Convergence of a stochastic particle approximation of the stress tensor for the FENE-P model

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We are interested in the FENE-P micro-macro model of polymeric fluid. This model couples a nonlinear stochastic differential equation ruling the evolution of the polymers at the microscopic level and a partial differential equation prescribing the evolution of the velocity and pressure at the macroscopic level. In this paper, we suppose that the velocity field is known and we analyse the nonlinear stochastic differential equation. We prove existence and convergence of a stochastic particle approximation and deduce the convergence of the approximate macroscopic stress tensor.

Keywords: FENE-P model, particle approximations, nonlinear stochastic differential equation, propagation of chaos.

1 Introduction

Some models of polymeric fluids are based on the coupling of a stochastic differential equation (SDE) which rules the evolution of a vector representing the polymer in the flow (a microscopic parameter), and a partial differential equation (typically Navier-Stokes equation with an additional term depending on the microscopic parameter) modelling the evolution of macroscopic quantities in the fluid (velocity, pressure) (see [2, 15]). In these so-called micro-macro models, the microscopic quantities influence the macroscopic ones through the stress tensor, and the macroscopic quantities intervene in the evolution of the microscopic unknowns through transport and friction.

The vector $\boldsymbol{X}$ representing the polymer in the flow at the microscopic level gives the orientation and the length of the polymer (see Figure 1) which is modelled by two beads linked by a spring. Three forces act on each bead: a drag force, an entropic force modelled by the spring, and a Brownian force due to the thermal agitation of the molecules of the solvent.

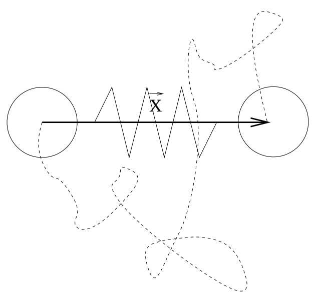

Figure 1: The polymer (in dashed line) is modelled by two beads linked by a spring, and the length and orientation of the polymer is given by the so-called end-to-end vector $\boldsymbol{X}$.

Writing down the Langevin equation on each bead in a given velocity field $\boldsymbol{u}(t,\boldsymbol{x})$, one obtains by difference the following stochastic partial differential equation on the vector $\boldsymbol{X}_t(\boldsymbol{x})$ (see [2, 15]):

$$d\boldsymbol{X}_t(\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}).\nabla \boldsymbol{X}_t(\boldsymbol{x})\,dt = \left(\nabla \boldsymbol{u}(t,\boldsymbol{x})\boldsymbol{X}_t(\boldsymbol{x}) - \frac{1}{2\mathrm{We}}\boldsymbol{F}(\boldsymbol{X}_t(\boldsymbol{x}))\right)dt + \frac{1}{\sqrt{\mathrm{We}}}d\boldsymbol{W}_t \quad (1)$$

where $\boldsymbol{x}$ denotes the space variable, We the Weissenberg number, $\boldsymbol{W}$ a Brownian motion not depending on $\boldsymbol{x}$ and $\boldsymbol{F}$ the entropic force between the two beads. Here and in the following, we write all the equations in a non-dimensional form and we suppose that $\boldsymbol{X} \in \mathbb{R}^N$ with $N = 2$ or $N = 3$. The contribution $\boldsymbol{\tau}_p$ of the polymers to the stress tensor is then:

$$\boldsymbol{\tau}_p(t,\boldsymbol{x}) = \frac{\varepsilon}{\mathrm{We}}\left(\mathbb{E}\left(\boldsymbol{X}_t(\boldsymbol{x}) \otimes \boldsymbol{F}(\boldsymbol{X}_t(\boldsymbol{x}))\right) - \mathrm{Id}\right) \quad (2)$$

where ε denotes the ratio of the viscosity due to the polymer over the total viscosity of the liquid.

In the following, we consider that the velocity field $\boldsymbol{u}$ is regular enough (say $\mathcal{C}^1$) so that one can use the characteristic method (by integrating the vector field $\boldsymbol{u}$) to rewrite equation (1) in the following form, for each characteristic:

$$d\boldsymbol{X}_t = \left(\boldsymbol{G}(t)\boldsymbol{X}_t - \frac{1}{2\mathrm{We}}\boldsymbol{F}(\boldsymbol{X}_t)\right)dt + \frac{1}{\sqrt{\mathrm{We}}}d\boldsymbol{W}_t \quad (3)$$

where $\boldsymbol{G} : \mathbb{R}_+ \to \mathbb{R}^{N\times N}$. In the following, we suppose that

$$\boldsymbol{G} \text{ is a locally bounded fonction of time on } \mathbb{R}_+. \quad (4)$$

Notice that the process $\boldsymbol{X}_t$ in (3) is labelled by the characteristic (see [7]). In the following, we therefore consider one process $\boldsymbol{X}_t$, on a fixed characteristic, for a fixed velocity field (see also Remark 3).

A typical example for the entropic force is a linear force, $\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{X}$: this leads to the the so-called Hookean dumbbell model. For an analysis of the coupling of the SDE in this case with a PDE on the velocity of the fluid $\boldsymbol{u}$, we refer to [9]. In order to take into account the finite extensibility of the polymer, Warner (see [19]) introduced the Finite Extensible Nonlinear Elastic (FENE) model which consists in choosing the following force law in the dumbbell:

$$\boldsymbol{F}(\boldsymbol{X}) = \frac{\boldsymbol{X}}{1 - ||\boldsymbol{X}||^2/b}. \quad (5)$$

This model gives better results than the Hookean model compared to experimental results. In the case of the FENE model, the SDE for the end-to-end vector $\boldsymbol{X}_t$ of a dumbbell is then:

$$d\boldsymbol{X}_t = \left(\boldsymbol{G}(t)\boldsymbol{X}_t - \frac{1}{2\mathrm{We}}\frac{\boldsymbol{X}_t}{1 - ||\boldsymbol{X}_t||^2/b}\right)dt + \frac{1}{\sqrt{\mathrm{We}}}d\boldsymbol{W}_t \quad (6)$$

where $\sqrt{b}$ denotes the maximal extensibility of the polymer. For an analysis of this SDE, we refer to [8].

The contribution $\boldsymbol{\tau}_p$ of the polymers to the stress tensor is then:

$$\boldsymbol{\tau}_p = \frac{\varepsilon}{\mathrm{We}}\left(\mathbb{E}\left(\frac{\boldsymbol{X}_t \otimes \boldsymbol{X}_t}{1 - ||\boldsymbol{X}_t||^2/b}\right) - \mathrm{Id}\right). \quad (7)$$

To date, it is believed (see [2] p. 89) that it is not possible to obtain a closed consitutive equation for the FENE model. In other words, one cannot find a partial differential equation (PDE) on $\boldsymbol{\tau}_p$. This is a problem when one wants to compare these micro-macro models with classical macroscopic models (for example Oldroyd B, PTT or Giesekus models, see [1]) which are usually based on a PDE written on the stress tensor. From a numerical point of view, these macroscopic models are also interesting since the computational cost to simulate them is less than for models based on microscopic equations. Following the ideas of Peterlin (see [17]), Bird et al. (see [3]) then suggested to consider a force law with the square of the length of the polymer in the denominator of (5) replaced by its expectation:

$$\boldsymbol{F}(\boldsymbol{X}_t) = \frac{\boldsymbol{X}_t}{1 - \mathbb{E}(||\boldsymbol{X}_t||^2)/b}, \tag{8}$$

where $\boldsymbol{X}_t$ is therefore solution of the following SDE:

$$d\boldsymbol{X}_t = \left(\boldsymbol{G}(t)\boldsymbol{X}_t - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{X}_t}{1 - \mathbb{E}\left(||\boldsymbol{X}_t||^2\right)/b} \right) dt + \frac{1}{\sqrt{\mathrm{We}}} d\boldsymbol{W}_t. \tag{9}$$

This is the so-called FENE-P model (see [12, 3, 6]). Notice that this SDE is nonlinear in the sense of Mc Kean because of the presence of the expectation of the square of the solution norm in the denominator of the drift coefficient. In Section 2, we prove existence of solution to (9).

With this closure approximation, one can thus show that the stress tensor $\boldsymbol{\tau}_p$ obtained by the following formula:

$$\boldsymbol{\tau}_p = \frac{\varepsilon}{\mathrm{We}} \left(\frac{\mathbb{E}\left(\boldsymbol{X}_t \otimes \boldsymbol{X}_t\right)}{1 - \mathbb{E}\left(||\boldsymbol{X}_t||^2\right)/b} - \mathrm{Id} \right) \tag{10}$$

can be obtained equivalently by solving a nonlinear PDE. Indeed, one can easily show (at least formally) that $\boldsymbol{A}(t) = \mathbb{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t)$ is solution of:

$$\frac{d\boldsymbol{A}(t)}{dt} = \boldsymbol{G}(t)\boldsymbol{A}(t) + \boldsymbol{A}(t)\boldsymbol{G}(t)^T - \frac{1}{\mathrm{We}} \frac{\boldsymbol{A}(t)}{1 - \mathrm{tr}(\boldsymbol{A}(t))/b} + \frac{1}{\mathrm{We}} \mathrm{Id} \tag{11}$$

and then recover $\boldsymbol{\tau}_p = \frac{\varepsilon}{\mathrm{We}} \left(\frac{\boldsymbol{A}}{1 - \mathrm{tr}(\boldsymbol{A})/b} - \mathrm{Id} \right)$.

For the computation of the FENE-P model, some authors (see for example [12]) have compared the PDE approach and the Monte-Carlo approach based on the following system of SDEs (see details about the CONNFFESSIT method in [13, 14, 16] for example): $\forall 1 \leq i \leq M$,

$$d\boldsymbol{X}_t^{i,M} = \left(\boldsymbol{G}(t)\boldsymbol{X}_t^{i,M} - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{X}_t^{i,M}}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2/b} \right) dt + \frac{1}{\sqrt{\mathrm{We}}} d\boldsymbol{W}_t^i \tag{12}$$

where $(\boldsymbol{W}^i)_{1 \leq i \leq M}$ denotes a collection of M independent Brownian motions. The stress tensor is then approximated by the empirical mean:

$$\boldsymbol{\tau}_p^M = \frac{\varepsilon}{\mathrm{We}} \left(\frac{\frac{1}{M}\sum_{i=1}^M \left(\boldsymbol{X}_t^{i,M} \otimes \boldsymbol{X}_t^{i,M} \right)}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2/b} - \mathrm{Id} \right). \tag{13}$$

After studying (12) in Section 3, we prove our main result in Section 4:

Theorem 1 *Assuming (4) and under suitable hypothesis on b and the initial conditions $\boldsymbol{X}_0$ and $\boldsymbol{X}_0^i$ (see (35) and (38)), we have, $\forall t \geq 0$,*

$$\lim_{M \to \infty} \mathbb{E} \left| \boldsymbol{\tau}_p^M(t) - \boldsymbol{\tau}_p(t) \right| = 0, \tag{14}$$

where $\boldsymbol{\tau}_p^M$ is defined by (13) and $\boldsymbol{\tau}_p$ is defined by (10).

Remark 1 *These kinds of mean-field interactions in the framework of fluid mechanics occur not only in the case of closure approximations, but can also be relevant from the physical point of view, for example in the case of liquid crystal polymers (cf. [15] p. 114 and 252-255, or chapter 10 of [5]). We intend to generalize the results obtained here for the FENE-P model also for these models.*

Remark 2 *Since for the FENE-P model, the stress tensor can be computed from the PDE it solves, the FENE-P particle approximation of the stress tensor can be used as a control variate for the computation of the stress tensor in the FENE model. Of course, one then takes the same driving Brownian motions for the FENE and the FENE-P SDEs (see [4] for the use of control variates in CONNFFESSIT simulations).*

Remark 3 (Homogeneous flows) *If $\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{\kappa}(t)\boldsymbol{x}$, together with a pressure $p(t,\boldsymbol{x})$, solve the original Navier-Stokes equations (without the term $\mathrm{div}\,(\boldsymbol{\tau}_p)$), then $\nabla \boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{\kappa}(t)$ does not depend on space. Therefore, (1) reduces to (3) with $\boldsymbol{G}(t) = \boldsymbol{\kappa}(t)$ and the tensor $\boldsymbol{\tau}_p$ does not depend on space. As $\mathrm{div}\,(\boldsymbol{\tau}_p) = 0$, the momentum equation on $\boldsymbol{u}$ reduces to the original Navier-Stokes equations. Therefore, $(\boldsymbol{u}, p, \boldsymbol{X}_t)$ solves the micro-macro system (see formula (1.8) in [9]). In other words, in the homogeneous flow case, there is only a one-way coupling : the velocity field is not influenced by the microscopic variables.*

2 Continuous level: the nonlinear SDE

Definition 1 *A $(\mathcal{F}_t)$-adapted process $(\boldsymbol{X}_t)_{t\geq 0}$ is a global solution to (9) if, $\forall t \geq 0$,*

$$\begin{cases} \sup\limits_{0\leq s\leq t} \mathbb{E}\left(||\boldsymbol{X}_s||^2\right) < b, \ \text{and, } \mathbb{P}\text{-a.s.}, \\ \displaystyle\int_0^t \frac{||\boldsymbol{X}_s||}{1-\mathbb{E}\left(||\boldsymbol{X}_s||^2\right)/b}\, ds < \infty, \\ \boldsymbol{X}_t = \boldsymbol{X}_0 + \displaystyle\int_0^t \left(\boldsymbol{G}(s)\boldsymbol{X}_s - \frac{1}{2\,We}\frac{\boldsymbol{X}_s}{1-\mathbb{E}\left(||\boldsymbol{X}_s||^2\right)/b}\right) ds + \frac{1}{\sqrt{We}}\boldsymbol{W}_t. \end{cases} \tag{15}$$

We shall prove the following result:

Proposition 1 *Assume (4) and $\mathbb{E}(||\boldsymbol{X}_0||^2) < b$ then (9) admits a unique solution in the sense of Definition 1.*

Remark 4 *Going through the proof of this Proposition, one can notice that uniqueness for solutions such that $\mathbb{E}(||\boldsymbol{X}_0||^2) < b$ also holds when replacing the first requirement on $(\boldsymbol{X}_t)$ in (15) by the weaker requirement:*

$$\sup_{0\leq s\leq t} \mathbb{E}\left(||\boldsymbol{X}_s||^2\right) < \infty.$$

To prove Proposition 1, we proceed in two steps. First we prove an existence result on the equation verified by $\mathbb{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t)$. Then we prove existence and uniqueness of solutions to (9).

Let us introduce the following ordinary differential equation, defined for a time-dependent $N \times N$ matrix $\boldsymbol{A}(t)$:

$$\begin{cases} \dfrac{d\boldsymbol{A}(t)}{dt} = \boldsymbol{G}(t)\boldsymbol{A}(t) + \boldsymbol{A}(t)\boldsymbol{G}(t)^T - \dfrac{1}{\mathrm{We}}\dfrac{\boldsymbol{A}(t)}{1-\mathrm{tr}(\boldsymbol{A}(t))/b} + \dfrac{1}{\mathrm{We}}\mathrm{Id}, \\ \boldsymbol{A}(0) = \mathbb{E}(\boldsymbol{X}_0 \otimes \boldsymbol{X}_0). \end{cases} \tag{16}$$

Definition 2 *Let $\boldsymbol{X}_0$ be such that $\mathbb{E}(||\boldsymbol{X}_0||^2) < b$. A continuous $(\mathbb{R}^{N\times N})$-valued function $(\boldsymbol{A}(t))$ is a solution to (16) on $[0,T)$ if, $\forall 0 \le t < T$,*

$$\begin{cases} \sup\limits_{0\le s\le t} \operatorname{tr}(\boldsymbol{A}(s)) < b, \\ \boldsymbol{A}(t) = \mathbb{E}(\boldsymbol{X}_0 \otimes \boldsymbol{X}_0) + \displaystyle\int_0^t \boldsymbol{G}(s)\boldsymbol{A}(s) + \boldsymbol{A}(s)\boldsymbol{G}(s)^T - \frac{1}{We}\frac{\boldsymbol{A}(s)}{1-\operatorname{tr}(\boldsymbol{A}(s))/b}\,ds + \frac{t}{We}Id. \end{cases} \tag{17}$$

Proposition 2 *Assume (4) and $\mathbb{E}(||\boldsymbol{X}_0||^2) < b$. There exists a global-in-time solution $(\boldsymbol{A}(t))_{t\ge 0}$ to (16) such that, $\forall t \ge 0$, $\boldsymbol{A}(t)$ is a symmetric non-negative matrix. Moreover, any other solution on a time interval $[0,T)$ coincides with $\boldsymbol{A}(t)$ on $[0,T)$. In addition, $\boldsymbol{A}(t)$ is the unique solution of the following ordinary differential equation:*

$$\overline{\boldsymbol{A}}(0) = \boldsymbol{A}(0)\,;\ \frac{d\overline{\boldsymbol{A}}(t)}{dt} = \boldsymbol{G}(t)\overline{\boldsymbol{A}}(t) + \overline{\boldsymbol{A}}(t)\boldsymbol{G}(t)^T - \frac{1}{We}\frac{\overline{\boldsymbol{A}}(t)}{1-\operatorname{tr}(\boldsymbol{A}(t))/b} + \frac{1}{We}Id. \tag{18}$$

Proof : Let us consider the ordinary differential equation (16). Using hypothesis (4), it is clear that the application

$$(t,\boldsymbol{A}) \mapsto \boldsymbol{G}(t)\boldsymbol{A} + \boldsymbol{A}\boldsymbol{G}(t)^T - \frac{1}{\mathrm{We}}\frac{\boldsymbol{A}}{1-\operatorname{tr}(\boldsymbol{A})/b} + \frac{1}{\mathrm{We}}\mathrm{Id}$$

is locally Lipschitz w.r.t. $\boldsymbol{A}$ on $\mathbb{R}_+ \times \left(\mathbb{R}^{N\times N} \setminus \{\boldsymbol{M} \in \mathbb{R}^{N\times N},\ \operatorname{tr}(\boldsymbol{M}) = b\}\right)$. Therefore, by the Cauchy theorem, there exists a unique solution to (16) in the sense of Definition 2 on a maximum time interval $[0,T^*)$ such that:

$$\text{if } T^* < \infty, \text{ then } \lim_{t\to T^*} \operatorname{tr}(\boldsymbol{A}(t)) = b. \tag{19}$$

By the uniqueness result of the Cauchy theorem, it is clear that this solution is symmetric ($\boldsymbol{A} = \boldsymbol{A}^T$) since if $\boldsymbol{A}(t)$ is solution of (16), $\boldsymbol{A}(t)^T$ is solution of the same equation, with the same initial condition.

We shall now prove that the solution $\boldsymbol{A}(t)$ to (16) is necessarily non negative.

Let us introduce the process $\boldsymbol{Y}_t$ such that $\boldsymbol{Y}_0 = \boldsymbol{X}_0$ and $\boldsymbol{Y}_t$ is solution to the following SDE:

$$d\boldsymbol{Y}_t = \left(\boldsymbol{G}(t)\boldsymbol{Y}_t - \frac{1}{2\mathrm{We}}\frac{\boldsymbol{Y}_t}{1-\operatorname{tr}(\boldsymbol{A}(t))/b}\right)dt + \frac{1}{\sqrt{\mathrm{We}}}d\boldsymbol{W}_t. \tag{20}$$

It is clear that there exists a solution to (20) on the time interval $[0,T^*)$ and that $t \mapsto \mathbb{E}(||\boldsymbol{Y}_t||^2)$ is locally bounded on $[0,T^*)$ since the application

$$(t,\boldsymbol{Y}) \mapsto \boldsymbol{G}(t)\boldsymbol{Y} - \frac{1}{2\mathrm{We}}\frac{\boldsymbol{Y}}{1-\operatorname{tr}(\boldsymbol{A}(t))/b}$$

is Lipschitz and with linear growth w.r.t. $\boldsymbol{Y}$ locally in time on $[0,T^*)$. Let us now consider $\overline{\boldsymbol{A}}(t) = \mathbb{E}(\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)$. One can easily check by Itô's calculus that $\overline{\boldsymbol{A}}(t)$ is solution of (18) on $[0,T^*)$. As the right-hand side of (18) is affine w.r.t. $\overline{\boldsymbol{A}}$, with coefficients bounded locally in time on $[0,T^*)$, uniqueness of solution to (18) on $[0,T^*)$ holds and thus

$$\boldsymbol{A}(t) = \mathbb{E}(\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)$$

and therefore $\boldsymbol{A}(t)$ is a symmetric and non negative matrix on $[0,T^*)$.

We shall now prove that $T^* = +\infty$. Let us suppose that $T^* < \infty$ and obtain a contradiction. We notice that $\mathrm{Tr}(t) = \operatorname{tr}(\boldsymbol{A}(t))$ is solution of the following ODE:

$$\frac{d\mathrm{Tr}(t)}{dt} = \operatorname{tr}\left((\boldsymbol{G}(t) + \boldsymbol{G}(t)^T)\boldsymbol{A}(t)\right) - \frac{1}{\mathrm{We}}\frac{\mathrm{Tr}(t)}{1-\mathrm{Tr}(t)/b} + \frac{N}{\mathrm{We}}. \tag{21}$$

One can check that there exists $C > 0$ such that, for any symmetric non negative matrix $\boldsymbol{A}$,

$$\operatorname{tr}\left((\boldsymbol{G}(t) + \boldsymbol{G}(t)^T)\boldsymbol{A}\right) \leq C||\boldsymbol{G}(t)||\operatorname{tr}(\boldsymbol{A}).$$

This can be proved using the fact that, for a symmetric non negative matrix $\boldsymbol{A}$, $\sqrt{\operatorname{tr}(\boldsymbol{A}^2)} \leq \operatorname{tr}(\boldsymbol{A})$. Therefore,we have:

$$\frac{d\mathrm{Tr}(t)}{dt} \leq C||\boldsymbol{G}(t)||\mathrm{Tr}(t) - \frac{1}{\mathrm{We}}\frac{\mathrm{Tr}(t)}{1 - \mathrm{Tr}(t)/b} + \frac{N}{\mathrm{We}}. \tag{22}$$

By (19), we have $\forall t \in [0, T^*), \mathrm{Tr}(t) \in [0, b)$ and $\lim_{t\to T^*} \mathrm{Tr}(t) = b$ so that,

$$\lim_{t\to T^*} C||\boldsymbol{G}(t)||\mathrm{Tr}(t) - \frac{1}{\mathrm{We}}\frac{\mathrm{Tr}(t)}{1 - \mathrm{Tr}(t)/b} + \frac{N}{\mathrm{We}} = -\infty,$$

and therefore there exists $\epsilon > 0$ such that $\forall t \in (T^* - \epsilon, T^*)$, $\frac{d\mathrm{Tr}(t)}{dt} < 0$. This is the desired contradiction and shows that $T^* = \infty$.

One can finally easily show uniqueness of solution to (16) on any interval $[0, T)$ by the uniqueness result of the Cauchy theorem.

$\diamondsuit$

The proof of Proposition 1 is now straightforward. One considers the solution $\boldsymbol{A}(t)$ to (16) we have built in Proposition 2. It is then easy to find a solution $\boldsymbol{Y}_t$ to (20) defined on $\mathbb{R}_+$. By uniqueness of solutions to (18), we obtain $\boldsymbol{A}(t) = \mathbb{E}(\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)$. This shows that $\boldsymbol{Y}_t$ is solution of (9) in the sense of Definition 1. Uniqueness can then be deduced from the fact that if $\boldsymbol{X}_t$ is solution to (9) in the sense of Definition 1, then, by Itô's calculus, $\mathbb{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t)$ is solution to (16). Therefore, $\mathbb{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t) = \boldsymbol{A}(t)$ where $\boldsymbol{A}(t)$ is the unique solution to (16). Uniqueness of the solution $\boldsymbol{X}_t$ to (9) therefore follows from uniqueness of solutions to (20).

3 Discrete level: the particle system

We prove existence of solutions to (12) and study some properties of these solutions, which will be useful for the proof of Theorem 1 in Section 4. Henceforth, $(\boldsymbol{W}_t^i)_{i\geq 1}$ denote a collection of independent Brownian motions, and $(\boldsymbol{X}_0^i)_{i\geq 1}$ an independent collection of initial random variables. Notice that the case $M = 1$ coincides with the FENE model, studied in [8, 10].

3.1 Existence of solutions to (12)

We consider the system of SDEs (12) and prove existence and uniqueness of solution.

Definition 3 *We shall say that the $(\mathcal{F}_t)$-adapted process $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ (with value in $\mathbb{R}^{N\times M}$) is solution to (12) if, $\mathbb{P}-a.s., \forall t \geq 0$,*

$$\begin{cases} \displaystyle\int_0^t \left|\frac{\sum_{i=1}^M ||\boldsymbol{X}_s^{i,M}||}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2/b}\right| ds < \infty \qquad \textit{and, } \forall 1 \leq i \leq M, \\ \displaystyle\boldsymbol{X}_t^{i,M} = \boldsymbol{X}_0^i + \int_0^t \left(\boldsymbol{G}(s)\boldsymbol{X}_s^{i,M} - \frac{1}{2\,We}\frac{\boldsymbol{X}_s^{i,M}}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_s^{i,M}||^2/b}\right) ds + \frac{1}{\sqrt{We}}\boldsymbol{W}_t^i. \end{cases}$$

Proposition 3 *Assume (4), $Mb \geq 2$ and that, a.s., $\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_0^i||^2 < b$. There exists a unique solution $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ to (12) in the sense of Definition 3. In addition, this solution is such that*

$$\mathbb{P}\left(\exists t \geq 0, \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2 = b\right) = 0.$$

Moreover, for any $r > 1$, if

$$b > 2(r+1) \quad \text{and} \quad \sup_{M\geq 1}\mathbb{E}\left(\frac{1}{1-\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_0^i||^2/b}\right)^p < \infty, \text{ for some } p > r, \tag{23}$$

then

$$t \mapsto \sup_{M\geq 1}\mathbb{E}\left(\frac{1}{1-\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2/b}\right)^r \text{ is locally bounded.} \tag{24}$$

Remark 5 *If moreover the initial conditions $(\boldsymbol{X}_0^i)_{i\geq 1}$ are i.i.d., then the particles $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ are exchangeable and, $\forall t \geq 0$,*

$$\mathbb{E}(||\boldsymbol{X}_t^{1,M}||^2) = \mathbb{E}\left(\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2\right) \leq b. \tag{25}$$

Proof : Let us introduce the process $(\boldsymbol{X}_t^{i,M,n})_{1\leq i\leq M}$ solution of

$$\boldsymbol{X}_t^{i,M,n} = \boldsymbol{X}_0^i + \int_0^t \left(\boldsymbol{G}(s)\boldsymbol{X}_s^{i,M,n} - \frac{1}{2\mathrm{We}}\frac{\boldsymbol{X}_s^{i,M,n}}{\max\left(1-\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M,n}||^2/b, \frac{1}{n}\right)}\right) ds + \frac{1}{\sqrt{\mathrm{We}}}\boldsymbol{W}_t^i,$$

and the stopping time $\tau_n = \inf\left\{t, \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M,n}||^2 \geq b(1-1/n)\right\}$. By continuation of the solutions $(\boldsymbol{X}_t^{i,M,n})_{1\leq i\leq M}$ considered on $[0,\tau_n)$, one easily obtains a solution $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ to (12) on $[0, \lim_{n\to\infty}\tau_n)$.

Let us now introduce

$$R_t^M = \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2. \tag{26}$$

By Itô's calculus, one obtains that R_t^M is solution of the following SDE on $[0, \lim_{n\to\infty}\tau_n)$:

$$\begin{aligned} R_t^M &= R_0^M + \int_0^t \left(\frac{2}{M}\sum_{i=1}^M \boldsymbol{X}_s^{i,M}.(\boldsymbol{G}(s)\boldsymbol{X}_s^{i,M}) - \frac{1}{\mathrm{We}}\frac{R_s^M}{1-R_s^M/b} + \frac{N}{\mathrm{We}}\right) ds \\ &\quad + \frac{2}{\sqrt{\mathrm{We}\,M}}\int_0^t \sqrt{R_s^M}\,dB_s, \end{aligned} \tag{27}$$

with (B_t) a Brownian motion defined by

$$B_t = \int_0^t \frac{1_{\{\sum_{i=1}^M ||\boldsymbol{X}_s^{i,M}||^2>0\}}}{\sqrt{\sum_{i=1}^M ||\boldsymbol{X}_s^{i,M}||^2}}\sum_{i=1}^M \boldsymbol{X}_s^{i,M} d\boldsymbol{W}_s^i + \int_0^t 1_{\{\sum_{i=1}^M ||\boldsymbol{X}_s^{i,M}||^2=0\}} d\boldsymbol{W}_s^1.$$

We will use the fact that $\forall \boldsymbol{X}, |\boldsymbol{X}.(\boldsymbol{G}(t)\boldsymbol{X})| \leq \frac{C}{2}||\boldsymbol{G}(t)||||\boldsymbol{X}||^2$, with $C > 0$ a constant, and we denote in the following $g(t) = C||\boldsymbol{G}(t)||$.

Let us now introduce ρ_t^M a stochastic process solution of the following SDE:

$$\rho_t^M = R_0^M + \int_0^t \left(g(s)\rho_s^M\,ds - \frac{1}{\mathrm{We}}\frac{\rho_s^M}{1-\rho_s^M/b} + \frac{N}{\mathrm{We}}\right) ds + \frac{2}{\sqrt{\mathrm{We}\,M}}\int_0^t \sqrt{\rho_s^M}\,dB_s. \tag{28}$$

Notice that one can build a process ρ_t^M weak solution to (28) by considering the solution $(\boldsymbol{Y}_t^{i,M})_{1\le i\le M}$ of the following system of SDEs:

$$\boldsymbol{Y}_t^{i,M} = \boldsymbol{X}_0^i + \int_0^t \left(g(s)\boldsymbol{Y}_s^{i,M} - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{Y}_s^{i,M}}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{Y}_t^{i,M}||^2/b} \right) ds + \frac{1}{\sqrt{\mathrm{We}}}\boldsymbol{W}_t^i, \qquad (29)$$

and then considering $\rho_t^M = \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{Y}_t^{i,M}||^2$. By using the fact that $Mb \ge 2$ and that g is locally square integrable and by following exactly the lines of the proof of the existence of a solution for the FENE model (see Section 2 in [10]), it is easy to show that (29) admits a solution $(\boldsymbol{Y}_t^{i,M})_{1\le i\le M}$ defined on $\mathbb{R}_+$, and such that

$$\mathbb{P}(\exists t \ge 0, \rho_t^M = b) = \mathbb{P}(\exists t \ge 0, \rho_t^M = 0) = 0.$$

Moreover, using (23) and following the proof in Section 2 of [10], we have

$$t \mapsto \mathbb{E}\left(\frac{1}{1-\rho_t^M/b}\right)^r \text{ is locally bounded, uniformly in } M.$$

Existence of a strong solution to (28) follows then from Yamada-Watanabe theorem since pathwise uniqueness holds for solutions with paths in $\mathcal{C}(\mathbb{R}^+, (0,b))$.

Let us prove this last point. Let us consider two solutions (ρ_t) and $(\tilde\rho_t)$ to (28) with same initial condition and same Brownian motion, and with paths in $\mathcal{C}(\mathbb{R}^+, (0,b))$. By Corollary 3.4 page 360 in [18], we know that $L^0(\rho-\tilde\rho) = 0$. Therefore, by Tanaka's formula (see Theorem 1.2 p. 207 in [18]):

$$\begin{aligned}(\rho_t - \tilde\rho_t)^+ &= \int_0^t 1_{\rho_s\ge\tilde\rho_s} d(\rho-\tilde\rho)_s \\ &\le \int_0^t g(s)(\rho_s - \tilde\rho_s)^+\, ds + \frac{2}{\sqrt{\mathrm{We}\, M}} \int_0^t 1_{\rho_s\ge\tilde\rho_s}(\sqrt{\rho_s} - \sqrt{\tilde\rho_s})\, dB_s\end{aligned}$$

where we have used the fact that $r \mapsto \dfrac{r}{1-r/b}$ is increasing on $(0,b)$. We know that ρ and $\tilde\rho$ are bounded so that:

$$\mathbb{E}(\rho_t - \tilde\rho_t)^+ \le \int_0^t g(s)\mathbb{E}(\rho_s - \tilde\rho_s)^+\, ds.$$

Using Gronwall Lemma and the fact that g is locally integrable, we therefore obtain that $\rho \le \tilde\rho$. It is clear then that we also have $\tilde\rho \le \rho$ and therefore pathwise uniqueness holds for (29).

Using exactly the same proof as above on $[0, \lim_{n\to\infty}\tau_n)$ with R^M and ρ^M replacing ρ and $\tilde\rho$, it is easy to show that $\forall t \in [0, \lim_{n\to\infty}\tau_n)$,

$$R_t^M \le \rho_t^M. \qquad (30)$$

Since a.s. $\rho^M \in \mathcal{C}(\mathbb{R}_+, (0,b))$, we deduce from (30) that a.s.:

$$\lim_{n\to\infty} \tau_n = \infty.$$

This also shows (24).

To conclude the proof, one needs to show pathwise uniqueness of solutions to (12), but this can be easily done using the same proof as in Lemma 1 of [10]. ♢

3.2 Properties of solutions to (12)

Let us now introduce ρ_t^∞ solution of the following ODE:

$$\rho_t^\infty = \mathbb{E}(||\boldsymbol{X}_0^1||^2) + \int_0^t \left(g(s)\rho_s^\infty \, ds - \frac{1}{\mathrm{We}} \frac{\rho_s^\infty}{1-\rho_s^\infty/b} + \frac{N}{\mathrm{We}} \right) ds. \tag{31}$$

We suppose that $\mathbb{E}(||\boldsymbol{X}_0^1||^2) < b$. Using the fact that $t \mapsto g(t)$ is locally bounded and the fact that, $\forall t \geq 0$, uniformly in $s \in [0,t]$, $\rho \mapsto g(s)\rho - \frac{1}{\mathrm{We}}\frac{\rho}{1-\rho/b} + \frac{N}{\mathrm{We}}$ goes to $-\infty$ when ρ tends to b^-, and to $\frac{N}{\mathrm{We}} > 0$ when ρ tends to 0^+, it is easy to show that (31) admits a unique solution defined on $\mathbb{R}_+$ and with values in $(0,b)$. We will use the following lemma:

Lemma 1 *Let us suppose (4), $b \geq 2$, $\mathbb{E}(||\boldsymbol{X}_0^1||^2) < b$, $\mathbb{E}(||\boldsymbol{X}_0^1||^4) < \infty$ and that the initial conditions $\boldsymbol{X}_0^i$ are i.i.d.. We have, $\forall t > 0$,*

$$\mathbb{E}\left(\sup_{s \leq t} ((R_s^M - \rho_s^\infty)^+)^2 \right) \leq \frac{C(t)}{M}, \tag{32}$$

where $C(t)$ depends on b, $\mathrm{Var}(||\boldsymbol{X}_0^1||^2)$ and $||\boldsymbol{G}||$.

Proof : By Itô's calculus, one obtains:

$$\begin{aligned}
(\rho_t^M - \rho_t^\infty)^2 &= \left(\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_0^i||^2 - \mathbb{E}(||\boldsymbol{X}_0^1||^2) \right)^2 + 2\int_0^t (\rho_s^M - \rho_s^\infty)\, d(\rho^M - \rho^\infty)_s + \frac{4}{\mathrm{We}\, M}\int_0^t \rho_s^M \, ds \\
&\leq \left(\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_0^i||^2 - \mathbb{E}(||\boldsymbol{X}_0^1||^2) \right)^2 + 2\int_0^t g(s)(\rho_s^M - \rho_s^\infty)^2 \, ds + \frac{4}{\mathrm{We}\, M}\int_0^t \rho_s^M \, ds \\
&\quad + \frac{4}{\sqrt{\mathrm{We}\, M}} \int_0^t (\rho_s^M - \rho_s^\infty)\sqrt{\rho_s^M}\, dB_s.
\end{aligned}$$

So that:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq t} (\rho_s^M - \rho_s^\infty)^2 \right) &\leq \frac{\mathrm{Var}(||\boldsymbol{X}_0^1||^2)}{M} + 2\int_0^t g(s)\mathbb{E}\left(\sup_{0\leq r\leq s} (\rho_r^M - \rho_r^\infty)^2 \right) ds + \frac{4b}{\mathrm{We}\, M} t \\
&\quad + \frac{4}{\sqrt{\mathrm{We}\, M}} \mathbb{E}\left(\sup_{0\leq s\leq t} \int_0^s (\rho_r^M - \rho_r^\infty)\sqrt{\rho_r^M}\, dBr \right).
\end{aligned}$$

Using Burkholder-Davis-Gundy inequality, we have:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq t} \int_0^s (\rho_r^M - \rho_r^\infty)\sqrt{\rho_r^M}\, dBr \right) &\leq C\mathbb{E}\left(\sqrt{\int_0^t (\rho_r^M - \rho_r^\infty)^2 \rho_r^M \, dr} \right) \\
&\leq C\sqrt{b}\sqrt{\int_0^t \mathbb{E}\left(\sup_{0\leq r\leq s} (\rho_r^M - \rho_r^\infty)^2 \right) ds} \\
&\leq C\sqrt{b}\left(\frac{1}{\sqrt{M}} + \frac{\sqrt{M}}{4} \int_0^t \mathbb{E}\left(\sup_{0\leq r\leq s} (\rho_r^M - \rho_r^\infty)^2 \right) ds \right)
\end{aligned}$$

so that,

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq s\leq t} (\rho_s^M - \rho_s^\infty)^2 \right) &\leq \int_0^t \left(2g(s) + \frac{C\sqrt{b}}{\sqrt{\mathrm{We}}} \right) \mathbb{E}\left(\sup_{0\leq r\leq s} (\rho_r^M - \rho_r^\infty)^2 \right) ds \\
&\quad + \frac{1}{M}\left(\frac{4b}{\mathrm{We}} t + \frac{4C\sqrt{b}}{\sqrt{\mathrm{We}}} + \mathrm{Var}(||\boldsymbol{X}_0^1||^2) \right).
\end{aligned}$$

Using the fact that g is locally integrable, we then obtain, by Gronwall Lemma,

$$\mathbb{E}\left(\sup_{s\leq t}(\rho_s^M - \rho_s^\infty)^2\right) \leq \frac{C(t)}{M}. \tag{33}$$

It is then easy to prove (32) using (30):

$$(R_s^M - \rho_s^\infty)^+ \leq (R_s^M - \rho_s^M)^+ + (\rho_s^M - \rho_s^\infty)^+ \leq (\rho_s^M - \rho_s^\infty)^+.$$

$\diamondsuit$

Let us now generalize (25) and control the moments of $\boldsymbol{X}_t^{1,M}$:

Lemma 2 *Let us assume (4). Let $p \geq 1$ and suppose that $\mathbb{E}(||\boldsymbol{X}_0^1||^{2p}) < b$. Then $\forall t \geq 0$,*

$$\sup_{M\geq 1}\sup_{s\leq t}\mathbb{E}\left(||\boldsymbol{X}_s^{1,M}||^{2p}\right) \leq C_p(t). \tag{34}$$

where $C_p(t)$ depends on $||\boldsymbol{G}||$ and b.

Proof : By Itô's calculus, we have:

$$\begin{aligned} ||\boldsymbol{X}_t^{1,M}||^{2p} &= ||\boldsymbol{X}_0^{1,M}||^{2p} + \int_0^t 2p||\boldsymbol{X}_s^{1,M}||^{2p-2}\boldsymbol{X}_s^{1,M}.\boldsymbol{G}(s)\boldsymbol{X}_s^{1,M}\,ds \\ &+\frac{1}{\mathrm{We}}\int_0^t p(2p+N-2)||\boldsymbol{X}_s^{1,M}||^{2p-2}\,ds \\ &-\frac{p}{\mathrm{We}}\int_0^t \frac{||\boldsymbol{X}_s^{1,M}||^{2p}}{1-R_s^M/b}\,ds + \frac{1}{\sqrt{\mathrm{We}}}\int_0^t 2p||\boldsymbol{X}_s^{1,M}||^{2p-2}\boldsymbol{X}_s^{1,M}.d\boldsymbol{W}_s \end{aligned}$$

We then obtain formally (using the fact that $\mathbb{E}(||\boldsymbol{X}_s^{1,M}||^{2p-2}) \leq C\left(1+\mathbb{E}(||\boldsymbol{X}_s^{1,M}||^{2p})\right)$):

$$\begin{aligned} \mathbb{E}(||\boldsymbol{X}_t^{1,M}||^{2p}) &\leq \mathbb{E}(||\boldsymbol{X}_0^{1,M}||^{2p}) + \int_0^t 2p||\boldsymbol{G}(s)||\mathbb{E}(||\boldsymbol{X}_s^{1,M}||^{2p})\,ds \\ &+\frac{1}{\mathrm{We}}\int_0^t p(2p+N-2)\mathbb{E}(||\boldsymbol{X}_s^{1,M}||^{2p-2})\,ds \\ &\leq \mathbb{E}(||\boldsymbol{X}_0^{1,M}||^{2p}) + \int_0^t C(s)\mathbb{E}(||\boldsymbol{X}_s^{1,M}||^{2p})\,ds + Ct \end{aligned}$$

where the constants depend on $||\boldsymbol{G}||$, p but not on M. One can then conclude by Gronwall Lemma. One uses a localization argument to complete the proof rigorously. $\diamondsuit$

4 Convergence of $\boldsymbol{\tau}_p^M$ towards $\boldsymbol{\tau}_p$

This section is devoted to the proof of Theorem 1. From now on, we suppose that (4) holds and that:

$$\begin{array}{c} \text{The initial variable } \boldsymbol{X}_0 \text{ is such that} \\ \mathbb{P}\text{-a.s. } ||\boldsymbol{X}_0||^2 < b \text{ and } \mathbb{E}\left(\frac{1}{1-||\boldsymbol{X}_0||^2/b}\right)^p < \infty \text{ for some } p > 2. \\ \text{Moreover, initial conditions } \boldsymbol{X}_0^i \text{ are independent with the same distribution as } \boldsymbol{X}_0. \end{array} \tag{35}$$

We consider the solutions $\boldsymbol{X}_t^{i,M}$ to (12), R_t^M defined by (26) for $M \geq 1$. We will also use ρ_t^∞ defined by (31). Notice that under these assumptions, the results stated in Lemma 1 and in Lemma 2 (for any p) hold. In particular, we have $\forall p \geq 1$, $\forall t \geq 0$,

$$\sup_{M\geq 1}\sup_{s\leq t}\mathbb{E}\left(||\boldsymbol{X}_s^{1,M}||^{2p}\right) \leq C_p(t). \tag{36}$$

Notice that for the exponent p defined in (35), by convexity of $x \in (0,b) \mapsto \left(\frac{1}{1-x/b}\right)^p$ and Jensen inequality, we also have:

$$\begin{aligned}\sup_{M\geq 1}\mathbb{E}\left(\frac{1}{1-\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_0^i||^2/b}\right)^p &\leq \sup_{M\geq 1}\mathbb{E}\left(\frac{1}{M}\sum_{i=1}^M\left(\frac{1}{1-||\boldsymbol{X}_0^i||^2/b}\right)^p\right)\\ &\leq \mathbb{E}\left(\frac{1}{1-||\boldsymbol{X}_0||^2/b}\right)^p < \infty.\end{aligned} \tag{37}$$

Moreover, we suppose in the following that:

$$b > 6. \tag{38}$$

Therefore, by (37) and (38), using (24) in Proposition 3, we know that $\exists r > 2$,

$$t \mapsto \sup_{M\geq 1}\mathbb{E}\left(\frac{1}{1-\frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_t^{i,M}||^2/b}\right)^r \text{ is locally bounded.} \tag{39}$$

Remark 6 *The parameter b is in practice of the order of 100 (see [15] page 217). The hypothesis (38) is therefore not a constraint from the physical point of view.*

Let us consider the solution $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ to (12) and $\boldsymbol{\tau}_p^M$ defined by (13). We want to show that, in the limit $M \to \infty$, $\boldsymbol{\tau}_p^M$ converges towards $\boldsymbol{\tau}_p$ defined by (10) with $\boldsymbol{X}_t$ solution to (9).

We introduce the random variable μ_M with values in $\mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N))$ defined by:

$$\mu_M = \frac{1}{M}\sum_{i=1}^M \delta_{\boldsymbol{X}^{i,M}}. \tag{40}$$

We denote $\Pi_M \in \mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N))\right)$ the law of μ_M.

We will prove the convergence of $\boldsymbol{\tau}_p^M$ towards $\boldsymbol{\tau}_p$ (Section 4.3) by proving first the convergence (in probability) of μ_M towards the law of $\boldsymbol{X}_t$, solution to (9) in the sense of Definition 1 (Section 4.2). To perform the proof, we need to characterize the law of $\boldsymbol{X}_t$ as the solution of a martingale problem (Section 4.1).

In the following, we denote $(\boldsymbol{Y}_t)_{t\geq 0}$ the canonical process on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)$ and Q the canonical variable on $\mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N))$.

4.1 A nonlinear martingale problem

In order to prove the convergence of Π_M, we introduce the following martingale problem.

Definition 4 *We say that $Q \in \mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N))$, with marginals $(Q_t)_{t\geq 0}$ defined in $\mathcal{P}(\mathbb{R}^N)$, is solution of the martingale problem (MP) if:*

$$Q_0 \text{ is the law of } \boldsymbol{X}_0, \tag{41}$$

$$\forall t > 0, \sup_{0 \le s \le t} \left(\int ||\boldsymbol{Y}_s||^2 Q(d\boldsymbol{Y}) \right) < b, \tag{42}$$

$$\left\{ \begin{array}{l} \forall \phi \in \mathcal{C}_0^2(\mathbb{R}^N), \\ M_t^\phi \;=\; \phi(\boldsymbol{Y}_t) - \phi(\boldsymbol{Y}_0) - \int_0^t \left(\boldsymbol{G}(s)\boldsymbol{Y}_s - \frac{1}{2\,We} \frac{\boldsymbol{Y}_s}{1 - \int ||\boldsymbol{Y}_s||^2 Q(d\boldsymbol{Y})/b} \right) . \nabla\phi(\boldsymbol{Y}_s) \\ \qquad + \frac{1}{2\,We} \Delta\phi(\boldsymbol{Y}_s)\, ds \\ \textit{is a } Q\textit{-martingale}, \end{array} \right. \tag{43}$$

where $\mathcal{C}_0^2(\mathbb{R}^N)$ *denotes the set of twice continuously differentiable functions having compact support.*

Remark 7 *Writing the constancy of the expectation of the martingale* M_t^ϕ*, one obtains that if* Q *verifies (43), then* $t \mapsto Q_t$ *is a weak solution of the Fokker-Planck equation associated to (9) :*

$$\partial_t \psi(t, \boldsymbol{\xi}) = -\mathrm{div}_{\boldsymbol{\xi}} \left(\left(\boldsymbol{G}(t)\boldsymbol{\xi} - \frac{1}{2\,We} \frac{\boldsymbol{\xi}}{1 - \int ||\boldsymbol{\xi}||^2 \psi(t, \boldsymbol{\xi})\, d\boldsymbol{\xi}/b} \right) \psi(t, \boldsymbol{\xi}) \right) + \frac{1}{2\,We} \Delta_{\boldsymbol{\xi}} \psi(t, \boldsymbol{\xi}).$$

We have the following proposition:

Proposition 4 *The martingale problem (MP) admits a unique solution, which is the law of the process* $\boldsymbol{X}_t$ *solution to (9) in the sense of Definition 1.*

Proof : The fact that the law of the solution $\boldsymbol{X}_t$ to (9) solves the martingale problem (MP) is an easy consequence of Itô's formula. We refer to Proposition 1 for existence of the process $\boldsymbol{X}_t$.

Let us now consider uniqueness of solutions to (MP). We first notice that if Q is solution to (MP), then, according to Paul Lévy's characterization, the process $(\boldsymbol{B}_t)$ defined by:

$$\boldsymbol{B}_t = \sqrt{\mathrm{We}} \left(\boldsymbol{Y}_t - \boldsymbol{Y}_0 - \int_0^t \left(\boldsymbol{G}(s)\boldsymbol{Y}_s - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{Y}_s}{1 - \int ||\boldsymbol{Y}_s||^2 Q(d\boldsymbol{Y})/b} \right) ds \right)$$

is a Brownian motion under Q. Therefore, $((\boldsymbol{Y}_t)_{t \ge 0}, (\boldsymbol{B}_t)_{t \ge 0}, Q)$ is a weak solution to (9).

Let us now consider two solutions Q^1 and Q^2 to (MP) and set $\boldsymbol{A}^1(t) = \int \boldsymbol{Y}_t \otimes \boldsymbol{Y}_t\, Q^1(d\boldsymbol{Y})$ and $\boldsymbol{A}^2(t) = \int \boldsymbol{Y}_t \otimes \boldsymbol{Y}_t\, Q^2(d\boldsymbol{Y})$. We know from (42) that these two quantities are well defined. Moreover, it is easy to check that $\boldsymbol{A}^1$ and $\boldsymbol{A}^2$ are solution to the ODE (16). By Proposition 2, we therefore obtain, $\forall t \ge 0$,

$$\boldsymbol{A}^1(t) = \boldsymbol{A}^2(t).$$

From this we deduce that

$$\int ||\boldsymbol{Y}_t||^2 Q^1(d\boldsymbol{Y}) = \int ||\boldsymbol{Y}_t||^2 Q^2(d\boldsymbol{Y}),$$

so that, by (41-42), Q^1 and Q^2 are solutions of a martingale problem with (locally in time) Lipschitz coefficients and therefore are weak solutions of a SDE with (locally in time) Lipschitz coefficients. By Yamada-Watanabe theorem, uniqueness in law holds for such SDEs and thus we have:

$$Q^1 = Q^2.$$

$\diamondsuit$

4.2 Weak convergence of the empirical distribution

Theorem 2 *The sequence* $(\mu_M)_{M\geq 1}$ *converges in the probability sense on* $\mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))\right)$ *towards the constant* $P \in \mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))$ *where* P *is the unique solution of the martingale problem (MP).*

We prove this theorem in three steps:

Step 1: First we prove the tightness of the sequence $(\Pi_M)_{M\geq 1}$ on $\mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))\right)$ so that (up to the extraction of a subsequence) $\lim_{M\to\infty}\Pi_M = \Pi_\infty$ (in the weak sense) (see Section 4.2.1 and Lemma 3),

Step 2: Then we prove that $\Pi_\infty(dQ)$-a.s., Q verifies the properties (41) and (42) of the martingale problem (MP) (see Section 4.2.2 and Lemma 4).

Step 3: Finally, we show that $\Pi_\infty(dQ)$-a.s., Q is solution of the martingale problem (MP) by showing that $\Pi_\infty(dQ)$-a.s., Q verifies (43) (see Section 4.2.3 and Lemma 5).

After these three steps, we have that $\Pi_\infty(dQ)$-a.s., Q is solution of the martingale problem (MP). From this and the fact that the martingale problem (MP) admits a unique solution P (see Proposition 4), we deduce that $\Pi_\infty = \delta_P$ and therefore that the convergence $\lim_{M\to\infty}\mu_M = P$ holds in probability. This concludes the proof of Theorem 2.

The next three sections give the proof of each steps.

4.2.1 Step 1

Lemma 3 *The sequence* $(\Pi_M)_{M\geq 1}$ *on* $\mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))\right)$ *is tight. Therefore, there exists a subsequence of* $(\Pi_M)_{M\geq 1}$*, that we still denote* $(\Pi_M)_{M\geq 1}$ *for the sake of clarity, which converges in the weak sense towards* $\Pi_\infty \in \mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))\right)$.

Proof :

Let us first notice that, since the initial conditions $\boldsymbol{X}_0^i$ are i.i.d., the random variables $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$ on $\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N)$ are exchangeable so that to prove the tightness of the sequence $(\Pi_M)_{M\geq 1}$ on $\mathcal{P}\left(\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))\right)$, it suffices to prove the tightness of the sequence of laws of $(\boldsymbol{X}_t^{1,M})_{M\geq 1}$ in $\mathcal{P}(\mathcal{C}(\mathbb{R}_+,\mathbb{R}^N))$. This can be proved using Kolmogorov criterion (see Problem 4.11 p. 64 in [11]). Let us set $0<\beta<r/2-1$, where r is defined in (39), and $0\leq u<v\leq t$. We have:

$$
\begin{aligned}
&\mathbb{E}\left(||\boldsymbol{X}_v^{1,M}-\boldsymbol{X}_u^{1,M}||^{2(1+\beta)}\right)\\
&\leq 3^{(1+2\beta)}\mathbb{E}\left(\int_u^v ||\boldsymbol{G}(s)||\,||\boldsymbol{X}_s^{1,M}||\,ds\right)^{2(1+\beta)}\\
&\quad+\frac{3^{(1+2\beta)}}{(2\mathrm{We})^{2(1+\beta)}}\mathbb{E}\left(\int_u^v \frac{||\boldsymbol{X}_s^{1,M}||}{1-R_s^M/b}\,ds\right)^{2(1+\beta)}+\frac{3^{(1+2\beta)}}{\mathrm{We}^{(1+\beta)}}\mathbb{E}||\boldsymbol{W}_v-\boldsymbol{W}_u||^{2(1+\beta)},\\
&\leq C\left(\int_u^v ||\boldsymbol{G}(s)||^{2(1+\beta)}\,\mathbb{E}\left(||\boldsymbol{X}_s^{1,M}||^{2(1+\beta)}\right)ds(v-u)^{1+2\beta}\right.\\
&\quad\left.+\int_u^v \mathbb{E}\left(\frac{||\boldsymbol{X}_s^{1,M}||}{1-R_s^M/b}\right)^{2(1+\beta)}ds\,(v-u)^{1+2\beta}+(v-u)^{(1+\beta)}\right),\\
&\leq C\left(||\boldsymbol{G}||^{2(1+\beta)}_{L^\infty(0,t)}C_{(1+\beta)}(t)(v-u)^{2(1+\beta)}\right.\\
&\quad\left.+\sup_{s\leq t}\left(\mathbb{E}\left(\frac{1}{1-R_s^M/b}\right)^r\right)^{1/p}\sup_{s\leq t}\left(\mathbb{E}||\boldsymbol{X}_s^{1,M}||^{2q(1+\beta)}\right)^{1/q}(v-u)^{2(1+\beta)}+(v-u)^{(1+\beta)}\right),\\
&\leq C(t)(v-u)^{(1+\beta)},
\end{aligned}
$$

where $p = \frac{r}{2(1+\beta)}$, $q = \frac{p}{1-p}$, $C(t)$ is a constant depending on b, r, $\boldsymbol{G}$ but not on M, and where we have used (36) and (39). $\diamondsuit$

We now want to identify Π_∞ as δ_P, where P is the unique solution of the martingale problem (MP).

4.2.2 Step 2

As the initial variables are i.i.d. according to P_0, one easily obtains (41).

Let us now consider the second point (42) of (MP), namely the estimation of $\int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})$ under Π_∞. We prove the following lemma, which implies (42):

Lemma 4 *$\Pi_\infty(dQ)$-a.s., $\forall t \geq 0$, $\int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y}) \leq \rho_t^\infty$.*

Proof :

$$
\begin{aligned}
&\mathbb{E}^{\Pi_\infty}\left(\left(\int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y}) - \rho_t^\infty\right)^+\right) \\
&= \lim_{n\to\infty} \mathbb{E}^{\Pi_\infty}\left(\left(\int (||\boldsymbol{Y}_t||^2 \wedge n) Q(d\boldsymbol{Y}) - \rho_t^\infty\right)^+\right) && (44) \\
&= \lim_{n\to\infty}\lim_{M\to\infty} \mathbb{E}^{\Pi_M}\left(\left(\int (||\boldsymbol{Y}_t||^2 \wedge n) Q(d\boldsymbol{Y}) - \rho_t^\infty\right)^+\right) && (45) \\
&\leq \lim_{M\to\infty} \mathbb{E}\left(\left(R_t^M - \rho_t^\infty\right)^+\right) \leq \lim_{M\to\infty}\sqrt{\frac{C(t)}{M}} = 0. && (46)
\end{aligned}
$$

We have used the monotone convergence theorem for (44), the fact that $Q \mapsto \left(\int (||\boldsymbol{Y}_t||^2 \wedge n) Q(d\boldsymbol{Y}) - \rho_t^\infty\right)^+$ is continuous and bounded for (45), and estimation (32) for (46). This shows that $\forall t \geq 0$, $\Pi_\infty(dQ)$-a.s.,

$$\int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y}) \leq \rho_t^\infty.$$

The lower semi-continuity of $t \mapsto \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})$ (which follows from Fatou Lemma) and the continuity of ρ^∞ enable to conclude the proof. $\diamondsuit$

4.2.3 Step 3

We now want to show the last point (43) of (MP), namely that:

Lemma 5 *$\Pi_\infty(Q)$-a.s., $\forall \phi \in \mathcal{C}_0^2(\mathbb{R}^N)$, M_t^ϕ is a Q-martingale.*

Proof : Let $p \in \mathbb{N}$, $p \geq 1$ and $0 \leq s_1 \leq \dots \leq s_p \leq s \leq t$. Let us introduce $g \in \mathcal{C}_b(\mathbb{R}^{p\times N}, \mathbb{R})$ and $F_n : \mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)) \to \mathbb{R}$ defined by:

$$
\begin{aligned}
F_n(Q) = &\int \Bigg(\phi(\boldsymbol{Y}_t) - \phi(\boldsymbol{Y}_s) - \int_s^t \Bigg(\boldsymbol{G}(r)\boldsymbol{Y}_r - \frac{1}{2\mathrm{We}}\frac{\boldsymbol{Y}_r}{\max\left(1 - \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y})/b, \frac{1}{n}\right)}\Bigg) \\
&.\nabla\phi(\boldsymbol{Y}_r) + \frac{1}{2\mathrm{We}}\Delta\phi(\boldsymbol{Y}_r)\, dr\Bigg) g(\boldsymbol{Y}_{s_1}, \dots, \boldsymbol{Y}_{s_p}) Q(d\boldsymbol{Y}). \qquad (47)
\end{aligned}
$$

Notice that $F_n : \mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)) \to \mathbb{R}$ is continuous and bounded. We also define $F_\infty : \mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)) \to \mathbb{R}$ by:

$$\begin{aligned} F_\infty(Q) &= \int \Bigg(\phi(\boldsymbol{Y}_t) - \phi(\boldsymbol{Y}_s) - \int_s^t \Bigg(\boldsymbol{G}(r)\boldsymbol{Y}_r - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{Y}_r}{1 - \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y})/b} \Bigg) . \nabla\phi(\boldsymbol{Y}_r) \\ &\quad + \frac{1}{2\mathrm{We}} \Delta\phi(\boldsymbol{Y}_r)\, dr \Bigg) g(\boldsymbol{Y}_{s_1}, ..., \boldsymbol{Y}_{s_p}) Q(d\boldsymbol{Y}). \end{aligned} \tag{48}$$

We want to show that:

$$\mathbb{E}^{\Pi_\infty} |F_\infty(Q)| = 0. \tag{49}$$

Notice that when (49) will be proved, the proof of Lemma 5 will be complete, since we can intervert "$\forall \phi \in \mathcal{C}_0^2(\mathbb{R}^N)$" and "$\Pi_\infty(dQ) - a.s.$" using a countable dense subset of $\mathcal{C}_0^2(\mathbb{R}^N)$. We can also intervert "$\forall p \in \mathbb{N}, p \geq 1$, $\forall 0 \leq s_1 \leq ... \leq s_p \leq s \leq t$" and "$\Pi_\infty(dQ) - a.s.$" by using a countable dense subset of $\mathbb{R}_+$.

Let us now consider (49). We have:

$$\begin{aligned} \mathbb{E}^{\Pi_\infty} |F_\infty(Q)| &\leq \mathbb{E}^{\Pi_\infty} |F_\infty(Q) - F_n(Q)| + \mathbb{E}^{\Pi_\infty} |F_n(Q)|, \\ &\leq \mathbb{E}^{\Pi_\infty} |F_\infty(Q) - F_n(Q)| + \lim_{M\to\infty} \mathbb{E}^{\Pi_M} |F_n(Q)|, \\ &\leq \mathbb{E}^{\Pi_\infty} |F_\infty(Q) - F_n(Q)| + \limsup_{M\to\infty} \mathbb{E}^{\Pi_M} |F_n(Q) - F_\infty(Q)| + \limsup_{M\to\infty} \mathbb{E}^{\Pi_M} |F_\infty(Q)|. \end{aligned}$$

The last term is null. Indeed,

$$\begin{aligned} \mathbb{E}^{\Pi_M} |F_\infty(Q)| &= \frac{1}{M} \mathbb{E} \Bigg| \sum_{i=1}^M \Bigg(\phi(\boldsymbol{X}_t^{i,M}) - \phi(\boldsymbol{X}_s^{i,M}) - \int_s^t \Bigg(\boldsymbol{G}(r)\boldsymbol{X}_r^{i,M} - \frac{1}{2\mathrm{We}} \frac{\boldsymbol{X}_r^{i,M}}{1 - R_r^M/b} \Bigg) . \nabla\phi(\boldsymbol{X}_r^{i,M}) \\ &\quad + \frac{1}{2\mathrm{We}} \Delta\phi(\boldsymbol{X}_r^{i,M})\, dr \Bigg) g(\boldsymbol{X}_{s_1}^{i,M}, ..., \boldsymbol{X}_{s_p}^{i,M}) \Bigg|, \\ &= \frac{1}{\mathrm{We}\, M} \mathbb{E} \Bigg| \sum_{i=1}^M \Bigg(\int_s^t \nabla\phi(\boldsymbol{X}_r^{i,M}).d\boldsymbol{W}_r^i \Bigg) g(\boldsymbol{X}_{s_1}^{i,M}, ..., \boldsymbol{X}_{s_p}^{i,M}) \Bigg| \leq \frac{C}{\sqrt{M}}. \end{aligned}$$

We have therefore:

$$\mathbb{E}^{\Pi_\infty} |F_\infty(Q)| \leq \limsup_{n\to\infty} \mathbb{E}^{\Pi_\infty} |F_\infty(Q) - F_n(Q)| + \limsup_{n\to\infty} \limsup_{M\to\infty} \mathbb{E}^{\Pi_M} |F_n(Q) - F_\infty(Q)|. \tag{50}$$

We first analyse the term $|F_\infty(Q) - F_n(Q)|$. Since g is bounded and $\nabla\phi$ is bounded with compact support, we obtain:

$$\begin{aligned} &|F_\infty(Q) - F_n(Q)| \\ &\leq C \int_s^t \left| \frac{1}{1 - \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{\max\left(1 - \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y})/b, \frac{1}{n}\right)} \right| dr, \end{aligned} \tag{51}$$

where C is a constant depending on the datas but not on Q. Notice that the term between the absolute value signs in (51) is (a.s.) non negative under $\Pi_\infty(dQ)$ or $\Pi_M(dQ)$, since in this case, $\int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y}) < b$.

Let us now consider the first term in the r.h.s. of (50). Using Lemma 4, we know that $\exists \epsilon > 0, \forall s \leq r \leq t$, $\Pi_\infty(dQ)$-a.s.,

$$\int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y}) \leq b(1 - \epsilon).$$

From this, one can deduce that $\exists n_0$, $\forall n \geq n_0$, $\forall s \leq r \leq t$, $\Pi_\infty(dQ)$-a.s.,

$$\int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y}) \leq b(1-1/n).$$

Therefore, we have: $\forall n \geq n_0$, $\Pi_\infty(dQ)$-a.s.,

$$|F_\infty(Q) - F_n(Q)| \leq C \int_s^t \left(\frac{1}{1 - \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{1 - \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y})/b} \right) dr.$$

Moreover, by the monotone convergence theorem, $\Pi_\infty(dQ)$-a.s., $\forall r$,

$$\lim_{n\to\infty} \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y}) = \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y}).$$

Since we also have by Lemma 4, $\Pi_\infty(dQ)$-a.s.,

$$\left(\frac{1}{1 - \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{1 - \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y})/b} \right) \leq \frac{1}{1-\rho_r^\infty/b}$$

by the Lebesgue's theorem, we obtain:

$$\limsup_{n\to\infty} \mathbb{E}^{\Pi_\infty} \int_s^t \left(\frac{1}{1 - \int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{\max\left(1 - \int (||\boldsymbol{Y}_r||^2 \wedge n) Q(d\boldsymbol{Y})/b, \frac{1}{n}\right)} \right) dr = 0. \tag{52}$$

Equation (52) shows that the first term in the r.h.s. of (50) is zero. Let us now consider the second term in the r.h.s. of (50). Using the bound (51) on $|F_n(Q) - F_\infty(Q)|$, exchangeability of the random variables $(\boldsymbol{X}_t^{i,M})_{1\leq i\leq M}$, (36) and (39), we have:

$$\begin{aligned}
&\mathbb{E}^{\Pi_M}|F_n(Q) - F_\infty(Q)| \\
&\leq C \int_s^t \mathbb{E} \left(\frac{1}{1 - \frac{1}{M}\sum_{i=1}^M ||\boldsymbol{X}_r^{i,M}||^2/b} - \frac{1}{\max\left(1 - \frac{1}{M}\sum_{i=1}^M (||\boldsymbol{X}_r^{i,M}||^2 \wedge n)/b, \frac{1}{n}\right)} \right) dr \\
&\leq C \int_s^t \mathbb{E} \left(\frac{\max\left(1 - \frac{1}{M}\sum_{i=1}^M (||\boldsymbol{X}_r^{i,M}||^2 \wedge n)/b, \frac{1}{n}\right) - (1 - R_r^M/b)}{(1 - R_r^M/b)\max\left(1 - \frac{1}{M}\sum_{i=1}^M (||\boldsymbol{X}_r^{i,M}||^2 \wedge n)/b, \frac{1}{n}\right)} \right) dr \\
&\leq C \int_s^t \mathbb{E} \left(\left(\frac{1}{1 - R_r^M/b} \right)^2 \left(\max\left(1 - \frac{1}{M}\sum_{i=1}^M (||\boldsymbol{X}_r^{i,M}||^2 \wedge n)/b, \frac{1}{n}\right) - \max\left(1 - R_r^M/b, \frac{1}{n}\right) \right) \right) dr \\
&\quad + C \int_s^t \mathbb{E} \left(\left(\frac{1}{1 - R_r^M/b} \right)^2 \left(\max\left(1 - R_r^M/b, \frac{1}{n}\right) - (1 - R_r^M/b) \right) \right) dr \\
&\leq C \int_s^t \mathbb{E} \left(\left(\frac{1}{1 - R_r^M/b} \right)^2 \left(\frac{1}{Mb}\sum_{i=1}^M \left(||\boldsymbol{X}_r^{i,M}||^2 - (||\boldsymbol{X}_r^{i,M}||^2 \wedge n)\right) + \frac{1}{n} \right) \right) dr \\
&\leq \frac{C}{b} \int_s^t \mathbb{E} \left(\left(\frac{1}{1 - R_r^M/b} \right)^2 \left(||\boldsymbol{X}_r^{1,M}||^2 - (||\boldsymbol{X}_r^{1,M}||^2 \wedge n)\right) \right) dr + \frac{C(t)}{n} \\
&\leq \frac{C}{b} \int_s^t \left(\mathbb{E} \left(\frac{1}{1 - R_r^M/b} \right)^r \right)^{1/p} \left(\mathbb{E} \left(||\boldsymbol{X}_r^{1,M}||^2 - (||\boldsymbol{X}_r^{1,M}||^2 \wedge n)\right)^q \right)^{1/q} dr + \frac{C(t)}{n} \\
&\leq \frac{C(t)}{b} \int_s^t \left(\mathbb{E} \left(||\boldsymbol{X}_r^{1,M}||^{2q} 1_{||\boldsymbol{X}_r^{1,M}||^2 \geq n} \right) \right)^{1/q} dr + \frac{C(t)}{n} \\
&\leq \frac{C(t)}{b} \int_s^t \left(\mathbb{E} \left(\frac{||\boldsymbol{X}_r^{1,M}||^{2(q+1)}}{n} \right) \right)^{1/q} dr + \frac{C(t)}{n} \leq \frac{C(t)}{n^{1/q}},
\end{aligned}$$

where $C(t)$ does not depend on M, $p = r/2$ and $q = \frac{p}{p-1}$. We therefore obtain:

$$\limsup_{n\to\infty} \limsup_{M\to\infty} \mathbb{E}^{\Pi_M} |F_n(Q) - F_\infty(Q)| = 0. \tag{53}$$

Using (52,53), we obtain (49) and this ends the proof of Lemma 5.

$\diamondsuit$

4.3 Convergence of the stress tensor

We can now prove Theorem 1. For $t \geq 0$, we consider

$$\mathbb{E} \left| \boldsymbol{\tau}_p^M(t) - \boldsymbol{\tau}_p(t) \right| = \mathbb{E}^{\Pi^M} \left| \boldsymbol{T}_\infty(Q) \right|,$$

where $\boldsymbol{T}_\infty : \mathcal{P}(\mathcal{C}(\mathbb{R}^+, \mathbb{R}^N)) \to \mathbb{R}^{N\times N}$ is defined by:

$$\boldsymbol{T}_\infty(Q) = \frac{\varepsilon}{\mathrm{We}} \left(\frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)\, Q(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} - \frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)\, P(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 P(d\boldsymbol{Y})/b} \right). \tag{54}$$

Let us also introduce, for $n \in \mathbb{N}$, $n \geq 1$, $\boldsymbol{T}_n : \mathcal{P}(\mathcal{C}(\mathbb{R}^+, \mathbb{R}^N)) \to \mathbb{R}^{N\times N}$ defined by:

$$\boldsymbol{T}_n(Q) = \frac{\varepsilon}{\mathrm{We}} \left(\frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t) * \frac{||\boldsymbol{Y}_t||^2 \wedge n}{||\boldsymbol{Y}_t||^2} Q(d\boldsymbol{Y})}{\max(1 - \int (||\boldsymbol{Y}_t||^2 \wedge n)\, Q(d\boldsymbol{Y})/b, \frac{1}{n})} - \frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)\, P(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 P(d\boldsymbol{Y})/b} \right). \tag{55}$$

Notice that $\boldsymbol{T}_n$ is a bounded and continuous function. We want to show that

$$\lim_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q)| = 0. \tag{56}$$

Using that $\mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_\infty(Q)| = 0$, we have:

$$\begin{aligned}
\limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q)| &\leq \limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| + \limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_n(Q)|, \\
&\leq \limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| + \mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_n(Q)|, \\
&\leq \limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| + \mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_n(Q) - \boldsymbol{T}_\infty(Q)|.
\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}
&\limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q)| \\
&\quad \leq \limsup_{n\to\infty} \limsup_{M\to\infty} \mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| + \limsup_{n\to\infty} \mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_n(Q) - \boldsymbol{T}_\infty(Q)|. \quad (57)
\end{aligned}$$

Let us consider the difference $|\boldsymbol{T}_n(Q) - \boldsymbol{T}_\infty(Q)|$. We have:

$$\begin{aligned}
|\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| &= \frac{\varepsilon}{\mathrm{We}} \left| \frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t)\, Q(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} - \frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t) * \frac{||\boldsymbol{Y}_t||^2 \wedge n}{||\boldsymbol{Y}_t||^2} Q(d\boldsymbol{Y})}{\max(1 - \int (||\boldsymbol{Y}_t||^2 \wedge n)\, Q(d\boldsymbol{Y})/b, \frac{1}{n})} \right| \\
&\leq \frac{\varepsilon}{\mathrm{We}} \left| \frac{\int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t) \left(1 - \frac{||\boldsymbol{Y}_t||^2 \wedge n}{||\boldsymbol{Y}_t||^2}\right) Q(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} \right| \\
&\quad + \frac{\varepsilon}{\mathrm{We}} \left| \int (\boldsymbol{Y}_t \otimes \boldsymbol{Y}_t) * \frac{||\boldsymbol{Y}_t||^2 \wedge n}{||\boldsymbol{Y}_t||^2} Q(d\boldsymbol{Y}) \right.
\end{aligned}$$

$$
* \left(\frac{1}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{\max(1 - \int (||\boldsymbol{Y}_t||^2 \wedge n)\, Q(d\boldsymbol{Y})/b, \frac{1}{n})} \right) \Bigg|
$$

$$
\leq C \left(\frac{\int (||\boldsymbol{Y}_t||^2 - (||\boldsymbol{Y}_t||^2 \wedge n))\, Q(d\boldsymbol{Y})}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} \right) \tag{58}
$$

$$
+C \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y}) \left| \frac{1}{1 - \int ||\boldsymbol{Y}_t||^2 Q(d\boldsymbol{Y})/b} - \frac{1}{\max(1 - \int (||\boldsymbol{Y}_t||^2 \wedge n)\, Q(d\boldsymbol{Y})/b, \frac{1}{n})} \right| . \tag{59}
$$

Notice that the term between the absolute value signs in (59) is (a.s.) non negative under $\Pi_\infty(dQ)$ or $\Pi_M(dQ)$, since in this case, $\int ||\boldsymbol{Y}_r||^2 Q(d\boldsymbol{Y}) < b$.

Let us consider the second term in (57). Using the bound (58-59), we have:

$$
\mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_n(Q) - \boldsymbol{T}_\infty(Q)| \leq C \left(\frac{\mathbb{E}\left(||\boldsymbol{X}_t||^2 - ||\boldsymbol{X}_t||^2 \wedge n\right)}{1 - \mathbb{E}\left(||\boldsymbol{X}_t||^2\right)/b} \right)
$$

$$
+C\mathbb{E}\left(||\boldsymbol{X}_t||^2\right) \left(\frac{1}{1 - \mathbb{E}\left(||\boldsymbol{X}_t||^2\right)/b} - \frac{1}{\max(1 - \mathbb{E}\left(||\boldsymbol{X}_t||^2 \wedge n\right)/b, \frac{1}{n})} \right).
$$

Using Lebesgue's theorem and the fact that $\sup_{0 \leq s \leq t} \mathbb{E}\left(||\boldsymbol{X}_s||^2\right) < b$, we then easily obtain:

$$
\limsup_{n \to \infty} \mathbb{E}^{\Pi^\infty} |\boldsymbol{T}_n(Q) - \boldsymbol{T}_\infty(Q)| = 0. \tag{60}
$$

Let us prove that the first term in (57) is zero. Using the bound (58-59), exchangeability of the random variables $(\boldsymbol{X}_t^{i,M})_{1 \leq i \leq M}$ and arguing as in the proof of (53) we have:

$$
\begin{aligned}
&\mathbb{E}^{\Pi^M} |\boldsymbol{T}_\infty(Q) - \boldsymbol{T}_n(Q)| \\
&\leq \; C\mathbb{E}\left(\left(||\boldsymbol{X}_t^{1,M}||^2 - (||\boldsymbol{X}_t^{1,M}||^2 \wedge n) \right) \frac{1}{1 - R_t^M/b} \right) \\
&\quad +C\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2 \left(\frac{1}{1 - R_t^M/b} - \frac{1}{\max\left(1 - \frac{1}{M}\sum_{i=1}^M \left(||\boldsymbol{X}_t^{i,M}||^2 \wedge n\right)/b, \frac{1}{n}\right)} \right) \right) \\
&\leq \; C\mathbb{E}\left(\left(||\boldsymbol{X}_t^{1,M}||^2 - (||\boldsymbol{X}_t^{1,M}||^2 \wedge n) \right) \frac{1}{1 - R_t^M/b} \right)
\end{aligned} \tag{61}
$$

$$
+C\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2 \left(\frac{1}{1 - R_t^M/b} \right)^2 \left(\frac{1}{Mb} \sum_{i=1}^M \left(||\boldsymbol{X}_r^{i,M}||^2 - (||\boldsymbol{X}_r^{i,M}||^2 \wedge n) \right) + \frac{1}{n} \right) \right). \tag{62}
$$

We first consider the term (61). Using (36) and (39), we have:

$$
\begin{aligned}
&\mathbb{E}\left(\left(||\boldsymbol{X}_t^{1,M}||^2 - (||\boldsymbol{X}_t^{1,M}||^2 \wedge n) \right) \frac{1}{1 - R_t^M/b} \right) \\
&\leq \; \sqrt{\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2 - (||\boldsymbol{X}_t^{1,M}||^2 \wedge n) \right)^2} \sqrt{\mathbb{E}\left(\frac{1}{1 - R_t^M/b} \right)^2} \\
&\leq \; C\sqrt{\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^4 1_{||\boldsymbol{X}_t^{1,M}||^2 \geq n} \right)} \leq \frac{C}{\sqrt{n}} \sqrt{\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^6 \right)} \leq \frac{C}{\sqrt{n}},
\end{aligned} \tag{63}
$$

where C does not depend on M.

Let us now consider the term (62). We have, using (36) and (39):

$$
\begin{aligned}
&\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2\left(\frac{1}{1-R_t^M/b}\right)^2\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_r^{i,M}||^2-(||\boldsymbol{X}_r^{i,M}||^2\wedge n)\right)+\frac{1}{n}\right)\right)\\
&\le \left(\mathbb{E}||\boldsymbol{X}_t^{1,M}||^{2p}\right)^{1/p}\left(\mathbb{E}\left(\left(\frac{1}{1-R_t^M/b}\right)^{2q}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_r^{i,M}||^2-(||\boldsymbol{X}_r^{i,M}||^2\wedge n)\right)+\frac{1}{n}\right)^q\right)\right)^{1/q}\\
&\le C\left(\mathbb{E}\left(\left(\frac{1}{1-R_t^M/b}\right)^{2q}\left(\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_r^{i,M}||^2-(||\boldsymbol{X}_r^{i,M}||^2\wedge n)\right)\right)^q+\frac{1}{n^q}\right)\right)\right)^{1/q}\\
&\le C\left(\mathbb{E}\left(\left(\frac{1}{1-R_t^M/b}\right)^{2q}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_r^{i,M}||^2-(||\boldsymbol{X}_r^{i,M}||^2\wedge n)\right)\right)^q\right)+\frac{C}{n^q}\right)^{1/q}
\end{aligned}
$$

with $1<q<r/2$ and $p=\frac{q}{q-1}$, where r is defined in (39). Using (39), we have:

$$
\begin{aligned}
&\mathbb{E}\left(\left(\frac{1}{1-R_t^M/b}\right)^{2q}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_t^{i,M}||^2-||\boldsymbol{X}_t^{i,M}||^2\wedge n\right)\right)^q\right)\\
&\le \left(\mathbb{E}\left(\frac{1}{1-R_t^M/b}\right)^r\right)^{1/p'}\left(\mathbb{E}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_t^{i,M}||^2-||\boldsymbol{X}_t^{i,M}||^2\wedge n\right)\right)^{qq'}\right)^{1/q'}\\
&\le C\left(\mathbb{E}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_t^{i,M}||^2-||\boldsymbol{X}_t^{i,M}||^2\wedge n\right)\right)^{qq'}\right)^{1/q'}
\end{aligned}
$$

with $p'=\frac{r}{2q}$, $q'=\frac{p'}{p'-1}$, and C is a constant not depending on M. Finally, we observe that, using (36) and arguing as in the proof of (63):

$$
\begin{aligned}
&\mathbb{E}\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_t^{i,M}||^2-||\boldsymbol{X}_t^{i,M}||^2\wedge n\right)\right)^{qq'}\\
&\le \left(\frac{1}{b}\right)^{qq'}\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2-||\boldsymbol{X}_t^{1,M}||^2\wedge n\right)^{qq'}\\
&\le \left(\frac{1}{bn}\right)^{qq'}\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^{4qq'}\right)\le\left(\frac{1}{bn}\right)^{qq'}C_{2qq'}(t).
\end{aligned}
$$

Therefore we obtain:

$$
\mathbb{E}\left(||\boldsymbol{X}_t^{1,M}||^2\left(\frac{1}{1-R_t^M/b}\right)^2\left(\frac{1}{Mb}\sum_{i=1}^M\left(||\boldsymbol{X}_r^{i,M}||^2-(||\boldsymbol{X}_r^{i,M}||^2\wedge n)\right)+\frac{1}{n}\right)\right)\le\frac{C}{n}, \quad (64)
$$

where C does not depend on M. Using the bound (61-62), by (63) and (64), we have:

$$
\limsup_{n\to\infty}\limsup_{M\to\infty}\mathbb{E}^{\Pi^M}\left|\boldsymbol{T}_\infty(Q)-\boldsymbol{T}_n(Q)\right|=0. \quad (65)
$$

By (60) and (65), we have (56) which is equivalent to (14).

5 Conclusion

We have analyzed the SDE arising in the FENE-P model of polymeric fluids. We have shown that both the nonlinear SDE at the continuous level and the particle system at the discrete level admit a solution, and that the stress tensor obtained with the particle system converges towards the stress tensor obtained with the nonlinear SDE, in the limit of an infinite number of particles. This theoretical result confirms numerical experiments performed in [12].

From a mathematical point of view, we can summarize the results obtained by the following: the solutions to the FENE-P model behave like Hookean dumbbells (see [9]) with a time-variable spring constant at the continuous level (they can reach infinite extensibility), and rather like FENE dumbbells (see [10, 8]) at the discrete level, once the problem is discretized in a particle system (since they cannot reach infinite extensibility).

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