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Abstract

In this paper, in the particular case of a concave flux function, we are interested in the long time behaviour of the nonlinear process associated in [8] to the one-dimensional viscous scalar conservation law. We also consider the particle system obtained by replacing the cumulative distribution function in the drift coefficient of this nonlinear process by the empirical cumulative distribution function. We first obtain a trajectorial propagation of chaos estimate which strengthens the weak convergence result obtained in [8] without any convexity assumption on the flux function. Then Poincaré inequalities are used to get explicit estimates concerning the long time behaviour of both the nonlinear process and the particle system.
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Introduction

In this paper, we are interested in the viscous scalar conservation law with C^1 flux function $-A$

$$\partial_t F_t(x) = \frac{\sigma^2}{2}\partial_{xx}F_t(x) + \partial_x(A(F_t(x))),\ F_0(x) = H * m(x). \tag{0.1}$$

where m is a probability measure on the real line and $H(x) = 1_{\{x\geq 0\}}$ denotes the Heaviside function. Since A appears in this equation through its derivative, we suppose without restriction that $A(0) = 0$. According to [8], one may associate the following nonlinear process with the conservation law:

$$\begin{cases} X_t = X_0 + \sigma B_t - \int_0^t A'(H * P_s(X_s))ds, \\ \forall t \geq 0, \text{the law of } X_t \text{ is } P_t. \end{cases} \tag{0.2}$$

where $(B_t)_{t\geq 0}$ is a real Brownian motion independent from the initial random variable X_0 with law m and σ a positive constant. More precisely, according to [8], this nonlinear stochastic differential equation admits a unique weak solution. Moreover, $H * P_t(x)$ is the unique bounded

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weak solution of (0.1). For $t > 0$, by Girsanov theorem, P_t admits a density p_t with respect to the Lebesgue measure on the real line.

We want to address the long time behaviour of the nonlinear process solving (0.2) by studying convergence of the density p_t. Since the cumulative distribution function $x \to H * P_s(x)$ which appears in the drift coefficient is non-decreasing, convexity of A is a natural assumption in order to ensure ergodicity. Then the flux function $-A$ in the conservation law (0.1) is concave.
In the first section of the paper, after recalling results obtained in [8], we show that trajectorial uniqueness holds for (0.2) under convexity of A. Then we introduce a simulable system of n particles obtained by replacing in the drift coefficient the cumulative distribution function by its empirical version and the derivative A' by a suitable finite difference approximation. When A is convex, existence and trajectorial uniqueness hold for this system. Moreover, we prove a trajectorial estimation of propagation of chaos which strengthens the weak convergence result obtained in [8]. Unfortunately, because the empirical cumulative distribution function is a step function and therefore not an increasing one, this estimation is not uniform in time.

The second and main section deals with the long time behaviour of both the nonlinear process and the particle system. In order to ensure that $|\mathbb{E}(X_t)|$ does not go to infinity with t, one has to assume $A(1) = 0$. We address the convergence of the density p_t of X_t by first studying the convergence of the associated solution $H * p_t$ of (0.1) to the solution F_∞ with the same expectation of the stationary equation $\frac{\sigma^2}{2}\partial_{xx}F_\infty(x) + \partial_x(A(F_\infty(x)) = 0$ obtained by removing the time derivative in (0.1). For this result, no convexity hypothesis is made on A. Instead, one assumes $A(u) < 0$ for $u \in (0,1)$, $A'(0) < 0$ and $A'(1) > 0$. In contrast, to prove exponential convergence of the density of the particle system uniform in the number n of particles, we suppose that the function A is uniformly convex. The key step in the proof consists in obtaining a Poincaré inequality for the stationary density of the particle system uniform in n. This density has exponential-like tails and does not satisfy a logarithmic Sobolev inequality. So the derivation of the Poincaré inequality cannot rely on the curvature criterion, used for instance by Malrieu [12] [13] when dealing with the granular media equation. Instead we make a direct estimation of the Poincaré constant using the specific analytic form of the invariant density. To our knowledge, our study provides the first example of a particle system, for which a Poincaré inequality but no logarithmic Sobolev inequality holds uniformly in the number n of particles.

Assumption : In the whole paper, we assume that A is a C^1 function on $[0,1]$ s.t. $A(0) = 0$.

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1 Propagation of chaos

1.1 The nonlinear process

Let us first state existence and uniqueness for the nonlinear stochastic differential equation (0.2).

Theorem 1.1 *The nonlinear stochastic differential equation* (0.2) *admits a unique weak solution* $((X_t, P_t))_{t\geq 0}$. *For* $t > 0$, P_t *admits a density* p_t *with respect to the Lebesgue measure on* $\mathbb{R}$. *The function* $(t,x) \mapsto H * P_t(x)$ *is the unique bounded weak solution of the viscous scalar conservation law* (0.1). *Moreover,*

$$\forall t \geq 0,\ X_t - X_0 \textit{ is integrable and } \mathbb{E}(X_t - X_0) = -A(1)t. \tag{1.1}$$

Last, if the function A is convex on $[0,1]$, (0.2) admits a unique strong solution.

Proof . The first and third statements are consequences of Proposition 1.2 and Theorem 2.1 [8] (uniqueness follows from uniqueness for (0.1) and existence is obtained by a propagation of chaos result).
According to Yamada-Watanabe's theorem, to deduce the last statement, it is enough to check that when A is convex, then trajectorial uniqueness holds for the standard stochastic differential equation

$$dX_t = \sigma dB_t - A'(H * Q_t(X_t))dt$$

where $(Q_t)_{t\geq 0}$ is the flow of time-marginals of a probability measure Q on $C([0,+\infty),\mathbb{R})$. Since for each $t \geq 0$ the function $x \mapsto A'(H * Q_t(x))$ is non-decreasing, if $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ both solve this standard SDE, one has

$$|X_t - Y_t| = |X_0 - Y_0| + \int_0^t \mathrm{sign}(X_s - Y_s)(A'(H * Q_s(Y_s)) - A'(H * Q_s(X_s)))ds \leq |X_0 - Y_0|.$$

Existence of the density p_t for $t > 0$, follows from the boundedness of the drift coefficient and Girsanov theorem. To prove (1.1), one first remarks that by boundedness of the drift coefficient, for each $t \geq 0$, the random variable $X_t - X_0$ is integrable and

$$\mathbb{E}(X_t - X_0) = -\int_0^t \mathbb{E}(A'(H * P_s(X_s)))ds = -\int_0^t \int_{\mathbb{R}} A'\Big(\int_{-\infty}^x P_s(dy)\Big)P_s(dx)ds.$$

For $s > 0$, since by Girsanov theorem P_s does not weight points,

$$\int_{\mathbb{R}} A'\Big(\int_{-\infty}^x P_s(dy)\Big)P_s(dx) = [A(H * P_s(x))]_{-\infty}^{+\infty} = A(1).$$

■

Corollary 1.2 *Assume that A is C^2 on $[0,1]$. Then the function $H*P_t(x)$ is $C^{1,2}$ on $(0,+\infty)\times\mathbb{R}$ and solves (0.1) in the classical sense on this domain.*

Proof . By Girsanov theorem, for $t_0 > 0$, the law P_{t_0} of X_{t_0} admits a density with respect to the Lebesgue measure on $\mathbb{R}$. Hence $(t,x) \mapsto H * P_t(x)$ is a continuous function on $(0,+\infty)\times\mathbb{R}$ with values in $[0,1]$. According to [11] Theorem 8.1 p. 495, Remark 8.1 p.495 and Theorem 2.5 p. 18, there exists a function u with values in $[0,1]$, continuous on $[0,+\infty)\times\mathbb{R}$ and $C^{1,2}$ on $(0,+\infty)\times\mathbb{R}$ such that

$$\forall x \in \mathbb{R},\ u_0(x) = H * P_{t_0}(x) \text{ and } \forall t > 0,\ \partial_t u(t,x) = \frac{\sigma^2}{2}\partial_{xx}u(t,x) + \partial_x(A(u(t,x))).$$

By the uniqueness result for bounded weak solutions of this viscous scalar conservation law recalled in Theorem 1.1, $\forall t \geq t_0$, $H * P_t(x) = u(t - t_0, x)$. The conclusion follows since t_0 is arbitrary. ■

1.2 Study of the particle system

For $n \in \mathbb{N}^*$, let $(a_n(i))_{1\leq i\leq n}$ be a sequence of real numbers. In this section, we are interested in the n-dimensional stochastic differential equation

$$dX_t^{i,n} = \sigma dB_t^i - a_n\left(\sum_{j=1}^n 1_{\{X_t^{j,n}\leq X_t^{i,n}\}}\right)dt,\ X_0^{i,n} = X_0^i,\ 1\leq i\leq n \tag{1.2}$$

where $(B^i)_{i\geq 1}$ are independent standard Brownian motions independent from the sequence $(X_0^i)_{i\geq 1}$ of initial random variables.
In the next section devoted to the approximation of the nonlinear stochastic differential equation (0.2), we will choose $a_n(i)$ equal to the finite difference approximation $n(A(i/n) - A((i-1)/n)$ of $A'(\frac{i}{n})$. For this particular choice, the non-decreasing assumption made in the following proposition is implied by convexity of A.

Proposition 1.3 *Assume that the sequence $(a_n(i))_{1\leq i\leq n}$ is non-decreasing. Then the stochastic differential equation (1.2) has a unique strong solution. Let $(Y_t^{1,n},\ldots,Y_t^{n,n})$ denote another solution starting from $(Y_0^1,\ldots,Y_0^n)$ and driven by the same Brownian motion $(B^1,\ldots,B^n)$. Then*

$$a.s.,\ \forall t\geq 0,\quad \sum_{i=1}^n (X_t^{i,n} - Y_t^{i,n})^2 \leq \sum_{i=1}^n (X_0^i - Y_0^i)^2. \tag{1.3}$$

In addition, if the initial conditions $(X_0^1,\ldots,X_0^n)$ and $(Y_0^1,\ldots,Y_0^n)$ are such that a.s., $\forall i\in\{1,\ldots,n\}$, $X_0^i < Y_0^i$ (resp. $X_0^i\leq Y_0^i$) then

$$a.s.,\ \forall t\geq 0,\ \forall i\in\{1,\ldots,n\},\quad X_t^{i,n} < Y_t^{i,n}\ (\textit{resp. } X_t^{i,n}\leq Y_t^{i,n}). \tag{1.4}$$

Existence of a weak solution to (1.2) is a consequence of Girsanov theorem. Therefore, according to Yamada-Watanabe's theorem, it is enough to prove (1.3) which implies trajectorial uniqueness to obtain existence of a unique strong solution. To do so, we will need the following Lemma.

Lemma 1.4 *Let $(a(i))_{1\leq i\leq n}$ and $(b(i))_{1\leq i\leq n}$ denote two non-decreasing sequences of real numbers. Then for any permutation $\tau\in\mathcal{S}_n$, $\sum_{i=1}^n a(i)b(\tau(i)) \leq \sum_{i=1}^n a(i)b(i)$.*

Proof . For $n = 2$, the result is an easy consequence of the inequality

$$(a(2) - a(1))(b(2) - b(1)) \geq 0.$$

For $n > 2$, we define τ_1 as τ if $\tau(1) = 1$ and as τ composed with the transposition between 1 and $\tau^{-1}(1)$ otherwise. This way, $\tau_1(1) = 1$. In addition, using the result for $n = 2$, we get $\sum_{i=1}^n a(i)b(\tau(i)) \leq \sum_{i=1}^n a(i)b(\tau_1(i))$.
For $2\leq j\leq n-1$, we define inductively τ_j as τ_{j-1} if $\tau_{j-1}(j) = j$ and to τ_{j-1} composed with the transposition between j and $\tau_{j-1}^{-1}(j)$ otherwise. This way, for $1\leq i\leq j$, $\tau_j(i) = i$. Again by the result for $n = 2$, one has

$$\sum_{i=1}^n a(i)b(\tau(i)) \leq \sum_{i=1}^n a(i)b(\tau_1(i)) \leq \sum_{i=1}^n a(i)b(\tau_2(i)) \leq \ldots \leq \sum_{i=1}^n a(i)b(\tau_{n-1}(i)).$$

We conclude by remarking that τ_{n-1} is the identity. ∎

We are now ready to complete the proof of Proposition 1.3.
Proof of Proposition 1.3. Let $(X^{1,n},\ldots,X^{n,n})$ and $(Y^{1,n},\ldots,Y^{n,n})$ denote two solutions. The difference

$$\sum_{i=1}^{n}(X_t^{i,n}-Y_t^{i,n})^2-\sum_{i=1}^{n}(X_0^i-Y_0^i)^2$$

is equal to

$$2\int_0^t\sum_{i=1}^{n}(X_s^{i,n}-Y_s^{i,n})\Bigg(a_n\Bigg(\sum_{j=1}^{n}1_{\{Y_s^{j,n}\leq Y_s^{i,n}\}}\Bigg)-a_n\Bigg(\sum_{j=1}^{n}1_{\{X_s^{j,n}\leq X_s^{i,n}\}}\Bigg)\Bigg)ds. \tag{1.5}$$

By Girsanov theorem, for any $s>0$ the distributions of $(X_s^{1,n},\ldots,X_s^{n,n})$ and $(Y_s^{1,n},\ldots,Y_s^{n,n})$ admit densities w.r.t. the Lebesgue measure on $\mathbb{R}^n$ and therefore $d\mathbb{P}\otimes ds$ a.e. the positions $X_s^{1,n},\ldots,X_s^{n,n}$ (resp. $Y_s^{1,n},\ldots,Y_s^{n,n}$) are distinct and there is a unique permutation $\tau_s^X\in\mathcal{S}_n$ (resp. $\tau_s^Y\in\mathcal{S}_n$) such that $X_s^{\tau_s^X(1),n}<X_s^{\tau_s^X(2),n}<\ldots<X_s^{\tau_s^X(n),n}$ (resp. $Y_s^{\tau_s^Y(1),n}<Y_s^{\tau_s^Y(2),n}<\ldots<Y_s^{\tau_s^Y(n),n}$). Therefore $d\mathbb{P}\otimes ds$ a.e.,

$$\sum_{i=1}^{n}(X_s^{i,n}-Y_s^{i,n})\Bigg(a_n\Bigg(\sum_{j=1}^{n}1_{\{Y_s^{j,n}\leq Y_s^{i,n}\}}\Bigg)-a_n\Bigg(\sum_{j=1}^{n}1_{\{X_s^{j,n}\leq X_s^{i,n}\}}\Bigg)\Bigg)$$

is equal to

$$\sum_{i=1}^{n}a_n(i)\left((X_s^{\tau_s^Y(i),n}-Y_s^{\tau_s^Y(i),n})-(X_s^{\tau_s^X(i),n}-Y_s^{\tau_s^X(i),n})\right).$$

The sequence $(a_n(i))_{1\leq i\leq n}$ is non-decreasing. Applying Lemma 1.4 with $b(i)=X_s^{\tau_s^X(i),n}$ and $\tau=(\tau_s^X)^{-1}\circ\tau_s^Y$ then with $b(i)=Y_s^{\tau_s^Y(i),n}$ and $\tau=(\tau_s^Y)^{-1}\circ\tau_s^X$, one obtains that the integrand in the right-hand-side of (1.5) is non-positive $d\mathbb{P}\otimes ds$ a.e.. Hence (1.3) holds.

Let us now suppose that a.s. $\forall i\in\{1,\ldots,n\}$, $X_0^i<Y_0^i$ and define $\nu=\inf\{t>0:\exists i\in\{1,\ldots,n\},X_t^{i,n}\geq Y_t^{i,n}\}$ with the convention $\inf\emptyset=+\infty$. From now on, we restrict ourselves to the event $\{\nu<+\infty\}$. Let $i\in\{1,\ldots,n\}$ be such that $Y_\nu^{i,n}=X_\nu^{i,n}$. There is an increasing sequence $(s_k)_{k\geq1}$ of positive times with limit ν such that $\forall k\geq1$, $a_n\left(\sum_{j=1}^n 1_{\{X_{s_k}^{j,n}\leq X_{s_k}^{i,n}\}}\right)<a_n\left(\sum_{j=1}^n 1_{\{Y_{s_k}^{j,n}\leq Y_{s_k}^{i,n}\}}\right)$. Since $(a_n(i))_{1\leq i\leq n}$ is non-decreasing, by extracting a subsequence still denoted by $(s_k)_k$ for simplicity, one deduces the existence of $j\in\{1,\ldots,n\}$ with $j\neq i$ such that $\forall k\geq1$, $X_{s_k}^{i,n}<X_{s_k}^{j,n}$ and $Y_{s_k}^{j,n}\leq Y_{s_k}^{i,n}$. Since $s_k<\nu$, $X_{s_k}^{i,n}<X_{s_k}^{j,n}<Y_{s_k}^{j,n}\leq Y_{s_k}^{i,n}$. By continuity of the paths, one obtains $X_\nu^{i,n}=X_\nu^{j,n}=Y_\nu^{j,n}=Y_\nu^{i,n}$. Now since

$$\mathbb{P}\Bigg(\exists i_1,i_2,i_3\text{ distinct in }\{1,\ldots,n\},\ \exists t>0,\ X_0^{i_1}+\sigma B_t^{i_1}=X_0^{i_2}+\sigma B_t^{i_2}=X_0^{i_3}+\sigma B_t^{i_3}\Bigg)=0,$$

Girsanov theorem implies that a.s. $\forall l\in\{1,\ldots,n\}\setminus\{i,j\}$, $X_\nu^{l,n}\neq X_\nu^{i,n}=X_\nu^{j,n}$. In the same way, $Y_\nu^{l,n}\neq Y_\nu^{i,n}=Y_\nu^{j,n}$. By continuity of the paths and definition of ν one deduces that for k large enough,

$$\forall t\in[s_k,\nu],\ \sum_{\substack{l=1\\l\neq i,j}}^{n}1_{\{Y_t^{l,n}\leq Y_t^{i,n}\}}\leq\sum_{\substack{l=1\\l\neq i,j}}^{n}1_{\{X_t^{l,n}\leq X_t^{i,n}\}}\text{ and }\sum_{\substack{l=1\\l\neq i,j}}^{n}1_{\{Y_t^{l,n}\leq Y_t^{j,n}\}}\leq\sum_{\substack{l=1\\l\neq i,j}}^{n}1_{\{X_t^{l,n}\leq X_t^{j,n}\}}.$$

Since a.s. dt a.e., $Y_t^{i,n} \neq Y_t^{j,n}$ and $(a_n(i))_{1\leq i\leq n}$ is non-decreasing, one obtains that a.s. dt a.e. on $[s_k, \nu]$,

$$a_n\left(\sum_{l=1}^n 1_{\{Y_t^{l,n}\leq Y_t^{i,n}\}}\right) + a_n\left(\sum_{l=1}^n 1_{\{Y_t^{l,n}\leq Y_t^{i,n}\}}\right) \leq a_n\left(\sum_{l=1}^n 1_{\{X_t^{l,n}\leq X_t^{i,n}\}}\right) + a_n\left(\sum_{l=1}^n 1_{\{X_t^{l,n}\leq X_t^{i,n}\}}\right).$$

By integration with respect to t on $[s_k, \nu]$, this implies that a.s. $Y_\nu^{i,n} - X_\nu^{i,n} + Y_\nu^{j,n} - X_\nu^{j,n} \geq Y_{s_k}^{i,n} - X_{s_k}^{i,n} + Y_{s_k}^{j,n} - X_{s_k}^{j,n} > 0$. Therefore $\mathbb{P}(\nu < +\infty) = 0$.
When a.s. for $i \in \{1,\dots,n\}$, $X_0^i \leq Y_0^i$, one obtains that for $\varepsilon > 0$ the solution $(Y_t^{1,n,\varepsilon},\dots,Y_t^{n,n,\varepsilon})$ to (1.2) starting from $(Y_0^1+\varepsilon,\dots,Y_0^n+\varepsilon)$ is such that

$$a.s., \ \forall t\geq 0,\ \forall i\in\{1,\dots,n\},\ X_t^{i,n} < Y_t^{i,n,\varepsilon}.$$

Since by (1.3), $Y_t^{i,n,\varepsilon} \leq Y_t^{i,n} + \sqrt{n}\varepsilon$, one easily concludes by letting $\varepsilon \to 0$. ∎

1.3 Trajectorial propagation of chaos

>From now on, we set

$$\forall n\in\mathbb{N}^*,\ \forall i\in\{1,\dots,n\},\quad a_n(i) = n\left(A\left(\frac{i}{n}\right) - A\left(\frac{i-1}{n}\right)\right) \tag{1.6}$$

and assume that the initial positions $(X_0^i)_{i\geq 1}$ of the particles are independent and identically distributed according to m.
In the present section, we also suppose that A is a convex function on $[0,1]$. By Theorem 0.2, for each $i \geq 1$, the nonlinear stochastic differential equation

$$\begin{cases} X_t^i = X_0^i + \sigma B_t^i - \int_0^t A'(H * P_s(X_s^i))ds, \\ \forall t\geq 0, \text{the law of } X_t^i \text{ is } P_t. \end{cases}$$

has a unique solution and for all $t \geq 0$, the law P_t of X_t^i does not depend on i. Under a Lipschitz regularity assumption on A', we obtain the following trajectorial propagation of chaos estimation.

Theorem 1.5 *If $A : [0,1] \to \mathbb{R}$ is convex and A' is Lipschitz continuous with constant K then*

$$\forall n\geq 1,\ \forall 1\leq i\leq n,\ \forall t\geq 0,\quad \mathbb{E}\left(\sup_{s\in[0,t]}(X_s^{i,n} - X_s^i)^2\right) \leq \frac{K^2t^2}{6n}.$$

Proof . One has

$$\sum_{i=1}^n (X_t^{i,n} - X_t^i)^2 = 2\int_0^t \sum_{i=1}^n (X_s^{i,n} - X_s^i)\Bigg(a_n\Bigg(\sum_{j=1}^n 1_{\{X_s^j\leq X_s^i\}}\Bigg) - a_n\Bigg(\sum_{j=1}^n 1_{\{X_s^{j,n}\leq X_s^{i,n}\}}\Bigg)\Bigg)ds$$
$$+ 2\int_0^t \sum_{i=1}^n (X_s^{i,n} - X_s^i) C(s, X_s^1,\dots,X_s^n)\, ds$$

where $C(s, X_s^1,\dots,X_s^n)$ is equal to

$$A'(H * P_s(X_s^i)) - n\Bigg(A\Bigg(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j\leq X_s^i\}}\Bigg) - A\Bigg(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j\leq X_s^i\}} - \frac{1}{n}\Bigg)\Bigg).$$

Like in the proof of trajectorial uniqueness for (1.2), because of the convexity of A, the first term of the r.h.s. is non-positive. Moreover, by Lipschitz continuity of A',

$$\begin{aligned}
\Bigg(A'(H*P_s(X_s^i)) - n\bigg(A\Big(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j\le X_s^i\}}\Big) - A\Big(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j\le X_s^i\}} - \frac{1}{n}\Big)\bigg)\Bigg)^2 \\
&= \bigg(\int_0^1 A'(H*P_s(X_s^i)) - A'\Big(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j\le X_s^i\}} + \frac{\theta-1}{n}\Big)d\theta\bigg)^2 \\
&\le \frac{K^2}{n^2}\int_0^1 \bigg(\sum_{j\neq i}(H*P_s(X_s^i) - 1_{\{X_s^j\le X_s^i\}}) + (H*P_s(X_s^i)-\theta)\bigg)^2 d\theta.
\end{aligned}$$

For $s>0$, as the variables X_s^i are i.i.d. with common law P_s which does not weight points and $H*P_s(X_s^i)$ is uniformly distributed on $[0,1]$,

$$\begin{aligned}
\int_0^1 \mathbb{E}\bigg(\Big(\sum_{j\neq i}(H*P_s(X_s^i) - 1_{\{X_s^j\le X_s^i\}}) + (H*P_s(X_s^i)-\theta)\Big)^2\bigg)d\theta \\
&= \sum_{j\neq i}\mathbb{E}((H*P_s(X_s^i) - 1_{\{X_s^j\le X_s^i\}})^2) + \int_0^1 \mathbb{E}((H*P_s(X_s^i)-\theta)^2)d\theta \\
&= (n-1)\mathbb{E}\Big((H*P_s(X_s^i)(1-H*P_s(X_s^i))\Big) + 1/6 \\
&= n/6.
\end{aligned}$$

Using Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
\mathbb{E}\left(\sup_{s\in[0,t]}\sum_{i=1}^n (X_s^{i,n}-X_s^i)^2\right) &\le 2\int_0^t \sqrt{\frac{K^2}{6n}\mathbb{E}\bigg(\Big(\sum_{i=1}^n (X_s^{i,n}-X_s^i)\Big)^2\bigg)}ds \\
&\le \frac{2K}{\sqrt{6}}\int_0^t \sqrt{\mathbb{E}\Big(\sup_{u\in[0,s]}\sum_{i=1}^n (X_u^{i,n}-X_u^i)^2\Big)}ds.
\end{aligned}$$

By comparison with the ordinary differential equation $\alpha'(t) = 2K\sqrt{\frac{\alpha(t)}{6}}$, one concludes that

$$\forall t\ge 0,\quad \mathbb{E}\left(\sup_{s\in[0,t]}\sum_{i=1}^n (X_s^{i,n}-X_s^i)^2\right) \le \frac{K^2t^2}{6}.$$

Exchangeability of the couples $(X^{i,n}, X^i)$, $i\in\{1,\dots,n\}$ completes the proof. ∎

Remark 1.6 *One could think that assuming that A is uniformly convex:*

$$\exists\alpha>0,\ \forall 0\le x\le y\le 1,\quad A'(y)-A'(x)\ge\alpha(y-x) \tag{1.7}$$

would lead to a better estimation. Indeed, then

$$\forall i\in\{1,\dots,n-1\},\ a_n(i+1)-a_n(i) = n\int_{i/n}^{(i+1)/n}\left[A'(x) - A'\left(x-\frac{1}{n}\right)\right]dx \ge \frac{\alpha}{n}.$$

But since even in this situation, the non-positive term

$$\sum_{i=1}^n (X_s^{i,n}-X_s^i)\bigg(a_n\Big(\sum_{j=1}^n 1_{\{X_s^j\le X_s^i\}}\Big) - a_n\Big(\sum_{j=1}^n 1_{\{X_s^{j,n}\le X_s^{i,n}\}}\Big)\bigg)$$

vanishes as soon as the order between the coordinates of $(X_s^{1,n},\dots,X_s^{n,n})$ *is the same as the order between the coordinates of* $(X_s^1,\dots,X_s^n)$*, we were not able so far to improve the estimation.*

Corollary 1.7 *Under the hypotheses of Theorem 1.5, let* $\tilde{m}$ *be a probability measure on* $\mathbb{R}$ *such that* $\forall x\in\mathbb{R}$, $H*\tilde{m}(x)\leq H*m(x)$*. If for some random variable* U_1 *uniform on* $[0,1]$ *independent from* $(B^i)_{i\geq 1}$, $X_0^1=\inf\{x : H*m(x)\geq U_1\}$ *and* $(Y_t^1)_{t\geq 0}$ *denotes the solution of the nonlinear stochastic differential equation*

$$\begin{cases} Y_t^1 = Y_0^1+\sigma B_t^1-\int_0^t A'(H*\tilde{P}_s(Y_s^1))ds,\\ \forall t\geq 0, \textit{the law of } Y_t^1 \textit{ is } \tilde{P}_t. \end{cases} \tag{1.8}$$

with $Y_0^1=\inf\{x : H*\tilde{m}(x)\geq U_1\}$*, then*

$$\mathbb{P}(\forall t\geq 0,\ X_t^1\leq Y_t^1)=1.$$

Moreover $\forall t\geq 0$, $\forall x\in\mathbb{R}$, $H*\tilde{P}_t(x)\leq H*P_t(x)$*. Last, the function* $t\mapsto \mathbb{E}|Y_t^1-X_t^1|$ *is constant.*

Remark 1.8 *At least when* m *and* $\tilde{m}$ *do not weight points, one has a.s.* $A'(H*P_0(X_0^1)) = A'(H*\tilde{P}_0(Y_0^1))$ *since* $H*m(X_0^1)=H*\tilde{m}(Y_0^1)=U_1$*. Therefore a.s.* $d(Y^1-X^1)_0=0$ *and one may wonder whether a.s.* $Y_t^1-X_t^1$ *does not depend on* t*. If this property holds, necessarily, a.s.* dt *a.e.* $A'(H*P_t(X_t^1))=A'(H*\tilde{P}_t(Y_t^1))$*. If* A' *is increasing, a.s. for all* $t>0$, $H*p_t(X_t^1)=H*\tilde{p}_t(Y_t^1)$ *with* p_t *and* $\tilde{p}_t$ *denoting the respective densities of* P_t *and* $\tilde{P}_t$*. If* A *is* C^2*, the Brownian contribution in* $d\left(H*P_t(X_t^1)-H*\tilde{P}_t(Y_t^1)\right)$ *given by Itô's formula vanishes i.e.* $p_t(X_t^1)=\tilde{p}_t(Y_t^1)$ *and* $\forall u\in]0,1[$, $p_t((H*p_t)^{-1}(u))=\tilde{p}_t((H*\tilde{p}_t)^{-1}(u))$ *or equivalently* $((H*p_t)^{-1})'(u)=((H*\tilde{p}_t)^{-1})'(u)$*. Hence* $Y_t^1=X_t^1+c$ *for a deterministic constant* c *which does not depend on* t *according to (1.1). Letting* $t\to 0$*, one obtains* $Y_0^1=X_0^1+c$*. This necessary condition turns out to be sufficient as* $(X_t^1+c)_{t\geq 0}$ *obviously solves the nonlinear stochastic differential equation (0.2) starting from* X_0^1+c*.*

Proof . For $(U_i)_{i\geq 2}$ a sequence of independent uniform random variables independent from $(U_1,(B^i)_{i\geq 1}))$, we set

$$\forall i\geq 2,\ X_0^i=\inf\{x : H*m(x)\geq U_i\} \text{ and } Y_0^i=\inf\{x : H*\tilde{m}(x)\geq U_i\}.$$

Since $H*\tilde{m}\leq H*m$, a.s. $\forall i\geq 1$, $Y_0^i\geq X_0^i$. From Proposition 1.3, one deduces that the solutions $(X_t^{1,n},\dots,X_t^{n,n})$ and $(Y_t^{1,n},\dots,Y_t^{n,n})$ to (1.2) respectively starting from $(X_0^1,\dots,X_0^n)$ and $(Y_0^1,\dots,Y_0^n)$ are such that

$$a.s.,\ \forall n\geq 1,\ \forall i\in\{1,\dots,n\},\ \forall t\geq 0,\ Y_t^{i,n}\geq X_t^{i,n}.$$

Since, by Theorem 1.5, for fixed $t\geq 0$, one may extract from $(X_t^{1,n},Y_t^{1,n})_{n\geq 1}$ a subsequence almost surely converging to (X_t^1,Y_t^1), one easily deduce that $\mathbb{P}(\forall t\geq 0,\ X_t^1\leq Y_t^1)=1$. Hence

$$\forall t\geq 0,\ \forall x\in\mathbb{R},\ H*\tilde{P}_t(x)=\mathbb{P}(Y_t^1\leq x)\leq \mathbb{P}(X_t^1\leq x)=H*P_t(x).$$

Since $|Y_t^1-X_t^1|-|Y_0^1-X_0^1|=Y_t^1-Y_0^1-(X_t^1-X_0^1)$, (1.1) ensures that $\mathbb{E}|Y_t^1-X_t^1|\in[0,+\infty]$ does not depend on t. ∎

2 Long time behaviour

In this section we are interested in the long time behaviour of both the nonlinear process and the particle system. According to (1.1) and the equality $\sum_{i=1}^{n} a_n(i) = nA(1)$, we have to suppose $A(1) = 0$ in order to obtain convergence of the densities as t tends to infinity. We address the convergence of the density p_t of X_t by first studying the convergence of the associated cumulative distribution function F_t. Then, in addition to the weak condition $A(u) < 0$ for $u \in (0, 1)$, it is enough to make assumptions on the behaviour of A near the boundaries 0 and 1 of the interval $[0, 1]$ (namely $A'(0) < 0$ and $A'(1) > 0$) that determine the spatial behaviour at infinity of the drift coefficient in (0.2).
To prove exponential convergence of the density of the particle system uniform in the number n of particles, we make the stronger assumption of uniform convexity on A. The key step in the proof is to obtain a Poincaré inequality uniform in n for the stationary density of the particle system. This density has exponential-like tails and does not satisfy a logarithmic Sobolev inequality. So the derivation of the Poincaré inequality cannot rely on the curvature criterion, used for instance by Malrieu [12] [13] when dealing with the granular media equation. Instead, we take advantage of the following nice feature : up to reordering of the coordinates, the stationary density is the density of the image by a linear transformation of a vector of independent exponential variables. And it turns out that the control of the constant in the n-dimensional Poincaré inequality relies on the Hardy inequality stated in Lemma 2.16 which is a one-dimensional Poincaré-like inequality. To our knowledge, our study provides the first example of a particle system, for which a Poincaré inequality but no logarithmic Sobolev inequality holds uniformly in the number n of particles.

2.1 The nonlinear process

In this section, we are first going to obtain necessary and sufficient conditions on the function A ensuring existence for the stationary Fokker-Planck equation obtained by removing the time-derivative in the nonlinear Fokker-Planck equation

$$\partial_t p_t = \frac{\sigma^2}{2} \partial_{xx} p_t + \partial_x (A'(H * p_t) p_t) \tag{2.1}$$

satisfied by the density of the solution of (0.2). Under a slightly stronger condition, the solutions satisfy a Poincaré inequality.

Lemma 2.1 *A necessary and sufficient condition for the existence of a probability measure μ solving the stationary Fokker-Planck equation $\frac{\sigma^2}{2}\partial_{xx}\mu + \partial_x(A'(H * \mu(x))\mu) = 0$ in the distribution sense is $A(1) = 0$ and $A(u) < 0$ for all $u \in (0, 1)$. Under that condition, all the solutions are the spatial translations of a probability measure with a C^1 density f which satisfies*

$$\forall x \in \mathbb{R},\ f(x) = -\frac{2}{\sigma^2} A(H * f(x)) \text{ and } f'(x) = -\frac{2}{\sigma^2} A'(H * f(x)) f(x). \tag{2.2}$$

If $A'(0) < 0$ and $A'(1) > 0$, then

$$\begin{matrix} \text{when } x \to -\infty \\ \text{when } x \to +\infty \end{matrix},\ f(x) \sim \begin{cases} -\frac{2A'(0)}{\sigma^2} \int_{-\infty}^{x} f(y)dy \\ \frac{2A'(1)}{\sigma^2} \int_{x}^{+\infty} f(y)dy \end{cases} \quad \text{and} \quad \int_0^x \frac{dy}{f(y)} \sim \begin{cases} \frac{-\sigma^2}{2A'(0)f(x)} \\ \frac{\sigma^2}{2A'(1)f(x)} \end{cases}, \tag{2.3}$$

and all the solutions satisfy a Poincaré inequality and have a finite expectation. Last, if the function A is C^2 on $[0, 1]$, then f is C^2 and satisfies

$$f''(x) = -\frac{2}{\sigma^2} A''(H * f(x)) f^2(x) + \frac{f'^2(x)}{f(x)}. \tag{2.4}$$

Proof . Let μ be a probability measure on $\mathbb{R}$ solving the stationary Fokker-Planck equation. The equality $\frac{\sigma^2}{2}\partial_{xx}\mu = -\partial_x(A'(H*\mu(x))\mu)$ ensures that μ does not weight points. Hence the stationary equation is equivalent to $\partial_{xx}(\frac{\sigma^2}{2}\mu + A(H*\mu(x))) = 0$. One deduces that μ possesses a C^1 density f such that

$$\forall x \in \mathbb{R}, \quad f(x) = -\frac{2}{\sigma^2}A(H*f(x)) + \alpha x + \beta, \tag{2.5}$$

for some constants α and β. Since $A(0) = 0$, letting $x \to -\infty$ then $x \to +\infty$ in the last equality, one obtains $\alpha = \beta = A(1) = 0$. For $u \in (0,1)$, since $u = H*f(x)$ for some $x \in \mathbb{R}$ and $H*f$ is not constant and equal to u, the Cauchy-Lipschitz theorem and (2.5) imply that $A(u) \neq 0$. Since f is non-negative, $A(u) < 0$. Hence $A(1) = 0$ and $A(u) < 0$ for all $u \in (0,1)$ is a necessary condition.

Under that condition, a probability measure μ solves the stationary Fokker-Planck equation iff its cumulative distribution function $H*\mu(x)$ is a C^2 solution to the differential equation

$$\varphi'(x) = -\frac{2}{\sigma^2}A(\varphi(x)),\ x \in \mathbb{R}. \tag{2.6}$$

By the Cauchy-Lipschitz theorem, for each $v \in [0,1]$ this equation admits a unique solution φ_v with values in $[0,1]$ such that $\varphi_v(0) = v$. Moreover, as $A(0) = A(1) = 0$, $\varphi_0 \equiv 0$ and $\varphi_1 \equiv 1$ and

$$\forall v \in (0,1),\ \forall x \in \mathbb{R},\ 0 < \varphi_v(x) < 1. \tag{2.7}$$

For $v \in (0,1)$, since φ_v is non-decreasing and $\varphi_v(x) = v - \frac{2}{\sigma^2}\int_0^x A'(\varphi_v(y))dy$, necessarily $\lim_{y\to+\infty}\varphi_v(y) = 1$. In the same way, $\lim_{y\to-\infty}\varphi_v(y) = 0$ and φ_v is an increasing function from $\mathbb{R}$ to $(0,1)$ with inverse denoted by φ_v^{-1}. The uniqueness result for (2.6) implies that $\forall v \in (0,1),\ \forall x \in \mathbb{R}, \varphi_v(x) = \varphi_{\frac{1}{2}}(x + \varphi_{\frac{1}{2}}^{-1}(v))$. Therefore the solutions to the stationary Fokker-Planck equation are the probability measures obtained by spatial translation of the probability measure with density $f(x) = \varphi'_{\frac{1}{2}}(x)$ which satisfies (2.2) according to (2.6).
Let us now suppose that $A'(0) < 0$ and $A'(1) > 0$. When $x \to +\infty$,

$$f(x) = -\frac{2}{\sigma^2}A\left(1 - \int_x^{+\infty} f(y)dy\right) \sim \frac{2A'(1)}{\sigma^2}\int_x^{+\infty} f(y)dy.$$

By (2.2), $\frac{f'(x)}{f(x)} = (\log f(x))' = -\frac{2}{\sigma^2}A'(\varphi_{\frac{1}{2}}(x))$ converges to $-\frac{2A'(1)}{\sigma^2}$ as $x \to +\infty$. This implies that $\frac{\log(f(x))}{x}$ converges to $-\frac{2A'(1)}{\sigma^2}$ and that $xf(x)1_{\{x\geq 0\}}$ is integrable. Moreover, since $\int_0^{+\infty}\frac{dy}{f(y)} = +\infty$, $\int_0^x \frac{dy}{f(y)} \sim \frac{\sigma^2}{2A'(1)}\int_0^x -\frac{f'(y)}{f^2(y)}dy \sim \frac{\sigma^2}{2A'(1)f(x)}$, as $x \to +\infty$. In the same way, one obtains the equivalents given in (2.3) when $x \to -\infty$ and checks the integrability of the function $xf(x)1_{\{x\leq 0\}}$.
From (2.3), one has

$$\lim_{x\to-\infty}\int_{-\infty}^x f(y)dy\int_x^0\frac{dy}{f(y)} = \frac{\sigma^4}{4(A'(0))^2} \text{ and } \lim_{x\to+\infty}\int_x^{+\infty} f(y)dy\int_0^x\frac{dy}{f(y)} = \frac{\sigma^4}{4(A'(1))^2}.$$

By Theorem 6.2.2 p.99 [1], one concludes that the measure with density f satisfies a Poincaré inequality.
By (2.2), the function f is C^2 as soon as the function A is C^2 on $[0,1]$. Moreover, $f''(x) = -\frac{2}{\sigma^2}A''(H*f(x))f^2(x) - \frac{2}{\sigma^2}A'(H*f(x))f'(x)$ which combined with (2.2) implies (2.4).

∎

Remark 2.2 *When A is a C^1 convex function on $[0,1]$ such that $A(0)=A(1)=0$ and $A'(u)<0$ for some $u\in(0,1)$, then the necessary and sufficient condition in Lemma 2.1 is obviously satisfied. Moreover, since (2.5) with $\alpha=\beta=0$ implies*

$$(\log f(x))''=\left(\frac{f'(x)}{f(x)}\right)'=\left(\frac{-\frac{2}{\sigma^2}A'(H*f(x))f(x)}{f(x)}\right)'=-\frac{2}{\sigma^2}A''(H*f(x))f(x)\leq 0,$$

the probability measures solving the stationary Fokker-Planck equation admit log-concave densities with respect to the Lebesgue measure.

Example . The following three choices for A lead to exact computations and different tails for the stationary densities:

- if $A(x)=\frac{1}{2}x(x-1)$, one gets $\log\left(\frac{F_{\frac{1}{2}}(x)}{1-F_{\frac{1}{2}}(x)}\right)=x/\sigma^2$ i.e.

$$F_{\frac{1}{2}}(x)=\frac{e^{x/\sigma^2}}{1+e^{x/\sigma^2}}\quad\text{and}\quad F'_{\frac{1}{2}}(x)=\frac{1}{4\sigma^2\cosh^2(x/2\sigma^2)};$$

- if $A(x)=x^3-x=x(x-1)(x+1)$,

$$F_{\frac{1}{2}}(x)=\frac{1}{\sqrt{1+e^{-4x/\sigma^2}}}\quad\text{and}\quad F'_{\frac{1}{2}}(x)=\frac{2e^{-4x/\sigma^2}}{\sigma^2(1+e^{-4x/\sigma^2})^{3/2}};$$

- if $A(x)=(1-x)\log(1-x)$ and $\sigma^2=2$, one gets $\log\left(\frac{\log(1-F(x))}{\log(1-F(0))}\right)=\frac{2x}{\sigma^2}$ i.e.

$$F_{\frac{1}{2}}(x)=1-\exp\left(-\log(2)e^{2x/\sigma^2}\right)\quad\text{and}\quad F'_{\frac{1}{2}}(x)=\log(2)\exp\left(2x/\sigma^2-\log(2)e^{2x/\sigma^2}\right).$$

In the third example, the C^1 assumption on A is relaxed and one remarks that when the derivative A' is infinite at 0 or 1, then the corresponding tail of the invariant densities can be really small. ∎

When $A(1)=0$ and $A(u)<0$ for all $u\in(0,1)$, a natural question is how to link the translation parameter of the candidate long time limit of the marginal P_t solving the stationary Fokker-Planck equation to the initial marginal m. When $\int_{\mathbb{R}}|x|m(dx)<+\infty$, by (1.1), for all $t\geq 0$, $\mathbb{E}(X_t^1)=\mathbb{E}(X_0^1)$. Therefore the translation parameter is chosen in order to ensure that the invariant measure has the same mean as the initial measure m.
Let us denote by p_t the density of P_t and by $F_t=H*P_t$ its cumulative distribution function.

Theorem 2.3 *Let A be C^2 on $[0,1]$ and such that $A(1)=0$, $\forall u\in(0,1)$, $A(u)<0$, $A'(0)<0$ and $A'(1)>0$. Assume that m admits a density p_0 such that $\int_{\mathbb{R}}|x|p_0(x)dx<+\infty$ and $\int_{\mathbb{R}}\frac{(p_0-p_\infty)^2}{p_\infty}$ is small enough where p_∞ denotes the stationary distribution with same expectation as p_0. Last, we suppose that A and p_0 are such that p is a smooth solution of (2.1). Then $\int_{\mathbb{R}}\frac{(p_t-p_\infty)^2}{p_\infty}$ converges to 0 exponentially fast as $t\to+\infty$.*

By a smooth solution of (2.1), we mean that p possesses enough regularity and integrability so that the formal computations made in the proof below are justified.

Example . When $A(x) = \frac{1}{2}(x^2 - x)$, one easily checks that $\phi(t,x) = -F_t(x + \frac{t}{2})$ solves the Burgers equation

$$\partial_t \phi = \frac{\sigma^2}{2}\partial_{xx}\phi - \frac{1}{2}\partial_x \phi^2,\ \phi(0,x) = -F_0(x).$$

According to the Cole-Hopf transformation, $\psi(t,x) = \exp\left(-\frac{1}{\sigma^2}\int_{-\infty}^x \phi(t,y)dy\right)$ solves the heat equation

$$\partial_t \psi = \frac{\sigma^2}{2}\partial_{xx}\psi,\ \psi(0,x) = \exp\left(\frac{1}{\sigma^2}\int_{-\infty}^x F_0(y)dy\right).$$

Since $F_t(x) = \sigma^2 \frac{\partial_x \psi}{\psi}(t, x - \frac{t}{2})$, one deduces that

$$F_t(x) = \frac{\int_{\mathbb{R}} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} F_0(y)\psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}}{\int_{\mathbb{R}} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} \psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}}. \tag{2.8}$$

If $\bar{x}$ denotes the expectation associated with the cumulative distribution function F_0, one has $\int_{-\infty}^{\bar{x}} F_0(z)dz = \int_{\bar{x}}^{+\infty}(1 - F_0(z))dz$. Since

$$\int_{-\infty}^x F_0(z)dz = \int_{-\infty}^{\bar{x}} F_0(z)dz - \int_{\bar{x}}^x (1 - F_0(z))dz + (x - \bar{x}),$$

one deduces that the function $\tilde{\psi}(0,x) = e^{-\frac{x-\bar{x}}{\sigma^2}}\psi(0,x)$ (resp. $\psi(0,x)$) is bounded on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) and converges to 1 as x tends to $+\infty$ (resp. $-\infty$).
Let us deduce the limit of $F_t(x)$ as $t \to +\infty$. Writing the integral for $y \in \mathbb{R}$ as the sum of the integrals for $y \in \mathbb{R}_-$ and for $y \in \mathbb{R}_+$, and making the change of variables $z = \frac{y-x+\frac{t}{2}}{\sigma\sqrt{t}}$ (resp. $z = \frac{y-x-\frac{t}{2}}{\sigma\sqrt{t}}$) in the first (resp. second) integral, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-\frac{(y-x+\frac{t}{2})^2}{2\sigma^2 t}} F_0(y)\psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}} \\
&\quad = \int_{\mathbb{R}} e^{-\frac{z^2}{2}} 1_{\{z \le \frac{\sqrt{t}}{2\sigma} - \frac{x}{\sigma\sqrt{t}}\}} F_0(\sigma\sqrt{t}z + x - \frac{t}{2})\psi(0, \sigma\sqrt{t}z + x - \frac{t}{2})\frac{dz}{\sqrt{2\pi}} \\
&\quad + e^{\frac{x-\bar{x}}{\sigma^2}} \int_{\mathbb{R}} e^{-\frac{z^2}{2}} 1_{\{z \ge -\frac{\sqrt{t}}{2\sigma} - \frac{x}{\sigma\sqrt{t}}\}} F_0(\sigma\sqrt{t}z + x + \frac{t}{2})\tilde{\psi}(0, \sigma\sqrt{t}z + x + \frac{t}{2})\frac{dz}{\sqrt{2\pi}}.
\end{aligned}$$

By Lebesgue theorem, the first term of the right-hand-side converges to 0 whereas the second term converges to $e^{\frac{x-\bar{x}}{\sigma^2}}$. Replacing F_0 by 1 in the above computation, one obtains that the denominator in (2.8), converges to $1 + e^{\frac{x-\bar{x}}{\sigma^2}}$. Therefore

$$\forall x \in \mathbb{R},\ \lim_{x \to +\infty} F_t(x) = \frac{e^{\frac{x-\bar{x}}{\sigma^2}}}{1 + e^{\frac{x-\bar{x}}{\sigma^2}}}.$$

Notice that in the same way, one may also obtain the limit of the density

$$p_t(x) = \frac{\int_{\mathbb{R}} \frac{y+\frac{t}{2}-x}{\sigma^2 t} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} F_0(y)\psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}}{\int_{\mathbb{R}} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} \psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}} - \frac{1}{\sigma^2}\left(\frac{\int_{\mathbb{R}} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} F_0(y)\psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}}{\int_{\mathbb{R}} e^{-\frac{(x-\frac{t}{2}-y)^2}{2\sigma^2 t}} \psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}}}\right)^2.$$

One easily checks

$$\forall x \in \mathbb{R},\ \lim_{t\to+\infty} p_t(x) = \frac{1}{\sigma^2}\left(\frac{e^{\frac{x-\bar{x}}{\sigma^2}}}{1+e^{\frac{x-\bar{x}}{\sigma^2}}} - \frac{e^{\frac{2(x-\bar{x})}{\sigma^2}}}{\left(1+e^{\frac{x-\bar{x}}{\sigma^2}}\right)^2}\right) = \frac{1}{4\sigma^2\cosh^2(\frac{x-\bar{x}}{2\sigma^2})}.$$

∎

In order to prove Theorem 2.3, we are first going to check exponential convergence of F_t to the cumulative distribution function F_∞ of p_∞. Let $G_t = F_t - F_\infty$. Since for a random variable X with cumulative distribution function F, $\mathbb{E}(X) = \int_0^{+\infty}(1-F(x))dx - \int_{-\infty}^0 F(x)dx$ the equality of the expectations associated to F_t and F_∞ writes $\int_{\mathbb{R}} G_t = 0$. This very convenient expression of the link between p_t and p_∞ is one main reason for first considering the convergence of G_t to 0. In order to prove this convergence, we need the following result.

Lemma 2.4 *One has*

$$\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \le c\int_{\mathbb{R}}\left(\frac{G_t}{p_\infty}\right)'^2 p_\infty \tag{2.9}$$

where c denotes the constant in the Poincaré inequality satisfied by p_∞. Moreover

$$\int_{\mathbb{R}} \frac{(p_t-p_\infty)^2}{p_\infty} = \int_{\mathbb{R}}\left(\frac{G_t}{p_\infty}\right)'^2 p_\infty + \frac{2}{\sigma^2}\int_{\mathbb{R}} G_t^2 A''(F_\infty) \tag{2.10}$$

$$\textit{and}\quad \int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \le \tilde{c}\int_{\mathbb{R}} \frac{(p_t-p_\infty)^2}{p_\infty}. \tag{2.11}$$

Remark 2.5 *When A is convex,* (2.11) *is a consequence of* (2.10) *and* (2.9).

Proof . As $\int_{\mathbb{R}} G_t = 0$, (2.9) is nothing but the Poincaré inequality satisfied by p_∞ written for the function G_t/p_∞.
Since $\left(\frac{G_t}{p_\infty}\right)' = \frac{G_t'}{p_\infty} - \frac{G_t p_\infty'}{p_\infty^2}$, one has

$$\begin{aligned}\int_{\mathbb{R}}\left(\frac{G_t}{p_\infty}\right)'^2 p_\infty &= \int_{\mathbb{R}} \frac{(p_t-p_\infty)^2}{p_\infty} - \int_{\mathbb{R}} \frac{G_t^{2\prime} p_\infty'}{p_\infty^2} + \int_{\mathbb{R}} \frac{G_t^2 p_\infty'^{\,2}}{p_\infty^3}\\ &= \int_{\mathbb{R}} \frac{(p_t-p_\infty)^2}{p_\infty} + \int_{\mathbb{R}} \frac{G_t^2 p_\infty''}{p_\infty^2} - \int_{\mathbb{R}} \frac{G_t^2 p_\infty'^{\,2}}{p_\infty^3}.\end{aligned}$$

Since p_∞ solves (2.4), one easily deduces (2.10).
Writing $G_t^2(y) = 2\left(1_{\{y\le 0\}}\int_{-\infty}^y G_t(p_t-p_\infty)(x)dx - 1_{\{y>0\}}\int_y^{+\infty} G_t(p_t-p_\infty)(x)dx\right)$, one obtains

$$\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} = -2\int_{\mathbb{R}} G_t(p_t-p_\infty)(x)\int_0^x \frac{1}{p_\infty(y)}dydx. \tag{2.12}$$

By (2.3), and since $\frac{1}{p_\infty}$ is bounded from below and above on each compact subset of the real line,

$$\exists C>0, \forall x\in\mathbb{R},\ \left|\int_0^x \frac{1}{p_\infty(y)}dy\right| \le \frac{C}{p_\infty(x)}.$$

Using Cauchy-Schwarz inequality in (2.12), and inserting the latter bound, one obtains

$$\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \leq 2C \left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \right)^{1/2} \left(\int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty} \right)^{1/2}.$$

One easily deduces (2.11). ∎

According to (2.11), the exponential convergence of $\int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty}$ to zero is a stronger result than the exponential convergence stated in the next Lemma.

Lemma 2.6 *There is a positive constant C such that if $\int_{\mathbb{R}} \frac{G_0^2}{p_\infty}$ is small enough, then $\forall t \geq 0$, $\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \leq \frac{e^{-Ct}}{C} \int_{\mathbb{R}} \frac{G_0^2}{p_\infty}$.*

Proof . According to (2.2), one has $\frac{\sigma^2}{2} F_\infty'' + (A(F_\infty))' = 0$ which also writes $\frac{p_\infty'}{p_\infty} = -\frac{2}{\sigma^2} A'(F_\infty)$. Combining these equations with (0.1), then using Young's inequality, one easily obtains for $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} &= -\frac{\sigma^2}{2}\int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty - \int_{\mathbb{R}} (A(F_t) - A(F_\infty) - A'(F_\infty)G_t) \left(\frac{G_t}{p_\infty}\right)' \\ &\leq (\varepsilon - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty + \frac{\|A''\|_\infty^2}{16\varepsilon} \int_{\mathbb{R}} \frac{G_t^4}{p_\infty}. \end{aligned} \tag{2.13}$$

Since

$$\|G_t\|_\infty^2 \leq \left(\int_{\mathbb{R}} \frac{|p_t - p_\infty|}{\sqrt{p_\infty}} \sqrt{p_\infty} \right)^2 \leq \int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty}, \tag{2.14}$$

$|G_t|$ is bounded by 1 and $p_\infty A''(F_\infty) = -\frac{2}{\sigma^2} A \times A''(F_\infty)$ is bounded, one deduces from (2.10) that

$$\|G_t\|_\infty^2 \leq \frac{4}{\sigma^4} \|AA''\|_\infty \int_{\mathbb{R}} \frac{G_t^2}{p_\infty} + \left(1 \wedge \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty \right).$$

Inserting this bound in (2.13) and using Young's inequality, one deduces that for $\eta > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} &\leq (\varepsilon - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty + \frac{\|AA''\|_\infty \|A''\|_\infty^2}{4\varepsilon\sigma^4} \left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \right)^2 \\ &\quad + \eta \left(1 \wedge \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty \right)^2 + \frac{\|A''\|_\infty^4}{1024\varepsilon^2\eta} \left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \right)^2 \\ &\leq (\varepsilon + \eta - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty + \left(\frac{\|AA''\|_\infty \|A''\|_\infty^2}{4\varepsilon\sigma^4} + \frac{\|A''\|_\infty^4}{1024\varepsilon^2\eta} \right) \left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty} \right)^2.\end{aligned}$$

One easily concludes with (2.9) and Lemma 2.8 below. ∎

Remark 2.7 • *After reading this proof, one may wonder whether one could replace the upper-bound in* (2.13) *by*

$$(\varepsilon - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}\right)'^2 p_\infty + \frac{\|A''\|_\infty^2}{16\varepsilon} \int_{\mathbb{R}} \frac{G_t^2}{p_\infty}$$

using $\|G_t\|_\infty \leq 1$. If the constant c in the Poincaré inequality (2.9) was smaller than $\frac{\sigma^4}{\|A''\|_\infty^2}$, one could deduce exponential convergence of $\int \frac{G_t^2}{p_\infty}$ to 0 even for large values of $\int \frac{G_0^2}{p_\infty}$. In case $A(x) = \frac{1}{2}(x^2 - x)$, one has $\|A''\|_\infty = 1$ and

$$c \geq \int_{\mathbb{R}} x^2 p_\infty - \left(\int_{\mathbb{R}} x p_\infty \right)^2 = \int_0^{+\infty} \frac{x^2}{2\sigma^2 \cosh^2(\frac{x}{2\sigma^2})} > 4\sigma^4 \int_0^{+\infty} y^2 e^{-2y} = \sigma^4 = \frac{\sigma^4}{\|A''\|_\infty^2}$$

and this approach does not work.

- *Convexity of A implies non-negativity of the term $A(F_t) - A(F_\infty) - A'(F_\infty)G_t$ which appears in the right-hand-side of the first displayed equality in the proof. One may wonder if one could exploit this property to obtain exponential convergence of p_t to p_∞ even if p_0 is not close to p_∞. We have not been able to do so.*

Proof of Theorem 2.3. By (2.2), $p'_\infty = -\frac{2}{\sigma^2} A'(F_\infty) p_\infty$ and $\|p_\infty\|_\infty \leq \frac{2\|A\|_\infty}{\sigma^2}$. Using moreover the Fokker-Planck equation (2.1) for p_t then Young's inequality and (2.14), one easily checks that for $\varepsilon, \eta > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty} &= -\frac{\sigma^2}{2} \int_{\mathbb{R}} \left(\frac{p_t}{p_\infty} \right)'^2 p_\infty - \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty))(p_t - p_\infty) \left(\frac{p_t}{p_\infty} \right)' \\
&\quad - \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty)) p_\infty \left(\frac{p_t}{p_\infty} \right)' \\
&\leq (\eta + \varepsilon - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{p_t}{p_\infty} \right)'^2 p_\infty + \frac{1}{4\varepsilon} \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty))^2 \frac{(p_t - p_\infty)^2}{p_\infty} \\
&\quad + \frac{1}{4\eta} \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty))^2 p_\infty \\
&\leq (\eta + \varepsilon - \frac{\sigma^2}{2}) \int_{\mathbb{R}} \left(\frac{p_t}{p_\infty} \right)'^2 + \frac{\|A''\|_\infty^2}{4\varepsilon} \left(\int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty} \right)^2 \\
&\quad + \frac{\|A''\|_\infty^2}{4\eta} \times \frac{4\|A\|_\infty^2}{\sigma^4} \int_{\mathbb{R}} \frac{G_t^2}{p_\infty}.
\end{aligned}$$

By (2.11) and Lemma 2.6, for $\int_{\mathbb{R}} \frac{(p_0 - p_\infty)^2}{p_\infty}$ small enough, the last term of the r.h.s. is smaller than $\frac{\tilde{c} e^{-Ct}}{C} \int_{\mathbb{R}} \frac{(p_0 - p_\infty)^2}{p_\infty}$. Since $\int_{\mathbb{R}} \left(\frac{p_t}{p_\infty} \right)'^2 p_\infty \geq \frac{1}{c} \int_{\mathbb{R}} \frac{(p_t - p_\infty)^2}{p_\infty}$, one easily concludes by Lemma 2.8 below. ∎

Lemma 2.8 *Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies*

$$\forall t \geq 0, \ \frac{du}{dt}(t) \leq \beta u(t)(u(t) - \alpha) + \gamma e^{-\delta t}$$

for some constants $\alpha, \beta, \delta > 0$ and $\gamma \geq 0$.
If $\gamma = 0$ and $u(0) < \alpha$ then $\forall t \geq 0, \ u(t) \leq \dfrac{\alpha u(0) e^{-\alpha\beta t}}{\alpha + u(0)(e^{-\alpha\beta t} - 1)}$.
If $u(0) < \frac{\alpha}{2}$ and $\gamma < \frac{\beta\alpha^2}{4}$ then $u(t)$ converges to 0 exponentially fast as $t \to +\infty$.

Proof . When $\gamma = 0$, as long as $u(t) \in (0, \alpha)$, one has $\frac{du}{dt}(t) \left(\frac{1}{u(t)} + \frac{1}{\alpha - u(t)} \right) \leq -\alpha\beta$ and after

integration one obtains the desired estimation. Since the upper-bound is not greater than $u(0)$ and $u(t)=0 \Rightarrow \forall s \geq t,\ u(s)=0$ one easily concludes.
Now when $\gamma \in (0, \frac{\beta\alpha^2}{4})$, one has $\beta a(\alpha - a) = \gamma$ for some $a \in (0, \frac{\alpha}{2})$ and

$$\frac{d}{dt}(u(t) \wedge \frac{\alpha}{2} - a)^+ = 1_{\{a<u(t)<\frac{\alpha}{2}\}} \frac{du}{dt}(t) \leq 0.$$

Hence when $u(0) < \frac{\alpha}{2}$, $\forall t \geq 0$, $u(t) \leq u(0) \vee a$ and

$$\frac{du}{dt}(t) \leq -\beta(\alpha - u(0) \vee a)u(t) + \gamma e^{-\delta t}.$$

For $v(t) = e^{\beta(\alpha-u(0)\vee a)t}u(t)$ one deduces

$$\frac{dv}{dt}(t) \leq \gamma e^{(\beta(\alpha-u(0)\vee a)-\delta)t}$$

and one concludes by integration of this inequality that $u(t) \leq Ce^{-[(\beta(\alpha-u(0)\vee a))\wedge\delta]t}$. ∎

2.2 The particle system

Let us suppose that $A(1)=0$ and that the first order moment associated with the initial probability measure m is defined and equal to $\bar{x}$. As in the case of the granular media equation considered by Malrieu [12] [13], the direction $(v, v, \ldots, v)$ is quite singular for the particle system. Indeed,

$$d(X_t^{1,N} + \ldots + X_t^{N,N}) = \sigma \sum_{i=1}^{n} dB_t^i,$$

which prevents the law of $(X_t^{1,N}, \ldots, X_t^{N,N})$ from converging as $t \to +\infty$. Following [12] [13], one introduces the hyperplane $\mathcal{M}_n = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n :\ y_1 + \ldots + y_n = n\bar{x}\}$ orthogonal to this singular direction and denotes by $\bar{P}$ the orthogonal projection on $\mathcal{M}_n$ and by P the orthogonal projection on $\{y = (y_1, \ldots, y_n) \in \mathbb{R}^n :\ y_1 + \ldots + y_n = 0\}$. Since $\sum_{i=1}^n a_n(i) = n(A(1) - A(0)) = 0$, the orthogonal projection $(Y_t^{i,n} = \bar{x} + X_t^{i,n} - \frac{1}{n}\sum_{j=1}^n X_t^{j,n})_{1\leq i\leq n}$ of the original particle system on $\mathcal{M}_n$ is a diffusion on this hyperplane solving

$$dY_t^{i,n} = \sigma\frac{n-1}{n}dB_t^i - \frac{\sigma}{n}\sum_{j\neq i} dB_t^j - a_n\left(\sum_{j=1}^n 1_{\{Y_t^{j,n} \leq Y_t^{i,n}\}}\right)dt. \tag{2.15}$$

Propagation of chaos for the projected system is a consequence of the following estimate.

Proposition 2.9 *Assume that A is convex, such that A' is Lipschitz continuous with constant K and $A(1)=0$ and that the initial measure m has a finite second order moment. Then*

$$\forall i \in \{1, \ldots, n\},\ \forall t \geq 0,$$
$$\mathbb{E}\Big((X_t^i - Y_t^{i,n})^2\Big) \leq \frac{1}{n}\left(\frac{K^2t^2}{6} + \mathbb{E}((X_0 - \bar{x})^2) + \sigma^2 t + 2\int_0^t \int_{\mathbb{R}} A(F_s(x))dxds\right).$$

Proof . Denoting

$$X_1^n(t) = (X_t^1, \ldots, X_t^n),\ X_1^{n,n}(t) = (X_t^{1,n}, \ldots, X_t^{n,n}) \text{ and } Y_1^{n,n}(t) = (Y_t^{1,n}, \ldots, Y_t^{n,n}),$$

one has

$$|X_1^n(t)-Y_1^{n,n}(t)|^2 = |X_1^n(t)-\bar{P}X_1^{n,n}(t)|^2 = |X_1^n(t)-\bar{P}X_1^n(t)|^2 + |\bar{P}X_1^n(t)-\bar{P}X_1^{n,n}(t)|^2$$
$$\leq \frac{1}{n}\left(\sum_{i=1}^n (X_t^i-\bar{x})\right)^2 + \sum_{i=1}^n (X_t^i - X_t^{i,n})^2. \quad (2.16)$$

Since $(X_t-\bar{x})^2 \leq 3\left((X_0-\bar{x})^2+\sigma^2 B_t^2+\|A'\|_\infty^2 t^2\right)$, the variable X_t is square integrable. As

$$\forall x>0,\ |(x-\bar{x})A(F_t(x))| \leq \|A'\|_\infty (1-F_t(x))(x+|\bar{x}|) \leq \|A'\|_\infty \left(\frac{\mathbb{E}(X_t^2)}{x}+|\bar{x}|(1-F_t(x))\right),$$

one has $\lim_{x\to+\infty}(x-\bar{x})A(F_t(x))=0$. Similarly $(x-\bar{x})A(F_t(x))$ also vanishes as $x\to-\infty$ and $\int_{\mathbb{R}}(x-\bar{x})A'(F_t(x))p_t(x)dx = -\int_{\mathbb{R}} A(F_t(x))dx$. Computing $(X_t-\bar{x})^2$ by Itô's formula and taking expectations, one deduces that

$$\mathbb{E}((X_t-\bar{x})^2) = \mathbb{E}((X_0-\bar{x})^2)+\sigma^2 t + 2\int_0^t \int_{\mathbb{R}} A(F_s(x))dxds.$$

Moreover, by (1.1), $\mathbb{E}(X_t-\bar{x}) = -A(1)t = 0$. One concludes by taking expectations in (2.16) then using Theorem 1.5 and exchangeability of the particles. ∎

Let us now study the long time behaviour of the projected particle system.

Theorem 2.10 *Assume that the function A is uniformly convex on $[0,1]$ with constant α (see (1.7)) and such that $A(1)=0$. Then, the probability measure with density*

$$p_\infty^n(y) = \frac{1}{Z_n} e^{-\frac{2}{\sigma^2}\sum_{i=1}^n a_n(i) y_{(i)}}$$

with respect to the Lebesgue measure dy on $\mathcal{M}_n$ is invariant for the projected dynamics. Here $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(n)}$ denotes the increasing reordering of the coordinates of $y=(y_1,\ldots,y_n)$ and $Z_n = \int_{\mathcal{M}_n} e^{-\frac{2}{\sigma^2}\sum_{i=1}^n a_n(i)y_{(i)}}dy$.
Moreover, if $(Y_0^{1,n},\ldots,Y_0^{n,n})$ admits a symmetric density $p_0^n(y)$ with respect to the Lebesgue measure on $\mathcal{M}_n$, then for all $t\geq 0$ $(Y_t^{1,n},\ldots,Y_t^{n,n})$ admits a symmetric density $p_t^n(y)$ which is such that

$$\forall t\geq 0,\ \int_{\mathcal{M}_n}\left(\frac{p_t^n}{p_\infty^n}-1\right)^2 p_\infty^n dy \leq e^{-\lambda_n t}\int_{\mathcal{M}_n}\left(\frac{p_0^n}{p_\infty^n}-1\right)^2 p_\infty^n dy \quad (2.17)$$

where the sequence $(\lambda_n)_n$ is bounded from below by $\frac{\alpha^2}{12^3\sigma^2}$.

In order to deduce long time properties of the nonlinear process from long time properties of the projected system, the symmetry hypothesis on p_0^n is not restrictive. But the lack of uniformity in time of the estimation given in Proposition 2.9 is a real problem.

Remark 2.11 *In case $n=2$, the process $Z_t = Y_t^{2,2}-Y_t^{1,2}$ solves the stochastic differential equation*

$$dY_t = \sigma(dB_t^2-dB_t^1) - \mathrm{sgn}(Y_t)(a_2(2)-a_2(1))dt$$

and the density of Y_t converges exponentially to $\frac{a_2(2)-a_2(1)}{2\sigma^2}e^{-\frac{a_2(2)-a_2(1)}{\sigma^2}|y|}$ when the density of Y_0 is close enough to this limit. As $(Y_t^{1,2},Y_t^{2,2}) = \frac{1}{2}(-Z_t,Z_t)$, one easily deduces exponential convergence of the density of $(Y_T^{1,2},Y_T^{2,2})$ on the straight line $\mathcal{M}_2$ to $\frac{a_2(2)-a_2(1)}{\sqrt{2}\sigma^2}e^{-\frac{a_2(2)}{\sigma^2}2y_{(2)}}e^{\frac{a_2(1)}{\sigma^2}(-2y_{(1)})}$.

The proof of Theorem 2.10 relies on the following Poincaré inequality.

Proposition 2.12 *Under the assumptions of Theorem 2.10, the density*

$$\tilde{p}^n_\infty(y) = \frac{n!1_{\{y_1\leq y_2\leq\ldots\leq y_n\}}}{Z_n}e^{-\frac{2}{\sigma^2}\sum_{i=1}^n a_n(i)y_i}$$

on $\mathcal{M}_n$ is such that for $f:\mathbb{R}^n\to\mathbb{R}$ regular enough,

$$\int_{\mathcal{M}_n}\left(f(y)-\int_{\mathcal{M}_n}f(y)\tilde{p}^n_\infty(y)dy\right)^2\tilde{p}^n_\infty(y)dy\leq\frac{\sigma^2}{\lambda_n}\int_{\mathcal{M}_n}P|\nabla f(y)|^2\tilde{p}^n_\infty(y)dy \tag{2.18}$$

where the sequence $(\lambda_n)_n$ is bounded from below by $\frac{\alpha^2}{12^3\sigma^2}$.

Proof of Theorem 2.10. Let us first check the following Green formula: for $f:\mathbb{R}^n\to\mathbb{R}$ and $u:\mathbb{R}^n\to\mathbb{R}^n$ regular enough,

$$\int_{\mathcal{M}_n}f\nabla\cdot Pu(y)dy=-\int_{\mathcal{M}_n}P\nabla f\cdot Pu(y)dy. \tag{2.19}$$

Let $\mathbf{1}\in\mathbb{R}^n$ denote the vector with all coordinates equal to 1. For $\varphi:\mathbb{R}\to\mathbb{R}$ and $v:\mathbb{R}^n\to\mathbb{R}^n$, one has

$$\begin{aligned}\int_{\mathbb{R}}\varphi(\sqrt{n}z)\int_{\mathcal{M}_n}\nabla\cdot Pv\left(y+\frac{z\mathbf{1}}{\sqrt{n}}\right)dydz&=\int_{\mathbb{R}^n}\varphi(x_1+\ldots+x_n-n\bar{x})\nabla\cdot Pv(x)dx\\&=-\int_{\mathbb{R}^n}\varphi'(x_1+\ldots+x_n-n\bar{x})\mathbf{1}\cdot Pv(x)dx=0.\end{aligned}$$

The function φ being arbitrary, one deduces that $\int_{\mathcal{M}_n}\nabla\cdot Pv(y)dy=0$. Since $\nabla\cdot P(fu)=\nabla f\cdot Pu+f\nabla\cdot Pu=P\nabla f\cdot Pu+f\nabla\cdot Pu$, (2.19) follows for the choice $v=fu$.
By weak uniqueness for (2.15), when $(Y_0^{1,n},\ldots,Y_0^{n,n})$ has a symmetric density p_0^n with respect to the Lebesgue measure on $\mathcal{M}_n$, the particles $Y^{i,n}$, $i\in\{1,\ldots,n\}$ are exchangeable and for each $t\geq0$, $(Y_t^{1,n},\ldots,Y_t^{n,n})$ has a symmetric density p_t^n. By composition with the projection $\bar{P}$, one obtains an extension of p_t^n on $\mathbb{R}^n$ that we still denote by p_t^n. Since $\sum_{i=1}^n a_n(i)=n(A(1)-A(0))=0$, setting

$$b(y)=\sum_{\tau\in\mathcal{S}_n}1_{\{y_{\tau(1)}\leq y_{\tau(2)}\leq\ldots\leq y_{\tau(n)}\}}\begin{pmatrix}a_n(\tau^{-1}(1))\\a_n(\tau^{-1}(2))\\\vdots\\a_n(\tau^{-1}(n))\end{pmatrix},$$

one has $Pb=b$ and the infinitesimal generator associated with (2.15) is $L\psi=\frac{\sigma^2}{2}\nabla\cdot(P\nabla\psi)-Pb\cdot\nabla\psi$. Computing $d\psi(Y_t^{1,n},\ldots,Y_t^{n,n})$ by Itô's formula and taking expectations then using (2.19), one obtains

$$\int_{\mathcal{M}_n}\psi\partial_t p_t^n dy=\int_{\mathcal{M}_n}L\psi p_t^n dy=\int_{\mathcal{M}_n}\psi\nabla\cdot P\left(\frac{\sigma^2}{2}\nabla p_t^n+bp_t^n\right)dy.$$

Hence the densities solve the Fokker-Planck equation $\partial_t p_t^n=\nabla\cdot P\left(\frac{\sigma^2}{2}\nabla p_t^n+bp_t^n\right)$. Now using

(2.19) and $b = -\frac{\sigma^2 \nabla p^n_\infty}{2p^n_\infty}$, one deduces

$$\partial_t \int_{\mathcal{M}_n} \left(\frac{p^n_t}{p^n_\infty} - 1\right)^2 p^n_\infty dy = 2\int_{\mathcal{M}_n} \frac{p^n_t}{p^n_\infty} \nabla \cdot P\left(\frac{\sigma^2}{2}\nabla p^n_t + bp^n_t\right) dy$$
$$= -\sigma^2 \int_{\mathcal{M}_n} P\nabla \frac{p^n_t}{p^n_\infty} \cdot P\frac{\nabla p^n_t + \frac{2bp^n_t}{\sigma^2}}{p^n_\infty} p^n_\infty dy$$
$$= -\sigma^2 \int_{\mathcal{M}_n} \left|P\nabla \frac{p^n_t}{p^n_\infty}\right|^2 p^n_\infty dy. \tag{2.20}$$

By symmetry of the function $\frac{p^n_t}{p^n_\infty}$ and (2.18),

$$\sigma^2 \int_{\mathcal{M}_n} \left|P\nabla \frac{p^n_t}{p^n_\infty}\right|^2 p^n_\infty dy = \sigma^2 \int_{\mathcal{M}_n} \left|P\nabla \frac{p^n_t}{p^n_\infty}\right|^2 \tilde{p}^n_\infty dy$$
$$\geq \lambda_n \int_{\mathcal{M}_n} \left(\frac{p^n_t}{p^n_\infty} - 1\right)^2 \tilde{p}^n_\infty dy = \lambda_n \int_{\mathcal{M}_n} \left(\frac{p^n_t}{p^n_\infty} - 1\right)^2 p^n_\infty dy$$

and the conclusion follows. ∎

Notice that the computation in (2.20) is formal and can only be justified when p^n_t is a smooth solution of the Fokker-Planck equation.

Remark 2.13 *Let us denote by $Y^{(1),n}_t \leq \ldots \leq Y^{(n),n}_t$ the increasing reordering of the random variables $(Y^{1,n}_t, \ldots, Y^{n,n}_t)$. According to [9], the reordered system is a diffusion process normally reflected at the boundary of the closed convex set $\{y \in \mathcal{M}_n : y_1 \leq y_2 \leq \ldots \leq y_n\}$. More precisely,*

$$\begin{cases} dY^{(i),n}_t = \sigma d\beta^i_t - a_n(i)dt + (\gamma^i_t - \gamma^{i+1}_t)d|K|_t \\ (\int_0^t (\gamma^i_s - \gamma^{i+1}_s)d|K|_s, 1 \leq i \leq n)_{t\geq 0} \text{ is a continuous process with finite variation equal to } |K|_t \\ \gamma^1 \equiv \gamma^{n+1} \equiv 0 \text{ and } d|K|_t \text{ a.e. }, \forall 2 \leq i \leq n,\ \gamma^i_t \geq 0 \text{ and } \gamma^i_t(Y^{(i),n}_t - Y^{(i-1),n}_t) = 0 \end{cases} \tag{2.21}$$

where $(\beta^1, \ldots, \beta^n)$ is a Brownian motion such that $\frac{<\beta^i,\beta^j>_t}{t} = 1_{\{i=j\}} - \frac{\sigma^2}{n}$.
If the initial condition $(Y^{(1),n}_0 \leq \ldots \leq Y^{(n),n}_0)$ admits a density $\tilde{p}^n_0$ with respect to the Lebesgue measure on $\mathcal{M}_n$, then the law of $(Y^{(1),n}_t, \ldots, Y^{(n),n}_t)$ is the image of the symmetric law of the solution $(Y^{1,n}_t, \ldots, Y^{n,n}_t)$ to (2.15) starting from $(Y^{1,n}_0, \ldots, Y^{n,n}_0)$ with density p^n_0 obtained by symmetrization of $\tilde{p}^n_0$. Therefore $(Y^{(1),n}_t, \ldots, Y^{(n),n}_t)$ has the density $\tilde{p}^n_t(y) = n!p^n_t(y)1_{\{y_1\leq\ldots\leq y_n\}}$ and (2.17) holds with p^n replaced by $\tilde{p}^n$.

In order to prove Proposition 2.12, we take advantage of the specific form of the density $\tilde{p}^n_\infty$. Remarking that $\tilde{p}^n_\infty$ is the density of the image of a vector of independent exponential random variables by a linear transformation, one first obtains the following result.

Lemma 2.14 *The Poincaré inequality (2.18) holds with the constant λ_n greater than $\frac{\alpha^2}{4\sigma^2}$ multiplied by the the smallest eigenvalue $\tilde{\lambda}_n$ of the $(n-1)\times(n-1)$ matrix Q^n defined by $\forall 1 \leq i, j \leq n-1$,*

$Q^n_{ij} = b_n(i) L^n_{ij} b_n(j)$ *where*

$$b_n(i) = \frac{i(n-i)}{n} \text{ and } L^n = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 2 & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

The last statement in Proposition 2.12 then follows from the next lemma which is obtained by interpreting Q^n as a finite element rigidity matrix associated with the operator $-x(1-x)\partial_{xx}(x(1-x).)$ acting on functions on $(0,1)$. The Hardy inequality stated in Lemma 2.16, ensures that it is enough to bound the smallest eigenvalue of the corresponding mass matrix from below. The resort to this one-dimensional Poincaré-like inequality in order to estimate the constant in the n-dimensional Poincaré inequality (2.18) is striking.

Lemma 2.15 *The sequence* $(\tilde{\lambda}_n)_n$ *is bounded from below by* $1/(16 \times 27)$.

Proof of Lemma 2.14. Let f be such that $\int_{\mathcal{M}_n} f(y)\tilde{p}^n_\infty(y)dy = 0$. Since the left-hand-side in the Poincaré inequality (2.18) only depends on the restriction of f to $\mathcal{M}_n$, one may assume that $\forall x \in \mathbb{R}^n,\ f(x) = f(\bar{P}x)$, which ensures that for $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_1 + \dots + x_n = 0$, $f(\bar{x}+x_1, \dots, \bar{x}+x_n) = f(x_1, \dots, x_n)$ and $P\nabla f(\bar{x}+x_1, \dots, \bar{x}+x_n) = \nabla f(x_1, \dots, x_n)$. Therefore the Poincaré inequality (2.18) is equivalent to $I(f) \le \frac{\sigma^2}{\tilde{\lambda}_n} I(|\nabla f|)$ where

$$I(g) = \int_{\mathbb{R}^{n-1}} g^2 \tilde{p}^n_\infty(-(x_2 + \dots + x_n), x^n_2) dx^n_2 \text{ with } x^n_2 = (x_2, \dots, x_n).$$

To integrate the coordinates over independent domains, we make the change of variables $z^n_2 = Mx^n_2$ where

$$M = \begin{pmatrix} 2 & 1 & 1 & \dots & \dots & 1 \\ -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}.$$

One easily checks that for $2 \le i \le n$, $z_2 + \dots + z_i = x_2 + \dots + x_n + x_i$ and deduce that $(n-1)z_2 + (n-2)z_3 + \dots + 2z_{n-1} + z_n = n(x_2 + \dots + x_n)$. Therefore

$$M^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 2-n & 3-n & 4-n & \dots & -1 \\ 1 & 2 & 3-n & 4-n & \dots & -1 \\ 1 & 2 & 3 & 4-n & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & n-2 & -1 \\ 1 & 2 & 3 & \dots & \dots & n-1 \end{pmatrix}$$

and denoting

$$N = \begin{pmatrix} \frac{1-n}{n} & \frac{2-n}{n} & \dots -\frac{2}{n} & -\frac{1}{n} \\ & & M^{-1} & \end{pmatrix},$$

one has

$$I(f) = \int_{(\mathbb{R}_+)^{n-1}} f^2(Nz_2^n) e^{-\frac{2}{\sigma^2}\sum_{i=2}^n \beta_n(i) z_i} \frac{dz_2^n}{|M|}$$

with $\beta_n(i) = \frac{1}{n}\left((i-1)(a_n(i)+\ldots+a_n(n)) - (n+1-i)(a_n(1)+\ldots+a_n(i-1))\right)$ positive by α-convexity of A. Here $|M|$ denotes the determinant of the matrix M; it is equal to n by an easy computation. Tensorizing the Poincaré inequality satisfied by the one-dimensional exponential density [1], one obtains

$$\begin{aligned}
I(f) &\leq \int_{(\mathbb{R}_+)^{n-1}} \sum_{j=2}^n \frac{\sigma^4}{\beta_n^2(j)} \left(\sum_{k=1}^n N_{kj-1} \partial_k f(Nz_2^n)\right)^2 e^{-\frac{2}{\sigma^2}\sum_{i=2}^n \beta_n(i) z_i} \frac{dz_2^n}{|M|} \\
&= \sigma^4 \int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^n \left(\sum_{j=2}^n \frac{1}{\beta_n^2(j)} N_{kj-1} N_{lj-1}\right) \partial_k f \partial_l f \tilde{p}_\infty^n(-(x_2+\ldots+x_n), x_2^n) dx_2^n.
\end{aligned}$$

By uniform convexity of A, according to Remark 1.6,

$$\forall i \in \{1,\ldots,n-1\},\ \forall j \in \{1,\ldots,n-i\},\quad a_n(i+j) - a_n(i) \geq \alpha \frac{j}{n}.$$

Therefore, for $i \in \{2,\ldots,n\}$,

$$\begin{aligned}
\beta_n(i) &\geq \frac{\alpha}{n^2}\left((i-1)\sum_{j=i}^n j - (n+1-i)\sum_{j=1}^{i-1} j\right) = \frac{\alpha}{n^2}\left((i-1)\sum_{j=1}^n j - n\sum_{j=1}^{i-1} j\right) \\
&= \frac{\alpha}{n^2}\left((i-1)\frac{n(n+1)}{2} - n\frac{(i-1)i}{2}\right) = \alpha\frac{(i-1)(n-(i-1))}{2n} = \frac{\alpha}{2} b_n(i-1).
\end{aligned}$$

Therefore

$$\begin{aligned}
I(f) &\leq \frac{4\sigma^4}{\alpha^2} \int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^n \left(\sum_{j=1}^{n-1} \frac{1}{b_n^2(j)} N_{kj} N_{lj}\right) \partial_k f \partial_l f \tilde{p}_\infty^n(-(x_2+\ldots+x_n), x_2^n) dx_2^n \\
&\leq \frac{4\sigma^2}{\alpha^2 \tilde{\lambda}_n} I(|\nabla f|)
\end{aligned}$$

where $\tilde{\lambda}_n$ denotes the inverse of the largest eigenvalue of the symmetric non-negative matrix $\bar{N}\bar{N}^*$ defined by $\bar{N}_{ij} = \frac{N_{ij}}{b_n(j)}$. To prove Proposition 2.12 with a possibly modified lower bound, it is enough to check that the largest eigenvalue is bounded from above uniformly in n. Unfortunately, the trace of the matrix can be bounded from below by a positive constant multiplied by $\log(n)$. Therefore one has to be more precise.

Let w be an eigenvector associated with the largest eigenvalue : $\bar{N}\bar{N}^* w = \frac{1}{\tilde{\lambda}_n} w$. Of course $\bar{N}^* w$ is non-zero and multiplying the previous equality by $\bar{N}^*$, one obtains that $\bar{N}^* w$ is an eigenvector of $\bar{N}^*\bar{N}$ associated with the eigenvalue $\frac{1}{\tilde{\lambda}_n}$. By symmetry, $\frac{1}{\tilde{\lambda}_n}$ is also the largest eigenvalue of $\bar{N}^*\bar{N}$. We are going to check that the latter matrix is invertible with inverse equal to Q^n in order to conclude the proof. Because of the definition of $\bar{N}$, it is enough to check that N^*N is invertible with inverse equal to L_n.

By construction of the matrix N, for the equation $Nz_2^n = x$ where $x \in \mathbb{R}^n$ to have a solution z_2^n, it is necessary and sufficient that $x_1 = -(x_2+\ldots+x_n)$ and then $z_2^n = Mx_2^n$.

Now for fixed $y \in \mathbb{R}^{n-1}$, let us find $x_2^n \in \mathbb{R}^{n-1}$ such that $N^* x = y$ where $x = -(x_2+\ldots+x_n, x_2^n)$.

Denoting by $J \in \mathbb{R}^{(n-1)\times(n-1)}$ the matrix with all entries equal to 1, the equation writes

$$\left((M^{-1})^* - \begin{pmatrix} N_{11} \\ N_{12} \\ \vdots \\ N_{1n-1} \end{pmatrix} J \right) x_2^n = y.$$

One easily checks that the $(n-1)\times(n-1)$ matrix in the left-hand-side is equal to

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \text{ with inverse } R = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Combining $x_2^n = Ry$ with the solution of the previous problem, one obtains that the unique solution of the equation $N^*Nz_2^n = y$ is $z_2^n = MRy$. One concludes by checking that the matrix MR is equal to L_n. ∎

Proof of Lemma 2.15. The functions

$$u_i(x) = \begin{cases} 0 \text{ if } x \in (0,1) \setminus [\frac{i-1}{n}, \frac{i+1}{n}] \\ \frac{i(n-i)(x-\frac{i-1}{n})}{\sqrt{n}x(1-x)} \text{ if } x \in [\frac{i-1}{n}, \frac{i}{n}] \\ \frac{i(n-i)(\frac{i+1}{n}-x)}{\sqrt{n}x(1-x)} \text{ if } x \in [\frac{i}{n}, \frac{i+1}{n}] \end{cases}, i \in \{1, \dots, n-1\}$$

are such that

$$\forall i, j \in \{1, \dots, n-1\},\ Q_{ij}^n = \int_0^1 (x(1-x)u_i(x))'(x(1-x)u_j(x))'dx.$$

By the Hardy inequality stated in Lemma 2.16 below, the smallest eigenvalue of the matrix Q^n is greater than the smallest eigenvalue of the $(n-1)\times(n-1)$ tridiagonal matrix $R_{ij}^n = \int_0^1 u_i(x)u_j(x)dx$ divided by 16.
For $i \in \{1, \dots, n-2\}$, let $r_i^n = \int_{i/n}^{(i+1)/n} u_i(u_i - u_{i+1})(x)dx$ and

$$r_{n-1}^n = \int_{(n-1)/n}^1 u_{n-1}^2(x)dx = \frac{(n-1)^2}{n} \int_{(n-1)/n}^1 \frac{1}{x^2}dx = \frac{n-1}{n}.$$

Using the change of variables $y = 1-x$, one easily checks that

$$\forall i \in \{1, \dots, n-1\},\ R_{ii}^n - R_{ii-1}^n - R_{ii+1}^n = r_i^n + r_{n-i}^n$$

where by convention $R_{10}^n = R_{n-1n}^n = 0$. We are going to prove that

$$\forall n \geq 3, \forall i \in \{2, \dots, n-3\},\ r_i^n \geq \frac{1}{27}.$$

and that r_1^n and r_{n-2}^n are non-negative. For $y \in \mathbb{R}^{n-1}$, one deduces that

$$\begin{aligned} y^*R^n y &= \sum_{i=1}^{n-1} R_{ii}^n y_i^2 + 2\sum_{i=1}^{n-2} R_{ii+1}^n y_i y_{i+1} \\ &= \sum_{i=1}^{n-1} (R_{ii}^n - R_{ii-1}^n - R_{ii+1}^n) y_i^2 + \sum_{i=1}^{n-2} R_{ii+1}^n (y_i + y_{i+1})^2 \geq \frac{|y|^2}{27} \end{aligned}$$

and the conclusion follows.
Let us first suppose that $i \le \lfloor \frac{n}{2} \rfloor - 1$, which ensures that the function $f(x) = x^2(1-x)^2$ is increasing on $[i/n, (i+1)/n]$. Let $g(x) = u_i(u_i - u_{i+1})(x)$. One easily checks that

$$\int_{i/n}^{(i+1)/n} g(x)dx = \frac{i^2(n-i)^2}{n^4}\left(\frac{1}{3} - \frac{(i+1)(n-i-1)}{6i(n-i)}\right) \ge \begin{cases} 0 & \text{if } i = 1, \\ \frac{i^2(n-i)^2}{12n^4} & \text{if } i \ge 2. \end{cases}$$

Since there is some $x_i \in [i/n, (i+1)/n]$ such that the function $g(x)$ is non-negative on $[i/n, x_i]$ then non-positive on $[x_i, (i+1)/n]$, and f is positive and increasing, one deduces that for all $x \in [i/n, (i+1)/n]$, $\int_{i/n}^{x} \frac{g(y)}{f(y)} dy \ge 0$. This ensures that

$$\forall x \in [i/n, (i+1)/n], \quad \frac{d}{dx}\left(f(x)\int_{i/n}^{x} \frac{g(y)}{f(y)}dy\right) = f'(x)\int_{i/n}^{x} \frac{g(y)}{f(y)}dy + g(x) \ge g(x).$$

Therefore

$$r_i^n = \int_{i/n}^{(i+1)/n} \frac{g(y)}{f(y)}dy \ge \frac{1}{f((i+1)/n)}\int_{i/n}^{(i+1)/n} g(y)dy \ge \begin{cases} 0 & \text{if } i = 1, \\ \frac{i^2(n-i)^2}{12(i+1)^2(n-i-1)^2} \ge \frac{1}{27} & \text{if } i \ge 2. \end{cases}$$

Let us now suppose that $i \ge \lfloor \frac{n+1}{2} \rfloor$ so that the function f is decreasing on $[i/n, (i+1)/n]$. We deduce that

$$r_i^n \ge \frac{1}{f(i/n)}\int_{i/n}^{(i+1)/n} f u_i^2(x)dx - \frac{1}{f((i+1)/n)}\int_{i/n}^{(i+1)/n} f u_i u_{i+1}(x)dx = \frac{1}{3} - \frac{i(n-i)}{6(i+1)(n-i-1)}$$

and the left-hand-side is greater than 1/12 for $i \le n-3$ and non-negative for $i = n-2$.
We still have to deal with the case n odd and $i = (n-1)/2$. Then, f is not monotonic on $[i/n, (i+1)/n] = [1/2 - 1/2n, 1/2 + 1/2n]$. But by symmetry,

$$\begin{aligned} r_{(n-1)/2}^n &= \frac{(n-1)^2(n+1)^2}{16n}\int_{1/2-1/2n}^{1/2+1/2n} \frac{(1/2+1/2n-x)(1-2x)}{x^2(1-x)^2}dx \\ &= \frac{(n-1)^2(n+1)^2}{32n}\int_{1/2-1/2n}^{1/2+1/2n} \frac{(1-2x)^2}{x^2(1-x)^2}dx \\ &\ge \frac{(n-1)^2(n+1)^2}{2n}\int_{1/2-1/2n}^{1/2+1/2n} (1-2x)^2 dx = \frac{(n^2-1)^2}{6n^4}. \end{aligned}$$

∎

Lemma 2.16 *For all $u \in L^2(0,1)$ such that the distribution derivative $(x(1-x)u(x))'$ belongs to $L^2(0,1)$,*

$$\int_0^1 u^2(x)dx \le 16\int_0^1 \left((x(1-x)u(x))'\right)^2 dx.$$

Proof . For v a C^∞ function with compact support on $(0,1)$, by the integration by parts formula,

$$\begin{aligned} \int_0^{1/2} \frac{v^2(x)}{x^2(1-x)^2}dx \le 4\int_0^{1/2} \frac{v^2(x)}{x^2}dx &= 8\left(\int_0^{1/2} \frac{vv'(x)}{x}dx - v^2(1/2)\right) \\ &\le 8\left(\int_0^{1/2} \frac{v^2(x)}{x^2}dx\right)^{1/2}\left(\int_0^{1/2} (v'(x))^2 dx\right)^{1/2}. \end{aligned}$$

Dealing with the integral on $(1/2,1)$ in a symmetric way, one deduces

$$\int_0^1 \frac{v^2(x)}{x^2(1-x)^2}dx \leq 16\int_0^1 (v'(x))^2dx. \tag{2.22}$$

Now approximating $v \in H_0^1(0,1)$ by a sequence of C^∞ functions with compact support converging in the H^1 norm and almost everywhere, one deduces with Fatou lemma that the inequality still holds for $v \in H_0^1$.
For u satisfying the hypotheses in the Lemma, $v(x) = x(1-x)u(x)$ belongs to $H^1(0,1)$. According to Theorem VIII.2 p.122 [4], v admits a representative continuous on $[0,1]$ still denoted by v. Moreover, since $u(x) = \frac{v(x)}{x(1-x)}$ belongs to $L^2(0,1)$, necessarily, $v(0) = v(1) = 0$. By Theorem VIII.11 p.133 [4], v belongs to $H_0^1(0,1)$ and the conclusion follows from (2.22). ∎

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