Numerical analysis of micro-macro simulations of polymeric fluid flows : a simple case.

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We present in this article the numerical analysis of a simple micro-macro simulation of a polymeric fluid flow, namely the shear flow for the Hookean dumbbells model. Although restricted to this academic case (which is however used in practice as a test problem for new numerical strategies to be applied to more sophisticated cases), our study can be considered as a first step towards that of more complicated models. Our main result states the convergence of the fully discretized scheme (finite element in space, finite difference in time, plus Monte Carlo realizations) towards the coupled solution of a partial differential equation / stochastic differential equation system.

1. Introduction.

We are concerned here with the numerical analysis of a simple micro-macro simulation of a polymeric fluid flow. More precisely, we deal with the situation where the polymeric liquid, which is here supposed to be an infinitely diluted solution of polymers, experiences a pure shear flow and is modeled at the microscopic scale by the dynamics of stochastic Hookean dumbbells. To the best of our knowledge, such a study is new. We shall explain below why, despite the simplicity of the underlying model, our work can be seen as a first step towards the treatment of the more sophisticated models that are commonly used in the context of the so-called micro-macro approach in computational rheology.

Numerical simulations of the flow of complex fluids such as polymeric liquids is a long lasting challenge. The central difficulty is the rheology of these fluids, highly non Newtonian in nature : there is no simple linear relation linking the stress tensor $\boldsymbol{\tau}$ and the deformation tensor $\frac{1}{2}\left(\nabla \boldsymbol{u} +^t \nabla \boldsymbol{u}\right)$ as in the case of Newtonian fluids. This algebraic relation, the so-called constitutive equation of the fluid, is replaced in such fluid by a partial differential equation (abbreviated in PDE in the sequel) of the form

$$\frac{D\boldsymbol{\tau}}{Dt} = \boldsymbol{f}(\boldsymbol{\tau}, \nabla \boldsymbol{u}), \tag{1.1}$$

to be integrated along the Lagrangian trajectories of the particles, or by an integro-differential equation (the integral is taken along the trajectories)

$$\boldsymbol{\tau} = \int_{-\infty}^{t} m(t-s)\boldsymbol{S}_t(s)\, ds, \tag{1.2}$$

where m is a memory function (typically a decreasing exponential) and $\boldsymbol{S}_t(s)$ is a deformation-dependent tensor (typically a function of the Finger strain tensor).

The standard ("macroscopic") approach to simulate an incompressible flow of such polymeric liquids therefore consists in approximating the solution to a coupled system of the form

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \eta \Delta \boldsymbol{u} + \text{div } \boldsymbol{\tau}, \tag{1.3}$$

$$\text{div } \boldsymbol{u} = 0, \tag{1.4}$$

$$\left\{ \begin{array}{ccc} \frac{D\boldsymbol{\tau}}{Dt} & = & \boldsymbol{f}(\boldsymbol{\tau}, \nabla \boldsymbol{u}), \\ & \text{or} & \\ \boldsymbol{\tau} & = & \int_{-\infty}^{t} m(t-s)\boldsymbol{S}_t(s)\, ds, \end{array} \right. \tag{1.5}$$

together with convenient initial and boundary conditions. The derivative $\frac{D}{Dt}$ denotes the convective derivative $\frac{\partial}{\partial t} + \boldsymbol{u}.\nabla$, the vector $\boldsymbol{u}$ denotes the fluid velocity, p denotes the pressure. The two constants ρ and η denote respectively the density and the Newtonian viscosity of the solvent. We refer the reader to Refs. [13,14,15] for a general introduction to this type of simulations, and to Refs. [10,11,24,26,27] for examples of the numerous mathematical studies that have been devoted to such models. In this field, the most recent contribution is due to P.L. Lions and N. Masmoudi in Ref. [20].

Although very efficient, this purely macroscopic approach is now being questioned. The main concerns are indeed to find good constitutive equations (1.1) or (1.2) that could apply to the ever increasing number of non Newtonian fluids of interest in today's technology, and also to evaluate the impact of some closure hypothesis made to build these constitutive equations on the quality / validity of the final result. An alternative approach, which circumvents the bottleneck of making those closure hypothesis, has therefore been developed on the basis of kinetic theory. In a nutshell, this approach consists in finding an expression of the macroscopic stress tensor in terms of the microscopic dynamics of the polymer chains and in treating *explicitly* both scales in the simulation. On the contrary, constitutive laws are derived in a more or less rigorous way from the kinetic theory with the help of closure approximations, the kinetic foundation being next forgotten. Instead of (1.5), the system that has therefore to be treated is (1.3)-(1.4) together with the Fokker Planck equation describing the microscopic dynamics

$$\frac{\partial \psi}{\partial t} + \boldsymbol{u}.\nabla_{\boldsymbol{x}} \psi = -\text{div}_{\boldsymbol{Q}} \left((\nabla_x \boldsymbol{u} \boldsymbol{Q} - \frac{2}{\zeta} \boldsymbol{F}(\boldsymbol{Q}))\psi \right) + \frac{\sigma^2}{\zeta^2} \Delta_{\boldsymbol{Q}} \psi, \tag{1.6}$$

and the expression of the stress tensor as the average

$$\boldsymbol{\tau} = n \int (\boldsymbol{Q} \otimes \boldsymbol{F}(\boldsymbol{Q}))\psi(t, \boldsymbol{x}, \boldsymbol{Q})\, d\boldsymbol{Q} - nk_B T \mathrm{Id}. \tag{1.7}$$

The function $\psi(t, \boldsymbol{x}, \boldsymbol{Q})$ is the probability density function of the end-to-end vector $\boldsymbol{Q}$ of the polymer at time t and at position $\boldsymbol{x}$. The function $\boldsymbol{F}(\boldsymbol{Q})$ denotes the force within the spring which models the polymer, ζ denotes the friction, n is the number density of polymers and σ is defined by $\sigma^2 = 2k_B T\zeta$, where T is the temperature. We refer to Refs. [2,3,7,21] for more details about the derivation of such equations.

From the theoretical standpoint, this approach is clearly more satisfactory than the previous one. It is however not perfect : current research in the modeling of complex flows aims at going further the simple setting of "thermodynamics at equilibrium" upon which this approach is based (see Refs. [1,8,9]). From the mathematical standpoint, systems of the type (1.3), (1.4), (1.6) and (1.7) have been studied for instance in Refs. [6,25], and are therefore rather well known. However, this approach, as such, suffers from a severe drawback as far as numerical simulations are concerned : the Fokker Planck equation, typically set on a space of large dimension (say $\mathbb{R}^N$ with $N = 100$), is not tractable numerically. The idea has emerged in the early 90's to simulate the underlying stochastic differential equation (abbreviated in SDE in the sequel) rather than the Fokker Planck equation itself. This approach has been called CONFESSIT[17] which means Calculation of Non-Newtonian Fluids : Finite Elements and Stochastic Simulation Techniques.

The "modern" way of simulating an incompressible flow of an infinitely diluted solution of polymer is therefore to approximate

$$\left\{ \begin{array}{l} \rho(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u}) = -\nabla p + \eta \Delta \boldsymbol{u} + \mathrm{div}\ (\boldsymbol{\tau}), \\ \mathrm{div}\ (\boldsymbol{u}) = 0, \\ \boldsymbol{\tau} = n\mathbb{E}(\boldsymbol{Q} \otimes \boldsymbol{F}(\boldsymbol{Q})) - nk_B T \mathrm{Id}, \\ d\boldsymbol{Q} + \boldsymbol{u}.\nabla \boldsymbol{Q} dt = \left(\nabla \boldsymbol{u} \boldsymbol{Q} - \frac{2}{\zeta} \boldsymbol{F}(\boldsymbol{Q}) \right) dt + \frac{\sqrt{2}\sigma}{\zeta} d\boldsymbol{W}_t, \end{array} \right. \tag{1.8}$$

where $\mathbb{E}$ denotes the expectation and $\boldsymbol{W}_t$ is a standard (multidimensional) Brownian motion. This very lively field of numerical simulation can be approached by the reading of works such as Refs. [4,5,17,21,30] (see other references therein). It should be already clear in the reader's mind that such an approach raises hundreds of interesting questions, both theoretical and numerical, and all lying at the intersection of the world of PDEs and SDEs (or even SPDEs i.e. stochastic partial differential equations). So far as we know, no existing study deals with the existence of solution $(\boldsymbol{u}, \boldsymbol{\tau}, \boldsymbol{Q}_t)$ to the above system (1.8) or any system of the same family. Moreover, despite the numerous simulations done, no proof of convergence of a numerical scheme towards the "continuous" solution has ever been established.

Our present work aims at giving a complete mathematical and numerical analysis of a system such as (1.8). For reasons that will be clear below, we are bound to restrict ourselves to a very simple case, that we hope however to be instructive enough to motivate further studies.

2. The model and our main result.

The system we study here is the following

$$\frac{\partial u}{\partial t}(t,x) - \partial^2_{x,x} u(t,x) = \partial_x \tau(t,x) + f_{ext}(t,x), \tag{2.9}$$

$$\tau(t,x) = \mathbb{E}(P(t)Q(t,x)), \tag{2.10}$$

$$dP(t) = -\frac{P(t)}{2}dt + dV_t, \tag{2.11}$$

$$dQ(t,x) = \left(\partial_x u(t,x)P(t) - \frac{Q(t,x)}{2}\right)dt + dW_t, \tag{2.12}$$

complemented with *ad hoc* boundary and initial conditions, which will be both made precise below. It is obtained from (1.8) by making the following assumptions :

- (H1) We consider a shear flow in 2D : $\boldsymbol{u} = u_y(x)\boldsymbol{e}_y$ (see Figure 1). The function u_y is henceforth denoted by u. Consequently, the divergence free condition (1.4) is automatically fulfilled. Another striking consequence of this geometrical assumption is that the Navier term $\boldsymbol{u}.\nabla\boldsymbol{u}$ in (1.3) and the transport term $\boldsymbol{u}.\nabla$ in the stochastic equations both vanish. In equation (2.10), τ denotes the (x,y) components of the stress tensor $\boldsymbol{\tau}$. In equations (2.11) and (2.12), $(P(t),Q(t,x))$ (resp. (V_t,W_t)) are the two components of the end-to-end vector $\boldsymbol{Q}(t)$ (resp. the Brownian motion $\boldsymbol{W}_t$). In equation (2.9), f_{ext} denotes an external force.

- (H2) The force $\boldsymbol{F}(\boldsymbol{Q})$ in (1.8) is chosen to be a simple linear force $\boldsymbol{F}(\boldsymbol{Q}) = H\boldsymbol{Q}$ with H the coefficient of the Hookean spring which models the polymer (let us incidentally mention that such a force has nothing to do with the modeling of intra-molecular forces inside the polymer chain : it is only entropic in nature, and models the simple property stating that when the polymer chain stretches, the volume of the region of the configurations space visited by the polymer gets smaller). A consequence of this "Hookean dumbbell" assumption is that the model (1.8) is indeed equivalent (at least formally, but more can be said than that) to a purely macroscopic model of the type (1.1), namely the famous Oldroyd B model written here in its differential form :

$$\boldsymbol{\tau} + \lambda\frac{\delta\boldsymbol{\tau}}{\delta t} = nk_BT\lambda(\nabla\boldsymbol{u} +^t \nabla\boldsymbol{u}), \tag{2.13}$$

with the upper convected derivative $\frac{\delta}{\delta t}$ defined by

$$\frac{\delta\boldsymbol{\tau}}{\delta t} = \frac{\partial\boldsymbol{\tau}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{\tau} - \boldsymbol{\tau}^t\nabla\boldsymbol{u} - \nabla\boldsymbol{u}\boldsymbol{\tau},$$

where $\lambda = \frac{\zeta}{4H}$ is a characteristic time. In our simple case, (2.13) reduces to :

$$\frac{\partial\tau}{\partial t} + \tau = \partial_x u. \tag{2.14}$$

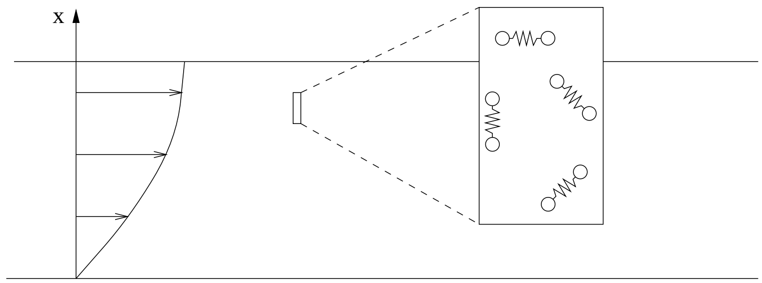

Fig. 1. Velocity profile in a shear flow of a dilute solution of polymers.

Let also notice that we have chosen units of time and length such that $\lambda = 1$ and $d = \sqrt{\frac{k_B T}{H}} = 1$. Moreover, we have taken the physical parameters in order to simplify the equations. All the results we give are of course also valid with different conventions.

The main two results of our work, which are respectively stated in Theorem 1 and Theorem 3 in a very precise way show :

(a) that there exists a solution (u, Q_t) to the system in the natural energy spaces associated to the problem,

(b) that the fully discretized solution $(\overline{u}_h^n, \frac{1}{M}\sum_{j=1}^M \overline{P}_n^j \overline{Q}_{h,n}^j)$ (the velocity being discretized over P1 finite elements in space, and by finite differences in time while the SDE being discretized by an Euler scheme in time and the stress tensor approximated by Monte Carlo realizations) converges up to a slight technical modification, which is linked to the stability of the SDE and that will be made precise in subsection 4.3, to the continuous solution at the order $O\left(h + \delta t + \frac{1}{\sqrt{M}}\right)$, where h is the space step, δt is the time step and M is the number of realizations of the SDEs (i.e. the number of dumbbells per cell).

The sequel of this paper is devoted to the proof of these two assertions. However, before we get to the heart of the matter, let us emphasize our goal, and also give some comments that we believe such results deserve.

The proof of the existence (and in fact uniqueness and regularity) of the continuous problem is reproduced here mainly for the sake of consistency. Although it does not appear as such in the literature, it could be derived in a rather straightforward way from the observation that our model is, as mentioned above, in fact equivalent to an Oldroyd B type model. The only (slight) novelty is that, with a view to tackle next the Galerkin approximation, we deliberately work in the natural energy (Sobolev) spaces. On the contrary, studies such as Ref. [25] take a much more regular setting and the study Ref. [20] considers another type of convective derivative (for details about convective derivatives and frame indifference, see for example Ref. [28]).

On the other hand, the "numerical analysis" part of our work seems completely new. We are aware that the case we deal with, precisely because of its simplicity and its formal equivalence with a purely macroscopic model (hypotheses (H1) and (H2)) cannot be considered as a prototype (in terms of the mathematical difficulty) of all models of type (1.8). However,

(a) the simple model (2.9)-(2.12) embodies many, if not all, difficulties of model of type (1.8) : the coupling between the stochastic part and the macroscopic flow part, the fact that at any fixed number of dumbbells the (apparently determinininistic) flow velocity is actually a stochastic variable. It is therefore expected that the mathematical toolbox used for its study will be useful and instructive for the analysis of the other cases. At least it is a preliminary matter for them.

(b) the simple model (2.9)-(2.12) is indeed used in the numerical practice (and coded as such) in order to serve as a test case for advanced numerical techniques that will be then extended to more sophisticated cases. This justifies (to our opinion at least) the need for the numerical analysis of this model *per se.*

Remark 1 *When the microscopic model is not that of Hookean dumbbells, the force* $\boldsymbol{F}(\boldsymbol{Q})$ *is no longer linear but can be*

$$\boldsymbol{F}(\boldsymbol{Q}) = \frac{H\boldsymbol{Q}}{1 - |\boldsymbol{Q}|^2/b},$$

or

$$\boldsymbol{F}(\boldsymbol{Q}) = \frac{H\boldsymbol{Q}}{1 - \mathbb{E}(|\boldsymbol{Q}|^2)/b},$$

which are respectively the case for the so-called FENE and FENE-P dumbbells case. The FENE-P model is derived from the FENE model via a closure approximation (the so-called Peterlin approximation), which enables to obtain a purely macroscopic equivalent of the microscopic model. In these models, b is a positive parameter which is the square of the maximum elongation of the dumbbells. The mathematical difficulty is then to ensure that $\boldsymbol{Q}$ does not leave the region $|\boldsymbol{Q}| \leq \sqrt{b}$ and does not even reach its boundary. Current research[12] is directed towards trying to extend the present analysis to this case.

Remark 2 *When the macroscopic flow is no longer a pure shear flow, (at least) four new difficulties arise :*

(*i*) *the divergence free constraint (1.4) has to be accounted for,*
(*ii*) *the Navier term has to be treated,*
(*iii*) *the term $\boldsymbol{u}.\nabla\boldsymbol{Q}_t$ in the left hand side of the SDE of (1.8) has to be dealt with,*

(*iv*) *product of two non autonomous stochastic processes arises in the definition of* $\boldsymbol{\tau}$. *(Note that in (2.10),* P_t *is autonomous, i.e. does not depend on the flow.)*

Of these four difficulties, difficulties (iii) and (iv) are so far as we understand the most embarrassing ones. Difficulty (i) is standard, and (ii) is a classical well-known difficulty of the mathematical analysis of incompressible (Newtonian) Navier-Stokes equation (and we cannot hope to go further in the analysis of the present models than in that of the Navier Stokes equation). Difficulty (iii), namely the appearance of a transport term in the SDE (which ipso facto becomes a SPDE), creates at once an interesting question : in what sense can we consider the SDE of the system (1.8) ? A way to circumvent the difficulty is to set the SDE in the Lagrangian setting, i.e. follow the characteristics of the flow and write the SDE along them. But as we have in mind to deal with a weak solution $\boldsymbol{u}$ *of the macroscopic flow equations (think of the 3d case), it is not an easy task to define these characteristics, and also to give a rigorous foundation to the Lagrangian form (because of the term* $\nabla \boldsymbol{u}\boldsymbol{Q}$ *in the right-hand-side which lacks of regularity with respect to* $\boldsymbol{Q}$ *if* $\boldsymbol{u}$ *is only* H^1*). We refer the interested reader to Ref.* [18] *where it is shown that one can adapt and complete the Di Perna-Lions theory of almost everywhere flows to accommodate for this new situation.*

Remark 3 *When the solution is no longer infinitely diluted, other models arise. For high densities, models like those issued from the theory of reptation (Doi-Edwards models) appear. Then again, macroscopic models and micro-macro models are two alternatives. Questions like those of simulation of reflected Brownian processes then come into the picture (see Ref.* [21]*), giving also rise to questions of interest for the numerical analysist. Let us also mention that what is expected to be the most challenging case with respect to the difficulty of its modeling is neither the infinitely dilute case, nor the polymer melt case, but the case in between !*

Let us end this section saying that we hope to complement the results of the present work at least in two directions :

(a) evaluate on the same toy-model both by numerical analysis and computational experiments the validity of well known and commonly used techniques of this field of computational rheology such as variance reduction methods,

(b) do the same analysis as that of the present paper for some of the more difficult cases mentioned in the above remarks.

We refer the reader to Refs. [12,19] for both aspects.

3. Brief mathematical analysis of the continuous problem.

3.1. *Precise setting of the equations and definition of solutions.*

As announced above, we complement system (2.9)-(2.12) with the following boundary conditions :

$$\begin{cases} u(t,0) = f_0(t), \\ u(t,1) = f_1(t), \end{cases} \tag{3.15}$$

together with the initial data :

$$\begin{cases} u(0,x) & = & u_0(x), \\ Q(0,x) & = & Q_0, \\ P(0) & = & P_0. \end{cases} \tag{3.16}$$

Let us also make precise the notations : P_0 and Q_0 denote two independent normal random variables (because we suppose that the polymers are initially at equilibrium), also independent of V_t and W_t which denote two standard independent Brownian motions. Notice that, as function of the space variable x, (V_t, W_t) is constant. In the following, we have $(t,x) \in (0,T) \times \mathcal{O}$ with $\mathcal{O} = (0,1)$.

The following regularity for the external forces and the initial velocity are supposed :

$$\begin{cases} f_{ext} \in L^1_t(H^1_x) \cap W^{1,1}_t(L^2_x), \\ f_{ext}(0,x) \in L^2_x, \\ u_0 \in H^2. \end{cases} \tag{3.17}$$

It is to be remarked that although the regularities (3.17) have been chosen for simplicity and because they are necessary for our result of convergence (Theorem 3), some parts of the arguments below may be done under less regular requirements. Let us also notice that all the results we give are also valid with other assumptions of regularity on f_{ext}.

We restrict ourselves to the case of homogeneous boundary conditions ($f_0 = f_1 = 0$), the modifications to deal with the other cases being only a technical matter. In the following, t, x and ω denote respectively the variable in time, space and probability. For example, $Q_t \in L^\infty_t(L^2_x(L^2_\omega))$ means that $\text{supess}_{t\in(0,T)}\{\int_{\mathcal{O}} \mathbb{E}(Q_t^2)\, dx\} < \infty$.

We are now in position to define the notion of solution we shall deal with.

We say that (u, Q) is a weak solution of the homogeneous problem if $u \in L^\infty_t(L^2_x) \cap L^2_t(H^1_{0,x})$ and $Q_t \in L^\infty_t(L^2_x(L^2_\omega))$ satisfy that for all $v \in H^1_0(\mathcal{O})$,

$$\frac{d}{dt}\int_{\mathcal{O}} uv + \int_{\mathcal{O}} \partial_x u \partial_x v = -\int_{\mathcal{O}} \mathbb{E}(P_t Q_t(x))\partial_x v + \int_{\mathcal{O}} f_{ext}(t,x)v, \tag{3.18}$$

for a.e. (x,ω), $\forall t \in (0,T)$, $Q_t(x) = e^{-\frac{t}{2}}Q_0 + \int_0^t e^{\frac{s-t}{2}}\, dW_s + \int_0^t e^{\frac{s-t}{2}}\partial_x u P_s\, ds,$ (3.19)

with

$$P_t = e^{-\frac{t}{2}}P_0 + \int_0^t e^{\frac{s-t}{2}}\, dV_s. \tag{3.20}$$

Equation (3.18) holds in the sense of distributions in time. As usual, one may equivalently use time dependent test functions $v \in L^\infty_t(L^2_x) \cap L^2_t(H^1_{0,x})$.

3.2. *Formal a priori estimates.*

We now establish formal *a priori* estimates on the solution (u, Q). These estimates will be made rigorous at the discrete level in the next subsection.

Multiplying (2.9) by u, next integrating over the domain and in time, we obtain

$$\frac{1}{2}\int_{\mathcal{O}} u(t,x)^2 - \frac{1}{2}\int_{\mathcal{O}} u_0(x)^2 + \int_0^t \int_{\mathcal{O}} (\partial_x u)^2 = -\int_0^t \int_{\mathcal{O}} \mathbb{E}(P_s Q_s(x))\partial_x u(s,x) + \int_0^t \int_{\mathcal{O}} f_{ext}(s,x) u(s,x).$$

Next we compute Q_t^2 by Itô's formula using (2.12), take expectations and integrate again on $\mathcal{O}$ and in time to obtain

$$\frac{1}{2}\int_{\mathcal{O}} \mathbb{E}(Q_t^2) - \frac{1}{2} = \int_0^t \int_{\mathcal{O}} \mathbb{E}(P_s Q_s(x))\partial_x u(s,x) - \frac{1}{2}\int_0^t \int_{\mathcal{O}} \mathbb{E}(Q_s^2) + \frac{1}{2}t.$$

Summing up these two equalities, we obtain

$$\frac{1}{2}||u||^2_{L^2_x}(t) + \int_0^t ||\partial_x u||^2_{L^2_x} + \frac{1}{2}\int_{\mathcal{O}} \mathbb{E}(Q_t^2) + \frac{1}{2}\int_0^t \int_{\mathcal{O}} \mathbb{E}(Q_s^2)\, ds = \frac{1}{2}||u_0||^2_{L^2_x} + \frac{1}{2}(1+t) + \int_0^t \int_{\mathcal{O}} f_{ext} u, \quad (3.21)$$

which yields *the first energy inequality* :

$$||u||^2_{L^\infty_t(L^2_x)} + ||u||^2_{L^2_t(H^1_{0,x})} + ||Q_t||^2_{L^\infty_t(L^2_x(L^2_\omega))} + ||Q_t||^2_{L^2_t(L^2_x(L^2_\omega))} \leq C\left(1 + ||u_0||^2_{L^2_x} + T + ||f_{ext}||^2_{L^1_t(L^2_x)}\right), \quad (3.22)$$

with C a constant independent of the data of the problem.

At this stage, it is to be remarked that using the same arguments as in the derivation of (3.21) or (3.22) with $u = u_1 - u_2$ and $Q = Q_1 - Q_2$, one can show the uniqueness of solution. This point should be not surprising for the reader as the system (2.9)-(2.12) (once written in terms of u only, using equation (2.10) on τ and equation (2.12)) on Q_t is indeed a linear system with respect to the variable u. This is obviously a consequence of our simplifying assumptions (H1) and (H2).

We must also notice that this energy estimate shows that the regularity of the solution is at least : $u \in L^2_t(H^1_x)$ and $\frac{\partial u}{\partial t} \in L^2_t(H^{-1}_x)$. This shows in fact that $u \in \mathcal{C}([0,T], L^2(\mathcal{O}))$ which allows us to define $u(0)$ (see Ref. [29] Chapter III, Lemma 1.2).

Let us now turn to the second energy inequality. This time, we multiply (2.9) by $-\partial^2_{x,x}u$ and integrate over the domain to obtain

$$\frac{d}{dt}\int_{\mathcal{O}} (\partial_x u)^2 + \int_{\mathcal{O}} (\partial^2_{x,x}u)^2 = -\int_{\mathcal{O}} \partial_x \mathbb{E}(P_t Q_t)\partial^2_{x,x}u - \int_{\mathcal{O}} f_{ext}\partial^2_{x,x}u.$$

We need to control the first term in the right-hand side. Computing $d(P_t Q_t)$ from (2.11) and (2.12) and taking expectations, we get the following equation (equivalent to (2.14)) :

$$\frac{\partial}{\partial t}\mathbb{E}(P_tQ_t) = -\mathbb{E}(P_tQ_t) + \partial_x u\,\mathbb{E}(P_t^2). \tag{3.23}$$

By a standard application of Gronwall's lemma, this yields the following bound

$$||\partial_x \mathbb{E}(P_tQ_t)||^2_{L^2_x} \leq \int_0^t ||\partial^2_{x,x} u||^2_{L^2_x},$$

which we use to finally obtain (using again Gronwall's lemma) *the second energy inequality* :

$$||u||_{L^\infty_t(H^1_x)\cap L^2_t(H^2_x)} \leq C\left(||u_0||_{H^1_x} + ||f_{ext}||_{L^1_t(H^1_x)}\right), \tag{3.24}$$

where C only depends on T.

Likewise, we multiply (2.9) by $\frac{\partial u}{\partial t}$ after derivating it in time (all this is done formally we recall), and we integrate over $\mathcal{O}$ to obtain

$$\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{L^\infty_t(L^2_x)} + \left|\left|\frac{\partial\partial_x u}{\partial t}\right|\right|^2_{L^2_t(L^2_x)} \leq C\left(\left|\left|\frac{\partial}{\partial t}\mathbb{E}(P_tQ_t)\right|\right|^2_{L^2_t(L^2_x)} + \left|\left|\frac{\partial f_{ext}}{\partial t}\right|\right|^2_{L^1_t(L^2_x)} + \left|\left|\left.\frac{\partial u}{\partial t}\right|_{t=0}\right|\right|^2_{L^2_x}\right).$$

Using again equation (3.23), we obtain (by Gronwall inequality)

$$\left|\left|\frac{\partial}{\partial t}\mathbb{E}(P_tQ_t)\right|\right|^2_{L^2_t(L^2_x)} \leq C\left(1 + ||u_0||^2_{L^2_x} + ||f_{ext}||^2_{L^1_t(L^2_x)}\right)$$

and we then derive another *regularity in time* :

$$\left|\left|\frac{\partial u}{\partial t}\right|\right|_{L^\infty_t(L^2_x)\cap L^2_t(H^1_x)} \leq C\left(1 + ||u_0||_{H^2} + ||f_{ext}||_{W^{1,1}_t(L^2_x)} + ||f_{ext}(0,x)||_{L^2_x}\right), \tag{3.25}$$

where C only depends on T.

3.3. *Existence.*

We can now show the existence of a solution of problem (3.18)-(3.19).

3.3.1. *Semi-discretized weak formulation.*

Let us define a Riesz basis $\{v_i\}_{i=1..\infty}$ of $H^1_0(\mathcal{O})$. We set $V_m = \mathrm{Vect}\{v_1,...,v_m\}$. The semi-discretized problem is the following :
Find $U^m \in (L^\infty_t(\mathbb{R}))^m$ and $Q^m_t \in L^\infty_t(L^2_x(L^2_\omega))$ such that, for all $1 \leq i \leq m$, the couple $(u^m(t,x), Q^m_t(x))$, where $u^m(t,x) = \sum_i U^m_i(t)v_i(x)$, satisfies :

$$\frac{d}{dt}\int_{\mathcal{O}} u^m v_i = -\int_{\mathcal{O}} \partial_x u^m \partial_x v_i - \int_{\mathcal{O}} \mathbb{E}(P_tQ^m_t(x))\partial_x v_i + \int_{\mathcal{O}} f_{ext} v_i, \tag{3.26}$$

$$Q^m_t = e^{-\frac{t}{2}}Q_0 + \int_0^t e^{\frac{s-t}{2}}\,dW_s + \int_0^t e^{\frac{s-t}{2}}\partial_x u^m P_s\,ds, \tag{3.27}$$

with $Q_0^m = Q_0$ and $u(t=0) = \Pi_m(u_0)$ where Π_m is the H^1-projection on V_m. Again, (3.27) has to make sense for a.e. (x,ω), for all $t \in (0,T)$.

3.3.2. *Exitence of a semi-discretized solution.*

It is standard to find a solution to the discretized problem (3.26)-(3.27) using e.g. a fixed-point argument on the function

$$F : \left\{ \begin{array}{ccc} X & \longrightarrow & X \\ \begin{pmatrix} U(t) \\ Q_t(x) \end{pmatrix} & \longmapsto & \begin{pmatrix} U_0 - A^{-1}\left(\int_0^t \left(BU(s) - \int_{\mathcal{O}} \mathbb{E}(P_s Q_s)\partial_x V + F_{ext}\right)\right) \\ e^{-\frac{t}{2}} Q_0 + \int_0^t e^{\frac{s-t}{2}}\, dW_s + \int_0^t e^{\frac{s-t}{2}} \sum_i U_i \partial_x v_i P_s \end{pmatrix} \end{array} \right. ,$$

where $X = \{(U, Q_t) \in (L_t^\infty(\mathbb{R}))^m \times L_t^\infty(L_x^2(L_\omega^2))\}$ is a Banach space for the norm $||(U,Q_t)||_X = ||U||_{L_t^\infty} + ||Q_t||_{L_t^\infty(L_x^2(L_\omega^2))}$, $A_{i,j} = \displaystyle\int_{\mathcal{O}} v_i v_j$, $B_{i,j} = \displaystyle\int_{\mathcal{O}} \partial_x v_i \partial_x v_j$, V is a field of components v_j and F_{ext} is a vector of components $\int_{\mathcal{O}} f_{ext} v_j$.

The point is the following result stating the regularity of the discretized solution.

Lemma 1 (Regularity of the space-discretized solution) *Assuming* $u_0 \in L_x^2$ *and* $f_{ext} \in L_t^1(L_x^2)$*, we have :*

$$\begin{aligned} ||u^m||^2_{L_t^\infty(L_x^2)} + ||u^m||^2_{L_t^2(H^1_{0,x})} + ||Q_t^m||^2_{L_t^\infty(L_x^2(L_\omega^2))} + ||Q_t^m||^2_{L_t^2(L_x^2(L_\omega^2))} \\ \leq C\left(1 + ||u_0||^2_{L_x^2} + T + ||f_{ext}||^2_{L_t^1(L_x^2)}\right), \qquad (3.28) \end{aligned}$$

with C *independent of the data of the problem.*

Assuming $u_0 \in H_x^2$ *and* $f_{ext} \in W_t^{1,1}(L_x^2)$*, we have :*

$$\left|\left|\frac{\partial u^m}{\partial t}\right|\right|_{L_t^\infty(L_x^2)\cap L_t^2(H_x^1)} \leq C\left(||u_0||_{H^2} + ||f_{ext}||_{W_t^{1,1}(L_x^2)} + ||f_{ext}(0,x)||_{L_x^2}\right), \quad (3.29)$$

$$\left|\left|\frac{\partial^2 u^m}{\partial t^2}\right|\right|_{L_t^2(H^{-1}_{*,x})} \leq C\left(||u_0||_{H^2} + ||f_{ext}||_{W_t^{1,1}(L_x^2)} + ||f_{ext}(0,x)||_{L_x^2}\right), \quad (3.30)$$

where C *only depends on* T*. By definition,* $||g||_{H^{-1}_{*,x}} = \sup_{w \in V^m} \dfrac{|\int_{\mathcal{O}} gw|}{||\partial_x w||_{L_x^2}}$.

Proof. To obtain the first two estimates (3.28) and (3.29) is a classical exercise : one just needs to reproduce in a more rigorous way the *a priori* estimates (3.22) and (3.25) of the former subsection. The last result (3.30) is obtained by writing the derivative in time of (3.26) and observing that $\frac{\partial}{\partial t}\mathbb{E}(P_t Q_t^m) = -\mathbb{E}(P_t Q_t^m) + \partial_x u^m \mathbb{E}(P_t^2)$ which ensures $\frac{\partial}{\partial t}\mathbb{E}(P_t Q_t^m) \in L_t^2(L_x^2)$. □

3.3.3. *Convergence towards a continuous solution.*

We assume $u_0 \in L^2_x$ and $f_{ext} \in L^1_t(L^2_x)$. According to the former lemma, we have $||u^m||_{L^\infty_t(L^2_x)\cap L^2_t(H^1_x)} + ||Q^m||_{L^\infty_t(L^2_x(L^2_\omega))} \leq C$ with C independent of m. The convergence of the sequence $(u^m, Q^m)_{m\in\mathbb{N}}$ then classically derives from this estimate (notice that there are only linear terms in u^n and Q^n in the equations (3.26) and (3.27), since P_t is autonomous), following the next three steps :

Step 1 Using the estimate on $(u^m)_{m\in\mathbb{N}}$, one can define a function $u \in L^\infty_t(L^2_x)\cap L^2_t(H^1_{0,x})$ such that u^m converges towards u weakly in $L^2_t(H^1_x)$ and for the weak-* topology of $L^\infty_t(L^2_x)$ (and therefore in $\mathcal{D}'((0,T)\times\mathcal{O})$). This function u satisfies the first energy inequality (3.22) (taking the inferior limit).

Step 2 One can then define $\tilde{Q}$ by $\tilde{Q} = e^{-\frac{t}{2}}Q_0 + \int_0^t e^{\frac{s-t}{2}}\,dW_s + \int_0^t e^{\frac{s-t}{2}}\partial_x u(s,x)P_s\,ds$ and check that $\tilde{Q} \in L^\infty_t(L^2_x(L^2_\omega))$.

Step 3 It remains to check the convergence of the terms of the equation (3.26) satisfied by u^m. The only non-trivial term is $\int_\mathcal{O}\mathbb{E}(P_tQ^m_t(x))\partial_x v_i$. We use that for $w \in L^2_x(\mathcal{O})$,

$$\int_\mathcal{O}\mathbb{E}(P_tQ^m_t(x))w = \int_\mathcal{O}\mathbb{E}\left(P_t\int_0^t e^{\frac{s-t}{2}}\partial_x u^m P_s\,ds\right)w = \int_\mathcal{O}\int_0^t \partial_x u^m e^{\frac{s-t}{2}}\mathbb{E}(P_sP_t)w\,ds\,dx,$$

and this last term goes to $\int_\mathcal{O}\int_0^t\partial_x u\, e^{\frac{s-t}{2}}\mathbb{E}(P_sP_t)w\,ds\,dx = \int_\mathcal{O}\mathbb{E}(P_t\tilde{Q}_t(x))w$ (because $\partial_x u^m$ converges weakly towards $\partial_x u$ in $L^2_t(L^2_x)$).

We have therefore obtained a solution of the problem (3.18)-(3.19). Let us show now the convergence of Q^m_t towards $\tilde{Q}_t$ as well as the strong convergence of u^m towards u.

Lemma 2 *Assume $u_0 \in H^2_x$ and $f_{ext} \in W^{1,1}_t(L^2_x)$. Set (u, Q_t) the solution of the problem (3.18)-(3.19). Set V_m a subspace of H^1_0 and (u^m, Q^m_t) the solution of the semi-discretized problem (3.26)-(3.27) with an initial velocity u^m_0. Then, we have for all $t \in [0,T]$,*

$$\begin{aligned}||u(t)-u^m(t)||^2_{L^2_x} + \int_0^t ||\partial_x(u-u^m)||^2_{L^2_x} + ||Q_t - Q^m_t||^2_{L^2_x(L^2_\omega)} + \frac{1}{2}\int_0^t ||Q_s - Q^m_s||^2_{L^2_x(L^2_\omega)}\\ \leq ||u_0 - u^m_0||^2_{L^2_x} + \inf_{w\in V^m}\left(3||\partial_x(u-w)||^2_{L^2_t(L^2_x)} + C||u-w||_{L^2_t(L^2_x)}\right),\end{aligned}$$

with a constant C which depends on the data of the problem : u_0, f_{ext} and T.

Proof. Let w be a function in V_m. One can easily obtain, using the linearity of the variational formulations, and integrating in time :

$$\begin{aligned}&\frac{1}{2}\int_\mathcal{O}(u-u^m)^2(t) + \int_0^t\int_\mathcal{O}|\partial_x(u-u^m)|^2 = \frac{1}{2}\int_\mathcal{O}(u_0-u^m_0)^2\\ &-\int_0^t\int_\mathcal{O}\mathbb{E}(P_s(Q_s-Q^m_s))\partial_x(u-u^m) + \int_0^t\int_\mathcal{O}\mathbb{E}(P_s(Q_s-Q^m_s))\partial_x(u-w)\\ &\qquad + \int_0^t\int_\mathcal{O}\frac{\partial}{\partial t}(u-u^m)(u-w) + \int_0^t\int_\mathcal{O}\partial_x(u-u^m)\partial_x(u-w). \qquad (3.31)\end{aligned}$$

Using the equations on Q_t and Q_t^m, one can show that :

$$\frac{1}{2}\int_{\mathcal{O}} \mathbb{E}(Q_t - Q_t^m)^2 = \int_0^t \int_{\mathcal{O}} \mathbb{E}(P_s(Q_s - Q_s^m))\partial_x(u-u^m) - \frac{1}{2}\int_0^t \int_{\mathcal{O}} \mathbb{E}(Q_s - Q_s^m)^2. \tag{3.32}$$

Summing up (3.31) and (3.32), we have :

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{O}}(u-u^m)^2(t)+\int_0^t\int_{\mathcal{O}}|\partial_x(u-u^m)|^2+\frac{1}{2}\int_{\mathcal{O}}\mathbb{E}(Q_t-Q_t^m)^2+\frac{1}{2}\int_0^t\int_{\mathcal{O}}\mathbb{E}(Q_s-Q_s^m)^2 \\ = \frac{1}{2}\int_{\mathcal{O}}(u_0-u_0^m)^2 + \int_0^t\int_{\mathcal{O}}\mathbb{E}(P_s(Q_s-Q_s^m))\partial_x(u-w) \\ + \int_0^t\int_{\mathcal{O}}\frac{\partial}{\partial t}(u-u^m)(u-w) + \int_0^t\int_{\mathcal{O}}\partial_x(u-u^m)\partial_x(u-w).\end{aligned} \tag{3.33}$$

Using Cauchy-Schwarz inequalities, we have

$$\int_0^t \int_{\mathcal{O}} \partial_x(u-u^m)\partial_x(u-w) \le \frac{1}{2}\int_0^t\int_{\mathcal{O}}|\partial_x(u-u^m)|^2 + \frac{1}{2}\int_0^t\int_{\mathcal{O}}|\partial_x(u-w)|^2,$$

and (using $\mathbb{E}(P_s^2)=1$)

$$\int_0^t\int_{\mathcal{O}}\mathbb{E}(P_s(Q_s-Q_s^m))\partial_x(u-w) \le \frac{1}{4}\int_0^t\int_{\mathcal{O}}\mathbb{E}(Q_s-Q_s^m)^2 + \int_0^t\int_{\mathcal{O}}|\partial_x(u-w)|^2.$$

The estimation of $\left|\left|\frac{\partial u^m}{\partial t}\right|\right|_{L^2_t(L^2_x)}$ given by Lemma 1 also holds for the continuous solution u (taking the inferior limit). This yields the final estimate : $\int_0^t\int_{\mathcal{O}}\frac{\partial}{\partial t}(u-u^m)(u-w) \le C||u-w||_{L^2_t(L^2_x)}$. □

In the former proof, we notice that we can assume that w also depends on the time variable. Choosing $w=\Pi_m(u)$ (we recall that Π_m is the operator of the H^1-projection on V_m), one can therefore show the strong convergence of u^m towards u in $L^\infty_t(L^2_x)\cap L^2_t(H^1_x)$ and the strong convergence of Q_t^m towards Q_t in $L^\infty_t(L^2_x(L^2_\omega))$. We have therefore proved the following result :

Theorem 1 (Existence of a continuous solution) *Let us assume $u_0\in L^2_x$ and $f_{ext}\in L^1_t(L^2_x)$. The problem (3.18)-(3.19) admits a unique solution $u\in\mathcal{C}([0,T],L^2_x(\mathcal{O}))\cap L^2_t(H^1_{0,x})$ and $Q_t\in L^\infty_t(L^2_x(L^2_\omega))$.*
The solution (u^m,Q_t^m) of the semi-discretized problem (3.26)-(3.27) is unique. Assuming $u_0\in H^2_x$ and $f_{ext}\in W^{1,1}_t(L^2_x)$, (u^m,Q_t^m) converges towards (u,Q_t) in the following sense : $u^m\longrightarrow u$ strongly in $L^\infty_t(L^2_x)\cap L^2_t(H^1_x)$ and $Q_t^m\longrightarrow Q_t$ strongly in $L^\infty_t(L^2_x(L^2_\omega))$.

Remark 4 *It is clear that, under the hypothesis $u_0\in H^2_x$ and $f_{ext}\in W^{1,1}_t(L^2_x)$, the continuous solution u is a function of $L^\infty_t(L^2_x)\cap L^2_t(H^1_x)$ which satisfies the inequality (3.25). Moreover, under the assumptions $u_0\in H^1_x$ and $f_{ext}\in L^1_t(H^1_x)$, we can also prove that the solution satisfies the second energy estimate (3.24), what* *will be used in the subection 4.1. To prove this result, one uses the uniqueness of the solution and the fact that one can construct a sequence of approximations of the solution which satisfies (3.24) by a Galerkin method on a special base (stable for the laplacian). One can then also obtain (3.24) for the solution, taking the inferior limit.*

4. Analysis of the numerical scheme.

In this section, we want to show the convergence of a standard discretization of the problem (2.9)-(2.12). As above, we will suppose $u_0 \in H^2_x$, $f_{ext} \in L^1_t(H^1_x)$ and $\frac{\partial f_{ext}}{\partial t} \in L^1_t(L^2_x)$, which yields, using the *a priori* estimates (3.24) and (3.25) : $u \in L^\infty_t(H^1_x) \cap L^2_t(H^2_x)$ and $\frac{\partial u}{\partial t} \in L^\infty_t(L^2_x) \cap L^2_t(H^1_x)$.

For the sake of simplicity, we also assume here homogeneous Dirichlet boundary conditions. Standard modifications of our arguments yield the same conclusions with non homogeneous Dirichlet boundary conditions (see e.g. Remark 6.2.2 in Ref. [23].)

The original problem is discretized in three steps : in space (by a Galerkin method), in time (by an Euler semi-implicit scheme) and finally using the Monte Carlo method. We choose a P1 discretization in space of the velocity : the velocity space functions V_h is the space of the piecewise polynomials of degree 1 on a mesh $\mathcal{T}_h$ where h is the space discretization step. The time interval $(0, T)$ is discretized with a constant step δt. We consider M realizations of the dumbbell processes (P_t, Q_t). The scheme we use will be made precise in the subsection 4.3 (see equations (4.48)-(4.50)).

The aim of this section is to show Theorem 3 which states that the order of convergence of this scheme is $O\left(h + \delta t + \frac{1}{\sqrt{M}}\right)$.

4.1. *Convergence of the space-discretized problem.*

We consider here the space-discretized problem which is (3.26)-(3.27) with $V^m = V_h \subset H^1_x$ (we use a Galerkin method). Notice that since the velocity u_h is a piecewise linear function (P1), the process Q_h (and therefore the stress $\tau_h = \mathbb{E}(PQ_h)$) is a discontinuous piecewise constant function (discontinuous P0). We have already shown in subsection 3.3 that this problem admits a unique solution. Moreover, Lemma 2, together with the standard finite elements approximation inequality $||u - \Pi_h(u)||^2_{L^2_x} + h^2||\partial_x(u - \Pi_h(u))||^2_{L^2_x} \leq Ch^4||u||^2_{H^2_x}$ yields :

Lemma 3 (Convergence of the space-discretized problem) *Let us assume $u_0 \in H^2_x$, $f_{ext} \in L^1_t(H^1_x)$ and $\frac{\partial f_{ext}}{\partial t} \in L^1_t(L^2_x)$. Set (u, Q_t) the solution of the problem (3.18)-(3.19). Let us assume a P1 space discretization for the velocity. Set V_h the velocity space functions and (u_h, Q_h) the solution of the semi-discretized problem (3.26)-(3.27) with an initial velocity $u_{h,0} = \Pi_h(u_0) \in V_h$. Then we have :*

$$||u(t) - u_h(t)||^2_{L^\infty_t(L^2_x)} + ||\partial_x(u - u_h)||^2_{L^2_t(L^2_x)} + ||Q_t - Q_{h,t}||^2_{L^\infty_t(L^2_x(L^2_\omega))} \quad \leq \quad Ch^2,$$

with C a constant which depends on the data of the problem : u_0, f_{ext} and T.

4.2. *Convergence of the time-discretized problem.*

We turn now to the semi-discretized problem in time and in space. We have already compared the continuous solution (u, Q) with the space-discretized solution (u_h, Q_h) and we want to estimate the error introduced by discretizing (u_h, Q_h) by an Euler scheme in time.

More precisely, we consider the following problem :
Being given $(u_h^n, Q_{h,n}, P_n)$, we compute $(u_h^{n+1}, Q_{h,n+1}, P_{n+1})$ by the following algorithm : u_h^{n+1} is such that $\forall v \in V_h$,

$$\frac{1}{\delta t}\int_{\mathcal{O}}(u_h^{n+1}-u_h^n)v+\int_{\mathcal{O}}\partial_x u_h^{n+1}\partial_x v = -\int_{\mathcal{O}}\mathbb{E}(P_n Q_{h,n}(x))\partial_x v+\int_{\mathcal{O}} f_{ext}(t_n)v. \quad (4.34)$$

$Q_{h,n+1}$ and P_{n+1} are then computed by :

$$Q_{h,n+1} - Q_{h,n} = \left(\partial_x u_h^{n+1} P_n - \frac{1}{2} Q_{h,n}\right)\delta t + W_{t_{n+1}} - W_{t_n}, \quad (4.35)$$

$$P_{n+1} - P_n = -\frac{1}{2} P_n \delta t + V_{t_{n+1}} - V_{t_n}. \quad (4.36)$$

This problem is complemented with the initial data $u_{h,0}$, P_0 and Q_0.

We will first show the stability of the scheme and then the convergence.

Lemma 4 (Stability of the space-time-discretized problem) *We assume that $f_{ext} \in L_t^\infty(L_x^2)$ and $u_0 \in L_x^2$. Under the assumption $\delta t < \frac{1}{2}$, we have : for all $n \leq \frac{T}{\delta t}$,*

$$||u_h^n||_{L_x^2}^2+||Q_{h,n}||_{L_x^2(L_\omega^2)}^2+\frac{\delta t}{2}\sum_{k=1}^n\int_{\mathcal{O}}|\partial_x u_h^k|^2 \leq 1+||u_{h,0}||_{L_x^2}^2+T\left(1+C||f_{ext}||_{L_t^\infty(L_x^2)}^2\right),$$

where C is a constant independent of the data of the problem.

Proof. In order to lighten the notations, we set $u_h^n = u_n$ and $Q_{h,n} = Q_n$. We also set $||f||_{H_*^{-1}} = \sup_{v\in V_h}\frac{\int_{\mathcal{O}} fv}{||\partial_x v||_{L_x^2}}$. If $f \in L^2$, one clearly has $||f||_{H_*^{-1}} \leq C||f||_{L^2}$.
We choose $v = u_{n+1}$ in (4.34), what yields

$$\frac{1}{\delta t}\int_{\mathcal{O}} u_{n+1}^2 + \int_{\mathcal{O}}(\partial_x u_{n+1})^2 = \frac{1}{\delta t}\int_{\mathcal{O}} u_n u_{n+1} + \int_{\mathcal{O}} f_{ext}(t_n)u_{n+1} - \int_{\mathcal{O}}\mathbb{E}(P_n Q_n)\partial_x u_{n+1}$$
$$\leq \frac{1}{2\delta t}\left(\int_{\mathcal{O}} u_n^2 + \int_{\mathcal{O}} u_{n+1}^2\right) + \frac{1}{10}\int_{\mathcal{O}}(\partial_x u_{n+1})^2 + C||f_{ext}(t_n)||_{H_*^{-1}}^2 - \int_{\mathcal{O}}\mathbb{E}(P_n Q_n)\partial_x u_{n+1}.$$

One multiplies next (4.35) with Q_n and takes the expectation value :

$$\mathbb{E}(Q_{n+1}Q_n) - \mathbb{E}(Q_n^2) = \left(\mathbb{E}(\partial_x u_{n+1} P_n Q_n) - \frac{1}{2}\mathbb{E}(Q_n^2)\right)\delta t,$$

$$\frac{1}{2}\left(\mathbb{E}(Q_{n+1}^2)-\mathbb{E}(Q_n^2)\right)+\frac{1}{2}\mathbb{E}(Q_n^2)\delta t=\partial_x u_{n+1}\mathbb{E}(P_nQ_n)\delta t+\frac{1}{2}\mathbb{E}((Q_{n+1}-Q_n)^2).$$

Summing this estimate multiplied by $\frac{1}{\delta t}$ and integrated in space, and the one on u_n, we get :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}}u_{n+1}^2-\int_{\mathcal{O}}u_n^2\right)+\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}(Q_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}(Q_n^2)\right)+\frac{9}{10}\int_{\mathcal{O}}(\partial_x u_{n+1})^2$$
$$+\frac{1}{2}\int_{\mathcal{O}}\mathbb{E}(Q_n^2)\le C||f_{ext}(t_n)||^2_{H_*^{-1}}+\frac{1}{2\delta t}\int_{\mathcal{O}}\mathbb{E}((Q_{n+1}-Q_n)^2).\quad(4.37)$$

It remains to estimate the last term in the right-hand side. This is done by taking the square of (4.35) and then the expectation value :

$$\begin{aligned}\mathbb{E}((Q_{n+1}-Q_n)^2) &= \mathbb{E}\left(\left(\partial_x u_{n+1}P_n-\frac{1}{2}Q_n\right)^2\right)\delta t^2+\delta t\\ &\le 2(\partial_x u_{n+1})^2\mathbb{E}(P_n^2)\delta t^2+\frac{1}{2}\mathbb{E}(Q_n^2)\delta t^2+\delta t.\end{aligned}$$

It is easy to show that $\mathbb{E}(P_n^2)$ is bounded by $\frac{4}{4-\delta t}$ (by induction, using $\mathbb{E}(P_{n+1}^2)=\left(1-\frac{\delta t}{2}\right)^2\mathbb{E}(P_n^2)+\delta t$). We obtain then :

$$\mathbb{E}((Q_{n+1}-Q_n)^2)\le(\partial_x u_{n+1})^2\frac{8\delta t^2}{4-\delta t}+\frac{1}{2}\mathbb{E}(Q_n^2)\delta t^2+\delta t.\quad(4.38)$$

Using (4.37) and (4.38), one threfore obtains :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}}u_{n+1}^2-\int_{\mathcal{O}}u_n^2\right)+\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}(Q_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}(Q_n^2)\right)+\frac{9}{10}\int_{\mathcal{O}}(\partial_x u_{n+1})^2$$
$$+\frac{1}{2}\int_{\mathcal{O}}\mathbb{E}(Q_n^2)\le C||f_{ext}(t_n)||^2_{H_*^{-1}}+\frac{4\delta t}{4-\delta t}\int_{\mathcal{O}}(\partial_x u_{n+1})^2+\frac{\delta t}{4}\int_{\mathcal{O}}\mathbb{E}(Q_n^2)+\frac{1}{2}.\quad(4.39)$$

Under the assumption $\delta t<\frac{1}{2}$, one has $\int_{\mathcal{O}}u_{n+1}^2-\int_{\mathcal{O}}u_n^2+\int_{\mathcal{O}}\mathbb{E}(Q_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}(Q_n^2)+A\delta t\int_{\mathcal{O}}(\partial_x u_{n+1})^2\le\delta t\left(2C||f_{ext}(t_n)||^2_{H_*^{-1}}+1\right)$ (with $A=\frac{46}{70}\ge\frac{1}{2}$). We conclude by summation over n. □

We are now going to show the convergence of this scheme. We will first show the convergence of P_n towards P_{t_n} and then, reproducing the proof of the energy estimate (3.22) at the discrete level, we will show the convergence of $(u_h^n,Q_{h,n})$ towards $(u_h(t_n),Q_h(t_n))$.

Let us begin with the convergence of P_n towards P_{t_n} (we recall that P_n and P_{t_n} are defined independently of any space discretization). Since the diffusion coefficient in the SDE satisfied by P_t is constant, the Euler scheme is in fact a Milshtein scheme on P_t. The convergence is therefore in δt (see Theorem 10.3.5 in Ref. [16]) :

Lemma 5 (Convergence of the Euler-Maruyama scheme) *There exists a constant C which depends only on T such that*

$$\mathbb{E}\left((P_n-P_{t_n})^2\right)\le C(\delta t)^2.$$

Remark 5 *We could have used a scheme exact in law for P_t. We have chosen a classical Euler scheme because this is the scheme used in more complicated cases (see Remarks 1 and 2), when P_t also depends on x.*

We can now show the following convergence theorem :

Theorem 2 (Convergence of the time-discretized problem) *Let us assume $u_0 \in H^2_x$ and $f_{ext} \in W^{1,1}_t(L^2_x)$. Under the assumption $\delta t < \frac{1}{2}$, one has :*

$$||u_h^n - u_h(t_n)||^2_{L^2_x} + ||Q_{h,n} - Q_h(t_n)||^2_{L^2_x(L^2_\omega)} \leq C(\delta t)^2,$$

with C independent of h and $n \leq \frac{T}{\delta t}$, but depends on the data of the problem : u_0, f_{ext} and T.

Proof. As in the former proof, we omit here the subscript h : $u_h^n = u_n$, $Q_{h,n} = Q_n$, $u = u_h$ and $Q = Q_h$. We introduce the processes $\tilde{P}$ defined by $d\tilde{P}_t = -\frac{1}{2}\tilde{P}_{\tau_t}\, dt + dV_t$ (with $\tau_t = \lfloor \frac{t}{\delta t} \rfloor \delta t$, where $\lfloor x \rfloor$ is the integer part of x, and $\tilde{P}_0 = P_0$) and $\tilde{Q}$ defined by $d\tilde{Q}_t = \left(\partial_x u_{n(t)+1}\tilde{P}_{\tau_t} - \frac{1}{2}\tilde{Q}_{\tau_t}\right) dt + dW_t$ (with $n(t) = \lfloor \frac{t}{\delta t} \rfloor$ and $\tilde{Q}_0 = Q_0$). One can check easily that $P_n = \tilde{P}_{t_n}$ and $Q_n = \tilde{Q}_{t_n}$. Moreover, we set $e_n = u_n - u(t_n)$.

The stability lemma 4 shows that $\int_{\mathcal{O}} \mathbb{E}(Q_n^2)$ is uniformly bounded (in h and n), hence $\int_{\mathcal{O}} \mathbb{E}(\tilde{Q}_s^2)$ is also uniformly bounded in s. We have also a uniform bound in n on $\mathbb{E}(P_n^2)$ and a uniform bound in s on $\mathbb{E}(\tilde{P}_s^2)$.

Equation on u :

One obtains by subtraction of the continuous formulation in time at time t_n (3.26) (we recall that $u \in \mathcal{C}([0,T], L^2_x(\mathcal{O}))$) and the discretized formulation (4.34) : for all $v \in V_h$,

$$\int_{\mathcal{O}}\left(\frac{u_{n+1}-u_n}{\delta t} - \frac{\partial u}{\partial t}(t_n)\right) v + \int_{\mathcal{O}}(\partial_x u_{n+1} - \partial_x u(t_n))\partial_x v = -\int_{\mathcal{O}} \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x v.$$

With similar computations as those used in the proof of Lemma 4, choosing $v = e_{n+1}$, we obtain :

$$\begin{aligned}\frac{1}{2\delta t}\left(||e_{n+1}||^2_{L^2_x} - ||e_n||^2_{L^2_x}\right) + \int_{\mathcal{O}} |\partial_x e_{n+1}|^2 \leq -\int_{\mathcal{O}} \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x e_{n+1} \\ -\int_{t_n}^{t_{n+1}}\int_{\mathcal{O}} \frac{\partial \partial_x u}{\partial t}\partial_x e_{n+1} - \frac{1}{\delta t}\int_{t_n}^{t_{n+1}}(t_{n+1}-s)\int_{\mathcal{O}}\frac{\partial^2 u}{\partial t^2}(s) e_{n+1}\, dx\, ds. \qquad (4.40)\end{aligned}$$

For the last two terms, using Cauchy-Schwarz and the inequality $ab \leq \delta t a^2 + \frac{1}{4\delta t} b^2$, we have

$$\int_{t_n}^{t_{n+1}}\int_{\mathcal{O}} \frac{\partial \partial_x u}{\partial t}\partial_x e_{n+1} \quad \leq \quad \delta t \int_{t_n}^{t_{n+1}} \left|\left|\frac{\partial \partial_x u}{\partial t}\right|\right|^2_{L^2_x} + \frac{1}{4}\,||\partial_x e_{n+1}||^2_{L^2_x}.$$

In the same way :

$$\int_{t_n}^{t_{n+1}}(t_{n+1}-s)\int_{\mathcal{O}}\frac{\partial^2 u}{\partial t^2}(s) e_{n+1}\, dx\, ds \leq C(\delta t)^2 \int_{t_n}^{t_{n+1}} \left|\left|\frac{\partial^2 u}{\partial t^2}(s)\right|\right|^2_{H^{-1}_{*,x}} + \frac{\delta t}{4}||\partial_x e_{n+1}||^2_{L^2_x}.$$

Therefore, we obtain finally :

$$\frac{1}{2}\left(||e_{n+1}||^2_{L^2_x} - ||e_n||^2_{L^2_x}\right) + \frac{\delta t}{2}\int_{\mathcal{O}} |\partial_x e_{n+1}|^2 \le -\delta t \int_{\mathcal{O}} \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x e_{n+1}$$
$$+(\delta t)^2 \int_{t_n}^{t_{n+1}} \left|\left|\frac{\partial \partial_x u}{\partial t}(s)\right|\right|^2_{L^2_x} ds + C(\delta t)^2 \int_{t_n}^{t_{n+1}} \left|\left|\frac{\partial^2 u}{\partial t^2}(s)\right|\right|^2_{H^{-1}_{*,x}} ds. \quad (4.41)$$

Equation on Q :

In order to estimate the first term on the right-hand side of (4.41), we reproduce the proof of the energy inequality (3.22) at the discrete level. We write the SDE satisfied by $(Q_t - \tilde{Q}_t)^2$:

$$\frac{1}{2}d((Q_t - \tilde{Q}_t)^2) = \left((Q_t - \tilde{Q}_t)(\partial_x u P_t - \partial_x u_{n(t)+1}\tilde{P}_{\tau_t}) - \frac{1}{2}(Q_t - \tilde{Q}_t)(Q_t - \tilde{Q}_{\tau_t})\right) dt.$$

We set in the following $f_n = Q_{t_n} - Q_n$. Integrating the last equation over (t_n, t_{n+1}), we have :

$$\begin{aligned}\frac{1}{2}(f^2_{n+1} - f^2_n) &= -\frac{1}{2}\int_{t_n}^{t_{n+1}} (Q_s - \tilde{Q}_s)(Q_s - Q_n) + \int_{t_n}^{t_{n+1}} (P_s \partial_x u(s) - P_n \partial_x u_{n+1})(Q_s - \tilde{Q}_s)\\ &= -\frac{1}{2}\int_{t_n}^{t_{n+1}} (Q_s - \tilde{Q}_s)^2 + \frac{1}{2}\int_{t_n}^{t_{n+1}} (Q_s - \tilde{Q}_s)(Q_n - \tilde{Q}_s)\\ &\quad + \int_{t_n}^{t_{n+1}} (P_s \partial_x u(s) - P_n \partial_x u_{n+1})(Q_s - \tilde{Q}_s).\end{aligned}$$

We introduce in the expectation of the last expression the term of (4.41) we want to eliminate, namely $\delta t \int_{\mathcal{O}} \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x e_{n+1}$. We obtain :

$$\frac{1}{2}\mathbb{E}(f^2_{n+1} - f^2_n) + \frac{1}{2}\int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s - \tilde{Q}_s)^2 = \delta t \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x e_{n+1} + A, \quad (4.42)$$

with

$$A = \frac{1}{2}\int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)(Q_n - \tilde{Q}_s)\right) + \int_{t_n}^{t_{n+1}} \mathbb{E}\left((P_s \partial_x u(s) - P_n \partial_x u_{n+1})(Q_s - \tilde{Q}_s)\right)$$
$$-\delta t \mathbb{E}(P_n Q_n - P_{t_n} Q_{t_n})\partial_x e_{n+1}.$$

We will show the following estimate on A :

Proposition 1

$$\begin{aligned}|A| \quad &\le \quad C\delta t^3 \left(1 + |\partial_x u_{n+1}|^2 + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \left|\frac{\partial \partial_x u}{\partial t}\right|^2 + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} |\partial_x u|^2 + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s^2)\right.\\ &\left. + \mathbb{E}(Q_n^2) + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \mathbb{E}(\tilde{Q}_s^2)\right) + \epsilon \int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2\right) + \epsilon \delta t |\partial_x e_{n+1}|^2,\end{aligned}$$

with ϵ arbitrarily small and C a constant which is independent of n and δt, but depends on ϵ and on the data of the problem : u_0, f_{ext} and T.

Let us postpone the proof of Proposition 1 after the end of the proof of Theorem 2. Summing up (4.41) and (4.42) (integrated in space), using the estimation of Propsition 1, we have :

$$||e_{n+1}||^2_{L^2_x} - ||e_n||^2_{L^2_x} + \int_{\mathcal{O}} \mathbb{E}(f^2_{n+1} - f^2_n) + (1-2\alpha) \int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s - \tilde{Q}_s)^2 + (1-2\beta)\,\delta t \int_{\mathcal{O}} |\partial_x e_{n+1}|^2$$
$$\leq C\delta t^3 \left(1 + ||\partial_x u_{n+1}||^2_{L^2_x} + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} ||\partial_x u||^2_{L^2_x} + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\int_{\mathcal{O}} \mathbb{E}(Q_s^2) + \int_{\mathcal{O}} \mathbb{E}(Q_n^2) \right.$$
$$\left. + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\int_{\mathcal{O}} \mathbb{E}(\tilde{Q}_s^2) + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \left|\left| \frac{\partial \partial_x u}{\partial t}(s) \right|\right|^2_{L^2_x} ds + \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} \left|\left| \frac{\partial^2 u}{\partial t^2}(s) \right|\right|^2_{H^{-1}_{*,x}} ds \right),$$

where α and β are arbitrarily small positive constants. Summing up over n and using the regularities proved in Lemmas 1 and 4, this concludes the proof. □

We now have to prove Proposition 1. We will need the next two lemmas.

Lemma 6 *Set $R(t,x)$ a process (possibly deterministic). We have the following inequalities :*

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} R(s,x)(\partial_x e_{n+1})\, ds \right) \right| \leq \frac{1}{4\epsilon}\int_{t_n}^{t_{n+1}} \left(\mathbb{E}(R(s,x))\right)^2 ds + \epsilon\delta t |\partial_x e_{n+1}|^2,$$

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} R(s,x)(Q_s - \tilde{Q}_s)\, ds \right) \right| \leq \frac{1}{4\epsilon}\int_{t_n}^{t_{n+1}} \mathbb{E}\left(R(s,x)^2\right) ds + \epsilon \int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2\right) ds.$$

Let $S(t,x)$ be an Itô process such that $dS_t = a(x,t)\,dt + b(x,t)\,dV_t + c(x,t)\,dW_t$ with b and c square integrable in t. We also have the following inequality :

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} (S(s,x) - S(t_n,x))(\partial_x e_{n+1})\, ds \right) \right| \leq \frac{1}{4\epsilon}\delta t^2 \int_{t_n}^{t_{n+1}} \left(\mathbb{E}(a(x,s))\right)^2 ds + \epsilon\delta t |\partial_x e_{n+1}|^2,$$

with ϵ arbitrarily small.

Proof. These results are easy to obtain by Cauchy-Schwarz inequality, noticing that $\partial_x e_{n+1}$ is deterministic and using the inequality $|ab| \leq \frac{1}{4\epsilon}a^2 + \epsilon b^2$. □

Lemma 7 *We have the following two inequalities :*

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} (Q_s - \tilde{Q}_s)(\tilde{Q}_s - \tilde{Q}_{t_n})\, ds \right) \right| \leq C\delta t^2 \left(\delta t + \delta t \mathbb{E}(Q_n^2) + \delta t |\partial_x u_{n+1}|^2 + \int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s^2) \right)$$
$$+ \epsilon \int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2\right) ds,$$

with ϵ arbitrarily small.

$$\left| \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha(Q_s - \tilde{Q}_s)(\tilde{P}_s - \tilde{P}_{t_n})\, ds \right) \right| \leq C\delta t^2 \left(\delta t + \delta t \alpha^2 + \int_{t_n}^{t_{n+1}} |\partial_x u|^2 \right)$$
$$+ \epsilon \int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2\right) ds,$$

with α a constant, ϵ arbitrarily small and C a constant independent of α. The constant C is independent of n and δt, but depends on ϵ and on the data of the problem : u_0, f_{ext} and T.

Proof. The proof of the first inequality mimics that of the second one. Therefore, we only prove the second inequality. For all $t \in (t_n, t_{n+1})$, one can write $d\tilde{P}_t = -\frac{1}{2}P_n dt + dV_t$. We have therefore :

$$\begin{aligned}
&\mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha(\tilde{P}(s,x) - \tilde{P}(t_n,x))(Q_s - \tilde{Q}_s)\,ds\right) \\
&= \int_{t_n}^{t_{n+1}} \mathbb{E}\left(-(s-t_n)\frac{\alpha}{2}P_n(Q_s - \tilde{Q}_s)\right)\,ds + \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha(V_s - V_{t_n})(Q_s - \tilde{Q}_s)\,ds\right). \qquad (4.43)
\end{aligned}$$

For the first term of the right-hand side of (4.43), we apply the second inequality of Lemma 6 (with $R(s,x) = -(s-t_n)\frac{\alpha}{2}P_n$) in order to obtain a bound in $C\delta t^3\alpha^2 + \epsilon\int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2\right)\,ds$. The aim of the remainder of the proof is to show the following estimation on the second term of the right-hand side of (4.43) :

$$\left|\mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha(V_s - V_{t_n})(Q_s - \tilde{Q}_s)\,ds\right)\right| \leq C\delta t^2\left(\delta t + \delta t\alpha^2 + \int_{t_n}^{t_{n+1}} |\partial_x u|^2\right). \qquad (4.44)$$

In order to show (4.44), we use the SDE satisfied by $Q_s - \tilde{Q}_s$:

$$\begin{aligned}
Q_s - \tilde{Q}_s &= Q_{t_n} - Q_n - \frac{1}{2}\int_{t_n}^{s}(Q_v - Q_n)\,dv + \int_{t_n}^{s}(\partial_x u P_v - \partial_x u_{n+1}P_n)\,dv \\
&= \left[\left(1 - \frac{s-t_n}{2}\right)(Q_{t_n} - Q_n) - (s-t_n)\partial_x u_{n+1}P_n + \int_{t_n}^{s}\partial_x u P_{t_n}\,dv\right] \\
&\quad -\frac{1}{2}\int_{t_n}^{s}(Q_v - Q_{t_n})\,dv + \int_{t_n}^{s}\partial_x u(P_v - P_{t_n})\,dv. \qquad (4.45)
\end{aligned}$$

Let us denote B the term in brackets. The random variable B is independent of $(V_s - V_{t_n})$, which implies :

$$\mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha(V_s - V_{t_n})B\,ds\right) = 0.$$

We still have to estimate the contributions of the last two terms of (4.45). These contributions will be denoted respectively by T_1 and T_2.
Let us first turn to the term T_1 which is :

$$\begin{aligned}
T_1 &= \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \frac{\alpha}{2}(V_s - V_{t_n})\int_{t_n}^{s}(Q_v - Q_{t_n})\,dv\,ds\right) \\
&= \frac{\alpha}{2}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathbb{E}\left((V_s - V_{t_n})(Q_v - Q_{t_n})\right)\,dv\,ds.
\end{aligned}$$

It is clear that :

$$|T_1| \leq \delta t^3 \alpha^2 + \frac{1}{\delta t^3} \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}\left((V_s - V_{t_n})(Q_v - Q_{t_n})\right) \, dv \, ds \right)^2 .$$

Using the expression of $Q_v - Q_{t_n} = -\frac{1}{2}\int_{t_n}^{v} Q_w \, dw + \int_{t_n}^{v} \partial_x u P_w \, dw + W_v - W_{t_n}$, one obtains :

$$\begin{aligned} \mathbb{E}\left((V_s - V_{t_n})(Q_v - Q_{t_n})\right) &= \frac{1}{2}\mathbb{E}\left((V_s - V_{t_n}) \int_{t_n}^{v} (-Q_w) \, dw \right) \\ &\quad + \mathbb{E}\left((V_s - V_{t_n}) \int_{t_n}^{v} \partial_x u P_w \, dw \right) \\ &\quad + \mathbb{E}\left((V_s - V_{t_n})(W_v - W_{t_n})\right). \end{aligned} \tag{4.46}$$

The third term of (4.46) is zero. For the second term of (4.46), we write (using Cauchy-Schwarz) :

$$\begin{aligned} &\frac{1}{\delta t^3} \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{v} \partial_x u \mathbb{E}\left((V_s - V_{t_n}) P_w\right) \, dw \, dv \, ds \right)^2 \\ &\quad \leq \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{v} |\partial_x u|^2 \left(\mathbb{E}((V_s - V_{t_n}) P_w)\right)^2 \, dw \, dv \, ds \leq C \delta t^3 \int_{t_n}^{t_{n+1}} |\partial_x u|^2 . \end{aligned}$$

For the first term of (4.46), we write in the same manner :

$$\begin{aligned} &\frac{1}{\delta t^3} \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{v} \mathbb{E}\left((V_s - V_{t_n}) Q_w\right) \, dw \, dv \, ds \right)^2 \\ &\qquad \leq \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{v} \left(\mathbb{E}\left((V_s - V_{t_n}) Q_w\right)\right)^2 \, dw \, dv \, ds \\ &\qquad\quad \leq \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{v} (s - t_n) \mathbb{E}(Q_w^2) \, dw \, dv \, ds \leq \delta t^3 \int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s^2). \end{aligned}$$

Let us now turn to the estimation of the term T_2 :

$$\begin{aligned} T_2 &= \mathbb{E}\left(\int_{t_n}^{t_{n+1}} \alpha (V_s - V_{t_n}) \int_{t_n}^{s} \partial_x u (P_v - P_{t_n}) \, dv \, ds \right) \\ &= \alpha \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \partial_x u \mathbb{E}\left((V_s - V_{t_n})(P_v - P_{t_n})\right) \, dv \, ds. \end{aligned}$$

It is clear that :

$$|T_2| \leq \delta t^3 \alpha^2 + \frac{1}{\delta t^3} \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \partial_x u \mathbb{E}\left((V_s - V_{t_n})(P_v - P_{t_n})\right) \, dv \, ds \right)^2 .$$

Using the expression of $P_v - P_{t_n} = -\frac{1}{2}\int_{t_n}^{v} P_w \, dw + V_v - V_{t_n}$, we obtain :

$$\mathbb{E}\left((V_s - V_{t_n})(P_v - P_{t_n})\right) = \frac{1}{2}\int_{t_n}^{v} \mathbb{E}\left(-(V_s - V_{t_n}) P_w\right) \, dw + \mathbb{E}\left((V_s - V_{t_n})(V_v - V_{t_n})\right). \tag{4.47}$$

For the second term of (4.47), we have therefore :

$$\begin{aligned}
\frac{1}{\delta t^3}\left(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\partial_x u(v-t_n)\,dv\,ds\right)^2 &\leq \frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}|\partial_x u|^2(v-t_n)^2\,dv\,ds\\
&\leq \delta t^2\int_{t_n}^{t_{n+1}}|\partial_x u|^2.
\end{aligned}$$

For the first term of (4.47), we obtain in the same way :

$$\begin{aligned}
&\frac{1}{\delta t^3}\left(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\partial_x u\int_{t_n}^{v}\mathbb{E}\left((V_s-V_{t_n})P_w\right)\,dw\,dv\,ds\right)^2\\
&\qquad\leq\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}|\partial_x u|^2\int_{t_n}^{s}\int_{t_n}^{v}\left(\mathbb{E}\left((V_s-V_{t_n})P_w\right)\right)^2 ds\leq\delta t^3\int_{t_n}^{t_{n+1}}|\partial_x u|^2.
\end{aligned}$$

This ends the proof. □

One can now prove Proposition 1.

Proof. The first inequality of Lemma 7 shows that :

$$|A|\leq C\delta t^2\left(\delta t+\delta t|\partial_x u_{n+1}|^2+\int_{t_n}^{t_{n+1}}\mathbb{E}(Q_s^2)+\delta t\mathbb{E}(Q_n^2)\right)+\epsilon\int_{t_n}^{t_{n+1}}\mathbb{E}\left((Q_s-\tilde{Q}_s)^2\right)+|A'|,$$

with

$$\begin{aligned}
A' &= \int_{t_n}^{t_{n+1}}\mathbb{E}\left((P_s\partial_x u(s)-P_n\partial_x u_{n+1})(Q_s-\tilde{Q}_s)\right)\,ds\\
&\quad-\delta t\mathbb{E}(P_nQ_n-P_{t_n}Q_{t_n})\partial_x(u_{n+1}-u(t_{n+1}))\\
&= \int_{t_n}^{t_{n+1}}\mathbb{E}\Big((P_s\partial_x u(s)-P_n\partial_x u_{n+1})(Q_s-\tilde{Q}_s)\\
&\qquad\qquad-(P_nQ_n-P_{t_n}Q_{t_n})(\partial_x u_{n+1}-\partial_x u(t_{n+1}))\Big)\,ds.
\end{aligned}$$

Using Lemmas 6 and 7, we will prove the following estimate on A' :

$$\begin{aligned}
|A'| \leq\ & C\delta t^3\left(1+|\partial_x u_{n+1}|^2+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\left|\frac{\partial\partial_x u}{\partial t}\right|^2+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}|\partial_x u|^2+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\mathbb{E}(Q_s^2)\right.\\
&\left.+\mathbb{E}(Q_n^2)+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\mathbb{E}(\tilde{Q}_s^2)\right)+\epsilon\int_{t_n}^{t_{n+1}}\mathbb{E}\left((Q_s-\tilde{Q}_s)^2\right)+\epsilon\delta t|\partial_x e_{n+1}|^2,
\end{aligned}$$

with ϵ arbitrarily small.

The third inequality of Lemma 6 applied successively to P_sQ_s and $\tilde{P}_s\tilde{Q}_s$ and the second inequality of Lemma 7 (applied with $\alpha=\partial_x u_{n+1}$) show that $|A'|$ is bounded by :

$$\begin{aligned}
&\left|\int_{t_n}^{t_{n+1}}\mathbb{E}\left((P_s\partial_x u(s)-\tilde{P}_s\partial_x u_{n+1})(Q_s-\tilde{Q}_s)-(\tilde{P}_s\tilde{Q}_s-P_sQ_s)(\partial_x u_{n+1}-\partial_x u(t_{n+1}))\right)\right|\\
&+C\delta t^3\left(1+|\partial_x u_{n+1}|^2+\tfrac{1}{\delta t}\int_{t_n}^{t_{n+1}}|\partial_x u|^2+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\mathbb{E}(Q_s^2)+\mathbb{E}(Q_n^2)+\frac{1}{\delta t}\int_{t_n}^{t_{n+1}}\mathbb{E}(\tilde{Q}_s^2)\right)\\
&+\epsilon\int_{t_n}^{t_{n+1}}\mathbb{E}\left((Q_s-\tilde{Q}_s)^2\right)+\epsilon\delta t|\partial_x e_{n+1}|^2.
\end{aligned}$$

Then, using the second inequality of Lemma 6 (with $R(s,x) = P_s(\partial_x u(s) - \partial_x u(t_{n+1}))$), we obtain the following bound on $|A'|$:

$$\left| \int_{t_n}^{t_{n+1}} \mathbb{E}\left((P_s \partial_x u(t_{n+1}) - \tilde{P}_s \partial_x u_{n+1})(Q_s - \tilde{Q}_s) - (\tilde{P}_s \tilde{Q}_s - P_s Q_s)(\partial_x u_{n+1} - \partial_x u(t_{n+1})) \right) \right|$$
$$+ C\delta t^3 \left(1 + |\partial_x u_{n+1}|^2 + \tfrac{1}{\delta t} \int_{t_n}^{t_{n+1}} |\partial_x u|^2 + \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \left| \frac{\partial \partial_x u}{\partial t} \right|^2 + \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \mathbb{E}(Q_s^2) + \mathbb{E}(Q_n^2) \right.$$
$$\left. + \tfrac{1}{\delta t} \int_{t_n}^{t_{n+1}} \mathbb{E}(\tilde{Q}_s^2) \right) + \epsilon \int_{t_n}^{t_{n+1}} \mathbb{E}\left((Q_s - \tilde{Q}_s)^2 \right) + \epsilon \delta t |\partial_x e_{n+1}|^2 .$$

Then, developing the expression under the integral, we obtain the following term :

$$- \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\partial_x u(t_{n+1}) \tilde{Q}_s (P_s - \tilde{P}_s) \right) + \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\partial_x u_{n+1} Q_s (P_s - \tilde{P}_s) \right)$$
$$= \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\partial_x u_{n+1} (P_s - \tilde{P}_s)(Q_s - \tilde{Q}_s) \right) + \int_{t_n}^{t_{n+1}} \mathbb{E}\left(\tilde{Q}_s (P_s - \tilde{P}_s) \left(\partial_x u_{n+1} - \partial_x u(t_{n+1}) \right) \right).$$

One can now conclude using the inequality $\mathbb{E}\left((P_s - \tilde{P}_s)^2 \right) < C\delta t^2$ and applying the first two inequalities of Lemma 6 to both terms of the above expression. □

4.3. *Convergence of the Monte Carlo discretized problem.*

We now turn to the last level of discretization : the Monte Carlo method. In the preceding subsections, we have shown that the space and time discretized problem $(u_h^n, Q_{h,n})$ converges towards the continuous solution at time $t_n = n\delta t$: $(u(t_n), Q_{t_n})$. We now want to estimate the error induced by the approximation of $\mathbb{E}(P_n Q_{h,n})$ by an empirical mean. All the results of this subsection hold under the assumption $u_0 \in L^2_x$ and $f_{ext} \in L^\infty_t(L^2_x)$.

We define the fully discretized problem :
Being given at time $t_n = n\delta t$, the velocity $\overline{u}_h^n$ and the random variables P_n^j, $\overline{P}_n^j$ and $\overline{Q}_{h,n}^j$, one finds $\overline{u}_h^{n+1} \in V_h$ such that $\forall v \in V_h$,

$$\frac{1}{\delta t} \int_{\mathcal{O}} (\overline{u}_h^{n+1} - \overline{u}_h^n) v + \int_{\mathcal{O}} \partial_x \overline{u}_h^{n+1} \partial_x v \quad = \quad - \int_{\mathcal{O}} \overline{S}_{h,n} \partial_x v + \int_{\mathcal{O}} f_{ext}(t_n) v. \quad (4.48)$$

with $\overline{S}_{h,n} = \frac{1}{M} \sum_{j=1}^M \overline{P}_n^j \overline{Q}_{h,n}^j$. Then, one computes P_{n+1}^j, $\overline{P}_{n+1}^j$ and $\overline{Q}_{h,n+1}^j$ using :

$$\overline{Q}_{h,n+1}^j - \overline{Q}_{h,n}^j \quad = \quad \left(\partial_x \overline{u}_h^{n+1} \overline{P}_{h,n}^j - \frac{1}{2} \overline{Q}_{h,n}^j \right) \delta t + \left(W_{t_{n+1}}^j - W_{t_n}^j \right), \quad (4.49)$$

$$\left\{ \begin{array}{lcl} P_{n+1}^j - P_n^j & = & -\frac{1}{2} P_n^j \, \delta t + (V_{t_{n+1}}^j - V_{t_n}^j), \\ \overline{P}_{n+1}^j & = & \sup(-A, \inf(A, P_{n+1}^j)). \end{array} \right. \quad (4.50)$$

The processes $(V_t^1, \dots, V_t^N)$ and $(W_t^1, \dots, W_t^N)$ are standard independent M-dimensional Brownian motions. Initial conditions are $\overline{u}_{h,0} = \Pi_h(u_0)$ (with Π_h the finite elements interpolation operator), P_0^j and Q_0^j, which are independent normal variables, independent of the Brownian motion V_t^j and W_t^j.

One can see that we have modified the standard Euler scheme on P_t by introducing a cut-off constant $A > 0$. In fact, we will show two types of results : results with cut-off ($A < \infty$) and results without cut-off ($A = \infty$). In the first case, we will require $0 < A < \sqrt{\frac{3}{5\delta t}}$ (and then use a constant $\gamma > 0$ such that $A > -\gamma \ln(\delta t)$). The choice of the upper bound will be justified in the proof of Lemma 10. In the second case ($A = \infty$), we have $\overline{P}_n^j = P_n^j$ and we will state the results on a subset of the probability space. This subset will tend to the entire probability space when $\delta t \to 0$ or $M \to \infty$. These difficulties are linked with usual stability problems encountered in the discretization of SDEs (see Ref. [22]). More precisely, let us introduce the subset $\mathcal{A}_n$ defined for all $n \leq \frac{T}{\delta t}$ by :

$$\mathcal{A}_n = \left\{ \forall k \leq n, \ \frac{1}{M} \sum_{j=1}^{M} (\overline{P}_k^j)^2 < \frac{13}{20} \frac{1}{\delta t} \right\}.$$

The value of the upper bound $\frac{13}{20}\frac{1}{\delta t}$ will be justified in the proof of the stability lemma 9. For the sake of concision, the results at time $n\delta t$ will be stated on the event $\mathcal{A}_n$ in the absence ($A = \infty$) as well as in the presence ($A < \sqrt{\frac{3}{5\delta t}}$) of the cut-off, but it is important to notice that in the latter case, the probability of $\mathcal{A}_n$ is equal to 1.

Lemma 8 (Properties of $\mathcal{A}_n$) *Let us assume $A = \infty$ (in which case $\overline{P}_n^j = P_n^j$). The sequence of sets $(\mathcal{A}_n)_{n \in \mathbb{N}}$ is decreasing. Moreover, we can estimate the probability of the event $\mathcal{A}_n$: assuming $\delta t < \frac{13}{40}$,*

$$\mathbb{P}(\mathcal{A}_n) \geq 1 - \frac{1}{\delta t} \exp\left(-\frac{M}{2} \left(\frac{13}{40\delta t} - 1 - \ln\left(\frac{13}{40\delta t} \right) \right) \right),$$

with C_1 and C_2 two constants independent of n. In particular, for any $t \in [0,T]$, $\mathbb{P}\left(\mathcal{A}_{\lfloor \frac{t}{\delta t} \rfloor} \right) \longrightarrow 1$ when $\delta t \longrightarrow 0$, or when $M \longrightarrow \infty$ with $\delta t < \frac{13}{40}$.

Proof. The first property is clear. For the second one, notice first that a simple calculation yields $\mathbb{E}(P_n^2) < 2$. Hence, if $(G_j)_{j \geq 1}$ denotes a sequence of i.i.d. normal random variables, $\mathbb{P}\left(\frac{1}{M}\sum_{j=1}^{M} (P_n^j)^2 > C \right) \leq \mathbb{P}\left(\frac{1}{M}\sum_{j=1}^{M} (G_j)^2 > \frac{C}{2} \right)$. By Chernoff inequality,

$$\mathbb{P}\left(\frac{1}{M} \sum_{j=1}^{M} (G_j)^2 > C \right) \leq \exp(-M(\lambda C - \Lambda(\lambda))),$$

for any $\lambda > 0$ where Λ denotes the Legendre transform : $\Lambda(\lambda) = \ln(\mathbb{E}(\exp(\lambda G_1^2)))$. We conclude by minimizing the right-hand side over λ using :

$$\sup_{\lambda > 0} (\lambda x - \Lambda(\lambda)) = \begin{cases} 0 & \text{if } x \leq 1, \\ \frac{1}{2}(x - 1 - \ln x) & \text{if } x > 1. \end{cases}$$

□

In the following, we omit the subscript h in order to lighten the notations. It is important to already notice that for all n, the couples $(\overline{P}_n^j, \overline{Q}_n^j)$ are exchangeable, i.e. the law of the M-uplet $((\overline{P}_n^1, \overline{Q}_n^1), ..., (\overline{P}_n^M, \overline{Q}_n^M))$ remains the same for any permutation on the indices $(1, ..., M)$. This allows one to write e.g. $\mathbb{E}\left(\frac{1}{M}\sum_{j=1}^M \overline{Q}_n^j\right) = \mathbb{E}\left(\overline{Q}_n^1\right)$ or $\mathbb{E}\left(\frac{1}{M}\sum_{j=1}^M (\overline{P}_n^j)^2 \partial_x \overline{u}_{n+1}^2\right) = \mathbb{E}\left((\overline{P}_n^1)^2 \partial_x \overline{u}_{n+1}^2\right)$. Let us introduce another notation, only used in the proofs. We define the function $\mathbb{E}^n$ by : for any random variable X, $\mathbb{E}^n(X) = \mathbb{E}(X 1_{\mathcal{A}_n})$. Notice that in the case $A < \sqrt{\frac{3}{5\delta t}}$ (with cut-off), one has $\mathbb{E}^n = \mathbb{E}$.

We start with the stability of the scheme.

Lemma 9 (Stability of the fully discretized problem) *We assume $\delta t < 2$. Moreover, we assume either $\delta t A^2 \leq \frac{13}{20}$, or $A = \infty$. We have then the following inequality : $\forall n \leq \frac{T}{\delta t}$,*

$$\int_{\mathcal{O}} \mathbb{E}(\overline{u}_n^2 1_{\mathcal{A}_n}) + \frac{\delta t}{2} \sum_{k=0}^{n-1} \int_{\mathcal{O}} \mathbb{E}\left((\partial_x \overline{u}_{k+1})^2 1_{\mathcal{A}_k}\right) + \int_{\mathcal{O}} \mathbb{E}\left(\frac{1}{M}\sum_{j=1}^M (\overline{Q}_n^j)^2 1_{\mathcal{A}_n}\right)$$
$$\leq 1 + ||u_0||_{L_x^2}^2 + T\left(1 + C||f_{ext}||_{L_t^\infty(L_x^2)}\right),$$

with C a constant independent of the data of the problem.

Proof. Choosing $v = \overline{u}_{n+1}$ as a test function in (4.48), we obtain (in the same way as in the preceding stability proofs) :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \overline{u}_{n+1}^2 - \int_{\mathcal{O}} \overline{u}_n^2\right) + \frac{9}{10}\int_{\mathcal{O}} \partial_x \overline{u}_{n+1}^2 \quad \leq \quad -\int_{\mathcal{O}} \overline{S}_n \partial_x \overline{u}_{n+1} + C||f_{ext}(t_n)||_{L_x^2}^2.$$

Multiplying the equation (4.49) with $\overline{Q}_n^j$ and $1_{\mathcal{A}_n}$, we obtain :

$$\frac{1}{2\delta t}\left(\mathbb{E}^n((\overline{Q}_{n+1}^j)^2) - \mathbb{E}^n((\overline{Q}_n^j)^2)\right) + \frac{1}{2}\mathbb{E}^n((\overline{Q}_n^j)^2)$$
$$= \mathbb{E}^n(\partial_x \overline{u}_{n+1} \overline{P}_n^j \overline{Q}_n^j) + \frac{1}{2\delta t}\mathbb{E}^n((\overline{Q}_{n+1}^j - \overline{Q}_n^j)^2).$$

Summing up these two relations and using exchangeability, one obtains :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n(\overline{u}_{n+1}^2) - \int_{\mathcal{O}} \mathbb{E}^n(\overline{u}_n^2)\right) + \frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n\left((\overline{Q}_{n+1}^1)^2\right) - \int_{\mathcal{O}} \mathbb{E}^n\left((\overline{Q}_n^1)^2\right)\right)$$
$$+\frac{9}{10}\int_{\mathcal{O}} \mathbb{E}^n(\partial_x \overline{u}_{n+1}^2) + \frac{1}{2}\int_{\mathcal{O}} \mathbb{E}^n\left((\overline{Q}_n^1)^2\right) \leq \frac{1}{2\delta t}\mathbb{E}^n\left((\overline{Q}_{n+1}^1 - \overline{Q}_n^1)^2\right) + C||f_{ext}(t_n)||_{L_x^2}^2.$$

We have now to estimate the term on the right-hand side. We use again :

$$\begin{aligned}
\mathbb{E}^n((\overline{Q}_{n+1}^1 - \overline{Q}_n^1)^2) &= \delta t^2 \mathbb{E}^n((\partial_x \overline{u}_{n+1} \overline{P}_n^1 - \frac{1}{2}\overline{Q}_n^1)^2) + \delta t \\
&\leq 2\delta t^2 \mathbb{E}^n((\partial_x \overline{u}_{n+1} \overline{P}_n^1)^2) + \frac{1}{2}\delta t^2 \mathbb{E}^n((\overline{Q}_n^1)^2) + \delta t.
\end{aligned}$$

This yields :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^n(\overline{u}_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}^n(\overline{u}_n^2)\right)+\int_{\mathcal{O}}\mathbb{E}^n\left(\left(\frac{9}{10}-\delta t\frac{1}{M}\sum_{j=1}^{M}(\overline{P}_n^j)^2\right)\partial_x\overline{u}_{n+1}^2\right)$$
$$+\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^n((\overline{Q}_{n+1}^1)^2)-\int_{\mathcal{O}}\mathbb{E}^n((\overline{Q}_n^1)^2)\right)+\frac{1}{2}\int_{\mathcal{O}}\left(1-\frac{\delta t}{2}\right)\mathbb{E}^n((\overline{Q}_n^1)^2)$$
$$\leq\frac{1}{2}+C||f_{ext}(t_n)||_{L_x^2}^2.$$

Using the following three properties : $\left(\left(\frac{9}{10}-\delta t\frac{1}{M}\sum_{j=1}^{M}(\overline{P}_n^j)^2\right)\partial_x\overline{u}_{n+1}^2\right)1_{\mathcal{A}_n}\geq\frac{1}{4}\partial_x\overline{u}_{n+1}^2 1_{\mathcal{A}_n}$ (this is the inequality which defines the upper bound in the definition of $\mathcal{A}_n$), $1_{\mathcal{A}_n}\geq 1_{\mathcal{A}_{n+1}}$ and $\delta t<2$, we get :

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^{n+1}(\overline{u}_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}^n(\overline{u}_n^2)\right)+\frac{1}{4}\int_{\mathcal{O}}\mathbb{E}^n\left(\partial_x\overline{u}_{n+1}^2\right)$$
$$+\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^{n+1}((\overline{Q}_{n+1}^1)^2)-\int_{\mathcal{O}}\mathbb{E}^n((\overline{Q}_n^1)^2)\right)\leq\frac{1}{2}+C||f_{ext}(t_n)||_{L_x^2}^2.$$

This yields the stability, by summing up over n. □

Let us now turn to the convergence of the solution of the fully discretized problem towards the solution of the problem discretized in space and time.

We need to introduce the random variables $Q_{h,n}^j$ (denoted Q_n^j in the following) :

$$Q_{h,n+1}^j-Q_{h,n}^j=\left(\partial_x u_h^{n+1}P_n^j-\frac{1}{2}Q_{h,n}^j\right)\delta t+W_{t_{n+1}}^j-W_{t_n}^j. \tag{4.51}$$

The couples (P_n^j,Q_n^j) are independent realizations of the couples (P_n,Q_n). They also are exchangeable random variables.

The aim of this section is to prove the following lemma.

Lemma 10 (Convergence of the Monte Carlo method) *We assume $\delta t<\frac{1}{2}$. Moreover, we assume either $0<A<\sqrt{\frac{3}{5\delta t}}$ (convergence with cut-off), or $A=\infty$ (convergence without cut-off). We have then the following inequality : $\forall n\leq\frac{T}{\delta t}$,*

$$\int_{\mathcal{O}}\mathbb{E}((u_n-\overline{u}_n)^2 1_{\mathcal{A}_n})+\int_{\mathcal{O}}\mathbb{E}\left(\frac{1}{M}\sum_{j=1}^{M}(Q_n^j-\overline{Q}_n^j)^2 1_{\mathcal{A}_n}\right)\leq C\left(\frac{1}{M}+\delta t^2\right).$$

The constant C is independent of n, h and δt, but depends on the data of the problem : u_0, f_{ext} and T. In the case $0<A<\sqrt{\frac{3}{5\delta t}}$, C also depends on $\gamma>0$ such that $A>-\gamma\ln(\delta t)$. In the case $A=\infty$ the estimation is in fact of order $\frac{C}{M}$.

In the following, we will need an estimate of the variance of $P_nQ_{h,n}$.

Lemma 11 (A variance estimate) *We assume $\delta t < 1$. Then, $\exists C$, $\forall n \leq \frac{T}{\delta t}$,*

$$\int_{\mathcal{O}} \mathbb{E}\left((P_n Q_{h,n} - \mathbb{E}(P_n Q_{h,n}))^2 \right) < C.$$

The constant C is independent of h and δt, but depends on the data of the problem : u_0, f_{ext} and T.

Proof. The proof is based on an explicit calculation of the variance. Recall that we omit the subscript h.

In the following, we set $W_{t_{n+1}} - W_{t_n} = \sqrt{\delta t} G_n$ and $V_{t_{n+1}} - V_{t_n} = \sqrt{\delta t} G'_n$. The random variables G_n, G'_n are independent normal random variables, independent of P_0 and Q_0.

We recall that P_n and Q_n are defined by :

$$P_{k+1} = \left(1 - \frac{\delta t}{2}\right) P_k + \sqrt{\delta t} G'_k \text{ and } Q_{k+1} = \left(1 - \frac{\delta t}{2}\right) Q_k + \delta t \partial_x u_{k+1} P_k + \sqrt{\delta t} G_k.$$

By induction, it is easy to show that

$$Q_n = \left(1 - \frac{\delta t}{2}\right)^n Q_0 + \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{n-k} \sqrt{\delta t} G_{k-1} + \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{n-k} \partial_x u_k P_{k-1} \delta t. \tag{4.52}$$

We set $X_n = \delta t \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{-k} \partial_x u_k P_{k-1} P_n$. We have the following equalities :

$$P_n Q_n = \left(1 - \frac{\delta t}{2}\right)^n \left(P_n Q_0 + \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{-k} \sqrt{\delta t} P_n G_{k-1} + X_n \right),$$

$$P_n Q_n - \mathbb{E}(P_n Q_n) = \left(1 - \frac{\delta t}{2}\right)^n \left(P_n Q_0 + \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{-k} \sqrt{\delta t} P_n G_{k-1} + X_n - \mathbb{E}(X_n) \right).$$

Using independence properties, we find :

$$\begin{aligned} &\mathbb{E}\left((P_n Q_n - \mathbb{E}(P_n Q_n))^2 \right) \\ &\quad = \left(1 - \frac{\delta t}{2}\right)^{2n} \left(\mathbb{E}(P_n^2) + \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{-2k} \delta t \mathbb{E}(P_n^2) + \mathbb{E}\left((X_n - \mathbb{E}(X_n))^2 \right) \right). \end{aligned}$$

A simple calculation yields $\mathbb{E}(P_n^2) < 2$ and therefore $\mathbb{E}(P_n P_m) < 2$. It remains now to estimate $\left(1 - \frac{\delta t}{2}\right)^{2n} \mathbb{E}\left((X_n - \mathbb{E}(X_n))^2 \right)$. One can show that

$$\left(1 - \frac{\delta t}{2}\right)^{2n} (X_n - \mathbb{E}(X_n))^2 \leq \delta t^2 n \sum_{k=1}^n \left(1 - \frac{\delta t}{2}\right)^{2(n-k)} |\partial_x u^k|^2 (P_{k-1} P_n - \mathbb{E}(P_{k-1} P_n))^2.$$

One can check that $\mathbb{E}\left((P_{k-1} P_n - \mathbb{E}(P_{k-1} P_n))^2 \right) < C$ with C independent of δt (this is deduced from $\mathbb{E}(P_k^4) < C$). We obtain then :

$$\left(1 - \frac{\delta t}{2}\right)^{2n} \mathbb{E}\left((X_n - \mathbb{E}(X_n))^2 \right) \leq C T \delta t \sum_{k=1}^n |\partial_x u^k|^2.$$

The stability lemma 4 has shown that $\sum_{k=1}^{n} \delta t ||\partial_x u_k||^2_{L^2_x} < C$, which leads to the result. $\square$

In order to prove Lemma 10 in the case $A < \infty$ (convergence with cut-off), we will also use the following estimates :

Lemma 12 *We assume $\delta t < \frac{1}{2}$. Moreover, we assume that the cut-off constant is such that $-\gamma \ln(\delta t) < A < \infty$, for some positive constant γ. We have then :*

$$\mathbb{E}\left((P^1_n - \overline{P}^1_n)^2\right) < C\delta t^4,$$

$$\int_{\mathcal{O}} \mathbb{E}\left(\left(Q^1_n(P^1_n - \overline{P}^1_n)\right)^2\right) < C\delta t^4,$$

with C a constant depending on γ and on the data of the problem : u_0, f_{ext} and T.

Proof. In the following, as in the former proof, we set $W^j_{t_{n+1}} - W^j_{t_n} = \sqrt{\delta t} G^j_n$ and $V^j_{t_{n+1}} - V^j_{t_n} = \sqrt{\delta t}(G^j_n)'$. The first estimate is deduced from an estimation on normal random variables. We know that for all n, the random variables P^1_n are normal variables of variance less than 2. One can therefore write : for all n,

$$\mathbb{E}\left((P^1_n - \overline{P}^1_n)^2\right) < \frac{1}{\sqrt{\pi}} \int_A^\infty (x-A)^2 e^{-\frac{x^2}{4}}\, dx.$$

A simple calculation yields

$$\frac{1}{\sqrt{\pi}} \int_A^\infty (x-A)^2 e^{-\frac{x^2}{4}}\, dx < C \exp\left(-C' \frac{A^2}{8}\right) < C_\alpha \exp(\alpha \ln(\delta t)),$$

for any exponent $\alpha > 0$. Taking $\alpha = 4$, we obtain the first estimate. One can show in the same way the following estimate which will be used at the end of this proof :

$$\mathbb{E}\left((P^1_n - \overline{P}^1_n)^4\right) < C\delta t^8. \tag{4.53}$$

For the second estimate, we use the former computation (4.52) of Q^1_n. We can then write :

$$\begin{aligned}
\int_{\mathcal{O}} \mathbb{E}\left(\left(Q^1_n(P^1_n - \overline{P}^1_n)\right)^2\right) \leq\; & 3\int_{\mathcal{O}} \mathbb{E}\left(\left(\left(1-\frac{\delta t}{2}\right)^n Q^1_0 (P^1_n - \overline{P}^1_n)\right)^2\right) \\
& + 3\int_{\mathcal{O}} \mathbb{E}\left(\left(\sum_{k=1}^n \left(1-\frac{\delta t}{2}\right)^{n-k} \sqrt{\delta t} G^1_{k-1}(P^1_n - \overline{P}^1_n)\right)^2\right) \\
& + 3\int_{\mathcal{O}} \mathbb{E}\left(\left(\sum_{k=1}^n \left(1-\frac{\delta t}{2}\right)^{n-k} \partial_x u_k P^1_{k-1}(P^1_n - \overline{P}^1_n)\delta t\right)^2\right).
\end{aligned}$$

For the first and second terms, we notice that the random variables Q_0^1, $(P_n^1 - \overline{P}_n^1)$ and G_n^1 are independent, which yields :

$$\int_{\mathcal{O}} \mathbb{E}\left(\left(\left(1-\frac{\delta t}{2}\right)^n Q_0^1(P_n^1 - \overline{P}_n^1)\right)^2\right) + \int_{\mathcal{O}} \mathbb{E}\left(\left(\sum_{k=1}^n \left(1-\frac{\delta t}{2}\right)^{n-k} \sqrt{\delta t} G_{k-1}^1 (P_n^1 - \overline{P}_n^1)\right)^2\right)$$
$$\leq \mathbb{E}\left((Q_0^1)^2\right) \mathbb{E}\left((P_n^1 - \overline{P}_n^1)^2\right) + \sum_{k=1}^n \delta t \, \mathbb{E}\left((G_{k-1}^1)^2\right) \mathbb{E}\left((P_n^1 - \overline{P}_n^1)^2\right)$$
$$\leq (1+T)\mathbb{E}\left((P_n^1 - \overline{P}_n^1)^2\right) \leq C\delta t^4.$$

For the third term, we write :

$$\int_{\mathcal{O}} \mathbb{E}\left(\left(\sum_{k=1}^n \left(1-\frac{\delta t}{2}\right)^{n-k} \partial_x u_k P_{k-1}^1 (P_n^1 - \overline{P}_n^1)\delta t\right)^2\right)$$
$$\leq \int_{\mathcal{O}} \mathbb{E}\left(\left(\sqrt{\sum_{k=1}^n \partial_x u_k^2}\sqrt{\sum_{k=1}^n (P_{k-1}^1)^2}(P_n^1 - \overline{P}_n^1)\delta t\right)^2\right)$$
$$\leq \delta t^2 \int_{\mathcal{O}} \sum_{k=1}^n \partial_x u_k^2 \mathbb{E}\left(\sum_{k=1}^n (P_{k-1}^1)^2 (P_n^1 - \overline{P}_n^1)^2\right).$$

We have shown in the stability lemma 4 that $\delta t \sum_{k=1}^n \int_{\mathcal{O}} \partial_x u_k^2 < C$. One last term remains :

$$\mathbb{E}\left(\sum_{k=1}^n (P_{k-1}^1)^2 (P_n^1 - \overline{P}_n^1)^2\right) \quad \leq \quad \sum_{k=1}^n \sqrt{\mathbb{E}\left((P_{k-1}^1)^4\right)}\sqrt{\mathbb{E}\left((P_n^1 - \overline{P}_n^1)^4\right)} \leq \frac{C}{\delta t}\delta t^4.$$

using the fact that $\mathbb{E}\left(\left(P_k^1\right)^4\right) \leq C$ and (4.53). $\square$

We can now prove Lemma 10.

Proof. We set $S_n = \frac{1}{M}\sum_{j=1}^M P_n^j Q_n^j$, $g_n = u_n - \overline{u}_n$ and $R_n^j = Q_n^j - \overline{Q}_n^j$. Using the same arguments as in the former proofs, we obtain

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}} g_{n+1}^2 - \int_{\mathcal{O}} g_n^2\right) + \int_{\mathcal{O}} \partial_x g_{n+1}^2 \leq - \int_{\mathcal{O}} (\mathbb{E}(P_n Q_n) - \overline{S}_n)\partial_x g_{n+1},$$
$$\frac{1}{2\delta t}\left(\mathbb{E}^n((R_{n+1}^j)^2) - \mathbb{E}^n((R_n^j)^2)\right) + \frac{1}{2}\mathbb{E}^n((R_n^j)^2) = \mathbb{E}^n\left((\partial_x u_{n+1} P_n^j - \partial_x \overline{u}_{n+1}\overline{P}_n^j)R_n^j\right)$$
$$+\frac{1}{2\delta t}\mathbb{E}^n((R_{n+1}^j - R_n^j)^2).$$

Summing up these two expressions, one finds :

$$\begin{aligned}
&\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n(g_{n+1}^2) - \int_{\mathcal{O}} \mathbb{E}^n(g_n^2)\right) + \int_{\mathcal{O}} \mathbb{E}^n(\partial_x g_{n+1}^2) \\
&+\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n\left((R_{n+1}^1)^2\right) - \int_{\mathcal{O}} \mathbb{E}^n\left((R_n^1)^2\right)\right) + \frac{1}{2}\int_{\mathcal{O}} \mathbb{E}^n\left((R_n^1)^2\right) \\
&\leq \frac{1}{2\delta t}\int_{\mathcal{O}} \mathbb{E}^n\left((R_{n+1}^1 - R_n^1)^2\right) + \int_{\mathcal{O}} \mathbb{E}^n\left((S_n - \mathbb{E}(P_nQ_n))\partial_x g_{n+1}\right) + \int_{\mathcal{O}} \mathbb{E}^n\left(I_n^1\right).
\end{aligned} \tag{4.54}$$

with

$$\begin{aligned}
I_n^1 &= (\partial_x u_{n+1} P_n^1 - \partial_x \overline{u}_{n+1} \overline{P}_n^1) R_n^1 - (P_n^1 Q_n^1 - \overline{P}_n^1 \overline{Q}_n^1)\partial_x g_{n+1} \\
&= \partial_x u_{n+1} \overline{Q}_n^1 (\overline{P}_n^1 - P_n^1) + \partial_x \overline{u}_{n+1} Q_n^1 (P_n^1 - \overline{P}_n^1).
\end{aligned}$$

For the second term on the right-hand side of (4.54), we use Lemma 11 :

$$\int_{\mathcal{O}} \mathbb{E}^n\left((S_n - \mathbb{E}(P_nQ_n))\partial_x g_{n+1}\right) \leq \frac{1}{10}\int_{\mathcal{O}} \mathbb{E}^n\left((\partial_x g_{n+1})^2\right) + 10\int_{\mathcal{O}} \mathbb{E}\left((S_n - \mathbb{E}(P_nQ_n))^2\right).$$

The first term is controlled on the left-hand side of (4.54), while the second term is estimated using the variance of P_nQ_n (see Lemma 11) :

$$\int_{\mathcal{O}} \mathbb{E}\left((S_n - \mathbb{E}(P_nQ_n)^2\right) = \int_{\mathcal{O}} \mathbb{E}\left(\left(\frac{1}{M}\sum_{j=1}^{M}(P_n^j Q_n^j - \mathbb{E}(P_n^j Q_n^j))\right)^2\right) \leq \frac{C}{M}.$$

For the first term on the right-hand side of (4.54), we write :

$$\begin{aligned}
(R_{n+1}^1 - R_n^1)^2 &= \left((\partial_x u_{n+1} P_n^1 - \partial_x \overline{u}_{n+1} \overline{P}_n^1) - \frac{1}{2} R_n^1\right)^2 \delta t^2 \\
&= \left(\partial_x u_{n+1}(P_n^1 - \overline{P}_n^1) + (\partial_x g_{n+1} \overline{P}_n^1) - \frac{1}{2} R_n^1\right)^2 \delta t^2.
\end{aligned}$$

In the case $A = \infty$, using $P_n^j = \overline{P}_n^j$, one notices that forall j, $I_n^j = 0$ and that

$$(R_{n+1}^j - R_n^j)^2 \leq 2\left(\partial_x g_{n+1} \overline{P}_n^j\right)^2 \delta t^2 + \frac{1}{2}\left(R_n^j\right)^2 \delta t^2.$$

Using the assumption $\frac{1}{2}\delta t < 1$, the second term is controlled on the left-hand side of (4.54). It follows that :

$$\begin{aligned}
\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n(g_{n+1}^2) - \int_{\mathcal{O}} \mathbb{E}^n(g_n^2)\right) + \int_{\mathcal{O}} \mathbb{E}^n\left(\left(\frac{9}{10} - \delta t\frac{1}{M}\sum_{j=1}^{M}(\overline{P}_n^j)^2\right)\partial_x g_{n+1}^2\right) \\
+\frac{1}{2\delta t}\left(\int_{\mathcal{O}} \mathbb{E}^n\left((R_{n+1}^1)^2\right) - \int_{\mathcal{O}} \mathbb{E}^n\left((R_n^1)^2\right)\right) \leq \frac{C}{M}.
\end{aligned}$$

Using the properties of $\mathcal{A}_n$, we easily derive

$$\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^{n+1}(g_{n+1}^2)-\int_{\mathcal{O}}\mathbb{E}^n(g_n^2)\right)+\frac{1}{2\delta t}\left(\int_{\mathcal{O}}\mathbb{E}^{n+1}\left((R_{n+1}^1)^2\right)-\int_{\mathcal{O}}\mathbb{E}^n\left((R_n^1)^2\right)\right)\leq\frac{C}{M}.$$

Summing up (4.55) on n, we obtain an estimation in $\frac{C}{M}$, using the stability lemmas 4 and 9.

In the case $A<\infty$, we have (notice that $P_n^1\neq\overline{P}_n^1$) :

$$(R_{n+1}^1-R_n^1)^2 \quad\leq\quad 3\left(\partial_x u_{n+1}(P_n^1-\overline{P}_n^1)\right)^2\delta t^2+3\left(\partial_x g_{n+1}\overline{P}_n^1\right)^2\delta t^2+\frac{3}{4}\left(R_n^1\right)^2\delta t^2.$$

Using the assumption $\frac{3}{2}A^2\delta t<\frac{9}{10}$ (this is the inequality which defines the upper bound of A in the case $A<\infty$) and $\frac{3}{4}\delta t<1$, the last two terms are controlled on the left-hand side of (4.54). We obtain a bound of order $C\delta t^2$ on the first term using $\delta t\sum_n\int_{\mathcal{O}}\partial_x u_{n+1}^2<C$ (see Lemma 4) and $\mathbb{E}\left((P_n^1-\overline{P}_n^1)^2\right)<C\delta t$ (see Lemma 12). For the third term on the right-hand side of (4.54) (which is $\mathbb{E}(I_n^1)$), we use twice Lemma 12. Indeed, for the first term of I_n^1, we write :

$$\begin{aligned}\int_{\mathcal{O}}\mathbb{E}\left(\partial_x u_{n+1}\overline{Q}_n^1(\overline{P}_n^1-P_n^1)\right)&\leq\sqrt{\int_{\mathcal{O}}(\partial_x u_{n+1})^2}\,\mathbb{E}\left(\sqrt{\int_{\mathcal{O}}(\overline{Q}_n^1)^2}(\overline{P}_n^1-P_n^1)\right)\\&\leq\sqrt{\int_{\mathcal{O}}(\partial_x u_{n+1})^2}\sqrt{\mathbb{E}\left(\int_{\mathcal{O}}(\overline{Q}_n^1)^2\right)}\sqrt{\mathbb{E}\left((\overline{P}_n^1-P_n^1)^2\right)},\end{aligned}$$

which yields after summation over n an estimate of order $C\delta t^2$. For the second term of I_n^1, we write :

$$\int_{\mathcal{O}}\mathbb{E}\left(\partial_x\overline{u}_{n+1}Q_n^1(\overline{P}_n^1-P_n^1)\right)\leq\sqrt{\int_{\mathcal{O}}\mathbb{E}\left((\partial_x\overline{u}_{n+1})^2\right)}\sqrt{\int_{\mathcal{O}}\mathbb{E}\left(\left(Q_n^1(\overline{P}_n^1-P_n^1)\right)^2\right)},$$

which also yields after summation over n a bound in $C\delta t^2$. We can again conclude summing up over n and using the stability lemmas 4 and 9. ◻

Remark 6 *One can estimate, in the case $A<\infty$, the probability that the cut-off is active during a simulation. Indeed, the probability that one of the $|P_n^j|$ (with $n\leq\frac{T}{\delta t}$) goes beyond A is roughly bounded by $\frac{M}{\delta t}\left(1-\frac{2}{\sigma\sqrt{2\pi}}\int_0^A e^{-\frac{x^2}{2\sigma^2}}\right)=\frac{M}{\delta t}\left(1-erf\left(\frac{A}{\sigma\sqrt{2}}\right)\right)$ with σ^2 an upper bound on the variance of the P_n^j (one can take $\sigma^2=\frac{4}{4-\delta t}$) and $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$. Choosing $M=\frac{1}{\delta t^2}$ (which is consistent with the order of convergence $O\left(h+\delta t+\frac{1}{\sqrt{M}}\right)$) and $A=\sqrt{\frac{3}{5\delta t}}$, this probability is bounded by $\frac{1}{\delta t^3}\left(1-erf\left(\sqrt{\frac{3(4-\delta t)}{40\delta t}}\right)\right)$. This upper bound is very close to 0 when δt is small (it is equal to 10^{-8} for $\delta t=0.01$).*

4.4. *Conclusion : convergence of the fully discretized problem.*

We now state our main result.

Theorem 3 (Convergence of the fully discretized problem) *We assume a P1 discretization of the velocity in space. We also make the following regularity hypothesis :* $u_0 \in H^2_x$, $f_{ext} \in L^1_t(H^1_x)$ *and* $\dfrac{\partial f_{ext}}{\partial t} \in L^1_t(L^2_x)$. *We assume either* $A = \infty$ *(without cut-off), or* $0 < A < \sqrt{\frac{3}{5\delta t}}$ *(with cut-off, in which case* $1_{\mathcal{A}_n} = 1$*). Assuming* $\delta t < \frac{1}{2}$, *we have :*

$$\left|\left|u(t_n) - \overline{u}_h^n 1_{\mathcal{A}_n}\right|\right|_{L^2_x(L^2_\omega)} + \left|\left|\mathbb{E}(P_{t_n}Q_{t_n}) - \frac{1}{M}\sum_{j=1}^M \overline{P}^j_n \overline{Q}^j_{h,n} 1_{\mathcal{A}_n}\right|\right|_{L^1_x(L^1_\omega)} \leq C\left(h + \delta t + \frac{1}{\sqrt{M}}\right),$$

where C *is independent of* h *and* δt, *but depends on the data of the problem :* u_0, f_{ext} *and* T. *In the case* $0 < A < \sqrt{\frac{3}{5\delta t}}$, C *also depends on* $\gamma > 0$ *such that* $A > -\gamma \ln(\delta t)$.

Proof. For the estimation on u, we write : $u(t_n) - \overline{u}_h^n 1_{\mathcal{A}_n} = (u(t_n) - u_h(t_n)) + (u_h(t_n) - u_h^n) + u_h^n (1 - 1_{\mathcal{A}_n}) + (u_h^n - \overline{u}_h^n) 1_{\mathcal{A}_n}$. We use Lemma 3 for the first term, Theorem 2 for the second term and Lemma 10 for the last term. In case $A < \sqrt{\frac{3}{5\delta t}}$, the third term is nul. In case $A = \infty$, we upper bound this term thanks to Lemmas 8 and 4.

For the estimation on $\mathbb{E}(P_t Q_t)$, we write : $\mathbb{E}(P_{t_n}Q_{t_n}) - \frac{1}{M}\sum_{j=1}^M \overline{P}^j_n \overline{Q}^j_{h,n} 1_{\mathcal{A}_n} = (\mathbb{E}(P_{t_n}Q_{t_n}) - \mathbb{E}(P_{t_n}Q_{h,t_n})) + (\mathbb{E}(P_{t_n}Q_{h,t_n}) - \mathbb{E}(P_n Q_{h,n})) + \mathbb{E}(P_n Q_{h,n})(1 - 1_{\mathcal{A}_n}) + \left(\mathbb{E}(P_n Q_{h,n}) - \frac{1}{M}\sum_{j=1}^M P^j_n Q^j_{h,n}\right) 1_{\mathcal{A}_n} + \left(\frac{1}{M}\sum_{j=1}^M P^j_n (Q^j_{h,n} - \overline{Q}^j_{h,n})\right) 1_{\mathcal{A}_n} + \left(\frac{1}{M}\sum_{j=1}^M \overline{Q}^j_{h,n}(P^j_n - \overline{P}^j_n)\right) 1_{\mathcal{A}_n}$. We use then Lemma 3 for the first term, Theorem 2 for the second term, Lemma 11 for the fourth term and Lemma 10 for the fifth term. The third term is nul when $A < \sqrt{\frac{3}{5\delta t}}$ and is estimated by Lemmas 8 and 4 in the case $A = \infty$. The last term is zero in the case $A = \infty$ and is estimated by Lemma 12 in the case $A < \infty$. $\square$

Remark 7 *We have actually shown the following convergence result on* Q_t *:* $\forall j \leq M$,

$$\left|\left|Q^j_{t_n} - \overline{Q}^j_{h,n}\right|\right|_{L^2_x(L^2_\omega)} \leq C\left(h + \delta t + \frac{1}{\sqrt{M}}\right),$$

where (P^j_t, Q^j_t) *are the processes defined by (3.20) and (3.19) with* (V_t, W_t) *replaced by* (V^j_t, W^j_t).

Remark 8 *In the space-discretized problem of our model, the* j^{th} *dumbbell in each cell is driven by the same Brownian motion* (V^j, W^j). *However, the first CONFESSIT simulations were made with driving Brownian motions independent from* *one cell to another. More generally, one could choose any correlation in space for these Brownian motions. In fact, the convergence result stated in Theorem 3 holds whatever the choice of the correlation in space (the constant C in front of the rate of convergence $C\left(h+\delta t+\frac{1}{\sqrt{M}}\right)$, does not depend on the correlation). In return, the convergence on $\overline{Q}^j_{h,n}$ stated in the previous remark no longer makes sense.*

5. Numerical results.

In this section, we show some numerical results about the latter step of discretization : the convergence of the Monte Carlo method. It is indeed the less classical one, and the model we use is simple enough to compute exactly $(u_h^{n+1}, \mathbb{E}(P_{n+1}Q_{h,n+1}))$ being given $(u_h^n, \mathbb{E}(P_nQ_{h,n}))$. We use (4.34) to compute u_h^{n+1} and the following explicite calculation of $\mathbb{E}(P_{n+1}Q_{h,n+1})$ derived from (4.35) and (4.36) (which is just a discretization of the equivalent macroscopic model for the stress tensor) :

$$\left\{\begin{array}{lll} \mathbb{E}(P_{n+1}Q_{h,n+1}) & = & \left(1-\frac{\delta t}{2}\right)^2\mathbb{E}(P_nQ_{h,n}) + \left(1-\frac{\delta t}{2}\right)\partial_x u_h^{n+1}\mathbb{E}(P_n^2)\delta t, \\ \mathbb{E}(P_{n+1}^2) & = & \left(1-\frac{\delta t}{2}\right)^2\mathbb{E}(P_n^2)+\delta t. \end{array}\right.$$

This enables us to compare numerically the deterministic variables $(u_h^n, \mathbb{E}(P_nQ_{h,n}))$ (which, we recall, are an approximation in space and time of $(u(t_n), \mathbb{E}(P_{t_n}Q_{t_n})))$ with the Monte Carlo approximation $\left(\overline{u}_h^n, \frac{1}{M}\sum_{j=1}^M \overline{P}_n^j\overline{Q}_{h,n}^j\right)$. All the tests have been done with the following values for the physical parameters : $\lambda = 1$, $nk_BT = 20$ and $T = 1$. In the following, I denotes the number of space steps, N denotes the number of time steps and M denotes the number of Monte Carlo realizations (i.e. the number of dumbbells in each cell).

Tests on the stability.

First, by a deterministic calculus yielding $(u_h^n, \mathbb{E}(P_nQ_{h,n}))$, we have checked that when δt is too large, the solution oscillates (see Figure 2). This result is to be related to the stability lemma 4, which states that stability holds for δt small enough.

Tests on the cut-off.

In order to illustrate the effect of the cut-off on the fully discretized problem, one needs to take a δt near the upper bound of stability given in Lemma 4. In practice, we have chosen δt such that the deterministic computation begins to oscillate. We have chosen the following parameters : I = 10, N = 8 and M = 100. We have performed for each simulation (with cut-off and without cut-off) one million runs. We have then analyzed the errors (on velocity and stress) :

$$\sup_{0\le n\le \frac{T}{\delta t}} ||u_h^n-\overline{u}_h^n||_{L^2_x} \text{ and } \sup_{0\le n\le \frac{T}{\delta t}} \left|\left|\mathbb{E}(P_nQ_{h,n}) - \frac{1}{M}\sum_{j=1}^M \overline{P}_n^j\overline{Q}_{h,n}^j\right|\right|_{L^1_x}. \tag{5.55}$$

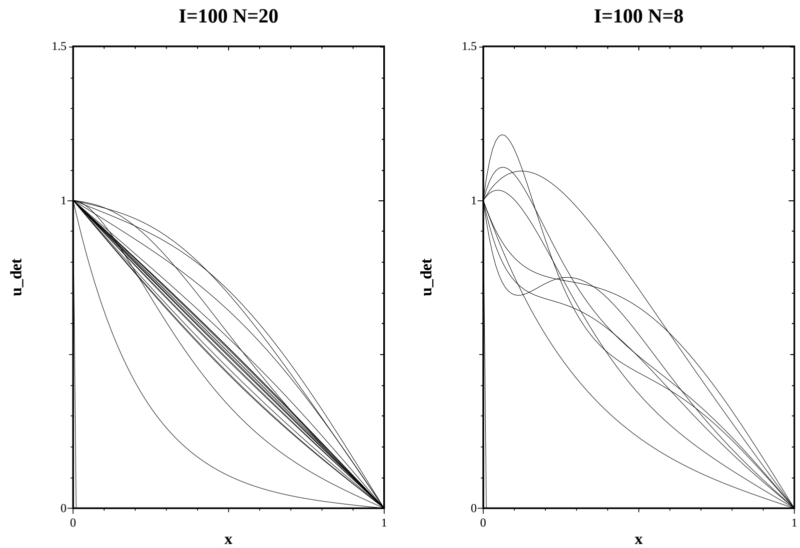

Fig. 2. Deterministic computation of velocity profile as time evolves.

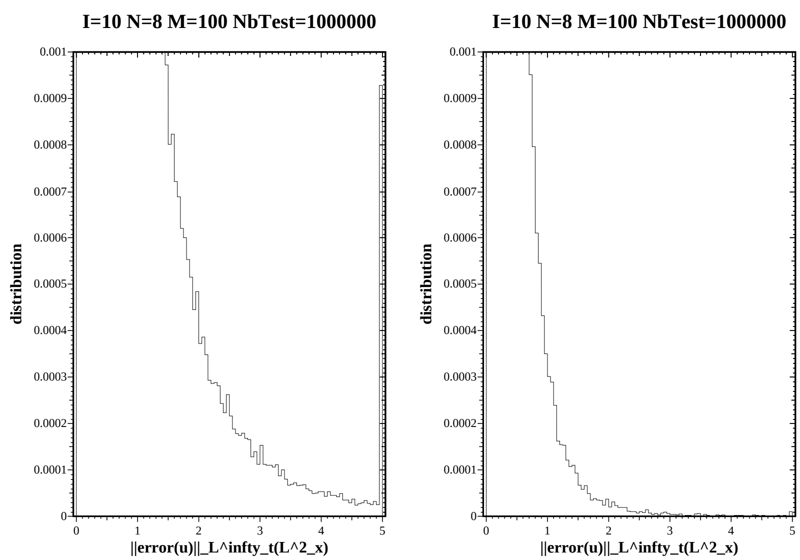

Fig. 3. Distribution of the errors on velocity (zoom) : on the left-hand side, simulation without cut-off and on the right-hand side, simulation with cut-off ($A = 2.3$).

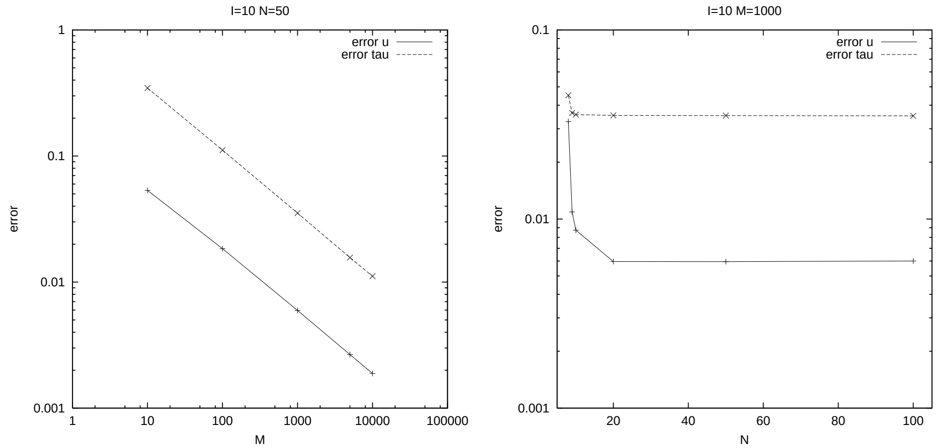

Fig. 4. Errors $L^\infty_t(L^2_x(L^2_\omega))$ on u and $L^\infty_t(L^1_x(L^1_\omega))$ on τ depending on the number of Monte Carlo realizations (M) and on the number of time steps (N).

These errors are in fact relative errors since u_h^n and $\mathbb{E}(P_n Q_{h,n})$ are of order 1. For the simulation with cut-off, the value of A has been chosen "optimally" in order to obtain the best numerical distribution of errors. It is greater than the theoretical upper bound $\sqrt{\frac{3}{5\delta t}}$ that we need in our convergence result (Theorem 3).

We have noticed that the errors are clearly reduced in the simulations with cut-off : for the set of parameters given above, the mean error on the velocity goes from 1.68×10^{-1} without cut-off to 7.56×10^{-2} with cut-off and the mean error on the stress goes from 0.19 to 0.13. Moreover, the empirical probability for the error on the velocity to be smaller than 0.01 goes from 72% without cut-off to more than 88% with cut-off.

In Figure 3, we give a zoom of an histogram representing the empirical distribution of the error on the velocity : $\sup_{0 \le n \le \frac{T}{\delta t}} ||u_h^n - \overline{u}_h^n||_{L^2_x}$. On the left figure, the bar on the far right contains all the simulations for which the error is greater than 4.95. One can clearly see on Figure 3 that the use of the cut-off reduces the empirical probability for the error to be large. This can be related to the fact that without cut-off, $\mathbb{P}(\mathcal{A}_n) < 1$ in the conclusion of the stability Lemma 9.

Tests on the space step, the number of realizations and the time step.

We have also checked that the means (computed without cut-off using 100 000 tests for each simulation) of the errors (5.55) on the velocity and the stress do not depend on the space step (at least when the solution does not oscillate, i.e. when δt is small enough for Lemma 4 to hold), which is in agreement with the result of Lemma 10. As usual in Monte Carlo methods, the error scales like $\frac{1}{\sqrt{M}}$, where M is the number of realizations, which confirms Lemma 10 (see Figure 4). Finally, we show the dependence of the error with repect to δt (see Figure 4). One can observe that there exists a bound on δt below which the error remains constant, which can be related to the result of Lemma 10.

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