Loss of martingality in asset price models with lognormal stochastic volatility

B.Jourdain*

July 7, 2004

Abstract

In this note, we prove that in asset price models with lognormal stochastic volatility, when the correlation coefficient between the Brownian motion driving the volatility and the one driving the actualized asset price is positive, this price is not a martingale.

Introduction

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, we consider the following risk-neutral model for the actualized asset price X_t :

$$\begin{cases} dX_t = \sigma_t X_t \, (\rho \, dB_t + \sqrt{1-\rho^2} \, dW_t), \; X_0 = x_0 \\ \sigma_t = e^{Y_t} \\ dY_t = \alpha dB_t + \mu dt - \gamma Y_t dt, \; Y_0 = y_0 \end{cases} \tag{1}$$

where $(B_t, W_t)_{t\geq 0}$ is a two-dimensional $\mathcal{F}_t$-Brownian motion, y_0, μ and γ belong to $\mathbb{R}$, x_0 and α are positive constants and the correlation coefficient ρ between the Brownian motion $\rho \, B_t + \sqrt{1-\rho^2} \, W_t$ driving the asset price and the Brownian motion B_t driving the volatility belongs to $[-1, 1]$.

In this model, first introduced by Scott [7] p.426, the volatility σ_t is the exponential of the Ornstein-Uhlenbeck process Y_t. When the elasticity coefficient γ is zero, σ_t evolves according to the Black-Scholes Stochastic Differential Equation

$$d\sigma_t = \sigma_t(\alpha \, dB_t + (\mu + \alpha^2/2)dt),$$

and (1) is the model introduced by Hull&White [3] and a special case of the very popular SABR model [1].

Since

$$X_t = x_0 \, \mathcal{E}_t \left(\int_0^{\cdot} e^{Y_s} (\rho \, dB_s + \sqrt{1-\rho^2} \, dW_s \right),$$

where $\mathcal{E}_t$ stands for Dooleans-Dade exponential, this process is a non-negative $\mathcal{F}_t$ local martingale and therefore a super-martingale. This leads to the following natural question : is $(X_t)_{t\geq 0}$ a

*ENPC-CERMICS, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne-la-Vallée Cedex 2, France and MATHFI project - e-mail : jourdain@cermics.enpc.fr

true martingale? The answer turns out to be affirmative if and only if the correlation coefficient ρ is non-positive (see Theorem 1 below).
In addition, when $\rho > 0$, it is possible to check that $t \to \mathbb{E}(X_t)$ is decreasing (see Proposition 4). As a consequence, the Call-Put parity relation does not hold whatever the maturity and strike of the options. The existence of such arbitrage opportunities invalidates the model.
For $\rho \leq 0$, $(X_t)_{t\geq 0}$ is a martingale and we investigate the integrability of X_t^δ for $t > 0$ and $\delta > 1$. This question is important for numerical considerations : for instance, integrability of X_T^2 is necessary to ensure that the convergence of a Monte-Carlo estimator of $\mathbb{E}((X_T - K)^+)$ is ruled by the central limit theorem. Basically, we obtain that X_t^δ is integrable when $\delta < 1/(1-\rho^2)$ and that $\mathbb{E}(X_t^\delta)$ is infinite when $\delta > 1/(1-\rho^2)$.

1 Study of martingale property

Our main result is the following one :

Theorem 1 *Process $(X_t)_{t\geq 0}$ is a martingale if and only if $\rho \leq 0$.*

Remark 2
- *The fact that $(X_t)_{t\geq 0}$ is not a martingale when $\rho > 0$ is not so bad since in order to modelize the increase of volatility in krach situations, ρ is generally chosen non-positive.*
- *For $t > 0$, by Jensen inequality and since $\int_0^t Y_s ds$ is a Gaussian random variable with positive variance,*

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^t e^{2Y_s} ds\right)\right) \geq \mathbb{E}\left(\exp\left(\frac{t}{2}\, e^{\frac{2}{t}\int_0^t Y_s ds}\right)\right) = +\infty.$$

Therefore we cannot rely on Novikov criterion and corollaries (see [5] p.198) to prove that $(X_t)_{t\geq 0}$ is a martingale in case $\rho \leq 0$.
In contrast, in the models proposed either in [7] p.421, [8] where the stochastic volatility σ_t in the first line of (1) solves

$$d\sigma_t = (\mu - \gamma\sigma_t)\, dt + \alpha dB_t$$

or in [4], [2] where

$$\sigma_t = \sqrt{Y_t} \text{ for } Y_t = y_0 + \int_0^t (\mu - \gamma Y_s) ds + \alpha \int_0^t \sqrt{Y_s} dB_s \text{ with } \mu, y_0 \geq 0$$

one easily checks that

$$\forall T > 0,\ \exists c_T > 0,\ \sup_{t\leq T} \mathbb{E}\left(e^{c_T \sigma_t^2}\right) < +\infty.$$

As a consequence if $0 \leq t_1 < t_2 \leq T$ and $t_2 - t_1 \leq 2c_T$, then by Jensen inequality,

$$\mathbb{E}\left(e^{\frac{1}{2}\int_{t_1}^{t_2} \sigma_t^2 dt}\right) \leq \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \mathbb{E}\left(e^{\frac{t_2-t_1}{2}\,\sigma_t^2}\right) dt < +\infty.$$

By [5] Corollary 5.14 p.199, one concludes that $(X_t)_{t\geq 0}$ is always a martingale in such models.

- *In order to deal with the general SABR model [1], let us consider* $(X_t)_{t\geq 0}$ *solving*

$$dX_t = e^{Y_t} X_t^{\beta}(\rho dB_t + \sqrt{1-\rho^2}\, dW_t),\ X_0 = x_0$$

with $\beta \in (0,1)$ *and* Y_t *like in* (1)*. Introducing* $\tau_n = \inf\{t \geq 0, |X_t| > n\}$*, one has*

$$\begin{aligned}\mathbb{E}(X_{t\wedge\tau_n}^2) &= x_0^2 + \mathbb{E}\left(\int_0^{t\wedge\tau_n} e^{2Y_s} X_s^{2\beta} ds\right)\\ &\leq x_0^2 + \int_0^t \left(\mathbb{E}\left(e^{2Y_s/(1-\beta)}\right)\right)^{1-\beta} (\mathbb{E}(X_{s\wedge\tau_n}^2))^{\beta} ds\\ &\leq x_0^2 + \int_0^t \left(\mathbb{E}\left(e^{2Y_s/(1-\beta)}\right)\right)^{1-\beta} (1+\mathbb{E}(X_{s\wedge\tau_n}^2)) ds.\end{aligned}$$

Remarking that $s \to \mathbb{E}\left(e^{2Y_s/(1-\beta)}\right)$ *is locally bounded, using Gronwall's Lemma, and letting* $n \to +\infty$*, one obtains that* $t \to \mathbb{E}\left(\int_0^t e^{2Y_s} X_s^{2\beta} ds\right)$ *is locally bounded which ensures that* $(X_t)_{t\geq 0}$ *is a martingale.*
In conclusion, in the SABR model, the actualized asset price may fail to be a martingale only in the limit case $\beta = 1$*.*

Proof : As $(X_t)_{t\geq 0}$ is a super-martingale, it is enough to check that the non-increasing function $t \to \mathbb{E}(X_t)/x_0$ is not constant if and only if $\rho > 0$.
Using the independence of W and B and the fact that $(Y_t)_{t\geq 0}$ is adapted to the natural filtration of $(B_t)_{t\geq 0}$, one obtains

$$\begin{aligned}\mathbb{E}(X_t) &= x_0 \mathbb{E}\left(\mathcal{E}_t\left(\rho \int_0^{\cdot} e^{Y_s} dB_s\right) \mathbb{E}\left(\mathcal{E}_t\left(\sqrt{1-\rho^2}\int_0^{\cdot} e^{Y_s} dW_s\right)\Big| B_s, s\leq t\right)\right)\\ &= x_0 \mathbb{E}\left(\mathcal{E}_t\left(\rho\int_0^{\cdot} e^{Y_s} dB_s\right)\right). \qquad (2)\end{aligned}$$

In case $\rho = 0$, one concludes that $\mathbb{E}(X_t) = x_0$ for any positive time t. To deal with the case $\rho \neq 0$, we are first going to use Girsanov theorem in order to be able to apply Exercice (2.10) p. 354 [6]. This way, $\mathbb{E}(X_t)/x_0$ turns out to be equal to the probability for the explosion time of a well-chosen stochastic differential equation to be greater than t. We will finally analyse whether this explosion time is finite with positive probability thanks to Feller's test for explosions [5] p.342-351.

Let us introduce the probability measure $\mathbb{Q}$ such that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t\left(-\int_0^{\cdot}\left(\frac{\gamma y_0 - \mu}{\alpha} + \gamma B_s\right) dB_s\right).$$

According to Girsanov theorem, $\tilde{B}_t = B_t + \int_0^t \left(\frac{\gamma y_0 - \mu}{\alpha} + \gamma B_s\right) ds$ is a Brownian motion under $\mathbb{Q}$. For any $t \geq 0$,

$$y_0 + \alpha B_t = y_0 + \alpha\tilde{B}_t - \gamma\int_0^t (y_0 + \alpha B_s) ds + \mu t.$$

As trajectorial uniqueness holds for the Ornstein-Uhlenbeck Stochastic Differential Equation, by Yamada Watanabe theorem, the law of $(\tilde{B}_t, y_0 + \alpha B_t)_{t\geq 0}$ under probability measure $\mathbb{Q}$ is the same as the law of $(B_t, Y_t)_{t\geq 0}$ under probability measure $\mathbb{P}$. As a consequence,

$$
\begin{aligned}
\frac{\mathbb{E}(X_t)}{x_0} &= \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}_t\left(\rho\int_0^{\cdot}\exp(y_0+\alpha B_s)d\tilde{B}_s\right)\right]\\
&= \mathbb{E}\Bigg[\mathcal{E}_t\left(\rho\int_0^{\cdot}\exp(y_0+\alpha B_s)dB_s\right)\exp\left(\rho\int_0^t\exp(y_0+\alpha B_s)\left(\frac{\gamma y_0-\mu}{\alpha}+\gamma B_s\right)ds\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\mathcal{E}_t\left(-\int_0^{\cdot}\left(\frac{\gamma y_0-\mu}{\alpha}+\gamma B_s\right)dB_s\right)\Bigg]\\
&= \mathbb{E}\left[\mathcal{E}_t\left(\int_0^{\cdot}b(B_s)dB_s\right)\right] \qquad (3)
\end{aligned}
$$

where

$$b(z)=\rho\exp(y_0+\alpha z)+\frac{\mu-\gamma y_0}{\alpha}-\gamma z.$$

Let us briefly recall the link made in [6] Exercice (2.10) p.354 between the last expectation and the probability for the explosion time of the Stochastic Differential Equation

$$Z_t = B_t + \int_0^t b(Z_s)ds \qquad (4)$$

to be greater than t. Since function b is locally Lipschitz continuous, for any $n\in\mathbb{N}^*$, there exists a bounded and globally Lipschitz continuous function b_n which coincides with b on interval $[-n,n]$. We denote by Z_t^n the solution of the equation similar to (4) with b replaced by b_n and introduce

$$\tau_n=\inf\{t\geq 0 : |Z_t^n|>n\}.$$

Then $Z_t=\sum_{n\in\mathbb{N}^*}1_{\{\tau_{n-1}\leq t<\tau_n\}}Z_t^n$ (convention : $\tau_0=0$) solves (4) on time-interval $[0,\tau_\infty)$ where $\tau_\infty=\lim_{n\to+\infty}\tau_n$. Let us also define

$$\sigma_n=\inf\{t\geq 0 : |B_t|>n\}.$$

By Girsanov theorem and since b_n coincides with b on interval $[-n,n]$,

$$\mathbb{P}(\tau_n>t)=\mathbb{E}\left(1_{\{\sigma_n>t\}}\mathcal{E}_t\left(\int_0^{\cdot}b_n(B_s)dB_s\right)\right)=\mathbb{E}\left(1_{\{\sigma_n>t\}}\mathcal{E}_t\left(\int_0^{\cdot}b(B_s)dB_s\right)\right).$$

Letting $n\to+\infty$ then using (3), we conclude that

$$\mathbb{P}(\tau_\infty>t)=\mathbb{E}\left(\mathcal{E}_t\left(\int_0^{\cdot}b(B_s)dB_s\right)\right)=\frac{\mathbb{E}(X_t)}{x_0}.$$

In order to analyse the explosion time τ_∞ of the stochastic differential equation (4) thanks to Feller's test for explosions, we introduce constants $\tilde{\rho}=2\rho e^{y_0}/\alpha$ and $\eta=2(\mu-\gamma y_0)/\alpha$ so that the drift coefficient of this equation writes

$$b(z)=\alpha\tilde{\rho}\exp(\alpha z)/2+\eta/2-\gamma z.$$

Notice that the sign of $\tilde{\rho}$ is the same as the one of ρ. Function

$$p(z)=e^{\tilde{\rho}}\int_0^z\exp\left(\gamma x^2-\eta x-\tilde{\rho}e^{\alpha x}\right)dx \qquad (5)$$

is a scale function. According to [5] Theorem 5.29 p.348, $\mathbb{P}(\tau_\infty = +\infty) = 1$ is equivalent to $v(+\infty) = v(-\infty) = +\infty$ where function v is given by

$$\begin{aligned}
v(z) &= \int_0^z p'(x) \int_0^x \frac{2}{p'(y)} dy dx \\
&= 2 \int_0^z \exp\left(\gamma x^2 - \eta x - \tilde{\rho} e^{\alpha x}\right) \int_0^x \exp\left(-\gamma y^2 + \eta y + \tilde{\rho} e^{\alpha y}\right) dy dx. \qquad (6)
\end{aligned}$$

To conclude the proof, we are going to check that $v(+\infty) < +\infty$ if $\rho > 0$ and that $v(+\infty) = v(-\infty) = +\infty$ if $\rho < 0$.

Case $\rho > 0$ **i.e.** $\tilde{\rho} > 0$ **:** let $x_1 > 0$ be such that

$$\forall y \geq x_1,\ -2\gamma y + \eta + \tilde{\rho}\alpha e^{\alpha y} > 0.$$

By integration by parts, one has for $x \geq x_1$,

$$\begin{aligned}
\int_{x_1}^x \exp\left(-\gamma y^2 + \eta y + \tilde{\rho} e^{\alpha y}\right) dy =& \frac{\exp\left(-\gamma x^2 + \eta x + \tilde{\rho} e^{\alpha x}\right)}{-2\gamma x + \eta + \tilde{\rho}\alpha e^{\alpha x}} - \frac{\exp\left(-\gamma x_1^2 + \eta x_1 + \tilde{\rho} e^{\alpha x_1}\right)}{-2\gamma x_1 + \eta + \tilde{\rho}\alpha e^{\alpha x_1}} \\
&+ \int_{x_1}^x \frac{(\tilde{\rho}\alpha^2 e^{\alpha y} - 2\gamma) \exp\left(-\gamma y^2 + \eta y + \tilde{\rho} e^{\alpha y}\right)}{(-2\gamma y + \eta + \tilde{\rho}\alpha e^{\alpha y})^2} dy.
\end{aligned}$$

One may choose x_1 large enough to ensure that

$$\forall y \geq x_1,\ \tilde{\rho}\alpha^2 e^{\alpha y} - 2\gamma \leq \frac{1}{2}\left(-2\gamma y + \eta + \tilde{\rho}\alpha e^{\alpha y}\right)^2.$$

Then for any $x \geq x_1$,

$$\int_{x_1}^x \exp\left(-\gamma y^2 + \eta y + \tilde{\rho} e^{\alpha y}\right) dy \leq \frac{2\exp\left(-\gamma x^2 + \eta x + \tilde{\rho} e^{\alpha x}\right)}{-2\gamma x + \eta + \tilde{\rho}\alpha e^{\alpha x}}$$

and one easily concludes that $v(+\infty) < +\infty$.

Case $\rho < 0$ **i.e.** $\tilde{\rho} < 0$ **:** then $p(+\infty) = +\infty$ which implies $v(+\infty) = +\infty$ according to [5] Problem 5.27 p.348. If $\gamma > 0$, then $p(-\infty) = -\infty$ and therefore $v(-\infty) = +\infty$. It only remains to check that $v(-\infty) = +\infty$ in case $\gamma \leq 0$. Since for non positive y, $\exp(\tilde{\rho} e^{\alpha y})$ belongs to $[\exp(\tilde{\rho}), 1)$, $v(-\infty) = +\infty$ is equivalent to $w(-\infty) = \infty$ where

$$w(z) = \int_0^z \exp(\gamma x^2 - \eta x) \int_0^x \exp(-\gamma y^2 + \eta y) dy dx$$

- If $\gamma = 0$, then $\eta = 2\mu/\alpha$ and

$$w(z) = \begin{cases} z^2 \text{ if } \mu = 0 \\ \frac{\alpha}{\mu}\left(z + \frac{\alpha}{2\mu}(e^{-2\mu z/\alpha} - 1)\right) \text{ if } \mu \neq 0 \end{cases}$$

  which ensures $w(-\infty) = +\infty$.
- If $\gamma < 0$, setting $x_1 < 0 \vee \eta/2\gamma$, one obtains by integration by parts that for $x \leq x_1$

$$\begin{aligned}
\int_{x_1}^x \exp(-\gamma y^2 + \eta y) dy &= \frac{\exp(-\gamma x^2 + \eta x)}{\eta - 2\gamma x} - \frac{\exp(-\gamma x_1^2 + \eta x_1)}{\eta - 2\gamma x_1} - 2\gamma \int_{x_1}^x \frac{\exp(-\gamma y^2 + \eta y)}{(\eta - 2\gamma y)^2} dy \\
&\leq \frac{\exp(-\gamma x^2 + \eta x)}{\eta - 2\gamma x} - \frac{\exp(-\gamma x_1^2 + \eta x_1)}{\eta - 2\gamma x_1}.
\end{aligned}$$

Hence for any $x \leq x_1$, one has

$$\int_0^x \exp(-\gamma y^2 + \eta y)dy \leq \frac{\exp(-\gamma x^2 + \eta x)}{\eta - 2\gamma x} + C,$$

where the constant C does not depend on x. One deduces that $w(-\infty) = +\infty$.

■

Remark 3 *The argument given at the end of the previous proof to check that $v(-\infty) = +\infty$ in case $\rho < 0$ also leads to the same conclusion in case $\rho > 0$.*

When the correlation coefficient ρ is positive, the non-increasing and non-negative function $t \to \mathbb{E}(X_t)$ is not constant. It is natural to wonder whether this function is decreasing and whether it tends to 0 as $t \to +\infty$. The next proposition answers both questions :

Proposition 4 *Assume that $\rho > 0$. Then $t \to \mathbb{E}(X_t)$ is decreasing. In addition, $\mathbb{E}(X_t)$ tends to 0 as t tends to $+\infty$ if and only if either* $\boxed{\gamma > 0}$ *or* $\boxed{\gamma = 0 \text{ and } \mu \geq 0}$.

Remark 5 *As a consequence, when $\rho > 0$,*

$$\forall T, K > 0,\ \mathbb{E}((X_T - K)^+) - \mathbb{E}((K - X_T)^+) < x_0 - K$$

i.e. the Call-Put parity relation does not hold.

Proof : Let us first deal with the limit of $\mathbb{E}(X_t) = x_0\mathbb{P}(\tau_\infty > t)$ as t tends to $+\infty$. One easily checks that the scale function $p(z)$ defined by (5) satisfies $p(-\infty) = -\infty$ if and only if either $\gamma > 0$ or $\gamma = 0$ and $\eta \geq 0$. Because $\eta = 2(\mu - \gamma y_0)/\alpha$, the latter condition is equivalent to $\gamma = 0$ and $\mu \geq 0$. Since $v(+\infty)$ is finite according to the proof of Theorem 1 and $v(-\infty) = +\infty$ according to Remark 3, by [5] Proposition 5.32 p.350, one concludes that $\mathbb{P}(\tau_\infty < +\infty) = 1$ and equivalently $\lim_{t\to+\infty} \mathbb{E}(X_t) = 0$ if and only if $\gamma > 0$ or $\gamma = 0$ and $\mu \geq 0$.

Let us now check that $t \to \mathbb{E}(X_t)$ is decreasing. As we need to emphasize the dependence on the initial conditions, we denote $(X_t^{x_0,y_0}, Y_t^{y_0})$ the solution of (1). One has

$$\forall t \geq 0,\ \forall x_0 > 0,\ \forall y_0 \in \mathbb{R},\ \mathbb{E}(X_t^{x_0,y_0}) = x_0\mathbb{E}(X_t^{1,y_0}).$$

Let us first check that for any positive T, the set $A_T = \{y \in \mathbb{R} : \mathbb{E}(X_T^{1,y}) < 1\}$ has positive Lebesgue measure. Indeed if $T > 0$ is such that the Lebesgue measure of A_T is zero, remarking that by the Markov property,

$$\mathbb{E}(X_{2T}^{1,y_0}) = \mathbb{E}\left(\mathbb{E}(X_{2T}^{1,y_0}|\mathcal{F}_T)\right) = \mathbb{E}\left(\mathbb{E}(X_T^{x,y})|_{(x,y)=(X_T^{1,y_0},Y_T^{y_0})}\right) = \mathbb{E}\left(X_T^{1,y_0}\mathbb{E}(X_T^{1,y})|_{y=Y_T^{y_0}}\right),$$

and that since the law of $Y_T^{y_0}$ is absolutely continuous with respect to the Lebesgue measure $\mathbb{P}(Y_T^{y_0} \in A_T) = 0$, we obtain $\mathbb{E}(X_{2T}^{1,y_0}) = \mathbb{E}(X_T^{1,y_0})$.
Therefore $A_{2T} = A_T$. By induction, for any $n \in \mathbb{N}^*$, $A_{nT} = A_T$ and for $y_0 \in \mathbb{R} \setminus A_T$, $(X_t^{1,y_0})_t$ is a martingale, which contradicts Theorem 1.
Let now $0 \leq s < t$. Again by the Markov Property,

$$\mathbb{E}(X_t^{1,y_0}) = \mathbb{E}\left(X_{(t+s)/2}^{1,y_0}\mathbb{E}\left(X_{(t-s)/2}^{1,y}\right)\Big|_{y=Y_{(t+s)/2}^{y_0}}\right).$$

Since the law of $Y^{y_0}_{(t+s)/2}$ is equivalent to the Lebesgue measure, $\mathbb{P}\left(Y^{y_0}_{(t+s)/2} \in A_{(t-s)/2}\right) > 0$. One deduces that $\mathbb{E}(X^{1,y_0}_t) < \mathbb{E}(X^{1,y_0}_{(t+s)/2})$. As the right-hand-side is not greater than $\mathbb{E}(X^{1,y_0}_s)$, one concludes that $t \to \mathbb{E}(X^{1,y_0}_t)$ is decreasing. ∎

2 Integrability of X^δ_t for $\delta > 1$

Proposition 6 *Let $t > 0$ and $\delta > 1$. If $\rho = 0$ then $\mathbb{E}(X^\delta_t) = +\infty$. If $\rho < 0$, then $\mathbb{E}(X^\delta_t) < +\infty$ if and only if one of the following conditions is satisfied :*

- $\delta < 1/(1-\rho^2)$
- $\delta = 1/(1-\rho^2)$ *and* $\gamma > 0$
- $\delta = 1/(1-\rho^2)$, $\gamma = 0$ *and* $\mu + \frac{\alpha^2}{2} \leq 0$.

Proof : Let us compute X_t to the power δ with $\delta > 1$:

$$X^\delta_t = x^\delta_0 \exp \delta \left(\rho \int_0^t e^{Y_s} dB_s + \frac{1}{2}(\delta(1-\rho^2) - 1) \int_0^t e^{2Y_s} ds\right) \mathcal{E}_t \left(\delta\sqrt{1-\rho^2} \int_0^\cdot e^{Y_s}\, dW_s\right).$$

Therefore, reasoning like in (2), one obtains

$$\mathbb{E}(X^\delta_t) = x^\delta_0 \mathbb{E}\left[\exp \delta \left(\rho \int_0^t e^{Y_s} dB_s + \frac{1}{2}(\delta(1-\rho^2) - 1) \int_0^t e^{2Y_s} ds\right)\right]. \tag{7}$$

In case $\rho = 0$, by Jensen inequality and since $\int_0^t Y_s ds$ is a Gaussian variable with positive variance,

$$\mathbb{E}(X^\delta_t) = x^\delta_0 \mathbb{E}\left[\exp\left(\frac{\delta(\delta-1)}{2} \int_0^t e^{2Y_s} ds\right)\right] \geq x^\delta_0 \mathbb{E}\left[\exp\left(\frac{\delta(\delta-1)t}{2} e^{\frac{2}{t}\int_0^t Y_s ds}\right)\right] = +\infty.$$

Let us now deal with the case $\rho < 0$. According to (1) and Itô's formula,

$$e^{Y_t} - e^{y_0} = \alpha \int_0^t e^{Y_s} dB_s + \int_0^t (\mu + \alpha^2/2 - \gamma Y_s) e^{Y_s} ds.$$

Inserting in (7) the expression of $\int_0^t e^{Y_s} dB_s$ obtained from this formula, one obtains

$$\mathbb{E}(X^\delta_t) = x^\delta_0 \mathbb{E}\left[\exp \delta \left(\frac{\rho}{\alpha}(e^{Y_t} - e^{y_0}) + \int_0^t \left(\frac{\rho}{\alpha}(\gamma Y_s - \mu - \alpha^2/2) + \frac{1}{2}(\delta(1-\rho^2) - 1)e^{Y_s}\right) e^{Y_s} ds\right)\right].$$

Under any of the three conditions stated in the Proposition, function

$$y \in \mathbb{R} \to \left(\frac{\rho}{\alpha}(\gamma y - \mu - \alpha^2/2) + \frac{1}{2}(\delta(1-\rho^2) - 1)e^y\right) e^y$$

is bounded from above by a finite constant C. As a consequence,

$$\delta \left(\frac{\rho}{\alpha}(e^{Y_t} - e^{y_0}) + \int_0^t \left(\frac{\rho}{\alpha}(\gamma Y_s - \mu - \alpha^2/2) + \frac{1}{2}(\delta(1-\rho^2) - 1)e^{Y_s}\right) e^{Y_s} ds\right) \leq \delta(Ct - \rho e^{y_0}/\alpha)$$

and for any $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}(X_t^\delta) \le x_0^\delta \exp \delta(CT - \rho e^{y_0}/\alpha).$$

Let us now suppose that none of the three conditions stated in the Proposition is satisfied. Then there is a positive constant ε such that function

$$y \in \mathbb{R} \to \left(\frac{\rho}{\alpha}(\gamma y - \mu - \alpha^2/2) + \frac{1}{2}(\delta(1-\rho^2) - 1)e^y\right) e^y - \varepsilon e^y$$

is bounded from below by a finite constant. As a consequence there is a positive constant C such as

$$\mathbb{E}(X_t^\delta) \ge C\mathbb{E}\left[\exp \delta \left(\frac{\rho}{\alpha} e^{Y_t} + \varepsilon \int_0^t e^{Y_s} ds\right)\right].$$

By Jensen inequality,

$$\mathbb{E}(X_t^\delta) \ge C\mathbb{E}\left[\exp \delta \left(\frac{\rho}{\alpha} e^{Y_t} + \varepsilon t e^{\frac{1}{t}\int_0^t Y_s ds}\right)\right] = C\mathbb{E}\left[\exp\left(\delta \rho e^{Y_t}/\alpha\right) \mathbb{E}\left(\exp\left(\delta \varepsilon t e^{\frac{1}{t}\int_0^t Y_s ds}\right) |Y_t\right)\right].$$

Since the covariance matrix of the Gaussian vector $(Y_t, \int_0^t Y_s ds)$ is non-degenerate,

$$\mathbb{E}\left(\exp\left(\delta \varepsilon t e^{\frac{1}{t}\int_0^t Y_s ds}\right) |Y_t\right) = +\infty \text{ almost surely}$$

and one concludes that $\mathbb{E}(X_t^\delta) = +\infty$. ∎

Remark 7 *For $\rho > 0$, when one of the following condition is satisfied*

- $\delta > 1/(1-\rho^2)$
- $\delta = 1/(1-\rho^2)$ *and* $\gamma > 0$
- $\delta = 1/(1-\rho^2)$ *and* $\gamma = 0$ *and* $\mu + \frac{\alpha^2}{2} \le 0$,

then function $y \in \mathbb{R} \to \left(\frac{\rho}{\alpha}(\gamma y - \mu - \alpha^2/2) + \frac{1}{2}(\delta(1-\rho^2) - 1)e^y\right) e^y$ is bounded from below. Therefore $\mathbb{E}(X_t^\delta) \ge C\mathbb{E}(\exp(\delta\rho e^{Y_t}/\alpha)) = +\infty$ when $t > 0$. But it does not seem easy to analyse whether $\mathbb{E}(X_t^\delta)$ is finite when none of the previous conditions holds.

References

[1] P.S. Hagan, D. Kumar, A.S. Lesniewski, and D.E. Woodward. Managing Smile Risk. *Preprint*, 2000.

[2] S.L. Heston. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6(2):327–343, 1993.

[3] J. Hull and A. White. The Pricing of Options on Assets with Stochastic Volatilities. *The Journal of Finance*, XLII(2):281–300, june 1987.

[4] J. Hull and A. White. An analysis of the bias in option pricing caused by a stochastic volatility. *Advances in Futures and Options Research*, 3:29–61, 1988.

[5] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, 1988.

[6] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Springer-Verlag, 1991.

[7] L. Scott. Option pricing when the variance changes randomly : theory, estimation and an application. *Journal of Financial and Quantitative Analysis*, 22:419–438, 1987.

[8] E.M. Stein and J.C. Stein. Stock Price Distributions with Stochastic Volatility: An Analytic Approach. *The Review of Financial Studies*, 4(4):727–752, 1991.