduce that the second term of the r.h.s. of (2.8) is smaller than $\delta\left(\kappa M^\alpha + \max_{1\leq i\leq M} \sum_{j=1}^{i-1} \tilde{F}_{i-j,j}\right)$ which concludes the proof. ∎

2.2.3 Identification of the limit

Theorem 2.9 *We assume $(K\alpha)$ and $(F\gamma)$ with $\gamma > \alpha \in (\frac{1}{2}, 1]$. The empirical measures μ^N converge in law to the unique solution P of the nonlinear martingale problem (MP) as $N \to +\infty$.*

Proof : Let π^N denote the law of the empirical measure μ^N. According to Proposition 2.8 the sequence $(\pi^N)_N$ is tight. Let π^∞ be the limit of a weakly convergent subsequence that we still index by N for notational simplicity. Denoting by Q with time-marginals $(Q_s)_{s\geq 0}$ the canonical variable on $\mathcal{P}(D([0,+\infty),\mathbb{N}^*))$, we are going to check that π^∞ a.s., Q solves the nonlinear martingale problem (MP). Since the coordinates of the initial vector $(Y_0^{1,N},\dots,Y_0^{N,N})_N$ are i.i.d. according to the probability measure ρ on $\mathbb{N}^*$, we easily check that π^∞ a.s., $Q_0=\rho$ i.e. Q satisfies condition (i) in definition 2.2. To conclude, we have to check that π^∞ a.s., condition (ii) is satisfied. Since a function $\varphi:\mathbb{N}^*\to\mathbb{R}$ such that $\forall l\geq m$, $\varphi(l)=\varphi(m)$ writes $\varphi(l)=\varphi(m)+\sum_{i=1}^{m-1}1_i(l)(\varphi(i)-\varphi(m))$ it is enough to prove that $\forall i\in\mathbb{N}^*$, π^∞ a.s.

$$\begin{aligned}\Phi_t^i(X,Q)=&1_i(X_t)-1_i(X_0)-\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{K}_{X_s,j}(1_i(X_s+j)-1_i(X_s))Q_s(j)ds\\&+\int_0^t\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}1_i(X_s)ds-\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(X_s)ds \text{ is a } Q\text{-martingale.}\end{aligned}\tag{2.9}$$

Using $(K\alpha)$, we bound the absolute value of the sum of the four first terms in the above expression of $\Phi_t^i(X,Q)$ by $1+(\kappa i^\alpha+\sum_{j=1}^{i-1}\tilde{F}_{i-j,j})t$.
Hence the integrability condition π^∞ a.s. $<Q,|\Phi_t^i(X,Q)|><+\infty$ can be proved by checking that $\mathbb{E}^{\pi^\infty}\left(\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}Q_s(i+j)ds\right)$ is finite. By continuity of $Q\to\int_0^t\sum_{j=1}^J\tilde{F}_{i,j}Q_s(i+j)ds$ for $J\in\mathbb{N}^*$ and exchangeability of the particles $Y^{1,N},\dots,Y^{N,N}$ we obtain that this expectation is smaller than the supremum over N of $\mathbb{E}\Big(\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(Y_s^{1,N})ds\Big)$. Since the number of jumps on $[0,t]$ leading from $Y_{s^-}^{1,N}>i$ to $Y_s^{1,N}=i$ is by construction smaller than the number of jumps on $[0,t]$ leading from $Y_{s^-}^{1,N}=i$ to $Y_s^{1,N}\neq i$ plus 1, taking expectations we conclude that

$$\mathbb{E}^{\pi^\infty}\left(\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}Q_s(i+j)ds\right)\leq\sup_N\mathbb{E}\bigg(\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(Y_s^{1,N})ds\bigg)\leq 1+\bigg(\kappa i^\alpha+\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}\bigg)t.\tag{2.10}$$

For $l\in\mathbb{N}^*$, $g:\mathbb{N}^{*l}\to\mathbb{R}_+$ bounded and $0\leq s_1\leq s_2\leq\dots\leq s_l\leq r\leq t$, we set

$$G:Q\in\mathcal{P}(D([0,+\infty),\mathbb{N}^*))\to<Q,(\Phi_t^i(X,Q)-\Phi_r^i(X,Q))g(X_{s_1},\dots,X_{s_l})>\in\mathbb{R}\cup\{-\infty\}.$$

Our aim is to prove that $\mathbb{E}^{\pi^\infty}|G(Q)|=0$. For $1\leq n\leq N$, the processes

$$\begin{aligned}M_t^{n,N}=1_i(Y_t^{n,N})-1_i(Y_0^{n,N})&-\int_0^t\frac{1}{N}\sum_{m=1}^N\tilde{K}_{Y_s^{n,N},Y_s^{m,N}}(1_i(Y_s^{n,N}+Y_s^{m,N})-1_i(Y_s^{n,N}))\\&+\int_0^t\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}1_i(Y_s^{n,N})ds-\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(Y_s^{n,N})ds\end{aligned}$$

are square integrable martingales with brackets

$$\begin{aligned}<M^{n,N},M^{n',N}>_t=1_{\{n=n'\}}\bigg(&\int_0^t\frac{1}{N}\sum_{m=1}^N\tilde{K}_{Y_s^{n,N},Y_s^{m,N}}(1_i(Y_s^{n,N}+Y_s^{m,N})-1_i(Y_s^{n,N}))^2ds\\&+\int_0^t\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}1_i(Y_s^{n,N})ds+\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(Y_s^{n,N})ds\bigg)\end{aligned}$$

satisfying $\mathbb{E}(<M^{n,N},M^{n',N}>_t)\leq 1_{\{n=n'\}}\bigg(1+2\bigg(\kappa i^\alpha+\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}\bigg)t\bigg)$ by $(K\alpha)$ and (2.10).

Since $G(\mu^N)=\frac{1}{N}\sum_{n=1}^N(M_t^{n,N}-M_r^{n,N})g(Y_{s_1}^{n,N},\dots,Y_{s_l}^{n,N})$, we deduce that

$$(\mathbb{E}^{\pi^N}|G(Q)|)^2\leq \mathbb{E}(G^2(\mu^N)))\leq \frac{C}{N}\to_{N\to+\infty}0. \tag{2.11}$$

The function G being neither continuous nor bounded, the convergence of the sequence $(\pi^N)_N$ to π^∞ is not enough to deduce that $\mathbb{E}^{\pi^\infty}|G(Q)|=0$. Weak convergence of a sequence $(Q^n)_n$ to Q implies that for $t\notin D_Q=\{s\geq 0,\ Q(\{X_{s^-}\neq X_s\})>0\}$, $\lim_{n\to+\infty}\sum_{i\in\mathbb{N}^*}|Q_t^n(i)-Q_t(i)|=0$. Hence for $s_1,\dots,s_l,r,t\notin D_Q$, the contribution in G of the first four terms in the definition (2.9) of Φ_t^i is continuous at Q. And the function G^J obtained by replacing the fifth term $\int_r^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(X_s)ds$ by $\int_r^t\sum_{j=1}^J\tilde{F}_{i,j}1_{i+j}(X_s)ds$ is bounded and continuous at Q. We fix $s_1,\dots,s_l,r,t$ outside of the at most countable set $\{s\geq 0,\ \pi^\infty(\{Q:s\in D_Q\})>0\}$. Then π^∞ gives full weight to continuity points of G^J and $\lim_N\mathbb{E}^{\pi^N}|G^J(Q)|=\mathbb{E}^{\pi^\infty}|G^J(Q)|$. With (2.11), we deduce

$$\mathbb{E}^{\pi^\infty}|G(Q)|\leq\limsup_{J\to+\infty}\mathbb{E}^{\pi^\infty}|G-G^J|(Q)+\limsup_{J\to+\infty}\limsup_{N\to+\infty}\mathbb{E}^{\pi^N}|G-G^J|(Q)$$

Applying Lebesgue theorem thanks to the upper-bound (2.10), we get that the first term of the r.h.s. is nil. To deal with the second one, we use successively the exchangeability of the processes $Y^{1,N},\dots,Y^{N,N}$, Cauchy-Schwarz inequality, the above definition of $M_t^{1,N}$ and bound of $\mathbb{E}(<M^{1,N},M^{1,N}>_t)$:

$$\begin{aligned}
\Big(\mathbb{E}^{\pi^N}|G-G^J|(Q)\Big)^2&\leq C\Big(\mathbb{E}\Big(1_{\{\sup_{s\leq t}Y_s^{1,N}>i+J\}}\int_0^t\sum_{j\geq J+1}\tilde{F}_{i,j}1_{i+j}(Y_s^{1,N})ds\Big)\Big)^2\\
&\leq C\mathbb{P}\Big(\sup_{s\leq t}Y_s^{1,N}>i+J\Big)\mathbb{E}\Big(\Big(\int_0^t\sum_{j\in\mathbb{N}^*}\tilde{F}_{i,j}1_{i+j}(Y_s^{1,N})ds\Big)^2\Big)\\
&\leq C\mathbb{P}\Big(\sup_{s\leq t}Y_s^{1,N}>i+J\Big)\mathbb{E}\Big(\Big(|M_t^{1,N}|+1+\Big(\kappa i^\alpha+\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}\Big)t\Big)^2\Big)\\
&\leq C\mathbb{P}\Big(\sup_{s\leq t}Y_s^{1,N}>i+J\Big)\text{ where }C\text{ does not depend on }N.
\end{aligned}$$

By (2.7), we deduce that for any $l\in\mathbb{N}^*$, $g:\mathbb{N}^{*l}\to\mathbb{R}_+$ bounded and $s_1,\dots,s_l,r,t$ outside of the at most countable set $\{s\geq 0,\ \pi^\infty(\{Q:s\in D_Q\})>0\}$, π^∞ a.s.,

$$<Q,(\Phi_t^i(X,Q)-\Phi_r^i(X,Q))g(X_{s_1},\dots,X_{s_l})>=0.$$

The process X being càdlàg, we deduce that π^∞ a.s. $(\Phi_t^i(X,Q))_t$ is a Q-martingale, which completes the proof. ∎

3 Coagulation kernel satisfying $(K1)$ and small initial data ($\sum_{i\in\mathbb{N}^*}i\rho(i)<\infty$):

We only suppose that $(K1)$ holds and do not make any assumption on the fragmentation kernel. Instead we assume that $\sum_{i\in\mathbb{N}^*}i\rho(i)<+\infty$ i.e. $\sum_{i\in\mathbb{N}^*}i^2\gamma(i)<+\infty$.

3.1 Existence for (0.2)

Since by $(K1)$, $K_{i,j} \leq \varphi(i)\varphi(j)$ for the linear function $\varphi(i) = \sqrt{\kappa}\, i$ and p_t solves (0.2) if and only if $c_t(i) = p_t(i)/i$ solves (0.1), the following definition of strong solutions to (0.2) is consistent with the definition of strong solutions of the non necessarily discrete Smoluchowski coagulation equation introduced by Norris [15]

Definition 3.1 *A solution $t \in [0,T) \to p_t$ of (0.2) in the sense of Definition 0.1 is called a strong solution if $\forall t < T$, $\int_0^t \sum_{i\in\mathbb{N}^*} i p_s(i) ds < +\infty$.*

Remark 3.2 *Any strong solution is mass-conserving. Indeed if p_t is a strong solution on $[0,T)$, integrating (0.2) on $[0,t]$ for $t < T$ and summing the obtained result for $1 \leq i \leq n$, we get*

$$\sum_{i=1}^n p_t(i) = \sum_{i=1}^n \rho(i) - \int_0^t \sum_{i=1}^n \sum_{j\geq n+1-i} W_{i,j}(p_s) ds \geq \sum_{i=1}^n \rho(i) - \int_0^t \sum_{i=1}^n \sum_{j\geq n+1-i} \tilde{K}_{i,j} p_s(i) p_s(j) ds.$$

By $(K1)$ and the strong solution assumption, the second term of the r.h.s. converges to 0 as $n \to +\infty$ and $\sum_{i\in\mathbb{N}^*} p_t(i) \geq \sum_{i\in\mathbb{N}^*} \rho(i) = 1$. The converse inequality holds according to Definition 0.1.

Proposition 3.3 *If $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$ then equation (0.2) admits a strong solution p_t on $[0, T_\rho)$ where $T_\rho = (\kappa \sum_{i\in\mathbb{N}^*} i\rho(i))^{-1}$ (κ is the constant in assumption $(K1)$) satisfying $\forall t \in [0, T_\rho)$, $\sum_{i\in\mathbb{N}^*} i p_t(i) \leq (\kappa(T_\rho - t))^{-1}$.*

The proof follows ideas developped in [7] and consists in taking the limit $n \to +\infty$ in the following n-dimensional density conserving truncation of (0.2) :

$$\forall i \leq n,\ p_0^n(i) = \rho(i) \text{ and } \partial_t p_t^n(i) = \sum_{j=1}^{i-1} W_{i-j,j}(p_t^n) - \sum_{j=1}^{n-i} W_{i,j}(p_t^n) \tag{3.1}$$

This system has a unique solution on $[0, +\infty)$ with $p_t^n(i) \geq 0$ and $\sum_{i=1}^n p_t^n(i) = \sum_{i=1}^n \rho(i)$. Indeed local existence and uniqueness can be proved by a standard fixed-point approach. Since $p_t^n(i)$ is a factor in all terms with sign minus in the right-hand-side of (3.1), $p_t^n(i)$ remains non-negative. With the mass conservation, which writes $\sum_{i=1}^n \partial_t p_t^n(i) = 0$ and is a consequence of the Lemma

Lemma 3.4 *For $1 \leq m \leq n$, $\sum_{i=m}^n \left(\sum_{j=1}^{i-1} a_{i-j,j} - \sum_{j=1}^{n-i} a_{i,j} \right) = \sum_{i=1}^{m-1} \sum_{j=m-i}^{n-i} a_{i,j}$.*

for the choice $m = 1$ and $a_{i,j} = W_{i,j}(p_t^n)$, we deduce that $\forall 1 \leq i \leq n,\ 0 \leq p_t^n(i) \leq \sum_{i=1}^n \rho(i)$. This bound allows to iterate the fixed-point technique to obtain a unique global solution.

Proof of Lemma 3.4 : Exchanging summations over i and j and setting $k = i - j$ yields

$$\sum_{i=m}^n \sum_{j=1}^{i-1} a_{i-j,j} = \sum_{j=1}^{n-1} \sum_{k=(m-j)\vee 1}^{n-j} a_{k,j} = \sum_{k=1}^n \sum_{j=(m-k)\vee 1}^{n-k} a_{k,j} = \sum_{k=1}^{m-1} \sum_{m-k}^{n-k} a_{k,j} + \sum_{k=m}^n \sum_{j=1}^{n-k} a_{k,j}.$$

and the conclusion follows readily. ∎

Before proving Proposition 3.3 let us check that the estimation given in this Proposition for p_t holds for p_t^n.

Lemma 3.5 $\forall t \in [0,T_\rho),\ \sum_{i=1}^n i p_t^n(i) \leq (\kappa(T_\rho - t))^{-1}$.

Proof : To prove this result, we bound $\sum_{i=1}^n i\partial_t p_t^n(i)$. Since the fragmentation terms have a non-positive contribution,

$$\begin{aligned}
\sum_{i=1}^n i\partial_t p_t^n(i) &\leq \sum_{i=1}^n i \sum_{j=1}^{i-1} \tilde{K}_{i-j,j} p_t^n(i-j) p_t^n(j) - \sum_{i=1}^{n-1} i \sum_{j=1}^{n-i} \tilde{K}_{i,j} p_t^n(i) p_t^n(j) \\
&= \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} (k+j) \tilde{K}_{k,j} p_t^n(k) p_t^n(j) - \sum_{i=1}^{n-1} i \sum_{j=1}^{n-i} \tilde{K}_{i,j} p_t^n(i) p_t^n(j) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} j \tilde{K}_{i,j} p_t^n(i) p_t^n(j) \leq \kappa (\sum_{i=1}^n i p_t^n(i))^2 \ \text{ since by } (K1),\ \tilde{K}_{i,j} = K_{i,j}/j \leq \kappa i.
\end{aligned}$$

We conclude by comparison with the solution of the O.D.E. $\partial_t y(t) = \kappa y^2(t),\ y(0) = \sum_{i=1}^n i\rho(i)$. ∎

Proof of Proposition 3.3 : We set $\forall i > n,\ \forall t \geq 0,\ p_t^n(i) = 0$.
According to Lemma 2.3 [3], for $m \in \mathbb{N}^*$, $\frac{d}{dt}\sum_{i=m}^n p_t^n(i)$ is smaller than a constant independent of $n \geq m$. Since $\sum_{i=m}^n p_t^n(i) \in [0,1]$, we deduce that for fixed m the functions $(\sum_{i=m}^n p_t^n(i))_{n \geq m}$ and consequently $(p_t^n(m))_n$ are of uniform bounded variation on $[0,+\infty)$. Combining Helly's theorem (see [5] p.130) and a diagonal extraction procedure, we obtain a subsequence, that we still index by n for notational simplicity, such that $\forall m \in \mathbb{N}^*,\ \forall t \geq 0,\ p_t^n(m) \to p_t(m)$. By Lemma 3.5 and Fatou lemma,

$$\forall t \geq 0,\ \sum_{i \in \mathbb{N}^*} p_t(i) \leq \liminf_n \sum_{i \in \mathbb{N}^*} p_t^n(i) = 1 \text{ and } \forall t \in [0,T_\rho), \sum_{i \in \mathbb{N}^*} i p_t(i) \leq (\kappa(T_\rho - t))^{-1}. \tag{3.2}$$

The remainder of the proof consists in checking that p_t is a solution of (0.2) on $[0,T_\rho)$.
Integration of (3.1) yields

$$p_t^n(i) = \rho(i) + \int_0^t \sum_{j=1}^{i-1} W_{i-j,j}(p_s^n) ds - \int_0^t p_s^n(i) \sum_{j \in \mathbb{N}^*} \tilde{K}_{i,j} p_s^n(j) ds + \int_0^t \sum_{j \in \mathbb{N}^*} \tilde{F}_{i,j} p_s^n(i+j) ds.$$

Since $\forall i \in \mathbb{N}^*,\ p_t^n(i) \in [0,1]$, according to Lebesgue theorem, the second term of the right-hand-side converges to $\int_0^t \sum_{j=1}^i W_{i-j,j}(p_s) ds$. Combining $(K1)$ and Lemma 3.5, we check that for $t \in [0,T_\rho)$ the series $(\int_0^t p_s^n(i) \tilde{K}_{i,j} p_s^n(j) ds)_{j \in \mathbb{N}^*}$ are summable over j uniformly in n. Hence $\forall t \in [0,T_\rho)$, the third term of the r.h.s. converges to $-\int_0^t p_s(i) \sum_{j \in \mathbb{N}^*} \tilde{K}_{i,j} p_s(j) ds$. As a consequence for $t \in [0,T_\rho)$ the last term of the r.h.s. has a limit $f_t(i)$ that we still have to identify. Let $t < T_\rho$. Since for $n - k \geq m \geq 1$,

$$\int_0^t \sum_{j=m}^{n-k} \tilde{F}_{k,j} p_s^n(k+j) ds = \int_0^t \sum_{j \in \mathbb{N}^*} \tilde{F}_{k,j} p_s^n(k+j) ds - \int_0^t \sum_{j=1}^{m-1} \tilde{F}_{k,j} p_s^n(k+j) ds,$$

$$\limsup_{m \to +\infty} \limsup_{n \to +\infty} \int_0^t \sum_{j=m}^{n-k} \tilde{F}_{k,j} p_s^n(k+j) ds = f_t(k) - \int_0^t \sum_{j \in \mathbb{N}^*} \tilde{F}_{k,j} p_s(k+j) ds.$$

We are going to prove that the l.h.s. is nil. We suppose that $n-k \geq m \geq k+1$.

$$\sum_{j=m}^{n-k} \tilde{F}_{k,j} p_s^n(k+j) \leq \sum_{i=1}^{m-1} \sum_{j=m-i}^{n-i} \tilde{F}_{i,j} p_s^n(i+j) = \sum_{i=m}^{n} \left(\sum_{j=1}^{i-1} \tilde{F}_{i-j,j} p_s^n(i) - \sum_{j=1}^{n-i} \tilde{F}_{i,j} p_s^n(i+j) \right) \text{ by Lemma 3.4.}$$

Integrating (3.1) w.r.t. the time variable, summing the result for $m \leq i \leq n$ and using Lemma 3.4, we deduce that

$$\begin{aligned} \int_0^t \sum_{j=m}^{n-k} \tilde{F}_{k,j} p_s^n(k+j) ds &\leq \sum_{i=m}^{n} \rho(i) + \int_0^t \sum_{i=1}^{m-1} \sum_{j=m-i}^{n-i} \tilde{K}_{i,j} p_s^n(i) p_s^n(j) ds \\ &\leq \sum_{i=m}^{n} \rho(i) + \kappa \int_0^t \sum_{i=1}^{m-1} i p_s^n(i) \sum_{j=m-i}^{n-i} p_s^n(j) ds \end{aligned} \tag{3.3}$$

Since, by the mass-conservation for (3.1), $\sum_{i=1}^{m-1} i p_s^n(i) \sum_{j=m-i}^{n-i} p_s^n(j) \leq m-1$, applying Fatou lemma, we obtain

$$\limsup_{n \to +\infty} \int_0^t \sum_{i=1}^{m-1} i p_s^n(i) \sum_{j=m-i}^{n-i} p_s^n(j) ds \leq \int_0^t \sum_{i=1}^{m-1} i p_s(i) \limsup_{n \to +\infty} \sum_{j=m-i}^{n-i} p_s^n(j) ds.$$

Using Lemma 3.5, we check that $\sum_{j=m-i}^{n-i} p_s^n(j) \to \sum_{j \geq m-i} p_s(j)$ as $n \to +\infty$. Hence

$$\limsup_{n \to +\infty} \int_0^t \sum_{i=1}^{m-1} i p_s^n(i) \sum_{j=m-i}^{n-i} p_s^n(j) ds \leq \int_0^t \sum_{i+j \geq m} i p_s(i) p_s(j) ds.$$

As by (3.2) $\forall t \in [0, T_\rho)$, $\int_0^t \sum_{i,j \in \mathbb{N}^*} i p_s(i) p_s(j) ds \leq \ln(1 + t/(T_\rho - t))/\kappa$,

$$\limsup_{m \to +\infty} \limsup_{n \to +\infty} \int_0^t \sum_{i=1}^{m-1} i p_s^n(i) \sum_{j=m-i}^{n-i} p_s^n(j) ds = 0.$$

By (3.3), we conclude that $\forall t \in [0, T_\rho)$, $\limsup_m \limsup_n \int_0^t \sum_{j=m}^{n-k} \tilde{F}_{k,j} p_s^n(k+j) ds = 0$ i.e. p_t solves (0.2) on $[0, T_\rho)$. ∎

Remark 3.6 *In case* $(K1)$ *is replaced by the stronger assumption* $K_{i,j} \leq \kappa(i+j)$, *using this bound in the proof of Lemma 3.5, we obtain* $\partial_t \sum_{i=1}^n i p_t^n(i) \leq 2\kappa \sum_{i=1}^n i p_t^n(i)$ *and conclude by Gronwall lemma that* $\forall t \geq 0$, $\sum_{i=1}^n i p_t^n(i) \leq e^{2\kappa t} \sum_{i \in \mathbb{N}^*} i\rho(i)$. *Following the proof of Proposition 3.3, we deduce that if* $\sum_{i \in \mathbb{N}^*} i\rho(i) < +\infty$, *then* (0.2) *has a strong solution on* $[0, +\infty)$ *satisfying* $\forall t \geq 0$, $\sum_{i \in \mathbb{N}^*} i p_t(i) \leq e^{2\kappa t} \sum_{i \in \mathbb{N}^*} i\rho(i)$, *which is also a consequence of the combination of [3] Theorem 2.4 and [6] Theorem 3.2 concerning the original Smoluchowski equation* (0.1).

3.2 Nonlinear process and uniqueness for (0.2)

Given $t \in [0, T) \to u_t \in \{q : \mathbb{N}^* \to \mathbb{R}_+ : \sum_{i \in \mathbb{N}^*} q(i) \leq 1\}$, let X^u denote a one-dimensional jump process starting from ρ and with transition function

$$\forall (s, i, j) \in \mathbb{R}_+ \times \mathcal{E} \times \mathcal{E},\ \lambda(s, i, j) = 1_{\{i < +\infty\}} \left(\tilde{F}_{j, i-j} 1_{\{1 \leq j < i\}} + \tilde{K}_{i, j-i} u_s(j-i) 1_{\{i < j < +\infty\}} \right).$$

Proposition 3.7 *When u_t solves* (0.2) *on* $[0,T)$*, the process X^u is nonlinear in the following sense :* $\forall t \in [0, T \wedge T_\rho)$, $\forall i \in \mathbb{N}^*$, $\mathbb{P}(X_t^u = i) = u_t(i)$.

This result is obtained by an adaptation of the proof of Proposition 2.5. The estimations given in Proposition 2.1 and Lemma 2.6 are replaced by the following one

$$\forall t \in [0, T \wedge T_\rho),\ \max\left(\sum_{i\in\mathbb{N}^*} i u_t(i), \sum_{i\in\mathbb{N}^*} i\mathbb{P}(X_t^u = i)\right) \leq (\kappa(T_\rho - t))^{-1}, \tag{3.4}$$

which is deduced from comparison with the mass-conserving solution $(v_t)_{t\in[0,T_\rho)}$ of the equation with multiplicative coagulation kernel κij and no fragmentation given by Proposition 3.3:

$$\forall i \in \mathbb{N}^*,\ v_t(i) = \rho(i) + \kappa \int_0^t \left(\sum_{j=1}^{i-1}(i-j)v_s(i-j)v_s(j) - iv_s(i)\sum_{j\geq 1} v_s(j)\right)ds. \tag{3.5}$$

Since

$$\sum_{i\in\mathbb{N}^*} iv_t(i) = \sum_{n\in\mathbb{N}^*}\sum_{i\geq n} v_t(i) = \sum_{n\in\mathbb{N}^*}(1 - \sum_{i=1}^{n-1} v_t(i))$$

and $\sum_{i\in\mathbb{N}^*} iu_t(i)$ (resp. $\sum_{i\in\mathbb{N}^*} i\mathbb{P}(X_t^u = i))$) is smaller than $\sum_{n\in\mathbb{N}^*}(1 - \sum_{i=1}^{n-1} u_t(i))$ (resp. $\sum_{n\in\mathbb{N}^*}(1 - \sum_{i=1}^{n-1}\mathbb{P}(X_t^u = i))$), estimation (3.4) is obtained by combination of the estimation $\sum_{i\in\mathbb{N}^*} iv_t(i) \leq (\kappa(T_\rho - t))^{-1}$ given in Proposition 3.3 and the following comparison between v_t, u_t and the law of X_t^u :

Lemma 3.8 *If u_t solves* (0.2) *on* $[0,T)$ *then*

$$\forall t \in [0, T \wedge T_\rho),\ \forall n \in \mathbb{N}^*,\ \sum_{i=1}^n v_t(i) \leq \min\left(\sum_{i=1}^n u_t(i), \sum_{i=1}^n \mathbb{P}(X_t^u = i)\right).$$

The proof of this lemma is postponed to the next section. We are now ready to state our main result concerning general fragmentation kernels and coagulation kernels satisfying $(K1)$:

Theorem 3.9 *Suppose that $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$. If u_t and p_t solve* (0.2) *on* $[0,T)$ *and p_t is a strong solution then $u_t = p_t$ on* $[0,T)$*. Moreover* (0.2) *admits a unique maximal strong solution p_t on $[0,\mathcal{T})$ with $\mathcal{T} \geq T_\rho = (\kappa\sum_{i\in\mathbb{N}^*} i\rho(i))^{-1}$ and the process X^p is nonlinear on $[0,\mathcal{T})$:* $\forall t \in [0,\mathcal{T})$, $\forall i \in \mathbb{N}^*$, $\mathbb{P}(X_t^p = i) = p_t(i)$.

Translated on Smoluchowski's equation (0.1), this result is similar to [15] Theorem 2.1 for the choice $\varphi(i) = \sqrt{\kappa}\, i$.
With remark 3.6, we easily deduce :

Corollary 3.10 *Assume that $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$ and that $K_{i,j} \leq \kappa(i+j)$ then uniqueness holds for* (0.2) *and therefore for* (0.1).

In their uniqueness result (Theorem 4.1), Ball and Carr [3] make more stringent assumptions on both the coagulation and the fragmentation kernels : they suppose that for some $\alpha \in [0, 1/2]$, $(K\alpha)$ holds and that $\exists C > 0$, $\forall i \geq 2$, $\sum_{j=1}^{[(i+1)/2]} j^{1-\alpha} F_{i-j,j} \leq Ci^{1-\alpha}$. On the other hand, we suppose that the initial data γ is such that $\sum_{i\in\mathbb{N}^*} i^2\gamma(i) < +\infty$ whereas they only assume $\sum_{i\in\mathbb{N}^*} i\gamma(i) < +\infty$.

Proof of Theorem 3.9 : The proof of the uniqueness statement is based on a coupling argument. Let $(X_t, Y_t)_{t\in[0,T)}$ denote a two-dimensional jump process starting from the image of ρ by $i \in \mathbb{N}^* \to (i,i) \in \mathbb{N}^* \times \mathbb{N}^*$ and with transition function equal to zero for transitions involving an infinite coordinate and defined otherwise by

$$\begin{aligned}
&\forall x \in \mathbb{N}^*,\ \forall 1 \le j \le x-1,\ \lambda(s,(x,x),(x-j,x-j)) = \tilde{F}_{x-j,j}\\
&\qquad\qquad \forall j \in \mathbb{N}^*,\ \lambda(s,(x,x),(x+j,x+j)) = \tilde{K}_{x,j}\min(p_s(j),u_s(j))\\
&\qquad\qquad\qquad\qquad \lambda(s,(x,x),(x+j,x)) = \tilde{K}_{x,j}(p_s(j)-\min(p_s(j),u_s(j)))\\
&\qquad\qquad\qquad\qquad \lambda(s,(x,x),(x,x+j)) = \tilde{K}_{x,j}(u_s(j)-\min(p_s(j),u_s(j)))\\
&\forall (x,y) \in \mathbb{N}^* \times \mathcal{E} \text{ with } x \neq y,\\
&\qquad\qquad \forall 1 \le j \le x-1,\ \lambda(s,(x,y),(x-j,y)) = \lambda(s,(y,x),(y,x-j)) = \tilde{F}_{x-j,j}\\
&\qquad\qquad \forall j \in \mathbb{N}^*,\ \lambda(s,(x,y),(x+j,y)) = \tilde{K}_{x,j}p_s(j)\\
&\qquad\qquad\qquad\qquad \lambda(s,(y,x),(y,x+j)) = \tilde{K}_{x,j}u_s(j).
\end{aligned}$$

We easily check that X (resp. Y) is a jump process starting from ρ with transition function $\lambda(s,i,j) = 1_{\{i<+\infty\}}\left(\tilde{F}_{j,i-j}1_{\{1\le j<i\}} + \tilde{K}_{i,j-i}p_s(j-i)1_{\{i<j<+\infty\}}\right)$ (resp. the previous one with p_s replaced by u_s). By the weak uniqueness result in Proposition 1.3 and by Proposition 3.7, we deduce that $\forall t \in [0, T\wedge T_\rho)$, $\forall i \in \mathbb{N}^*$, $\mathbb{P}(X_t = i) = p_t(i)$ and $\mathbb{P}(Y_t = i) = u_t(i)$.
Therefore for $t < T \wedge T_\rho$,

$$\begin{aligned}
\sum_{j\in\mathbb{N}^*} |p_t(j) - u_t(j)| &= \sum_{j\in\mathbb{N}^*} |\mathbb{P}(X_t = j) - \mathbb{P}(Y_t = j)| \le \sum_{j\in\mathbb{N}^*} \mathbb{P}(X_t = j, Y_t \neq j) + \mathbb{P}(X_t \neq j, Y_t = j)\\
&\le 2\mathbb{P}(X_t \neq Y_t) \le 2\mathbb{P}(\exists s \le t,\ X_s \neq Y_s).
\end{aligned}$$

For $t < T \wedge T_\rho$, the probability that for some $s \le t$, $X_s \neq Y_s$ is smaller than the expectation $\mathbb{E}\left(\int_0^t \sum_{i\in\mathbb{N}^*} 1_{\{X_s=Y_s=i\}} \sum_{j\in\mathbb{N}^*} \tilde{K}_{i,j}|p_s(j)-u_s(j)|ds\right)$ of the number of jumps on $[0,t]$ leading form $X_{s^-} = Y_{s^-}$ to $X_s \neq Y_s$, which by $(K1)$ is smaller than $\kappa\int_0^t \sum_{j\in\mathbb{N}^*}|p_s(j) - u_s(j)|\sum_{i\in\mathbb{N}^*} ip_s(i)ds$.

$$\text{Hence } \forall t \in [0, T\wedge T_\rho),\ \sum_{j\in\mathbb{N}^*} |p_t(j) - u_t(j)| \le 2\kappa \int_0^t \sum_{j\in\mathbb{N}^*} |p_s(j) - u_s(j)| \sum_{i\in\mathbb{N}^*} ip_s(i)ds.$$

As p_t is a strong solution, we conclude by Gronwall lemma that $\forall t \in [0, T\wedge T_\rho)$, $p_t = u_t$ and consequently $\mathbb{P}(X_t = Y_t) = 1$.
Let $t_u = \sup\{t < T : \forall s \in [0,t],\ \forall i \in \mathbb{N}^*,\ \mathbb{P}(X_s = i) = p_s(i) = u_s(i) = \mathbb{P}(Y_s = i) \text{ and } \mathbb{P}(X_s = Y_s) = 1\}$. Assuming that $t_u < T$, we are going to obtain a contradiction. We have $0 < T_\rho \le t_u < T$. Since p_t is a strong solution, $\int_0^{t_u} \sum_{i\in\mathbb{N}^*} ip_s(i)ds < +\infty$. Hence for some $s \in (0, t_u)$, $T_{p_s} = (\kappa\sum_{i\in\mathbb{N}^*} ip_s(i))^{-1} > t_u - s$. Both $t \to u_{s+t}$ and $t \to p_{s+t}$ solve (0.2) on $[0, T-s)$ with initial condition ρ replaced by p_s. Moreover, p_{s+t} is a strong solution and $\mathbb{P}(X_s = Y_s) = 1$. By the reasoning we have just made, we deduce that $\forall t \in [0, (T-s)\wedge T_{p_s})$, $\forall i \in \mathbb{N}^*$, $\mathbb{P}(X_{s+t} = i) = p_{s+t}(i) = u_{s+t}(i) = \mathbb{P}(Y_{s+t} = i)$ and $\mathbb{P}(X_{s+t} = Y_{s+t}) = 1$. Since $T \wedge (s + T_{p_s}) > t_u$ this gives the desired contradiction and $t_u = T$.
With Proposition 3.3 we easily deduce the last assertion in the Theorem. ∎

3.3 Proof of Lemma 3.8

For $t \in [0,T) \to q_t \in \{q : \mathbb{N}^* \to \mathbb{R}_+ : \sum_{i\in\mathbb{N}^*} q(i) \leq 1\}$, let Y^q denote a jump process starting from ρ and with transition function

$$\forall (s,i,j) \in [0,T) \times \mathcal{E} \times \mathcal{E},\ \lambda(s,i,j) = 1_{\{i<j<+\infty\}} \kappa\, i\, q_s(j-i).$$

We easily check that

$$\forall i \in \mathbb{N}^*,\ \mathbb{P}(Y^q_t = i) = \rho(i) + \kappa \int_0^t \left(\sum_{j=1}^{i-1} (i-j)\mathbb{P}(Y^q_s = i-j) q_s(j) - i\mathbb{P}(Y^q_s = i) \sum_{j\geq 1} q_s(j) \right) ds. \tag{3.6}$$

For the solution u_t of (0.2) on $[0,T)$ considered in Lemma 3.8, we have

Lemma 3.11 $\forall t \in [0,T)$, $\forall n \in \mathbb{N}^*$, $\mathbb{P}(Y^u_t \leq n) \leq \min\left(\sum_{i=1}^n u_t(i), \mathbb{P}(X^u_t \leq n)\right)$.

Proof : Combining (0.2) and (3.6), then using $(K1)$, we get

$$\begin{aligned}\partial_t \sum_{i=1}^n (\mathbb{P}(Y^u_t = i) - u_t(i)) &= \sum_{i=1}^n \sum_{j\geq n-i+1} \left(-\tilde{F}_{i,j} u_t(i+j) + \tilde{K}_{i,j} u_t(i) u_t(j) - \kappa i \mathbb{P}(Y^u_t = i) u_t(j) \right) \\ &\leq \kappa \sum_{i=1}^n (u_t(i) - \mathbb{P}(Y^u_t = i)) i \sum_{j\geq n-i+1} u_t(j). \end{aligned} \tag{3.7}$$

For $n = 1$, this equation writes $\partial_t(\mathbb{P}(Y^u_t = 1) - u_t(1)) \leq \kappa \sum_{j\in\mathbb{N}^*} u_t(j)(u_t(1) - \mathbb{P}(Y^u_t = 1))$. Since $u_0(1) = \rho(1) = \mathbb{P}(Y^u_t = 1)$, we deduce that $\forall t \in [0,T)$, $\mathbb{P}(Y^u_t = 1) \leq u_t(1)$.
Supposing inductively that for $1 \leq m \leq n-1$, $\forall t \in [0,T)$, $\sum_{i=1}^m (u_t(i) - \mathbb{P}(Y^u_t = i)) \geq 0$ and using that $i \to i \sum_{j\geq n-i+1} u_t(j)$ is non-decreasing on $\{1,\ldots,n\}$, we obtain

$$\begin{aligned}&\sum_{i=1}^n \left(n \sum_{j\in\mathbb{N}^*} u_t(j) - i \sum_{j\geq n-i+1} u_t(j) \right) (u_t(i) - \mathbb{P}(Y^u_t = i)) \\ &\qquad = \sum_{m=1}^{n-1} \left((m+1) \sum_{j\geq n-m} u_t(j) - m \sum_{j\geq n-m+1} u_t(j) \right) \sum_{i=1}^m (u_t(i) - \mathbb{P}(Y^u_t = i)) \geq 0\end{aligned}$$

and deduce from (3.7)

$$\partial_t \sum_{i=1}^n (\mathbb{P}(Y^u_t = i) - u_t(i)) \leq \kappa n \sum_{j\in\mathbb{N}^*} u_t(j) \sum_{i=1}^n (u_t(i) - \mathbb{P}(Y^u_t = i)).$$

With $\sum_{i=1}^n u_0(i) = \sum_{i=1}^n \mathbb{P}(Y^u_0 = i)$, we conclude that $\forall t \in [0,T)$, $\sum_{i=1}^n \mathbb{P}(Y^u_t = i) \leq \sum_{i=1}^n u_t(i)$. Replacing (0.2) by the linear equation analogous to (2.2) satisfied by the law of X^u_t and following the same line of reasoning, we get that $\sum_{i=1}^n \mathbb{P}(Y^u_t = i) \leq \sum_{i=1}^n \mathbb{P}(X^u_t = i)$. ∎

As a consequence, for $t \in [0,T)$, the function

$$u^1_t(i) = 1_{\{i=1\}} \mathbb{P}(Y^u_t = 1) + 1_{\{i=2\}} (u_t(2) + u_t(1) - \mathbb{P}(Y^u_t = 1)) + 1_{\{i>2\}} u_t(i)$$

belongs to $\{q:\mathbb{N}^*\to\mathbb{R}^+,\ \sum_{i\in\mathbb{N}^*}q(i)=\sum_{i\in\mathbb{N}^*}u_t(i)\}$ and satisfies $\forall n\in\mathbb{N}^*,\ \sum_{i=1}^n u_t^1(i)\leq\sum_{i=1}^n u_t(i)$. We assume inductively that we have constructed $u_t^1,\dots,u_t^k\in\{q:\mathbb{N}^*\to\mathbb{R}^+,\ \sum_{i\in\mathbb{N}^*}q(i)=\sum_{i\in\mathbb{N}^*}u_t(i)\}$ such that $\forall t\in[0,T)$, $\forall n\in\mathbb{N}^*$,

$$\mathbb{P}(Y_t^{u^{k-1}}\leq n)\leq\ldots\leq\mathbb{P}(Y_t^{u^1}\leq n)\leq\mathbb{P}(Y_t^{u}\leq n)\leq\min\left(\mathbb{P}(X_t^u\leq n),\sum_{i=1}^n u_t(i)\right),$$

$$\sum_{i=1}^n u_t^k(i)\leq\sum_{i=1}^n u_t^{k-1}(i)\leq\ldots\leq\sum_{i=1}^n u_t^1(i)\leq\sum_{i=1}^n u_t(i).$$

and for $0\leq l\leq k-1$ (convention $u^0=u$),

$$u_t^{l+1}(i)=\begin{cases}\mathbb{P}(Y_t^{u^l}=i)\text{ if }1\leq i\leq l+1\\ \left(\sum_{j=1}^{l+2}u_t(j)-\sum_{j=1}^{l+1}\mathbb{P}(Y_t^{u^l}=i)\right)\text{ if }i=l+2\\ u_t(i)\text{ if }i>l+2\end{cases}. \tag{3.8}$$

Lemma 3.12 $\forall t\in[0,T),\ \forall n\in\mathbb{N}^*,\ \mathbb{P}(Y_t^{u^k}\leq n)\leq\mathbb{P}(Y_t^{u^{k-1}}\leq n)$.

Proof : This result as well as the comparison between $\mathbb{P}(Y_t^u\leq n)$ and $\mathbb{P}(X_t^u\leq n)$ given in Lemma 3.11 could be proved by a probabilistic coupling argument but we give a shorter analytic proof. By the hypothesis concerning u^{k-1} and u^k, we have $\forall t\in[0,T)$, $\forall n\in\mathbb{N}^*$,

$$\sum_{j\geq n}u_t^{k-1}(j)=\sum_{j\in\mathbb{N}^*}u_t(j)-\sum_{j=1}^{n-1}u_t^{k-1}(j)\leq\sum_{j\in\mathbb{N}^*}u_t(j)-\sum_{j=1}^{n-1}u_t^{k}(j)=\sum_{j\geq n}u_t^k(j).$$

Using (3.6) and then the previous upper-bound, we obtain

$$\begin{aligned}\partial_t(\mathbb{P}(Y_t^{u^k}\leq n)-\mathbb{P}(Y_t^{u^{k-1}}\leq n))&=\kappa\sum_{i=1}^n i\left(\mathbb{P}(Y_t^{u^{k-1}}=i)\sum_{j\geq n-i+1}u_t^{k-1}(j)-\mathbb{P}(Y_t^{u^k}=i)\sum_{j\geq n-i+1}u_t^k(j)\right)\\ &\leq\kappa\sum_{i=1}^n\left(\mathbb{P}(Y_t^{u^{k-1}}=i)-\mathbb{P}(Y_t^{u^k}=i)\right)i\sum_{j\geq n-i+1}u_t^k(j).\end{aligned}$$

We conclude like in the proof of lemma 3.11. ∎

We deduce that the function u_t^{k+1} defined by (3.8) for $l=k$ belongs to $\{q:\mathbb{N}^*\to\mathbb{R}^+,\ \sum_{i\in\mathbb{N}^*}q(i)=\sum_{i\in\mathbb{N}^*}u_t(i)\}$ and satisfies $\forall t\in[0,T)$, $\forall n\in\mathbb{N}^*$, $\sum_{i=1}^n u_t^{k+1}(i)\leq\sum_{i=1}^n u_t^k(i)$.
By induction we obtain for $t\in[0,T)$ a sequence $(u_t^k)_{k\in\mathbb{N}}\in\{q:\mathbb{N}^*\to\mathbb{R}^+,\ \sum_{i\in\mathbb{N}^*}q(i)=\sum_{i\in\mathbb{N}^*}u_t(i)\}$ such that $\forall n\in\mathbb{N}^*$, $(\sum_{i=1}^n u_t^k(i))_k$ is non-increasing and for $k\geq n$, $\mathbb{P}(X_t^u\leq n)\geq\mathbb{P}(Y_t^{u^{k-1}}\leq n)=\sum_{i=1}^n u_t^k(i)$. We deduce that $\forall i\in\mathbb{N}^*$, $u_t^k(i)$ converges to a limit $u_t^\infty(i)$ such that

$$u_t^\infty\in\{q:\mathbb{N}^*\to\mathbb{R}^+,\ \sum_{i\in\mathbb{N}^*}q(i)\leq\sum_{i\in\mathbb{N}^*}u_t(i)\}\text{ and }\forall n\in\mathbb{N}^*,\ \sum_{i=1}^n u_t^\infty(i)\leq\min\left(\sum_{i=1}^n u_t(i),\mathbb{P}(X_t^u\leq n)\right).$$

Hence to conclude the proof of Lemma 3.8 it is enough to check that

Lemma 3.13 $\forall t\in[0,T\wedge T_\rho),\ u_t^\infty=v_t$.

Proof : For $k \geq i \in \mathbb{N}^*$, $\forall 1 \leq j \leq i,\ \mathbb{P}(Y_t^{u^{k-1}} = j) = u_t^k(j)$ and (3.6) writes

$$u_t^k(i) = \rho(i) + \kappa \int_0^t \sum_{j=1}^{i-1} (i-j) u_s^k(i-j) u_s^{k-1}(j) - i u_s^k(i) \sum_{j \in \mathbb{N}^*} u_s(j) ds.$$

Taking the limit $k \to +\infty$ in this equation, we obtain that $\forall t \in [0,T),\ \forall i \in \mathbb{N}^*$,

$$u_t^\infty(i) = \rho(i) + \kappa \int_0^t \sum_{j=1}^{i-1} (i-j) u_s^\infty(i-j) u_s^\infty(j) - i u_s^\infty(i) \sum_{j \in \mathbb{N}^*} u_s(j) ds.$$

This equation also writes

$$u_t^\infty(i) = \rho(i) e^{-\kappa i \int_0^t \sum_{j\in\mathbb{N}^*} u_r(j)dr} + \kappa \int_0^t e^{-\kappa i \int_s^t \sum_{j\in\mathbb{N}^*} u_r(j)dr} \sum_{j=1}^{i-1} (i-j) u_s^\infty(i-j) u_s^\infty(j) ds.$$

Since by Remark 3.2 v_t is mass-conserving on $[0,T_\rho)$, we have similarly

$$\forall t \in [0,T_\rho),\ \forall i \in \mathbb{N}^*,\ v_t(i) = \rho(i) e^{-\kappa i t} + \kappa \int_0^t e^{-\kappa i (t-s)} \sum_{j=1}^{i-1} (i-j) v_s(i-j) v_s(j) ds.$$

Using that $\forall r \in [0,T),\ \sum_{j\in\mathbb{N}^*} u_r(j) \leq 1$, we check by induction on $i \in \mathbb{N}^*$ that $\forall t \in [0, T \wedge T_\rho),\ v_t(i) \leq u_t^\infty(i)$.
By Fatou Lemma, $\forall t \in [0,T),\ \sum_{i\in\mathbb{N}^*} u_t^\infty(i) \leq \sum_{i\in\mathbb{N}^*} u_t(i) \leq 1$. Since for $t \in [0,T_\rho)$, $\sum_{i\in\mathbb{N}^*} v_t(i) = 1$, we conclude that $\forall t \in [0, T \wedge T_\rho),\ u_t^\infty = v_t$. ∎

3.4 Propagation of chaos in case $K_{i,j} \leq \kappa(i+j)$

Combining Remarks 3.2, 3.6, Theorem 3.9 and Corollary 3.10, we obtain :

Proposition 3.14 *Assume that $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$ and $K_{i,j} \leq \kappa(i+j)$. Then the nonlinear martingale problem (MP) (see Definition 2.2) has a unique solution P. Moreover, $t \in [0,+\infty) \to P_t$ is the unique solution of* (0.2).

If $Y^{1,N}, \ldots, Y^{N,N}$ denotes the system of N-particles introduced in Section 2.2.1, replacing Lemma 2.7 by the following estimation

Lemma 3.15 *If $K_{i,j} \leq \kappa(i+j)$, $\forall N \in \mathbb{N}^*$, $\forall t \geq 0$, $\mathbb{E}\left(\sup_{s\leq t} Y_s^{1,N}\right) \leq e^{2\kappa t} \sum_{i\in\mathbb{N}^*} i\rho(i)$.*

in the Proofs of Proposition 2.8 and Theorem 2.9, we get :

Theorem 3.16 *Assume that $\sum_{i\in\mathbb{N}^*} i\rho(i) < +\infty$ and $K_{i,j} \leq \kappa(i+j)$. Then as $N \to +\infty$, the empirical measures $\mu^N = \frac{1}{N}\sum_{n=1}^N \delta_{Y^{n,N}}$ considered as $\mathcal{P}(D([0,+\infty),\mathbb{N}^*))$ random variables converge in law to the constant P where P denotes the unique solution of the nonlinear martingale problem (MP).*

Proof of Lemma 3.15 : Let $M \in \mathbb{N}^*$, $\sup_{s\leq t} Y^{1,N}_s \wedge M$ is necessarily smaller than the sum of $Y^{1,N}_0$ and of the contributions of the a.s. finite number of jumps of $s \in [0,t] \to Y^{1,N}_s$ leading from $Y^{1,N}_{s^-} \leq M$ to $Y^{1,N}_s > Y^{1,N}_{s^-}$ i.e.

$$\begin{aligned}\sup_{s\leq t} Y^{1,N}_s \wedge M &\leq \left(Y^{1,N}_0 + \sum_{s\leq t} 1_{\{Y^{1,N}_{s^-} < Y^{1,N}_s\}} 1_{\{Y^{1,N}_{s^-} \leq M\}} (Y^{1,N}_s - Y^{1,N}_{s^-}) \right) \wedge M \\ &\leq Y^{1,N}_0 + \sum_{s\leq t} 1_{\{Y^{1,N}_{s^-} < Y^{1,N}_s\}} 1_{\{Y^{1,N}_{s^-} \leq M\}} ((Y^{1,N}_s - Y^{1,N}_{s^-}) \wedge M).\end{aligned}$$

Taking expectations, using (2.5) then $K_{i,j} \leq \kappa(i+j)$ and the exchangeability of the processes $(Y^{n,N})_{1\leq n\leq N}$, we deduce

$$\begin{aligned}\mathbb{E}\left(\sup_{s\leq t} Y^{1,N}_s \wedge M\right) &\leq \mathbb{E}(Y^{1,N}_0) + \int_0^t \mathbb{E}\left(1_{\{Y^{1,N}_s \leq M\}} \frac{1}{N} \sum_{n=1}^N \tilde{K}_{Y^{1,N}_s, Y^{n,N}_s} (Y^{n,N}_s \wedge M) \right) ds \\ &\leq \sum_{i\in\mathbb{N}^*} i\rho(i) + \kappa \int_0^t \mathbb{E}\left(1_{\{Y^{1,N}_s \leq M\}} Y^{1,N}_s + \frac{1}{N}\sum_{n=1}^N (Y^{n,N}_s \wedge M)\right) ds \\ &\leq \sum_{i\in\mathbb{N}^*} i\rho(i) + 2\kappa \int_0^t \mathbb{E}\left(\sup_{r\leq s} Y^{1,N}_r \wedge M\right) ds.\end{aligned}$$

We apply Gronwall's lemma then let $M \to +\infty$ to conclude. ∎

4 Existence for (0.2) in case $\forall i \in \mathbb{N}^*,\ \lim_{j\to+\infty}(K_{i,j} + F_{i,j})/j = 0$

The existence result that we are going to prove implies existence for (0.1). It is obtained by considering the limit behaviour as $N \to +\infty$ of the particle system $(Y^{1,N}, \ldots, Y^{N,N})$ introduced in section 2.2.1. We first check a tightness result.
We endow the space $D([0,+\infty), \mathcal{E})$ of càdlàg functions from $[0,+\infty)$ to $\mathcal{E}$ with the Skorokhod topology. Note that $\mathcal{D}_1 \subset D([0,+\infty), \mathcal{E})$.

Lemma 4.1 *Assume that $\forall i \in \mathbb{N}^*$, $\sup_{j\in\mathbb{N}^*}(K_{i,j} + F_{i,j})/j < +\infty$. Then the sequence $(\pi^N)_N$ of the laws of the empirical measures $\mu^N = \frac{1}{N}\sum_{n=1}^N \delta_{Y^{n,N}}$ considered as $\mathcal{P}(D([0,+\infty), \mathcal{E}))$ valued random variables is tight.*

Proof : Like in the Proof of Proposition 2.8, it is enough to check the tightness of the laws of the $D([0,+\infty), \mathcal{E})$-valued processes $(Y^{1,N})_N$ thanks to Aldous criterion.
We recall that $\mathcal{E} = \mathbb{N}^* \cup \{+\infty\}$ is endowed with the metric $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ where by convention $\frac{1}{+\infty} = 0$. Since this space is compact, for any $s \geq 0$ the laws of the $\mathcal{E}$ valued variables $(Y^{1,N}_s)_N$ are tight.
Let $T > 0$ and for $N \geq 1$, τ_N be a stopping time of the filtration $\mathcal{F}^N_t = \sigma((Y^{1,N}_s, \ldots, Y^{N,N}_s),\ s \leq$

t) smaller than T. For $\delta,\eta>0$,

$$\sup_{\theta\in[0,\delta]}\mathbb{P}(d(Y^{1,N}_{\tau_N+\theta},Y^{1,N}_{\tau_N})>\eta)\leq\mathbb{P}\left(\sup_{\theta\in[0,\delta]}d(Y^{1,N}_{\tau_N+\theta},Y^{1,N}_{\tau_N})>\eta\right)$$
$$\leq\mathbb{P}\left(Y^{1,N}_{\tau_N}\leq\frac{1}{\eta}\text{ and }\exists\theta\in[0,\delta]\text{ s.t. }Y^{1,N}_{\tau_N+\theta}\neq Y^{1,N}_{\tau_N}\right)+\mathbb{P}\left(Y^{1,N}_{\tau_N}>\frac{1}{\eta}\text{ and }\exists\theta\in[0,\delta]\text{ s.t. }Y^{1,N}_{\tau_N+\theta}\leq\frac{1}{\eta}\right)$$

Let $[1/\eta]$ denote the integer part of $1/\eta$. Like in the proof of Proposition 2.8, we upper-bound the first term of the right-hand-side by $\delta\max_{1\leq i\leq[1/\eta]}\left(\sup_{j\in\mathbb{N}^*}\tilde{K}_{i,j}+\sum_{j=1}^{i-1}\tilde{F}_{j,i-j}\right)$.
To deal with the second term, we introduce the stopping time $\sigma_N=\inf\{s\geq\tau_N:Y^{1,N}_s\leq 1/\eta\}$ and set $\varphi(y^1,\ldots,y^N)=1_{\{y^1\leq 1/\eta\}}$. For the jump process $(Y^{1,N},\ldots,Y^{N,N})$ with transition function defined by (2.5), the martingale M^φ_t given by Definition 1.2 is such that a.s. on $\{Y^{1,N}_{\tau_N}>1/\eta\}$,

$$M^\varphi_{\sigma_N\wedge(\tau_N+\delta)}-M^\varphi_{\tau_N}=1_{\{\sigma_N\leq\tau_N+\delta\}}-\int_{\tau_N}^{\sigma_N\wedge(\tau_N+\delta)}\sum_{j>[1/\eta]}1_j(Y^{1,N}_s)\sum_{i=1}^{[1/\eta]}\tilde{F}_{i,j-i}ds.$$

As $\mathbb{E}\left(1_{\{Y^{1,N}_{\tau_N}>1/\eta\}}(M^\varphi_{\sigma_N\wedge(\tau_N+\delta)}-M^\varphi_{\tau_N})\right)=0$, we easily deduce that the second term is smaller than $\delta\sup_{j>[1/\eta]}\sum_{i=1}^{[1/\eta]}\tilde{F}_{i,j-i}$.
By the assumption made on the kernels, we deduce that $\sup_{\theta\in[0,\delta]}\mathbb{P}(d(Y^{1,N}_{\tau_N+\theta},Y^{1,N}_{\tau_N})>\eta)$ is arbitrarily small uniformly in (N,τ_N) for δ small, which puts an end to the proof. ∎

Under more stringent assumptions on the kernels, we are able to give the following partial characterization for weak limits of the sequence $(\pi^N)_N$.

Proposition 4.2 *Assume that $\forall i\in\mathbb{N}^*$, $\lim_{j\to+\infty}(K_{i,j}+F_{i,j})/j=0$. Then any weak limit of the sequence $(\pi^N)_N$ gives full weight to the subset of $\mathcal{P}(D([0,+\infty),\mathcal{E}))$ consisting in probability measures Q with marginals $(Q_t)_t$ such that $Q_0=\rho$ and for any $\varphi:\mathbb{N}^*\to\mathbb{R}$ satisfying $\varphi(l)=\varphi(l\wedge m)$ for some $m\in\mathbb{N}^*$,*

$$\begin{aligned}M^\varphi_t=\varphi(X_t)-\varphi(X_0)-\int_0^t\Bigg(&\sum_{j\in\mathbb{N}^*}\tilde{K}_{X_s,j}(\varphi(X_s+j)-\varphi(X_s))Q_s(j)\\&+\sum_{j=1}^{X_s-1}\tilde{F}_{X_s-j,j}(\varphi(X_s-j)-\varphi(X_s))\Bigg)ds\text{ is a }Q\text{-martingale}\end{aligned}\quad(4.1)$$

where X_t denotes the canonical process on $D([0,+\infty),\mathcal{E})$.

Writing for $i\in\mathbb{N}^*$ the constancy of the expectation of the Q-martingale $(M^{1_i}_t)_t$, we deduce :

Corollary 4.3 *If $\forall i\in\mathbb{N}^*$, $\lim_{j\to+\infty}(K_{i,j}+F_{i,j})/j=0$, then any weak limit of the sequence $(\pi^N)_N$ gives full weight to the subset of $\mathcal{P}(D([0,+\infty),\mathcal{E}))$ consisting in probability measures Q such that $t\to Q_t$ solves (0.2) on $[0,+\infty)$.*

Translated in terms of the original Smoluchowski's coagulation fragmentation equation, this provides a global existence result.

Proof of Proposition 4.2 : Let π^∞ denote the weak limit of a converging subsequence of $(\pi^N)_N$ that we still index by N for simplicity. Like in the proof of Theorem 2.9, it is enough to check that $\forall i \in \mathbb{N}^*$, π^∞ a.s.,

$$\begin{aligned}\Phi_t^i(X,Q) =& 1_i(X_t) - 1_i(X_0) - \int_0^t \sum_{j\in\mathbb{N}^*} \tilde{K}_{X_s,j}(1_i(X_s+j) - 1_i(X_s))Q_s(j)ds \\ &+ \int_0^t \sum_{j=1}^{i-1} \tilde{F}_{i-j,j}1_i(X_s)ds - \int_0^t \sum_{j\in\mathbb{N}^*} \tilde{F}_{i,j}1_{i+j}(X_s)ds \text{ is a } Q\text{-martingale.}\end{aligned}$$

By the assumptions made on the kernels K and F, the function Φ_t^i is bounded on $D([0,+\infty),\mathcal{E}) \times \mathcal{P}(D([0,+\infty),\mathcal{E}))$:

$$\forall (X,Q),\ |\Phi_t^i(X,Q)| \le 1 + t\left(\max_{k\le i}\sup_{j\in\mathbb{N}^*}\tilde{K}_{k,j} + \max\left(\sum_{j=1}^{i-1}\tilde{F}_{i-j,j}, \sup_{j\in\mathbb{N}^*}\tilde{F}_{i,j}\right)\right).$$

For $l \in \mathbb{N}^*$, $g : \mathcal{E}^l \to \mathbb{R}$ continuous and bounded and $0 \le s_1 \le s_2 \le \ldots \le s_l \le r \le t$, we define the bounded function $G : \mathcal{P}(D([0,+\infty),\mathcal{E})) \to \mathbb{R}$ by $G(Q) =< Q, (\Phi_t^i(X,Q) - \Phi_r^i(X,Q))g(X_{s_1},\ldots,X_{s_l}) >$.
By a reasoning similar to the one made in the Proof of Theorem 2.9, we obtain that

$$\lim_{N\to+\infty} \mathbb{E}^{\pi^N}|G(Q)| = 0.$$

When Q^n converges weakly in $\mathcal{P}(D([0,+\infty),\mathcal{E}))$ to Q then for $t \notin D_Q = \{s \ge 0, Q(X_s \neq X_{s^-}) > 0\}$, Q_t^n converges weakly to Q_t in $\mathcal{P}(\mathcal{E})$ i.e. $\forall i \in \mathbb{N}^*$, $Q_t^n(i) \to Q_t(i)$ (but $Q_t^n(+\infty)$ does not necessarily converge to $Q_t(+\infty)$). With the assumptions on the kernels, we deduce that for $s_1,\ldots,s_l,r,t \notin D_Q$, G is continuous at Q. Hence for $s_1,\ldots,s_l,r,t$ outside the at most countable set $\{s \ge 0, \pi^\infty(\{Q : s \in D_Q\}) > 0\}$, $\mathbb{E}^{\pi^\infty}|G(Q)| = \lim_{N\to+\infty}\mathbb{E}^{\pi^N}|G(Q)| = 0$. The canonical process X being càdlàg, we easily deduce that π^∞ a.s., $(\Phi_t^i(X,Q))_t$ is a Q-martingale. ■

An interesting question is whether any weak limit of the sequence $(\pi^N)_N$ gives full weight to $\{Q \in \mathcal{P}(D([0,+\infty),\mathcal{E})) : Q(\mathcal{D}_1) = 1\}$. As $\mathcal{D}_1$ is not a closed subset of $D([0,+\infty),\mathcal{E})$, the answer is not obvious. But in case the sequence of total fragmentation rates $(\sum_{j=1}^{i-1} F_{j,i-j})_{i\in\mathbb{N}^*}$ is bounded, it turns out to be positive :

Lemma 4.4 *Assume that* $\forall i \in \mathbb{N}^*, \sup_{j\in\mathbb{N}^*} F_{i,j}/j < +\infty$ *and* $\sup_{i\in\mathbb{N}^*}\sum_{j=1}^{i-1} F_{j,i-j} < +\infty$. *Then any* $Q \in \mathcal{P}(D([0,+\infty),\mathcal{E}))$ *such that for any* $\varphi : \mathbb{N}^* \to \mathbb{R}$ *satisfying* $\varphi(l) = \varphi(l \wedge m)$ *for some* $m \in \mathbb{N}^*$ (4.1) *holds gives full weight to* $\mathcal{D}_1$.

Proof : We introduce the stopping times $\sigma = \inf\{s \ge 0,\ X_{s^-} \vee X_s = +\infty\}$, $\sigma_k = \inf\{s \ge 0,\ X_s \ge k\}$ and $\tau_k = \inf\{s \ge \sigma,\ X_s \le k\}$ where $k \in \mathbb{N}^*$. We also set $\tau = \lim_{k\to+\infty}\tau_k = \inf\{s \ge \sigma,\ X_s < +\infty\}$. Let $t > 0$ and $1 \le i < k$,

$$M^{1_i}_{\sigma\wedge t} - M^{1_i}_{\sigma_k\wedge t} \ge 1_{\{\sigma\le t\}}1_i(X_\sigma) - \sup_{j\in\mathbb{N}^*}\left(1_{\{j<i\}}\tilde{K}_{i-j,j} + 1_{\{j>i\}}\tilde{F}_{i,j-i}\right)(\sigma\wedge t - \sigma_k \wedge t).$$

By the optional stopping Theorem, the expectation under Q of the left-hand-side is nil. Therefore $Q(\sigma \le t, X_\sigma = i) \le \sup_{j\in\mathbb{N}^*}\left(1_{\{j<i\}}\tilde{K}_{i-j,j} + 1_{\{j>i\}}\tilde{F}_{i,j-i}\right) < Q, \sigma\wedge t - \sigma_k \wedge t >$. Letting $k \to +\infty$, we deduce that $Q(\sigma \leq t, X_\sigma = i) = 0$. Hence $Q(\sigma \leq t, X_\sigma < +\infty) = 0$. As a consequence setting $\varphi(l) = 1_{\{l \leq k\}}$ and using that $X_t = +\infty$ on (σ, τ), we get

$$Q \text{ a.s., } M^\varphi_{\tau_k \wedge t} - M^\varphi_{\sigma \wedge t} = 1_{\{\tau_k \leq t\}} - \int_{\tau \wedge t}^{\tau_k \wedge t} \sum_{i \geq k+1} 1_i(X_s) \sum_{j=1}^{k} \tilde{F}_{j,i-j} ds.$$

By the optional stopping Theorem, we deduce that

$$Q(\tau_k \leq t) \leq \sup_{k \in \mathbb{N}^*} \sup_{i \geq k+1} \sum_{j=1}^{k} \tilde{F}_{j,i-j} < Q, \tau_k \wedge t - \tau \wedge t > .$$

Using the definition of $\tilde{F}$, we obtain that

$$\sup_{k \in \mathbb{N}^*} \sup_{i \geq k+1} \sum_{j=1}^{k} \tilde{F}_{j,i-j} = \sup_{i \geq 2} \sup_{1 \leq k \leq i-1} \sum_{j=1}^{k} \tilde{F}_{j,i-j} = \frac{1}{2} \sup_{i \geq 2} \sum_{j=1}^{i-1} F_{j,i-j} < +\infty.$$

Letting $k \to +\infty$ we get $Q(\tau < t) = 0$. As t is arbitrary, we conclude that $Q(\tau < +\infty) = 0$. ∎

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