Probabilistic approximation and inviscid limits for 1-D fractional conservation laws

Benjamin JOURDAIN[1], Sylvie MÉLÉARD[2], Wojbor A. WOYCZYNSKI[3]

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Abstract

We study the one dimensional scalar conservation law $\partial_t u(t,x) = \nu D^\alpha u(t,x) - \partial_x A(u(t,x))$ with fractional viscosity operator $D^\alpha v(x) = \mathcal{F}^{-1}\Big(|\xi|^\alpha \mathcal{F}(v)(\xi)\Big)(x)$ when $u(0,x)$ is the cumulative distribution function of a signed measure on $I\!R$. We associate a nonlinear martingale problem with the Fokker-Planck equation obtained by spatial differentiation of the conservation law. After checking uniqueness for both the conservation law and the martingale problem, we prove existence thanks to a propagation of chaos result for systems of interacting particles with fixed intensity of jumps related to ν. The empirical cumulative distribution functions of the particles converge to the solution of the conservation law. Finally, when the intensity of jumps vanishes ($\nu \to 0$) as the number of particles tends to $+\infty$, we obtain that the empirical cumulative distribution functions converge to the unique entropy solution of the inviscid ($\nu = 0$) conservation law.

Introduction

Let $\alpha \in (1,2)$ and D^α denote the *symmetric fractional derivative (fractional Laplacian)* of order α on $I\!R$, that is an operator defined either via the Fourier transform

[1]ENPC-CERMICS, 6-8 avenue Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallée Cedex 2, e-mail:jourdain@cermics.enpc.fr

[2]Université Paris 10, MODALX, 200 av. de la République, 92000 Nanterre, e-mail: Sylvie.Meleard@u-paris10.fr

[3]Department of Statistics and Center for Stochastic and Chaotic Processes in Science and Technology, Case Western Reserve University, Cleveland, OH 44106, e-mail: waw@po.cwru.edu

$\mathcal{F}$:

$$D^\alpha v(x) = \mathcal{F}^{-1}\Big(|\xi|^\alpha \mathcal{F}(v)(\xi)\Big)(x), \tag{0.1}$$

or, equivalently, by its singular integral representation:

$$D^\alpha v(x) = K \int_{I\!R} \Big(v(x+y) - v(x) - \mathbf{1}_{\{|y|\le 1\}} v'(x) y\Big) \frac{dy}{|y|^{1+\alpha}} \tag{0.2}$$

$$= K \int_{|y|>1} (v(x+y) - v(x)) \frac{dy}{|y|^{1+\alpha}} + K \int_{|y|\le 1} \int_0^1 v''(x+zy)(1-z)\, dz \frac{dy}{|y|^{\alpha-1}} \tag{0.3}$$

for a suitable positive constant K.

We are interested in the initial-value problem for the following *one-dimensional scalar conservation law with fractional viscosity*:

$$\begin{aligned} \partial_t u(t,x) &= \nu D^\alpha u(t,x) - \partial_x A(u(t,x)), &(0.4)\\ u(0,x) &= u_0(x), &(0.5) \end{aligned}$$

where $(t,x) \in I\!R_+ \times I\!R$, $\nu > 0$, and $A : I\!R \to I\!R$ is a C^1-function. Equations of this type appear in the study of growing interfaces in the presence of selfsimilar hopping surface diffusion [14], and existence and uniqueness results for them, as well as their selfsimilar asymptotics and critical nonlinearities were found via analytic tools in [2].

We will call the above equation *fractional conservation law.* The initial condition u_0 is assumed to be a nonconstant function with bounded variation on $I\!R$. In other words, dx-a.e. on $I\!R$,

$$u_0(x) = c + \int_{-\infty}^x m(dy) = c + H * m(x)$$

with $c \in I\!R$, m being a nonzero, bounded signed measure on $I\!R$, and $H(y)$ denoting the unit step function $\mathbf{1}_{\{y\ge 0\}}$.

More precisely, we consider bounded *weak solutions* u of equation (0.4) such that, for any C^∞-function ψ with compact support on $I\!R_+ \times I\!R$, and for any $t \ge 0$,

$$\begin{aligned} \int_{I\!R} \psi(t,x) u(t,x) dx &= \int_{I\!R} \psi(0,x) u_0(x) dx &(0.6)\\ &\quad + \int_0^t \int_{I\!R} \Big(u(\partial_s \psi + \nu D^\alpha \psi) + A(u) \partial_x \psi\Big)(s,x) dx ds. \end{aligned}$$

Let $\|m\|$ denote the total mass of measure m. Observe that $u(t,x)$ is a bounded weak solution of conservation law (0.4) if and only if function $(u(t,x) - c)/\|m\|$

is a bounded weak solution of the same conservation law with $A(\,.\,)$ replaced by $A(c+\|m\|\,.\,)/\|m\|$ and initial condition $v_0(x) = H * m(x)/\|m\|$. Therefore, without loss of generality, from now onwards we will assume that

$$c = 0, \qquad \text{and} \qquad \|m\| = 1.$$

With this standing assumption, the total variation $|m|$ of measure m is a probability measure on $I\!R$. Denote by $h : I\!R \to \{-1, 1\}$ a density of m with respect to $|m|$.

To give a probabilistic interpretation to the fractional conservation law (0.4), we will use an approach introduced in [3] [4] for the viscous Burgers equation ($\alpha = 2$, $A(u) = u^2/2$) and generalized in [9] to any C^1 function A (but still for $\alpha = 2$) : we deduce from (0.4) that gradient $v(t, x) = \partial_x u(t, x)$ satisfies evolution equation

$$\partial_t v = \nu D^\alpha v - \partial_x\Big(A'(H * v)v\Big), \quad v(0, .) = m. \tag{0.7}$$

If m is a probability measure on $I\!R$, this equation is a nonlinear Fokker-Planck equation. The case of a general signed measure can be dealt with using the following approach developed in [9] to associate a nonlinear martingale problem with (0.7).

Let $\mathcal{P}$ and $(X_t)_{t\geq 0}$ denote, respectively, the space of probability measures and the canonical process on the space $D(I\!R_+, I\!R)$ of càdlàg functions from $I\!R_+$ to $I\!R$ endowed with the Skorokhod topology. We associate with each probability measure $P \in \mathcal{P}$ a signed measure $\tilde{P}$ with density $h(X_0)$ with respect to P and denote, respectively, by $(P_t)_{t\geq 0}$ and $(\tilde{P}_t)_{t\geq 0}$, the flows of time-marginals of measures P and $\tilde{P}$. This way, for any $B \in \mathcal{B}(I\!R)$,

$$\tilde{P}_t(B) = I\!E^P\left(h(X_0)\mathbf{1}_B(X_t)\right).$$

Definition 0.1 We say that $P \in \mathcal{P}$ solves the nonlinear martingale problem (**MP**) if:

1. $P_0 = |m|$;
2. For any $\varphi(t, x)$ in the space $C_b^{1,2}(I\!R_+ \times I\!R)$ of functions which are continuously differentiable with respect to t and twice continuously differentiable with respect to x, and bounded together with their derivatives,

   $$M_t^\varphi \equiv \varphi(t, X_t) - \varphi(0, X_0) - \int_0^t \Big(\partial_s\varphi + \nu D^\alpha\varphi + A'(H * \tilde{P}_s(X_s))\partial_x\varphi\Big)(s, X_s)ds$$

   is a P-martingale.

When P solves this problem, $h(X_0)M_t^\varphi$ is a P-martingale. Writing the constancy of the expectation of this martingale, one obtains that $t \to \tilde{P}_t$ is a weak solution of (0.7).

The paper is organized as follows: in Section 1, we establish existence and uniqueness for martingale problems with linear generators belonging to a class which includes the generators of the particle systems that we study later in the paper.
In Section 2, we first check that the cumulative distribution functions of the signed marginals $\tilde{P}_t$ associated with any solution P of problem **(MP)** provide a bounded weak solution of the fractional conservation law (0.4). Since $t \to \tilde{P}_t$ is a weak solution of equation (0.7) obtained by spatial differentiation of (0.4), this result is not surprising. Then we prove uniqueness of bounded weak solutions of (0.4) and derive uniqueness for problem **(MP)**.
Section 3 is devoted to the convergence of systems of particles with jumps as the number of particles tends to $+\infty$. We first suppose that the intensity of jumps is constant and obtain existence for problem **(MP)** and therefore for (0.4) by proving a propagation of chaos result. As a consequence, the weighted empirical cumulative distribution functions of the particles converge to the solution of the fractional conservation law (0.4). Finally, we assume that the intensity of jumps vanishes ($\nu \to 0$) as the number of particles tends to $+\infty$: we then prove that the empirical cumulative distribution functions converge to the unique entropy solution of the inviscid ($\nu = 0$) conservation law (0.4). This last result can be related to [7] which is devoted to the convergence of the solution of the fractional conservation law to the unique entropy solution of the inviscid conservation law as $\nu \to 0$ in arbitrary space dimension D. More precisely, when the initial condition has bounded variation like the functions u_0 considered in the present paper, for any $T > 0$, the rate of convergence in $C([0,T], L^1_{loc}(I\!R^D))$ is proved to be $\mathcal{O}(\nu^{1/\alpha})$.

In [8], we construct probabilistic approximations for evolution equations involving the fractional Laplacian and a singular nonlinear operator of order similar to that of the term $-\partial_x((H * v)v)$ appearing in (0.7) in the case $A(u) = u^2/2$. The setting is d-dimensional, the Heaviside kernel H is replaced by a kernel $b : I\!R^d \to I\!R^d$ such that, for some $C > 0$ and $0 < \beta < +\infty$, and each $x \in I\!R^d$, $|b(x)| \leq C|x|^{\beta-d}$, and the initial measure m is assumed to be absolutely continuous with respect to the Lebesgue measure with a density belonging to $L^p(I\!R^d)$, where $p > 1$ is related to d, β and α. In [8], the study of the evolution equation of interest is based on the introduction of Lipschitz continuous and bounded cutoff versions of kernel b. In addition, the particles interact through these cutoff kernels. Here the approach is different : since the Heaviside kernel H is discontinuous but not singular at the origin, the cutoff procedure is not needed and are able to deal directly with general signed measures m. In the proof of the vanishing viscosity limit result it is important to consider particles interacting through the original kernel H.

We conclude the introduction by recalling some useful properties of the semigroup generated by the fractional Laplacian. Denote by p_t^α the convolution kernel of the Lévy semigroup $\exp(tD^\alpha)$ on $\mathbb{R}$. The kernel is selfsimilar, that is, for any positive t,

$$p_t^\alpha(x) = t^{-1/\alpha} p_1^\alpha(xt^{-1/\alpha}).$$

Moreover, there exists a constant $C_\alpha > 0$ (see, e.g., [1], Lemma 5.3.) such that

$$0 \le p_1^\alpha(x) \le \frac{C_\alpha}{1+|x|^{1+\alpha}}; \qquad |\partial_x p_1^\alpha(x)| \le \frac{C_\alpha |x|^\alpha}{(1+|x|^{1+\alpha})^2}.$$

If, for $n \ge 1$ and $t \ge 0$, we introduce product kernels

$$G_t^{\alpha,n} : \mathbb{R}^n \ni y = (y_1, \dots, y_n) \mapsto \prod_{i=1}^n p_t^\alpha(y_i),$$

then $G_t^{\alpha,1}(y) = p_t^\alpha(y)$, and the above properties of p_t^α immediately yield the following estimates for $G_t^{\alpha,n}$:

Lemma 0.2 *For any q, $1 \le q \le +\infty$, there is a constant $C > 0$ (depending on n, ν, α, and q) such that, for each $t > 0$, and $i = 1, \dots, n$,*

$$\|G_{\nu t}^{\alpha,n}\|_q \le Ct^{-n(q-1)/(\alpha q)}, \qquad \text{and} \qquad \|\partial_i G_{\nu t}^{\alpha,n}\|_q \le Ct^{-(n(q-1)+q)/(\alpha q)}.$$

Here ∂_i denotes the derivative with respect to the i-th spatial coordinate and $\| \,.\, \|_q$ stands for the usual Lebesgue space L^q.

1 Existence and uniqueness for a class of n-dimensional martingale problems

To construct particle systems whose empirical distributions approximate solutions of the fractional conservation law (0.4) we will initially prove the existence and uniqueness results for a class of martingale problems.

Let $\mathcal{P}_n$ and $(Y_t = (Y_t^1, \dots, Y_t^n))_{t\ge 0}$ denote, respectively, the set of probability measures and the canonical process on $D(\mathbb{R}_+, \mathbb{R}^n)$.

Definition 1.1 Let $b : I\!R_+ \times I\!R^n \mapsto I\!R^n$ be a bounded measurable function and $\eta \in \mathcal{P}(I\!R^n)$. We say that $Q \in \mathcal{P}_n$ solves the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b \cdot \nabla$ starting from η if the initial marginal Q_0 of Q is equal to η and, for any $\varphi \in C_b^{1,2}(I\!R_+ \times I\!R^n)$,

$$M_t^\varphi = \varphi(t, Y_t) - \varphi(0, Y_0) - \int_0^t \Big(\partial_s \varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi + b \cdot \nabla \varphi\Big)(s, Y_s) ds$$

is a Q martingale. Here, ∇ and D_i^α denote, respectively, the gradient with respect to the n spatial coordinates, and the symmetric fractional derivative of order α acting on the i-th spatial coordinate.

Proposition 1.2 *For any bounded measurable function $b = (b_1, \ldots, b_n) : I\!R_+ \times I\!R^n \mapsto I\!R^n$ and any probability measure η on $I\!R^n$, the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b \cdot \nabla$ starting from η admits a unique solution $Q \in \mathcal{P}_n$. Additionally, for any $t > 0$, the marginal Q_t has a density ρ_t with respect to the Lebesgue measure on $I\!R^n$ satisfying*

$$dy \text{ a.e. in } I\!R^n,\ \rho_t(y) = G_{\nu t}^{\alpha,n} * \eta(y) - \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i(s, \,.\,)\rho_s)(y)\, ds. \quad (1.1)$$

Remark: Since we do not assume any regularity of the drift coefficient b in the spatial variable, existence and uniqueness for the martingale problem cannot be proved by checking existence and trajectorial uniqueness for the corresponding stochastic differential equation. Moreover, the Lévy measure

$$K \sum_{i=1}^n \frac{dy_i}{|y_i|^{1+\alpha}}\, \delta_{(0,0,\ldots,0)}(dy_1, \ldots, dy_{i-1}, dy_{i+1}, \ldots dy_n)$$

corresponding to the operator $\sum_{i=1}^n D_i^\alpha$ is concentrated on the coordinate axes. Because of this singular feature, the general existence results given in the literature [12] do not apply to the generator $\nu \sum_{i=1}^n D_i^\alpha \varphi + b \cdot \nabla$.

Proof of Proposition 1.2: *Existence :* To prove existence we regularize the drift by setting, for each $\epsilon \in (0, 1]$,

$$b^\epsilon(t, y) = \int_{I\!R^n} b(t, y - \epsilon z) \frac{e^{-|z|^2/2}}{(2\pi)^{n/2}}\, dz.$$

Function b^ϵ is bounded by a constant independent of ϵ, and Lipschitz continuous with respect to the spatial variables with constant C_ϵ. Let now $Z_0 = (Z_0^1, \ldots, Z_0^n)$

be a random variable with law η and $(S_t = (S_t^1, \ldots, S_t^n))_{t\geq 0}$ an independent process whose coordinates are independent one-dimensional symmetric stable processes with index α, both defined on a probability space $(\Omega, \mathcal{F}, I\!\!P)$. The properties of b^ϵ imply existence and trajectorial uniqueness for the stochastic differential equation

$$Z_t^\epsilon = Z_0 + \nu^{1/\alpha} S_t + \int_0^t b^\epsilon(s, Z_s^\epsilon)\, ds \tag{1.2}$$

Let $Q^\epsilon \in \mathcal{P}_n$ denote the law of the process Z^ϵ. Since b^ϵ is bounded, uniformly in ϵ, the family $\{Q^\epsilon, \epsilon \in (0,1]\}$ is tight. Let now $(\epsilon_k)_{k\in I\!\!N^*}$ be a sequence converging to 0 and such that Q^{ϵ_k} converges weakly to some $Q \in \mathcal{P}_n$. Since, for any k, the initial marginal of Q^{ϵ_k} is equal to η, so is the initial marginal of Q. We are going to check that, for $\varphi \in C_b^{1,2}(I\!\!R_+ \times I\!\!R^n)$, M_t^φ is a Q martingale. To accomplish this we need uniform in ϵ estimates of the densities of the marginals Q_t^ϵ.

For $t > 0$, and ϕ a C^∞-function with compact support on $I\!\!R^n$, function $\varphi(s,y) = G_{\nu(t-s)}^{\alpha,n} * \phi(y)$ solves equation $\partial_s \varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi = 0$ on $[0,t] \times I\!\!R^n$. Consequently, computing $\varphi(s, Z_s^\epsilon)$ by Itô's formula, one checks that

$$\left(\varphi(s, Y_s) - \varphi(0, Y_0) - \int_0^s b^\epsilon \cdot \nabla\varphi(r, Y_r)\, dr\right)_{s\leq t}$$

is a Q^ϵ martingale vanishing at $s = 0$. Since the Q^ϵ expectation of this martingale at time $s = t$ is zero, we obtain

$$\int_{I\!\!R^n} \phi(y)\, Q_t^\epsilon(dy) = \int_{I\!\!R^n} G_{\nu t}^{\alpha,n} * \phi(y)\, \eta(dy) + \int_0^t \int_{I\!\!R^n} b^\epsilon(s,y) \cdot \nabla G_{\nu(t-s)}^{\alpha,n} * \phi(y)\, Q_s^\epsilon(dy)\, ds.$$

Since b is bounded the estimates given in Lemma 0.2 with $q = 1$ justify use of Fubini's Theorem which yields

$$\int_{I\!\!R^n} \phi(y)\, Q_t^\epsilon(dy) = \int_{I\!\!R} \phi(y) G_{\nu t}^{\alpha,n} * \eta(y)\, dy - \int_{I\!\!R} \phi(y) \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i^\epsilon(s,.) Q_s^\epsilon)(y)\, ds\, dy.$$

The sign minus appears because, for $s > 0$, and $1 \leq i \leq n$, the mapping $y \mapsto \partial_i G_s^{\alpha,n}(y)$ is an odd function. Since the equality holds for any test function ϕ we conclude that, for $t > 0$, Q_t^ϵ has a density ρ_t^ϵ with respect to the Lebesgue measure on $I\!\!R^n$ satisfying

$$\rho_t^\epsilon = G_{\nu t}^{\alpha,n} * Q_0 - \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i^\epsilon(s,.)\rho_s^\epsilon)\, ds.$$

By Lemma 0.2, and the uniform in ϵ boundedness of b^ϵ, one obtains that, for $1 \leq q < n/(n+1-\alpha)$, and any $t > 0$,

$$
\begin{aligned}
\|\rho_t^\epsilon\|_q &\leq \|G_{\nu t}^{\alpha,n}\|_q + \sum_{i=1}^n \int_0^t \|\partial_i G_{\nu(t-s)}^{\alpha,n}\|_q \cdot \|b_i^\epsilon\|_\infty \cdot \|\rho_s^\epsilon\|_1 \, ds \\
&\leq C\left(t^{-\frac{n(q-1)}{\alpha q}} + \int_0^t (t-s)^{-\frac{n(q-1)+q}{\alpha q}} ds\right) \\
&\leq C\left(t^{-\frac{n(q-1)}{\alpha q}} + t^{\frac{n-(n+1-\alpha)q}{\alpha q}}\right) \qquad (1.3)
\end{aligned}
$$

with a constant C independent on ϵ. Using the weak convergence of Q^{ϵ_k} to Q which implies the weak convergence of $Q_t^{\epsilon_k}$ to Q_t, for t outside of at most countable set $D_Q = \{r \geq 0,\ Q(|Y_r - Y_{r^-}| > 0) > 0\}$, and the right-continuity of the mapping $t \mapsto Q_t$, one obtains that, for any positive t, Q_t has a density ρ_t which satisfies the estimates given above for ρ_t^ϵ.

Let $\varphi \in C_b^{1,2}(I\!R_+ \times I\!R^n)$. To prove that M_t^φ is a Q-martingale it is sufficient to check that, for $l \in I\!N^*$, $g \in C_b(I\!R^{ln})$, and $0 \leq s_1 \leq s_2 \leq \ldots \leq s_l \leq s \leq t \notin D_Q$, we have $I\!E^Q(F(Y)) = 0$, where

$$
F(Y) = \left(\varphi(t, Y_t) - \varphi(s, Y_s) - \int_s^t (\partial_r \varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi + b \cdot \nabla\varphi)(r, Y_r)\, dr\right) g(Y_{s_1}, \ldots, Y_{s_l}).
$$

For $\epsilon \in (0,1]$, let F^ϵ be defined like F but with b^ϵ replacing b. Since F^ϵ is bounded and Q gives full weight to the continuity points of this mapping, one has

$$
\lim_{k\to+\infty} I\!E^{Q^{\epsilon_k}}(F^\epsilon(Y)) = I\!E^Q(F^\epsilon(Y)).
$$

In addition, $I\!E^{Q^{\epsilon_k}}(F^{\epsilon_k}(Y)) = 0$. Hence

$$
|I\!E^Q(F(Y))| \leq \limsup_{\epsilon\searrow 0} I\!E^Q(|F - F^\epsilon|(Y)) + \limsup_{\epsilon\searrow 0}\limsup_{k\to+\infty} I\!E^{Q^{\epsilon_k}}(|F^{\epsilon_k} - F^\epsilon|(Y)) \qquad (1.4)
$$

For $M > 0$, let $B(0, M)$ denote the open ball in $I\!R^n$ centered at the origin with radius M. Let $1 < q < n/(n+1-\alpha)$, which implies that $n(q-1)/(\alpha q) < 1 - 1/\alpha$. Then

$$
I\!E^{Q^{\epsilon_k}}(|F^{\epsilon_k} - F^\epsilon|(Y)) \leq C I\!E^{Q^{\epsilon_k}}\left(\int_s^t |b^{\epsilon_k}(r, Y_r) - b^\epsilon(r, Y_r)|\, dr\right)
$$

$$
\leq C\left(Q^{\epsilon_k}\left(\sup_{r\in[s,t]} |Y_r| \geq M\right) + \int_s^t \|b^{\epsilon_k}(r,.) - b^\epsilon(r,\,.)\|_{L^{q/(q-1)}(B(0,M))} \cdot \|\rho_r^{\epsilon_k}\|_q \, dr\right).
$$

Since, for any $r \geq 0$, by an easy adaptation of [5] Theorem IV.22 p.71, $b^\epsilon(r,.)$ converges to $b(r,.)$ in $L^{q/(q-1)}(B(0,M))$ as ϵ tends to 0, using (1.3) we obtain that

$$\limsup_{k\to+\infty} \int_s^t \|b^{\epsilon_k}(r,.) - b^\epsilon(r,.)\|_{L^{q/(q-1)}(B(0,M))} \cdot \|\rho_r^{\epsilon_k}\|_q \, dr$$

$$\leq C \int_s^t \|b(r,.) - b^\epsilon(r,.)\|_{L^{q/(q-1)}(B(0,M))} \left(r^{-\frac{n(q-1)}{\alpha q}} + r^{\frac{n-(n+1-\alpha)q}{\alpha q}} \right) dr.$$

Hence, for a fixed M,

$$\limsup_{\epsilon\to 0} \limsup_{k\to+\infty} \int_s^t \|b^{\epsilon_k}(r,.) - b^\epsilon(r,.)\|_{L^{q/(q-1)}(B(0,M))} \cdot \|\rho_r^{\epsilon_k}\|_q \, dr = 0.$$

In addition, for any ϵ, b_ϵ is bounded by $\|b\|_\infty$, so that

$$Q^\epsilon \left(\sup_{r\in[s,t]} |Y_r| \geq M \right) \leq I\!P \left(\sup_{r\in[s,t]} |Z_0 + \nu^{1/\alpha} S_r| \geq M - \|b\|_\infty t \right),$$

which implies that $Q^{\epsilon_k}(\sup_{r\in[s,t]} |Y_r| \geq M)$ is arbitrarily small, uniformly in k, for M sufficiently large. Hence the second term on the right-hand side of (1.4) vanishes. Similar arguments give that the first term is zero as well.

Uniqueness : For $r \geq 0$ and $\chi \in \mathcal{P}(I\!R^n)$, let Q^r and $\bar{Q}^r$ be two solutions of the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b(r + . , .) \cdot \nabla$ starting from χ. We are going to prove that Q^r and $\bar{Q}^r$ have the same time-marginals. Then uniqueness for the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b \cdot \nabla$ starting from η follows from an easy adaptation of [6], Theorem 4.2, p.184, to the case of time-dependent generators. By choosing test functions $\varphi(s,x) = G_{\nu(t-s)}^{\alpha,n} * \phi(x)$ as above, one obtains that, for $t > 0$, measure Q_t^r has a density ρ_t^r satisfying

$$\rho_t^r = G_{\nu t}^{\alpha,n} * \chi - \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i(r+s,\, .)\rho_s^r) \, ds.$$

For the choice $r = 0$ and $\chi = \eta$, we recognize (1.1). Similarly, $\bar{Q}_t^r$ has a density $\bar{\rho}_t^r$ satisfying the same equation. For $t > 0$, let $g(t) = \|\rho_t^r - \bar{\rho}_t^r\|_1$. By the above evolution equation and Lemma 0.2,

$$\begin{aligned} g(t) &= \left\| \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i(r+s,\, .)(\rho_s^r - \bar{\rho}_s^r)) ds \right\|_1 \\ &\leq \sum_{i=1}^n \int_0^t \|\partial_i G_{\nu(t-s)}^{\alpha,n}\|_1 \cdot \|b_i(r+s,.)\|_\infty \, g(s) \, ds \\ &\leq C \int_0^t g(s)(t-s)^{-1/\alpha} \, ds. \end{aligned}$$

Since $\alpha > 1$, we conclude that, for each $t > 0$, $g(t) = 0$ thanks to a version of Gronwall's Lemma which is provided below. ♠

Lemma 1.3 *Let $g : [0,T] \mapsto I\!\!R_+$ be an integrable function on $[0,T]$ such that, for positive constants A_0, C_0, and θ, and each $t \in [0,T]$,*

$$g(t) \le A_0 + C_0 \int_0^t g(s)(t-s)^{\theta-1}\, ds.$$

Then there exists a positive constant C independent of A_0 such that, for each $t \in [0,T]$,

$$g(t) \le CA_0.$$

Proof: Iterating the inequality satisfied by g and using Fubini's theorem one gets that, for each $t \in [0,T]$,

$$g(t) \le A_0 \left(1 + \frac{C_0 T^\theta}{\theta}\right) + C_0^2 \left(\int_0^1 u^{\theta-1}(1-u)^{\theta-1} du\right) \int_0^t g(s)(t-s)^{2\theta-1}\, ds.$$

Iterating inductively the successively obtained inequalities one gets after n steps that, for each $t \in [0,T]$,

$$g(t) \le A_n + C_n \int_0^t g(s)(t-s)^{2^n\theta-1} ds,$$

with $A_n = A_{n-1}\left(1 + C_{n-1}T^{2^{n-1}\theta}/(2^{n-1}\theta)\right)$, and $C_n = C_{n-1}^2 \int_0^1 u^{2^{n-1}\theta-1}(1-u)^{2^{n-1}\theta-1} du$. For sufficiently large n, $2^n\theta \ge 1$, and the standard Gronwall's Lemma can be applied to complete the proof. ♠

We complete this section by proving an estimate for two-point densities which will be useful later on.

Proposition 1.4 *Let $n \ge 2$ and Q be a solution of the martingale problem given by Proposition 1.2. For $1 \le i < j \le n$, and $t > 0$, denote by $\rho_t^{i,j}$ the density of $Q \circ (Y_t^i, Y_t^j)^{-1}$. Then, for all $q, 1 \le q < 2/(3-\alpha)$,*

$$\|\rho_t^{i,j}\|_q \le C\left(t^{-\frac{2(q-1)}{\alpha q}} + t^{\frac{2-(3-\alpha)q}{\alpha q}}\right),$$

where constant C depends only on ν, α, q, and $\|b_i\|_\infty + \|b_j\|_\infty$.

Proof: For simplicity's sake we assume that $i = 1$ and $j = 2$. Integrating (1.1) over $I\!R^{n-2}$ with respect to the $n-2$ last coordinates of y and setting $\eta^{1,2} = Q \circ (Y_0^1, Y_0^2)^{-1}$, we obtain

$$\rho_t^{1,2} = G^{\alpha,2}_{\nu(t-s)} * \eta^{1,2} - \sum_{i=1}^{2} \int_0^t \partial_i G^{\alpha,2}_{\nu(t-s)} * \bar{b}_i(s,.)\, ds,$$

where, for $1 \le i \le 2$ and $s > 0$,

$$\bar{b}_i(s, z_1, z_2) = \int_{I\!R^{n-2}} b_i(s, z_1, \ldots, z_n) \cdot \rho_s(z_1, \ldots, z_n)\, dz_3 \ldots dz_n.$$

Since $\|\bar{b}_i(s,.)\|_1 \le \|b_i\|_\infty$, for each $t > 0$,

$$\|\rho_t^{1,2}\|_q \le \|G^{\alpha,2}_{\nu(t-s)}\|_q + \sum_{i=1}^{2} \int_0^t \|\partial_i G^{\alpha,2}_{\nu(t-s)}\|_q \cdot \|b_i\|_\infty\, ds.$$

The proof can now be concluded by an application of estimates given in Lemma 0.2. ♠

2 Uniqueness for fractional conservation laws and the martingale problem (MP)

We begin this section by clarifying the connection between the martingale problem (**MP**) and the fractional conservation law (0.4). Since the martingale problem has been introduced by considering equation obtained by spatial differentiation of (0.4) as a Fokker-Planck equation, the following result is not surprising:

Lemma 2.1 *If P solves the martingale problem* (**MP**) *then $u(t,x) = H * \tilde{P}_t(x)$ is a bounded weak solution of the fractional conservation law (0.4).*

Proof: First, observe that $H * \tilde{P}_t(x)$ is bounded because

$$|H * \tilde{P}_t(x)| = |I\!E^P(h(X_0)\mathbf{1}_{\{X_t \le x\}})| \le I\!E^P|h(X_0)| = 1.$$

Since $b(t,x) = A'(H * \tilde{P}_t(x))$ is a bounded function, by Proposition 1.2, for any $t > 0$, measures P_t, and therefore $\tilde{P}_t$, are absolutely continuous with respect to the Lebesgue measure. Hence, for $t > 0$, the cumulative distribution function of measure $A'(H * \tilde{P}_t(x))\tilde{P}_t(dx)$ is $A(H * \tilde{P}_t(x)) - A(0)$. Let now $\psi(t,x)$ be a C^∞-function with compact support on $I\!R_+ \times I\!R$ and $\varphi(t,x) = \int_{-\infty}^x \psi(t,y)\,dy$. Process $(h(X_0)M_t^\varphi)_{t\geq 0}$ is a P-martingale so that

$$\int_{I\!R} \varphi(t,x)\tilde{P}_t(dx) = \int_{I\!R} \varphi(0,x)\,m(dx)$$
$$+\int_0^t \int_{I\!R} \Big(\partial_s\varphi + \nu D^\alpha \varphi + A'(H * \tilde{P}_s)\partial_x\varphi\Big)(s,x)\,\tilde{P}_s(dx)\,ds. \tag{2.1}$$

Integration by parts of the spatial integral involving D^α yields

$$\begin{aligned}\int_{I\!R} D^\alpha\varphi(s,x)\tilde{P}_s(dx) &= K\int_{|y|>1}\int_{I\!R}\int_x^{x+y} \psi(s,z)dz\tilde{P}_s(dx)\frac{dy}{|y|^{\alpha+1}} \\ &\quad +K\int_{|y|\leq 1}\int_0^1\int_{I\!R}\partial_x\psi(s,x+zy)\tilde{P}_s(dx)(1-z)dz\frac{dy}{|y|^{\alpha-1}} \\ &= -K\int_{|y|>1}\int_{I\!R}(\psi(s,x+y)-\psi(s,x))H*\tilde{P}_s(x)dx\frac{dy}{|y|^{\alpha+1}} \\ &\quad -K\int_{|y|\leq 1}\int_0^1\int_{I\!R}\partial_{xx}\psi(s,x+zy)H*\tilde{P}_s(x)dx(1-z)dz\frac{dy}{|y|^{\alpha-1}} \\ &= -\int_{I\!R} D^\alpha\psi(s,x)H*\tilde{P}_s(x)dx.\end{aligned}$$

Integrating by parts in the same way the other spatial integrals in (2.1), and using the fact that $H * \tilde{P}_t(+\infty) = \tilde{P}_t(I\!R) = I\!E^P(h(X_0))$ does not depend on t, we see that the weak equation (0.6) holds true for $u(t,x) = H * \tilde{P}_t(x)$. ♠

Proposition 2.2 *The fractional conservation law (0.4) has at most one bounded weak solution and the martingale problem* **(MP)** *has at most one solution.*

Proof: Let u be a weak solution of (0.4) bounded by M_u, and ϕ be a C^∞-function with compact support on $I\!R$. Function $\psi(s,x) = p^\alpha_{\nu(t-s)} * \phi(x)$ solves equation $\partial_s\psi + \nu D^\alpha\psi = 0$ for $(s,x) \in [0,t]\times I\!R$. By spatial truncation one can approximate ψ, its first order time derivative, and its first and second order spatial derivatives in $L^1([0,t]\times I\!R)$, by C^∞-functions ψ^n with compact support and their corresponding derivatives such that $\psi^n(t,.)$ and $\psi^n(0,.)$ converge, respectively, to $\psi(t,\,.\,)$ and $\psi(0,\,.\,)$ in $L^1(I\!R)$. Since

$$\begin{aligned}&|D^\alpha\psi(s,x) - D^\alpha\psi^n(s,x)| \\ \leq\ & K\int_{|y|>1}|\psi(s,x+y)-\psi^n(s,x+y)| + |\psi(s,x)-\psi^n(s,x)|\frac{dy}{|y|^{1+\alpha}} \\ &+K\int_{|y|\leq 1}\int_0^1 |\partial_{xx}\psi(s,x+zy) - \partial_{xx}\psi^n(s,x+zy)|(1-z)\,dz\,\frac{dy}{|y|^{\alpha-1}},\end{aligned}$$

$D^\alpha\psi^n$ also converges to $D^\alpha\psi$ in $L^1([0,t]\times I\!\!R)$. Writing the weak equation (0.6) with test function ψ^n and taking the limit $n\to+\infty$, in view of the boundedness of u one obtains that (0.6) holds true for the test function ψ. Using the partial differential equation satisfied by ψ, and then Fubini's Theorem, one deduces

$$\int_{I\!\!R}\phi(x)u(t,x)dx=\int_{I\!\!R}\phi(x)\,p^\alpha_{\nu t}*u_0(x)dx-\int_{I\!\!R}\phi(x)\int_0^t\partial_x p^\alpha_{\nu(t-s)}*A(u(s,\,.\,))(x)\,ds\,dx.$$

Since ϕ is arbitrary, function u solves the mild equation

$$u(t,\,.\,)=p^\alpha_{\nu t}*u_0-\int_0^t\partial_x p^\alpha_{\nu(t-s)}*A(u(s,\,.\,))ds, \tag{2.2}$$

for all $t\geq 0$.

Let u' be another weak solution of (0.4) bounded by $M_{u'}$. One can estimate $g(t)=\|u(t,\,.\,)-u'(t,\,.\,)\|_\infty$ by substracting from (2.2) the same equation written for u' to obtain

$$g(t)\leq\max_{|x|\leq M_u\vee M_{u'}}|A'(x)|\int_0^t\|\partial_x p^\alpha_{\nu(t-s)}\|_1\,g(s)\,ds.$$

Therefore, by Lemma 0.2, there is a constant C such that, for all $t\geq 0$,

$$g(t)\leq C\int_0^t g(s)(t-s)^{-1/\alpha}ds.$$

Since $\alpha>1$, we have $1/\alpha<1$ and, in view of Lemma 1.3, conclude that, for all $t\geq 0,\ g(t)=0$. Hence $u=u'$.

If P and Q both solve the nonlinear martingale problem, combining Lemma 2.1 and the just proved uniqueness result for the fractional conservation law (0.4), one gets that, for all $t\geq 0$, dx-a.e., $H*\tilde{P}_t(x)=H*\tilde{Q}_t(x)$. This equality holds for $(t,x)\in I\!\!R_+\times I\!\!R$ since, for fixed t, both sides are right-continuous with respect to x. Hence both P and Q solve the martingale problem for the generator $\nu D^\alpha+b(t,x)\partial_x$ with $b(t,x)=A'(H*\tilde{P}_t(x))$, starting from $|m|$. Since b is a bounded measurable function, by Proposition 1.2, we conclude that $P=Q$. ♠

3 Interacting particle systems

In this section we develop a Monte Carlo method for the fractional conservation laws. More precisely, we find a sequence of interacting particle systems such that their weighted cumulative empirical distribution functions converge, as the size of the system grows to infinity, to the solution of the conservation law. Results of this sort are also known as *propagation of chaos* results for the corresponding nonlinear, and in our case nonlocal, evolution equations.

3.1 Propagation of chaos for fixed fractional viscosity

Definition 3.1 For $n \in I\!N^*$, we say that $Q \in \mathcal{P}_n$ solves the martingale problem **(Pn)** if

1. $Q_0 = |m|^{\otimes n}$
2. For any $\varphi \in C_b^{1,2}(I\!R_+ \times I\!R^n)$,

$$\begin{aligned}\varphi(t, Y_t) - \varphi(0, Y_0) \quad &- \quad \int_0^t \left(\partial_s \varphi + \sum_{i=1}^n D_i^\alpha \varphi \right)(s, Y_s) \\ &+ \quad \sum_{i=1}^n A' \left(\frac{1}{n} \sum_{j=1}^n h(Y_0^j) \mathbf{1}_{\{Y_s^j \le Y_s^i\}} \right) \partial_i \varphi(s, Y_s)\, ds\end{aligned}$$

is a Q-martingale.

In the drift coefficient of the nonlinear martingale problem **(MP)** the argument of function A' is the cumulative distribution function of the weighted marginal $\tilde{P}_t$ of the solution. By comparison, in the martingale problem **(Pn)** the argument of function A' (which gives the drift coefficient of each particle) is the weighted cumulative empirical distribution function of the particle system.

Remark 3.2 In case m is a probability measure, for any $y \in I\!R$, $h(y) = 1$. Therefore, the existence and uniqueness for **(Pn)** is ensured by Proposition 1.2 for the time-homogeneous generator $\nu \sum_{i=1}^n D_i^\alpha + b^n.\nabla$, where

$$b^n : I\!R^n \ni y = (y_1, \ldots, y_n) \mapsto \left(A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{y_j \le y_1\}} \right), \ldots, A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{y_j \le y_n\}} \right) \right) \in I\!R^n \quad (3.1)$$

However, in general, the drift coefficient at time $t > 0$ depends on the initial position Y_0 through the signed weights $h(Y_0^j)$, $1 \le j \le n$. Because of this dependence, the martingale problem **(Pn)** is non-standard. If D^α is replaced by the usual Laplacian on $I\!R$, the existence and uniqueness for the analogous non-standard martingale problem is an easy consequence of the Girsanov theorem, as in the standard case. But in our case to obtain the existence result we have to proceed more cautiously.

To deal with signed weights, we remark that the function $I\!R_n \ni (y_1^0, \ldots, y_n^0) \mapsto (h(y_1^0), \ldots, h(y_n^0))$ takes its values in the finite set $\{-1,1\}^n$. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \{-1,1\}^n$, let us introduce mappings

$$b^{n,\gamma} : I\!R^n \ni y = (y_1, \ldots, y_n) \mapsto \left(A'\left(\frac{1}{n}\sum_{j=1}^n \gamma_j \mathbf{1}_{\{y_j \leq y_1\}}\right), \ldots, A'\left(\frac{1}{n}\sum_{j=1}^n \gamma_j \mathbf{1}_{\{y_j \leq y_1\}}\right)\right).$$

For $y^0 = (y_1^0, \ldots, y_n^0) \in I\!R^n$ let $Q^{y^0,\gamma}$ be the solution of the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b^{n,\gamma} \cdot \nabla$ starting from δ_{y^0} and given by Proposition 1.2. By adapting the proof of Theorem 4.6 p.188, [6], we obtain measurability of $y^0 \to Q^{y^0,\gamma}$ for a fixed $\gamma \in \{-1,1\}^n$.

Then

$$Q^n = \sum_{\gamma \in \{-1,1\}^n} \int_{I\!R^n} 1_{\{(h(y_1^0),\ldots,h(y_n^0))=\gamma\}} Q^{y^0,\gamma} |m|^{\otimes n}(dy^0),$$

solves the martingale problem **(Pn)**. Moreover, if σ denotes a permutation of $\{1, \ldots, n\}$, the uniqueness part of Proposition 1.2 ensures that, for $y^0 \in I\!R^n$ and $\gamma \in \{-1,1\}^n$, if $y_\sigma^0 = (y_{\sigma(1)}^0, \ldots, y_{\sigma(n)}^0)$ and $\gamma_\sigma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$, then

$$Q^{y^0,\gamma} \circ (Y_{\sigma(1)}, \ldots, Y_{\sigma(n)})^{-1} = Q^{y_\sigma^0, \gamma_\sigma}.$$

With the above definition of Q^n, we deduce that the particles $Y^1, \ldots, Y^n$ are exchangeable under this probability measure.

Finally, since, for $n \geq 1$ and $1 \leq i \leq n$, function $b_i^{n,\gamma}$ is bounded by the quantity $\max_{x \in [-1,1]} |A'(x)|$, we deduce from Proposition 1.4 that, for $n \geq 2$, $1 \leq i < j \leq n$ and $t > 0$, measure $Q^n \circ (Y_t^i, Y_t^j)^{-1}$ has a density $\rho_t^{n,i,j}$ with respect to the Lebesgue measure on $I\!R^2$ such that, for each $1 \leq q < \frac{2}{3-\alpha}$,

$$\|\rho_t^{n,i,j}\|_q \leq C\left(t^{-2(q-1)/(\alpha q)} + t^{(2-(3-\alpha)q)/(\alpha q)}\right) \tag{3.2}$$

where constant C is independent of n and t.

Let $\pi_n = Q^n \circ (\mu^n)^{-1}$, where for $Y = (Y^1, \ldots, Y^n) \in D(I\!R_+, I\!R^n)$, $\mu^n(Y) = \frac{1}{n}\sum_{i=1}^n \delta_{Y^i} \in \mathcal{P}$ denotes the empirical measure. The following propagation of chaos result implies existence for the nonlinear martingale problem **(MP)** :

Theorem 3.3 *Sequence $(\pi_n)_n$ converges weakly to δ_P, where P denotes the unique solution of the martingale problem* **(MP)**.

Proof: The proof is similar to the one given in [9] Theorem 2.1 where instead of the fractional laplacian D^α there appears the classical Laplacian ∂_{xx}. Hence we only show its main steps.

Since the particles $(Y^1, \ldots, Y^n)$ are exchangeable under Q^n, the tightness of the sequence $(\pi_n)_n$ is equivalent to the tightness of the sequence $Q^n \circ (Y^1)^{-1}$ of the distributions of the first particle. The latter follows from the fact that for each $n \in \mathbb{N}^*$, and $y^0, y \in \mathbb{R}^n$,

$$\left| A'\left(\frac{1}{n}\sum_{i=1}^n h(y_j^0)\mathbf{1}_{\{y_j \le y_1\}}\right)\right| \le \max_{x\in[-1,1]} |A'(x)|.$$

Now, let π_∞ denote the limit of a weakly convergent subsequence, for simplicity's sake also labeled $(\pi_n)_n$, and D_{π_∞} denote an at most countable set

$$\Big\{t \ge 0,\ \pi_\infty\Big(\ \{Q \in \mathcal{P};\ Q(|Y_t - Y_{t^-}| > 0) > 0\}\ \Big) > 0\Big\}.$$

Since, for any $n \in \mathbb{N}^*$, $Q_0^n = |m|^{\otimes n}$, we have that π_∞ a.s. $Q_0 = |m|$. Hence, to prove that π_∞ gives full weight to solutions of the nonlinear martingale problem **(MP)** it is enough to check that, for any $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, $l \in \mathbb{N}^*$, $g \in C_b(\mathbb{R}^l)$, and $0 \le s_1 \le s_2 \le \ldots \le s_l \le s \le t \notin D_{\pi_\infty}$, we have $\mathbb{E}^{\pi_\infty}|F(Q)| = 0$, where F associates with any $Q \in \mathcal{P}$

$$\Big\langle Q, \Big(\varphi(t,X_t) - \varphi(s,X_s) - \int_s^t (\partial_r\varphi + \nu D^\alpha\varphi + A'(H * \tilde{Q}_r(X_r))\partial_x\varphi)(r,X_r)dr\Big) \times g(X_{s_1}, \ldots, X_{s_l})\Big\rangle.$$

According to [9] Lemma 2.2, for any $k \in \mathbb{N}^*$, there exists a Lipschitz continuous function h_k such that $|m|(\{x : h_k(x) \ne h(x)\}) \le 1/k$. Let us also approximate the Heaviside function H by Lipschitz continuous functions $H_k(x) = (1+kx)1_{[-1/k,0]}(x) + 1_{\{x>0\}}$, and define F_k like F but with $A'(H * \tilde{Q}_r(.))$ replaced by $A'(\langle Q, H_k(. - X_r)h_k(X_0)\rangle)$.

If $(Q^j)_{j\ge1}$ converges weakly to Q in $\mathcal{P}$, as $j \to +\infty$, for $r \ge 0$ outside of an at most countable set $D_Q = \{r \ge 0 :\ Q(|X_r - X_{r^-}| > 0) > 0\}$, measure $Q^j \circ (X_0, X_r)^{-1}$ converges weakly to $Q \circ (X_0, X_r)^{-1}$. Therefore, by the continuity of A', mapping $x \mapsto A'(\langle Q^j, H_k(x - X_r)h_k(X_0)\rangle)$ converges uniformly to mapping $x \mapsto A'(< Q, H_k(x - X_r)h_k(X_0) >)$. We thus deduce that F_k is continuous at any Q such that $s_1, \ldots, s_l, s, t \notin D_Q$. Hence, π_∞ gives full weight to the continuity points of F_k. Now the boundedness of this mapping implies that

$$\mathbb{E}^{\pi_\infty}|F_k(Q)| = \lim_{n\to+\infty} \mathbb{E}^{\pi_n}|F_k(Q)| = \lim_{n\to+\infty} \mathbb{E}^{Q^n}|F_k(\mu^n)|.$$

Hence

$$\begin{aligned}\mathbb{E}^{\pi_\infty}|F(Q)| &\leq \limsup_{k\to+\infty} \mathbb{E}^{\pi_\infty}|F-F_k|(Q) + \limsup_{n\to+\infty} \mathbb{E}^{Q^n}|F(\mu_n)| \\ &\quad + \limsup_{k\to+\infty}\limsup_{n\to+\infty} \mathbb{E}^{Q^n}|F-F_k|(\mu^n).\end{aligned}$$

The same arguments as in the proof of Theorem 2.1 [9] imply that the sum of the two first terms of the right-hand side is zero and that the third term vanishes as long as

$$\limsup_{k\to+\infty}\limsup_{n\to+\infty} \mathbb{E}^{Q^n}\left(\int_s^t 1_{\{|Y_r^1-Y_r^2|\leq 1/k\}}dr\right) = 0. \tag{3.3}$$

To prove this equality we use the key estimate (3.2) of the two-particle density which replaces the one obtained via Girsanov's theorem in [9]. Let $1 < q < 2/(3-\alpha)$, which implies that $2(q-1)/(\alpha q) < 1 - 1/\alpha$. By Hölder's inequality and (3.2),

$$\begin{aligned}&\mathbb{E}^{Q^n}\left(\int_s^t \mathbf{1}_{\{|Y_r^1-Y_r^2|\leq 1/k\}}dr\right) \\ \leq\ & Q^n\left(\sup_{r\in[s,t]}|Y_r^1| \geq \sqrt{k}\right) + \int_s^t Q^n\left(|Y_r^1|\leq\sqrt{k}, |Y_r^1-Y_r^2|\leq \frac{1}{k}\right)dr \\ \leq\ & Q^n\left(\sup_{r\in[s,t]}|Y_r^1| \geq \sqrt{k}\right) \\ & + C\left(\int_{\mathbb{R}^2} 1_{\{|y_1|\leq\sqrt{k},\ |y_1-y_2|\leq 1/k\}}\,dy_1\,dy_2\right)^{\frac{q-1}{q}} \int_s^t r^{-\frac{2(q-1)}{\alpha q}} + r^{\frac{2-(3-\alpha)q}{\alpha q}}dr.\end{aligned}$$

The second term of the right-hand side does not depend on n and converges to 0, as $k \to +\infty$, because

$$\int_{\mathbb{R}^2} \mathbf{1}_{\{|y_1|\leq\sqrt{k},\ |y_1-y_2|\leq 1/k\}}\,dy_1\,dy_2 = \frac{4}{\sqrt{k}}.$$

Also, by the tightness of sequence $Q^n \circ (Y_1)^{-1}$, we have

$$\limsup_{k\to+\infty}\limsup_{n\to+\infty} Q^n\left(\sup_{r\in[s,t]}|Y_r^1|\geq\sqrt{k}\right) = 0.$$

Hence (3.3) holds true. ♠

Remark 3.4 Let $(S^i)_{i\geq 1}$ be a sequence of independent one-dimensional symmetric α-stable processes and $(Z_0^i)_{i\geq 1}$ be an independant sequence of initial variables with independent and identical distributions $|m|$. If function A' is locally Lipschitz-continuous then it is possible to define the n-particle system as the unique solution of equations

$$Z_t^{i,n} = Z_0^i + \nu^{1/\alpha} S_t^i + \int_0^t A'\left(\frac{1}{n}\sum_{j=1}^n h(Z_0^j) H_{\epsilon_n}(Z_s^{i,n} - Z_s^{j,n})\right) ds, \qquad 1 \leq i \leq n,$$

where $\epsilon_n > 0$ and, for $\epsilon > 0$, $H_\epsilon(x) = (1 + x/\epsilon)\mathbf{1}_{[-\epsilon,0]}(x) + \mathbf{1}_{\{x>0\}}$ is a Lipschitz-continuous regularization of the Heaviside function. If π_n denotes the law of the empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{Z^{i,n}}$, then the propagation of chaos result stated in Theorem 3.3 holds true as long as $\lim_{n\to+\infty} \epsilon_n = 0$.

The propagation of chaos result implies convergence of the weighted empirical cumulative distribution function of a system with n particles to the unique bounded weak solution of (0.4) as $n \to +\infty$.

Corollary 3.5 *Under Q^n, the approximate solution $\frac{1}{n}\sum_{j=1}^n h(Y_0^j)H(x - Y_t^j)$ converges to the unique bounded weak solution $u(t,x) = H * \tilde{P}_t(x)$ of (0.4) in the following sense : for each $T > 0$,*

$$\lim_{n\to+\infty} \sup_{t\leq T} \int_{I\!R} I\!E^{Q^n} \left| \frac{1}{n}\sum_{j=1}^n h(Y_0^j)H(x - Y_t^j) - u(t,x) \right| \frac{dx}{1+x^2} = 0.$$

Proof: Let

$$N^Y(ds, dy) = \sum_t \mathbf{1}_{\{\Delta Y_t \neq 0\}} \delta_{(t,\Delta Y_t)}(ds, dy),$$

and, respectively,

$$N^i(ds, dx) = \sum_t \mathbf{1}_{\{\Delta Y_t^i \neq 0\}} \delta_{(t,\nu^{-1/\alpha}\Delta Y_t^i)}(ds, dx)$$

denote the jump measure on $I\!R_+ \times I\!R^n$ (resp., $I\!R_+ \times I\!R$) associated with the canonical process $Y = (Y^1, \ldots, Y^n)$ on $D(I\!R_+, I\!R^n)$ (resp., with $\nu^{-1/\alpha} Y^i$). According to [11] Theorem 2.42, p.86, under Q^n, the predictable compensator of N^Y is

$$K\nu \sum_{i=1}^n \frac{dy_i}{|y_i|^{1+\alpha}}\, \delta_{(0,0,\ldots,0)}(dy_1, \ldots, dy_{i-1}, dy_{i+1}, \ldots dy_n)\, ds.$$

As a consequence, measures N^i are independent Poisson random measures on $I\!R_+ \times I\!R$ with common intensity $K\, ds\, dx/|x|^{1+\alpha}$. Therefore, processes

$$S_t^i = \int_{(0,t]\times I\!R} x \mathbf{1}_{\{|x|\leq 1\}} \left(N^i(dsdx) - \frac{K\, ds\, dx}{|x|^{1+\alpha}} \right) + \int_{(0,t]\times I\!R} x \mathbf{1}_{\{|x|>1\}} N^i(dsdx),$$

$1 \leq i \leq n$, are independent symmetric α-stable processes independent of the initial variables Y_0^i, $1 \leq i \leq n$, which are i.i.d. with common distribution $|m|$. Additionally, combining [11] Theorem 2.42, p. 86, and Theorem 2.34, p.84, we obtain that, for $1 \leq i \leq n$, and $t \geq 0$,

$$Y_t^i = Y_0^i + \nu^{1/\alpha} S_t^i + \int_0^t A' \left(\frac{1}{n} \sum_{j=1}^n h(Y_0^j) \mathbf{1}_{\{Y_s^j \leq Y_s^i\}} \right) ds.$$

Similarly, under the solution P of the nonlinear martingale problem **(MP)**,

$$X_t = X_0 + \nu^{1/\alpha} S_t + \int_0^t A'(H * \tilde{P}_s(X_s)) ds$$

where S_t is a symmetric α-stable process independent from X_0 which is distributed according to $|m|$.

The scaling property of the α-stable process and the boundedness of the drift coefficients imply that, for $0 \leq r \leq t$, and $1 \leq i \leq n$,

$$I\!E^{Q^n} \left(\sup_{s\in[r,t]} |Y_s^i - Y_r^i| \right) \leq C \left(\nu^{1/\alpha} (t-r)^{1/\alpha} + (t-r) \right) \tag{3.4}$$

$$I\!E^{P} \left(\sup_{s\in[r,t]} |X_s - X_r| \right) \leq C \left(\nu^{1/\alpha} (t-r)^{1/\alpha} + (t-r) \right). \tag{3.5}$$

We set $k \in I\!N^*$. Let h_k and H_k be the Lipschitz continuous approximations of functions h and H introduced in the proof of Theorem 3.3 and $\lfloor . \rfloor$ denote the integer part. One has

$$\begin{aligned}
&\sup_{t\leq T} \int_{I\!R} I\!E^{Q^n} \left| \frac{1}{n} \sum_{j=1}^n h(Y_0^j) H(x - Y_t^j) - H * \tilde{P}_t(x) \right| \frac{dx}{1+x^2} \\
&\leq \sup_{t\leq T} \int_{I\!R} I\!E^{Q^n} \left| h(Y_0^1) H(x - Y_t^1) - h_k(Y_0^1) H_k(x - Y^1_{T\lfloor k^2 t/T \rfloor / k^2}) \right| \frac{dx}{1+x^2} \\
&+ \max_{0\leq j\leq k^2-1} I\!E^{\pi_n} \left(\sup_{\frac{jT}{k^2} \leq t \leq \frac{(j+1)T}{k^2}} \int_{I\!R} \left| < Q, h_k(X_0) H_k(x - X_{\frac{jT}{k^2}}) > - H * \tilde{P}_t(x) \right| \frac{dx}{1+x^2} \right)
\end{aligned}$$

$$- \max_{0\leq j\leq k^2-1} I\!\!E^{\delta_P}\left(\sup_{\frac{jT}{k^2}\leq t\leq\frac{(j+1)T}{k^2}}\int_{I\!\!R}\left|< Q, h_k(X_0)H_k(x - X_{\frac{jT}{k^2}}) > -H * \tilde{P}_t(x)\right|\frac{dx}{1+x^2}\right)$$

$$+ \sup_{t\leq T}\int_{I\!\!R}\left|< P, h_k(X_0)H_k(x - X_{T\lfloor k^2t/T\rfloor/k^2}) - h(X_0)H(x - X_t) >\right|\frac{dx}{1+x^2}. \quad (3.6)$$

Since $|h(Y_0^1)H(x - Y_t^1) - h_k(Y_0^1)H_k(x - Y^1_{T\lfloor k^2t/T\rfloor/k^2})|$ is smaller than

$$|h - h_k|(Y_0^1) + \mathbf{1}_{\{x-1/k\leq Y_t^1\leq x\}} + k|Y_t^1 - Y^1_{T\lfloor k^2t/T\rfloor/k^2}|,$$

using (3.4) and the bound $\forall y \in I\!\!R$, $\int_{I\!\!R}\mathbf{1}_{\{x-1/k\leq y\leq x\}}\frac{dx}{1+x^2} \leq 1/k$ one obtains that the first term of the right-hand-side of (3.6) is smaller than $2\pi|m|(\{x : h_k(x) \neq h(x)\}) + 1/k + Ck^{1-2/\alpha}$ and vanishes as $k \to +\infty$. In the same way, the fourth term also converges to 0. Now for fixed $k \in I\!\!N^*$ and $1 \leq j \leq k^2$, one deduces from (3.5) that δ_P gives full weight to continuity points of the bounded mapping

$$Q \in \mathcal{P} \to \sup_{\frac{jT}{k^2}\leq t\leq\frac{(j+1)T}{k^2}}\int_{I\!\!R}\left|< Q, h_k(X_0)H_k(x - X_{\frac{jT}{k^2}}) > -H * \tilde{P}_t(x)\right|\frac{dx}{1+x^2}.$$

Hence for fixed k, Theorem 3.3 implies that the second term of the right-hand-side of (3.6) converges to minus the third term as n tends to ∞. ♠

3.2 The vanishing viscosity limit

In this subsection, we assume that m is a probability measure and introduce a sequence ν_n of positive numbers such that $\lim_{n\to+\infty}\nu_n = 0$. We are going to let the fractional viscosity vanish as the number n of particles tends to $+\infty$. We recall that uniqueness of bounded weak solutions fails to hold for the inviscid ($\nu = 0$) scalar conservation law (0.4):

$$\partial_t u(t, x) + \partial_x A(u(t, x)) = 0, \quad u(0, x) = u_0(x). \quad (3.7)$$

However, in view of Kruzhkov's theorem [13] [15], this equation admits a unique bounded entropy solution $u \in C(I\!\!R_+, L^1_{loc}(I\!\!R))$ characterized by the following entropy inequalities : for any $c \in I\!\!R$, and any non-negative C^∞-function ψ with compact support on $I\!\!R_+ \times I\!\!R$,

$$\int_{I\!\!R}|u_0(x) - c|\psi(0, x)dx \quad (3.8)$$

$$+ \int_0^\infty\int_{I\!\!R}\left(|u - c|\partial_t\psi + \mathrm{sgn}(u - c)(A(u) - A(c))\partial_x\psi\right)(t, x)\,dx\,dt \geq 0.$$

For $n \geq 1$, let $Q^n \in \mathcal{P}_n$ be the solution of the martingale problem with generator $\nu_n \sum_{i=1}^n D_i^\alpha + b^n \cdot \nabla$ where b^n is defined in (3.1) starting from $m^{\otimes n}$ and given by Proposition 1.2, and $\pi_n = Q^n \circ (\mu^n)^{-1}$ where for $Y = (Y^1, \ldots, Y^n) \in D(I\!R_+, I\!R^n)$, $\mu^n(Y) = \frac{1}{n}\sum_{i=1}^n \delta_{Y^i} \in \mathcal{P}$. Since $\max_{x\in[0,1]} |A'(x)| < +\infty$, the sequence $(\pi_n)_n$ is tight.

Theorem 3.6 *Any weak limit of the sequence $(\pi_n)_n$ gives full weight to the set*

$$\{Q \in \mathcal{P},\ t \to H * Q_t(.) \text{ is equal to } t \to u(t,.)\}.$$

In addition, for each $T > 0$

$$\lim_{n\to+\infty} \sup_{t\leq T} \int_{I\!R} I\!E^{Q^n} \left| \frac{1}{n} \sum_{j=1}^n H(x - Y_t^j) - u(t,x) \right| \frac{dx}{1+x^2} = 0.$$

Remark 3.7 If D^α is replaced by the Laplacian on $I\!R$ as the generator of the particle system, a similar result is known [10]. In that case the situation when initial m is a signed measure can be handled by modifying the dynamics of the particle system by killing pairs of particles with opposite weights whenever they collide. Such a modification seems difficult to generalize for processes with jumps.

Proof of Theorem 3.6: Let us first remark that the first assertion in the Theorem implies the second one. Indeed from any subsequence of $(Q^n)_n$ one can extract a further subsequence $(Q^{n'})_{n'}$ such that $\pi_{n'}$ converges weakly to π_∞ giving full weight to $\{Q \in \mathcal{P},\ t \to H * Q_t(.)$ is equal to $t \to u(t,.)\}$. Since (3.4) holds with ν replaced by ν_n, first taking r and t outside of $\{s \geq 0 : \pi_\infty(\{Q : Q(|Y_s - Y_{s^-}|) > 0\}) > 0\}$ and then using the right-continuity of sample-paths, one obtains that, for $0 \leq r \leq t$,

$$I\!E^{\pi_\infty}\left(< Q, \sup_{s\in[r,t]} |X_s - X_r| > \right) \leq C(t - r).$$

With this bound replacing (3.5), the arguments given in the proof of Corollary 3.5 imply that $\sup_{t\leq T} \int_{I\!R} I\!E^{Q^{n'}} \left| \frac{1}{n'} \sum_{j=1}^{n'} H(x - Y_t^j) - u(t,x) \right| \frac{dx}{1+x^2}$ converges to 0.

Let now π_∞ be the limit of a converging subsequence of $(\pi_n)_n$, which we still index by n for notational simplicity's sake, ψ be a non-negative C^∞-function with compact support on $I\!R_+ \times I\!R$ and $c \in I\!R$. It is sufficient to prove that π_∞-a.s. the entropy inequality (3.8) holds true for $u(t,x) = H * Q_t(x)$, where Q denotes the canonical variable on $\mathcal{P}$. Indeed, we can then conclude by taking c and ψ in a countable dense subsets.

Let us observe that since for any $Q \in \mathcal{P}$, we have $(t,x) \to H * Q_t(x) \in [0,1]$, and the entropy inequality for $c = 1$ (resp. $c = 0$) implies the entropy inequality for any $c \geq 1$ (resp. $c \leq 0$). For this reason we assume that $c \in [0,1]$.

As in the proof of Corollary 3.5 we obtain that, for $1 \leq i \leq n$, and $t \geq 0$,

$$Y_t^i = Y_0^i + \nu_n^{1/\alpha} S_t^i + \int_0^t A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{Y_s^j \leq Y_s^i\}} \right) ds,$$

where $S^i, 1 \leq i \leq n$, are independent α-stable processes independent of the initial variables Y_0^i, $1 \leq i \leq n$, which are i.i.d. with common distribution m.

For $n \geq 1$, we set $c_n = \lfloor nc \rfloor / n$, where $\lfloor . \rfloor$ denotes the integer part. Our strategy, inspired by the proof of Lemma 2.1, is as follows: we want to integrate by parts in the spatial variable in order to evaluate $\int_{\mathbb{R}} \psi(t,x) |H * \mu_t^n(x) - c_n| dx$, for $t > 0$. The distributional derivative of the step function with bounded variation

$$x \mapsto |H * \mu_t^n(x) - c_n| = \frac{1}{n} \sum_{i=1}^{\lfloor cn \rfloor} \mathbf{1}_{\{x < Y_t^{\sigma_t(i)}\}} + \frac{1}{n} \sum_{i=\lfloor cn \rfloor + 1}^{n} \mathbf{1}_{\{Y_t^{\sigma_t(i)} \leq x\}}$$

is equal to $\frac{1}{n} \left(\sum_{i=\lfloor cn \rfloor+1}^n - \sum_{i=1}^{\lfloor cn \rfloor} \right) \delta_{Y_t^{\sigma_t(i)}}$, where σ_t denotes a permutation of $\{1, \ldots, n\}$ such that $Y_t^{\sigma_t(1)} \leq Y_t^{\sigma_t(2)} \leq \ldots \leq Y_t^{\sigma_t(n)}$. This justifies our interest in computing

$$\sum_{i=\lfloor cn \rfloor+1}^{n} \varphi(t, Y_t^{\sigma_t(i)}) - \sum_{i=1}^{\lfloor cn \rfloor} \varphi(t, Y_t^{\sigma_t(i)}),$$

where

$$\varphi(t,x) = \int_{-\infty}^x \psi(t,z) \, dz, \qquad \text{for} \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Because this calculation is delicate, we are going to approximate Y by a process with finite intensity of jumps by removing the small jumps of S^i, $1 \leq i \leq n$. More precisely, for $\epsilon > 0$, we set $S_t^{i,\epsilon} = \int_{(0,t] \times \mathbb{R}} y \, \mathbf{1}_{\{|x| > \epsilon\}} N^i(ds \, dx)$, and

$$Y_t^{i,\epsilon} = Y_0^i + \nu_n^{1/\alpha} S_t^{i,\epsilon} + \int_0^t A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{Y_s^j \leq Y_s^i\}} \right) ds.$$

Then introducing $Y_t^\epsilon = (Y_t^{1,\epsilon}, \ldots, Y_t^{n,\epsilon})$ and definining σ_t^ϵ (resp., σ_{t-}^ϵ) to be a permutation such that $Y_t^{\sigma_t^\epsilon(1),\epsilon} \leq \ldots \leq Y_t^{\sigma_t^\epsilon(n),\epsilon}$ (resp., the same inequality but with t replaced by t^-) one has, for all $t \geq 0$,

$$\sum_{i=1}^n (Y_t^{\sigma_t^\epsilon(i),\epsilon} - Y_t^{\sigma_t(i)})^2 \leq |Y_t^\epsilon - Y_t|^2 \ , \text{ and } \lim_{\epsilon \to 0} \mathbb{E}^{Q^n} \Big(\sup_{s \leq t} |Y_s^\epsilon - Y_s|^2 \Big) = 0. \quad (3.9)$$

Since according to Proposition 1.2, for any $s > 0$, Q^n_s has a density with respect to the Lebesgue measure, Q^n-a.s. and ds-a.e. the positions $Y^1_s, \dots Y^n_s$ are distinct. Therefore Q^n_s- a.s., for all $t \geq 0$,

$$Y^{i,\epsilon}_t = Y^i_0 + \nu_n^{1/\alpha} S^{i,\epsilon}_t + \int_0^t A'\left(\sigma_s^{-1}(i)/n\right) ds.$$

By considering successive jump times of the process Y^ϵ one obtains

$$\begin{aligned}
&\sum_{i=1}^{\lfloor cn \rfloor} \varphi(t, Y_t^{\sigma_t^\epsilon(i),\epsilon}) \\
= &\sum_{i=1}^{\lfloor cn \rfloor} \Big(\varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t \Big(\partial_s \varphi + A'(\sigma_s^{-1}(\sigma_s^\epsilon(i))/n)\partial_x \varphi\Big)(s, Y_s^{\sigma_s^\epsilon(i),\epsilon})\, ds \Big)
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&+ \int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j) \leq \lfloor cn \rfloor\}} \Big(\mathbf{1}_{\left\{\nu_n^{1/\alpha} y \leq Y_{s-}^{\sigma^\epsilon_{s-}(\lfloor cn \rfloor+1),\epsilon} - Y_{s-}^{j,\epsilon}\right\}} \Big(\varphi(s, Y_{s-}^{j,\epsilon} + \nu_n^{1/\alpha} y) \\
&\qquad\qquad - \varphi(s, Y_{s-}^{j,\epsilon}) \Big) \\
&+ \mathbf{1}_{\left\{\nu_n^{1/\alpha} y > Y_{s-}^{\sigma^\epsilon_{s-}(\lfloor cn \rfloor+1),\epsilon} - Y_{s-}^{j,\epsilon}\right\}} \Big(\varphi(s, Y_{s-}^{\sigma^\epsilon_{s-}([cn]+1),\epsilon}) - \varphi(s, Y_{s-}^{j,\epsilon}) \Big) \Big) \\
&+ \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j) > \lfloor cn \rfloor\}} \Big(\mathbf{1}_{\left\{\nu_n^{1/\alpha} y < Y_{s-}^{\sigma^\epsilon_{s-}(\lfloor cn \rfloor),\epsilon} - Y_{s-}^{j,\epsilon}\right\}} \Big(\varphi(s, Y_{s-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s-}^{\sigma^\epsilon_{s-}(\lfloor cn \rfloor),\epsilon}) \Big) \Big) \\
&\qquad N^j(dsdy)
\end{aligned}$$

Because ψ is non-negative $x \mapsto \varphi(s,x) = \int_{-\infty}^x \psi(s,y)dy$ is non-decreasing and

$$\begin{aligned}
&\sum_{i=1}^{\lfloor cn \rfloor} \Big(\varphi(t, Y_t^{\sigma_t^\epsilon(i),\epsilon}) - \varphi(0, Y_0^{\sigma_0(i)}) - \int_0^t (\partial_s \varphi + A'(\sigma_s^{-1}(\sigma_s^\epsilon(i))/n)\partial_x \varphi)(s, Y_s^{\sigma_s^\epsilon(i),\epsilon}) ds \Big) \\
&\leq \int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j) \leq \lfloor cn \rfloor\}} \Big(\varphi(s, Y_{s-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s-}^{j,\epsilon}) \Big) N^j(ds\, dy)
\end{aligned}$$

By a similar but easier computation,

$$\sum_{i=1}^{n} \Big(-\varphi(t, Y_t^{\sigma_t^\epsilon(i),\epsilon}) + \varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t (\partial_s \varphi + A'(\sigma_s^{-1}(\sigma_s^\epsilon(i))/n)\partial_x \varphi)(s, Y_s^{\sigma_s^\epsilon(i),\epsilon}) ds \Big)$$

$$= -\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\varphi(s, Y_{s^-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s^-}^{j,\epsilon})) N^j(ds\, dy)$$

Adding this equality to the preceding inequality multiplied by 2 one obtains that $T_1^{n,\epsilon} \leq T_2^{n,\epsilon}$, where

$$T_1^{n,\epsilon} = \frac{1}{n} \left(\sum_{i=\lfloor cn \rfloor+1}^{n} - \sum_{i=1}^{\lfloor cn \rfloor} \right) \Big(\varphi(0, Y_0^{\sigma_0(i)})$$

$$+ \int_0^t (\partial_s \varphi + A'(\sigma_s^{-1}(\sigma_s^\epsilon(i))/n) \partial_x \varphi)(s, Y_s^{\sigma_s^\epsilon(i),\epsilon}) ds - \varphi(t, Y_t^{\sigma_t^\epsilon(i),\epsilon}) \Big)$$

and

$$T_2^{n,\epsilon} = \frac{1}{n} \int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\mathbf{1}_{\{(\sigma_{s^-}^\epsilon)^{-1}(j) \leq \lfloor cn \rfloor\}} - \mathbf{1}_{\{(\sigma_{s^-}^\epsilon)^{-1}(j) > \lfloor cn \rfloor\}})$$

$$(\varphi(s, Y_{s^-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s^-}^{j,\epsilon})) N^j(ds\, dy)$$

According to Lemma 3.8 which is stated just after the proof,

$$\lim_{n \to +\infty} \sup_{\epsilon > 0} I\!E^{Q^n} |T_2^{n,\epsilon}| = 0.$$

Hence $\lim_{n\to+\infty} \sup_{\epsilon>0} I\!E^{Q^n}((T_1^{n,\epsilon})^+) = 0$. According to (3.9), one can construct a sequence $(\epsilon_k)_k$ converging to 0 and such that Q^n-a.s., $\sup_{s \leq t} \sum_{i=1}^n (Y_s^{\sigma_s^{\epsilon_k}(i),\epsilon_k} - Y_s^{\sigma_s(i)})^2 \to 0$ as $k \to +\infty$. Moreover, since Q^n-a.s., and ds-a.e. positions $Y_s^1, \ldots, Y_s^n$ are distinct, Q^n-a.s., and ds-a.e. $\sigma_s^{\epsilon_k}$ is equal to σ_s for k big enough. Hence Q^n-a.s., T_1^{n,ϵ_k} converges to

$$T^n = \frac{1}{n} \left(\sum_{i=\lfloor cn \rfloor+1}^{n} - \sum_{i=1}^{\lfloor cn \rfloor} \right) \left(\varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t (\partial_s \varphi + A'(i/n) \partial_x \varphi)(s, Y_s^{\sigma_s(i)}) ds - \varphi(t, Y_t^{\sigma_t(i)}) \right).$$

Since variables $T_1^{n,\epsilon}$ are uniformly bounded in ϵ, we have

$$I\!E^{Q^n}((T^n)^+) = \lim_{k \to +\infty} I\!E^{Q^n}((T_1^{n,\epsilon_k})^+),$$

so that we can conclude that

$$\lim_{n \to +\infty} I\!E^{Q^n}((T^n)^+) = 0. \tag{3.11}$$

We now choose t such that the support of ψ, and therefore of φ, is contained in $[0, t) \times I\!R$ which permits us to get rid of the terms involving $\varphi(t, .)$ and perform spatial integration by parts as planned at the beginning of the proof. We thus obtain

$$\begin{aligned}
T_n &= (1-c_n)\int_{I\!R}\psi(0,x)dx - \int_{I\!R}\psi(0,x)|H*\mu_0^n(x)-c_n|\,dx\\
&\quad +(1-c_n)\int_0^t\int_{I\!R}\partial_s\psi(s,x)\,dx\,ds\\
&\quad -\int_0^t\int_{I\!R}|H*\mu_s^n(x)-c_n|\partial_s\psi(s,x)\\
&\qquad +\left(A(c_n)-A(0)+\frac{1}{n}\sum_{i=1}^{nH*\mu_s^n(x)}(1_{\{i>\lfloor cn\rfloor\}}-1_{\{i\le\lfloor cn\rfloor\}})A'(i/n)\right)\partial_x\psi(s,x)dxds.
\end{aligned}$$

As far as last term is concerned, observe that the cumulative distribution function of the signed measure

$$\frac{1}{n}\left(\sum_{i=\lfloor cn\rfloor+1}^{n}-\sum_{i=1}^{\lfloor cn\rfloor}\right)A'(i/n)\delta_{Y_s^{\sigma_s(i)}}$$

is a function

$$x\mapsto\frac{1}{n}\sum_{i=1}^{nH*\mu_s^n(x)}(1_{\{i>\lfloor cn\rfloor\}}-1_{\{i\le\lfloor cn\rfloor\}})A'(i/n),$$

where $nH*\mu_s^n(x)$ counts the number of particles with coordinates not greater than x at time s, and that no boundary term appears since

$\lim_{x\to+\infty}\partial_x\varphi(s,x)=\lim_{x\to+\infty}\psi(s,x)=0$.

The sum of the first and third terms of the right-hand side is zero. Moreover, for all $0\le k\le n$,

$$\begin{aligned}
&\left|\mathrm{sgn}\left(\frac{k}{n}-c_n\right)\left(A\left(\frac{k}{n}\right)-A(c_n)\right)-A(c_n)+A(0)-\frac{1}{n}\sum_{i=1}^{k}(1_{\{i>[cn]\}}-1_{\{i\le[cn]\}})A'\left(\frac{i}{n}\right)\right|\\
&=\left|\sum_{i=1}^{k}(1_{\{i>\lfloor cn\rfloor\}}-1_{\{i\le\lfloor cn\rfloor\}})\left(A\left(\frac{i}{n}\right)-A\left(\frac{i-1}{n}\right)-\frac{1}{n}A'\left(\frac{i}{n}\right)\right)\right|\\
&\le\sup_{x,y\in[0,1],\,|x-y|\le\frac{1}{n}}|A'(x)-A'(y)|,
\end{aligned}$$

and, for each $u\in[0,1]$,

$$|\mathrm{sgn}(u-c)(A(u)-A(c))-\mathrm{sgn}(u-c_n)(A(u)-A(c_n))|\le\sup_{u\in[c_n,c]}|A(c_n)+A(c)-2A(u)|.$$

Hence the random variables

$$
\begin{aligned}
&T_n + \int_{I\!R} \psi(0,x)|H * \mu_0^n(x) - c|dx \\
&+ \int_0^t \int_{I\!R} |H * \mu_s^n(x) - c|\partial_s \psi(s,x) + \operatorname{sgn}(H * \mu_s^n(x) - c)(A(H * \mu_s^n(x)) \\
&\quad -A(c))\partial_x \psi(s,x)\, dx\, ds
\end{aligned}
$$

converge uniformly to 0 as $n \to +\infty$. With the help of (3.11) we conclude that for a continuous and bounded function G which associates with any $Q \in \mathcal{P}$

$$
\begin{aligned}
G(Q) = \int_{I\!R} \psi(0,x)|H * Q_0(x) - c|dx + \int_0^t \int_{I\!R} \Big(&|H * Q_s(x) - c|\partial_s \psi(s,x) \\
&+ \operatorname{sgn}(H * Q_s(x) - c)(A(H * Q_s(x)) - A(c))\partial_x \psi(s,x) \Big)\, dx\, ds \quad (3.12)
\end{aligned}
$$

we have

$$
I\!E^{\pi_\infty}((G(Q))^-) = \lim_{n\to+\infty} I\!E^{\pi_n}((G(Q))^-) = 0.
$$

We now can conclude the proof by observing that π_∞-a.s, $Q_0 = m$, and therefore $H * Q_0 = u_0$. ♠

Lemma 3.8 *Under the notation introduced in the above proof of Theorem 3.6, we have*

$$
\lim_{n\to+\infty} \sup_{\epsilon>0} I\!E^{Q^n}|T_2^{n,\epsilon}| = 0.
$$

Proof: For $1 \le j \le n$, let us denote by

$$
\tilde{N}^j(ds\, dy) = N^j(ds\, dy) - \frac{K\, ds\, dy}{|y|^{1+\alpha}}
$$

the compensated measure associated with N^j. We shall write $T_2^{n,\epsilon} = R_1 + R_2$, where

$$
\begin{aligned}
R_1 = \frac{1}{n}\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)\le\lfloor cn\rfloor\}} - \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)>\lfloor cn\rfloor\}})(\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha} y) \\
-\varphi(s, Y^{j,\epsilon}_{s^-}))\tilde{N}^j(ds, dy) \\
R_2 = \frac{1}{n}\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)\le\lfloor cn\rfloor\}} - \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)>\lfloor cn\rfloor\}})(\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha} y) \\
-\varphi(s, Y^{j,\epsilon}_{s^-}) - \partial_x\varphi(s, Y^{j,\epsilon}_{s^-})\nu_n^{1/\alpha} y 1_{\{\nu_n^{1/\alpha}|y|\le 1\}})\frac{K ds dy}{|y|^{1+\alpha}}
\end{aligned}
$$

Under this notation one has

$$\begin{aligned}
\mathbb{E}^{Q_n}((R_1)^2) &= \frac{1}{n^2}\sum_{j=1}^{n}\int_0^t\int_{|z|>\nu_n^{1/\alpha}\epsilon}\mathbb{E}^{Q_n}\left((\varphi(s,Y_{s^-}^{j,\epsilon}+z)-\varphi(s,Y_{s^-}^{j,\epsilon}))^2\right)\frac{\nu_n K ds dz}{|z|^{1+\alpha}}\\
&\leq \frac{C\nu_n}{n}\int_{\mathbb{R}}\frac{z^2\wedge 1}{|z|^{1+\alpha}}dz
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^{Q_n}|R_2| &\leq \frac{1}{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}}\mathbb{E}^{Q_n}\left|\varphi(s,Y_{s^-}^{j,\epsilon}+z)-\varphi(s,Y_{s^-}^{j,\epsilon})-\partial_x\varphi(s,Y_{s^-}^{j,\epsilon})z\mathbf{1}_{\{|z|\leq 1\}}\right|\\
&\qquad\times\frac{\nu_n K ds\,dz}{|z|^{1+\alpha}}\\
&\leq C\nu_n\int_{\mathbb{R}}\frac{z^2\wedge 1}{|z|^{1+\alpha}}dz.
\end{aligned}$$

♠

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