# Approximation of OT problems with marginal moments contraints

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# Summary

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# Kantorovich problem with two marginal laws

For any (open or compact) subset $\mathcal{X} \subset \mathbb{R}^d$ ($d \in \mathbb{N}^*$), let us denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$.

- Let $d_1, d_2 \in \mathbb{N}^*$, $\mathcal{X}_1 \subset \mathbb{R}^{d_1}$ and $\mathcal{X}_2 \subset \mathbb{R}^{d_2}$ be open or compact subsets.
- Fo $\nu_1 \in \mathcal{P}(\mathcal{X}_1)$ and $\nu_2 \in \mathcal{P}(\mathcal{X}_2)$, let

$$\Pi(\nu_1, \nu_2) = \left\{ \begin{array}{c} \gamma \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2), \\ \int_{\mathcal{X}_2} \mathrm{d}\gamma(x_1, x_2) = \mathrm{d}\nu_1(x_1), \int_{\mathcal{X}_1} \mathrm{d}\gamma(x_1, x_2) = \mathrm{d}\nu_2(x_2) \end{array} \right\}$$

- Let $c : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}_+ \cup \{+\infty\}$ be lower semi-continuous (l.s.c) cost function.

The Kantorovich optimal transport problem reads:

$$\boxed{\inf_{\gamma \in \Pi(\nu_1, \nu_2)} \int_{\mathcal{X}_1, \mathcal{X}_2} c(x_1, x_2) \mathrm{d}\gamma(x_1, x_2).}$$

# Multi-marginal Kantorovich problem

- Let $N \in \mathbb{N}^*$, and for all $1 \leq i \leq N$, let $d_i \in \mathbb{N}^*$, $\mathcal{X}_i \subset \mathbb{R}^{d_i}$ be an open or compact subset.
- For all $1 \leq i \leq N$, let $\nu_i \in \mathcal{P}(\mathcal{X}_i)$, and let

$$\Pi^N((\nu_i)_{1\leq i\leq N}) := \left\{\gamma \in \mathcal{P}(\mathcal{X}_1, ..., \mathcal{X}_N), \mathrm{d}\mu_\gamma^i(x_i) = \mathrm{d}\nu_i(x_i), \forall 1 \leq i \leq N\right\},$$

  where $\mu_\gamma^i \in \mathcal{P}(\mathcal{X}_i)$ denotes the $i^{\text{th}}$ marginal law of $\gamma$, defined by

$$\mathrm{d}\mu_\gamma^i(x_i) := \int_{\mathcal{X}_1\times\ldots\mathcal{X}_{i-1}\times\mathcal{X}_{i+1}\times\ldots\times\mathcal{X}_N} \mathrm{d}\gamma(x_1, ..., x_N).$$

- Let $c : \mathcal{X}_1 \times ... \times \mathcal{X}_N \to \mathbb{R}_+ \cup \{+\infty\}$ be lower semi-continuous (l.s.c.) cost function.

The multi-marginal Kantorovich optimal transport problem reads:

$$\boxed{I = \inf_{\gamma\in\Pi^N((\nu_i)_{1\leq i\leq N})} \int_{\mathcal{X}_1\times\ldots\times\mathcal{X}_N} c\mathrm{d}\gamma.}$$

# discretization

Let $M \in \mathbb{N}^*$, we discretize the measure $\nu_i \in \mathcal{P}(\mathcal{X}_i)$ on a fixed discretization grid of $\mathcal{X}_i$, $x_i^1, ..., x_i^M \in \mathcal{X}_i$.

$$\mathrm{d}\nu_i(x) \approx \sum_{j=1}^{M} \bar{\nu}_i^j \delta_{x_i^j},$$

for some $\bar{\nu}_i^j \in \mathbb{R}_+$ such that $\sum_{j=1}^{M} \bar{\nu}_i^j = 1$.

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In the two marginal laws case, introduce

$$\Gamma := \left\{ \begin{array}{c} (\bar{\gamma}^{j_1,j_2}) \in \mathbb{R}_+^{M^2} \\ \forall 1 \leq j \leq M, \sum_{j_1=1}^M \bar{\gamma}^{j_1,j} = \bar{\nu}_2^j, \sum_{j_2=1}^M \bar{\gamma}^{j,j_2} = \bar{\nu}_1^j \end{array} \right\}$$

so that

$$\gamma \approx \sum_{j_1,j_2} \bar{\gamma}^{j_1,j_2} \delta_{x_1^{j_1}, x_2^{j_2}}$$

and we solve the **linear problem under linear constraints in** $\mathbb{R}^{M^2}$

$$\inf_{(\bar{\gamma}_{j_1,j_2}) \in \Gamma} \sum_{j_1,j_2}^{M} c(x_1^{j_1}, x_2^{j_2}) \bar{\gamma}^{j_1,j_2}.$$

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In the multi-marginal case with $N$ marginal laws, introduce

$$\Gamma := \left\{ \begin{array}{c} (\bar{\gamma}^{j_1,\cdots,j_N}) \in \mathbb{R}_+^{M^N} \\ \displaystyle\sum_{1 \leq j_1,\dots,j_{i-1},j_{i+1},\dots,j_N \leq M} \bar{\gamma}^{j_1,\cdots,j_{i-1},j,j_{i+1},\cdots,j_N} = \bar{\nu}_i^j, \ \forall i,j \end{array} \right\}$$

so that

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Curse of dimensionality!

# Numerical methods for multi-marginal optimal transport problems

Several numerical methods have been introduced in the literature for the resolution of the multi-marginal optimal transport problems. We mention here two of them in the context of the symmetric multimarginal Kantorovich problem with Coulomb Cost:

- [Benamou,Carlier,Cuturi,Nenna,Peyre,2015], [Nenna,2016] : use of an entropic regularization (using the Kullback-Leibler entropy), together with an iterative algorithm called Sinkhorn algorithm.
- [Mendl,Lin,2013]: dual formulation of the Kantorovich problem, and clever treatment of the (infinite-dimensional) inequality constraint.

# Alternative discretization: Moments Constrained Optimal Transport Problem

Let $M \in \mathbb{N}^*$ and $\phi_1^i, ..., \phi_M^i \in \mathcal{C}_b(\mathcal{X}_i)$ be some continuous bounded functions on $\mathcal{X}_i$. They will be called hereafter **test functions**.

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For all $1 \leq i \leq N$, the marginal constraint

$$\mathrm{d}\mu_\gamma^i(x_i) = \mathrm{d}\nu_i(x_i)$$

is then approximated by the $M$ moment constraints: for all $1 \leq j \leq M$,

$$\int_{\mathcal{X}_i} \phi_j^i(x_i)\mathrm{d}\mu_\gamma^i(x_i) = \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} \phi_j^i(x_i)\mathrm{d}\gamma(x_1, ..., x_N) = \int_{\mathcal{X}_i} \phi_j^i(x_i)\mathrm{d}\nu_i(x_i) =: \bar{\nu}_i^j$$

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and we consider the following approximate problem (**MCOT problem**)

$$I^M = \inf_{\substack{\gamma \in \mathcal{P}(\mathcal{X}_1 \times \ldots \times \mathcal{X}_N) \\ \forall 1 \leq i \leq N, \forall 1 \leq j \leq M, \\ \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} \phi_j^i(x_i)\mathrm{d}\gamma(x_1, \ldots, x_N) = \bar{\nu}_i^j}} \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} c\mathrm{d}\gamma. \tag{1}$$

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**Remark**

$$I^M \leq I$$

# Outline of the talk

Study the conditions on the test functions $(\phi^i_j)_{\substack{1\le i\le N\\ 1\le j\le M}}$ and on the marginal laws $\nu_i$, which can provide answers to the following questions:

- Does $I^M \xrightarrow[M\to+\infty]{} I$
- Does $I^M$ admits a minimizer? Can we say something about a minimizer which could be interesting for numerics?
- Can we obtain rates of convergences?

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# Tchakaloff's theorem

The following theorem is the backbone of our analysis.

**Theorem (Tchakaloff [Bayer & Teichmann, 2006], [Berschneider & Sasvári, 2012])**

*Let $d \in \mathbb{N}^*$ and let $\gamma$ be a measure on $\mathbb{R}^d$ concentrated on a Borel set $A \in \mathcal{F}$, i.e. $\gamma(\mathbb{R}^d \setminus A) = 0$. Let $M_0 \in \mathbb{N}^*$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}^{M_0}$ a Borel measurable map. Assume that the first moments of $\Lambda\#\gamma$ exist, i.e.*

$$\int_{\mathbb{R}^{M_0}} \|u\| \mathrm{d}\Lambda\#\gamma(u) = \int_{\mathbb{R}^d} \|\Lambda(z)\| \mathrm{d}\gamma(z) < \infty,$$

*where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^{M_0}$. Then, there exist an integer $1 \leq K \leq M_0$, points $z_1, \dots, z_K \in A$ and weights $w_1, \dots, w_K > 0$ such that*

$$\forall 1 \leq j \leq M_0, \quad \int_{\mathbb{R}^d} \Lambda_j(z) \mathrm{d}\gamma(z) = \sum_{k=1}^{K} w_k \Lambda_j(z_k),$$

*where $\Lambda_j$ denotes the $j$-th component of $\Lambda$.*

# Compact set case

We consider that for all $1 \leq i \leq N$, $\mathcal{X}_i$ is compact.
Let for all $1 \leq i \leq N$, $(\phi_j^i)_{j\in\mathbb{N}^*} \subset \mathcal{C}(\mathcal{X}_i)$ such that

$$\forall j \in \mathbb{N}^*, \int_{\mathcal{X}_i} |\phi_j^i(x_i)| \mathrm{d}\nu_i < +\infty, \tag{2}$$

and satisfying the density condition

$$\forall f \in \mathcal{C}(\mathcal{X}_i), \inf_{f_M \in \mathrm{Span}(\phi_1^i, \ldots, \phi_M^i)} \|f - f_M\|_{L^\infty(\mathcal{X}_i)} \xrightarrow[M\to\infty]{} 0. \tag{3}$$

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**Theorem**

*For all $M \in \mathbb{N}^*$, it holds that $I^M \leq I$ and $I^M \xrightarrow[M \to +\infty]{} I$.*
*Besides, there exists an integer $1 \leq K \leq MN + 2$, and for all $1 \leq k \leq K$, points $z^k = (x_1^k, \dots, x_N^k) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_N$ and weights $w_k > 0$ such that*

$$\gamma^M = \sum_{k=1}^{K} w_k \delta_{x_1^k, \dots, x_N^k}$$

*is a minimizer of the MCOT Problem* (1).

# Idea of proof

- Fix $M \in \mathbb{N}^*$, if for all $1 \leq i \leq N$, $\mathcal{X}_i$ is compact, minimizing sequences $(\gamma_n)_{n\in\mathbb{N}}$ for the MCOT problem are tight. Thus, up to the extraction of a subsequence $\gamma_n \xrightarrow[n\to+\infty]{} \gamma_\infty \in \mathcal{P}(\mathcal{X}_1, ..., \mathcal{X}_N)$, where $\gamma_\infty$ is a minimizer of the MCOT Problem (due to the continuity of test functions and using Fatou's Lemma and Skorohod representation theorem)

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- Applying Tchakaloff's theorem to

$$\Lambda : \begin{cases} \mathcal{X}_1 \times ... \times \mathcal{X}_N & \to & \mathbb{R}^{MN+2} \\ (x_1, ..., x_N) & \mapsto & \begin{pmatrix} \phi_1^1(x_1) \\ \vdots \\ \phi_M^N(x_N) \\ 1 \\ c(x_1, ..., x_N) \end{pmatrix} \end{cases} \tag{4}$$

gives the finite discrete minimizer.

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  gives the finite discrete minimizer.
- Tightness allows a subsequence of minimizers to converge towards a minimizer of the OT Problem (using the density condition).

# Non compact case

We consider that for all $1 \leq i \leq N$, $\mathcal{X}_i$ can be non compact.
Let for all $1 \leq i \leq N$, $(\phi_j^i)_{j \in \mathbb{N}^*} \subset \mathcal{C}(\mathcal{X}_i)$ such that

$$\forall j \in \mathbb{N}^*, \int_{\mathcal{X}_i} |\phi_j^i(x_i)| \mathrm{d}\nu_i < +\infty, \tag{5}$$

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and let us assume that the probability measures $\nu_i$ are characterized by their moments

$$\forall \eta \in \mathcal{P}(\mathcal{X}_i), \left(\forall j \in \mathbb{N}^*, \int_{\mathcal{X}_i} \phi_j^i(x) \mathrm{d}\eta(x) = \bar{\nu}_i^j\right) \implies \eta = \nu_i. \tag{6}$$

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Let us assume that there exists non-decreasing functions $\theta_{\nu_i} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

- for all $1 \leq i \leq N$, $\theta_{\nu_i}(r) \xrightarrow[r\to+\infty]{} +\infty$
- $A_0 := \sum_{i=1}^N \int_{\mathcal{X}_i} \theta_{\nu_i}(|x_i|) \mathrm{d}\nu_i(x_i) < +\infty$
- for all $1 \leq i \leq N$, for all $j \in \mathbb{N}^*$, there exists $C_j^{\nu_i} > 0$ and $0 < s_j^{\nu_i} < 1$ such that
$$\forall x \in \mathcal{X}_i,\ |\phi_j^i(x)| \leq C_j^{\nu_i}(1 + \theta_{\nu_i}(|x|))^{s_j^{\nu_i}}.$$

# Non compact case

Then, consider for all $M \in \mathbb{N}^*$,

$$I_{A_0}^M = \inf_{\substack{\gamma \in \mathcal{P}(\mathcal{X}_1 \times \ldots \times \mathcal{X}_N) \\ \forall 1 \leq i \leq N, \forall 1 \leq j \leq M, \\ \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} \phi_j^i(x_i) \mathrm{d}\gamma(x_1, \ldots, x_N) = \bar{\nu}_i^j \\ \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} \sum_{i=1}^N \theta_{\nu_i}(|x_i|) \mathrm{d}\gamma(x_1, \ldots, x_N) \leq A_0}} \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} c \mathrm{d}\gamma. \quad (7)$$

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## Theorem

*For all $M \in \mathbb{N}^*$, it holds that $I_{A_0}^M \leq I$ and $I_{A_0}^M \xrightarrow[M \to +\infty]{} I$.*
*Besides, there exists an integer $1 \leq K \leq MN + 2$, and for all $1 \leq k \leq K$, points $z^k = (x_1^k, \ldots, x_N^k) \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$ and weights $w_k > 0$ such that*

$$\gamma^M = \sum_{k=1}^K w_k \delta_{x_1^k, \ldots, x_N^k}$$

*is a minimizer of the MCOT Problem* (7).

The additional inequality constraint ensures the tightness of the minimizing sequences.

# Numerical scheme

This suggests to consider the following problem (**particle problem**) as a numerical scheme

$$\inf_{\substack{w_1,\dots,w_{MN+2}\geq 0 \\ \sum_{k=1}^{MN+2} w_k=1 \\ (x_1^k,\dots,x_{MN+2}^k)_{1\leq k\leq MN+2}\in(\mathcal{X}_1\times\dots\times\mathcal{X}_N)^{MN+2} \\ \forall 1\leq i\leq N,\, \forall 1\leq j\leq M, \sum_{k=1}^{MN+2} w_k\phi_j^i(x_i^k)=\bar{\nu}_i^j}} \sum_{k=1}^{MN+2} w_k c(x_1^k,\dots,x_N^k). \tag{8}$$

# Symmetric multi-marginal OT problem with Coulomb Cost

This problem arises in quantum chemistry applications [Seidl, 1999], [Seidl, Gori-Giorgi, Savin, 2007], [Cotar, Friesecke, Klüppelberg, 2011], [Lewin, 2017], [Cotar, Friesecke, Klüppelberg, 2018], where $N$ is a number of electrons. (see M. Seidl, C. Cotar, G. Friesecke, M. Lewin, A. Gerolin and L. Nenna talks)

- Let $d = d_1 = ... = d_N$ and $\mathcal{X}_1 = ... = \mathcal{X}_N = \mathbb{R}^d$.
- Let $\rho = \nu_1 = ... = \nu_N$, and $\Pi^N(\rho) := \Pi(\rho, ..., \rho)$
- Let

$$c : \begin{cases} (\mathbb{R}^d)^N & \rightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ (x_1, ..., x_N) & \mapsto & \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|} \end{cases}$$

  be the Coulomb Cost.

The problem considered is

$$I = \inf_{\gamma \in \Pi^N(\rho)} \int_{(\mathbb{R}^d)^N} c \mathrm{d}\gamma.$$

# Symmetric multi-marginal OT problem with Coulomb Cost

Recall the associated MCOT Problem, where $\rho_j = \int_{\mathbb{R}^d} \phi_j(x)\mathrm{d}\rho(x)$

$$I_{A_0}^M = \inf_{\substack{\gamma \in \mathcal{P}((\mathbb{R}^d)^N) \\ \forall 1 \leq i \leq N, \forall 1 \leq j \leq M, \\ \int_{(\mathbb{R}^d)^N} \phi_j(x_i)\mathrm{d}\gamma(x_1,\dots,x_N) = \rho_j \\ \int_{(\mathbb{R}^d)^N} \sum_{i=1}^N \theta_\rho(|x_i|)\mathrm{d}\gamma(x_1,\dots,x_N) \leq A_0}} \int_{(\mathbb{R}^d)^N} c\mathrm{d}\gamma. \tag{9}$$

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And let us introduce

$$\tilde{I}_{A_0}^M = \inf_{\substack{\gamma \in \mathcal{P}((\mathbb{R}^d)^N) \\ \forall 1 \leq j \leq M, \\ \int_{(\mathbb{R}^d)^N} \left( \frac{1}{N} \sum_{i=1}^N \phi_j(x_i) \right) \mathrm{d}\gamma(x_1, \ldots, x_N) = \rho_j \\ \int_{(\mathbb{R}^d)^N} \sum_{i=1}^N \theta_\rho(|x_i|) \mathrm{d}\gamma(x_1, \ldots, x_N) \leq A_0}} \int_{(\mathbb{R}^d)^N} c \mathrm{d}\gamma. \quad (10)$$

# Symmetric multi-marginal OT problem with Coulomb Cost

**Theorem**

*For all $M \in \mathbb{N}^*$, it holds that $I_{A_0}^M = \tilde{I}_{A_0}^M$.*
*Besides, there exists an integer $1 \leq K \leq M + 2$, and for all $1 \leq k \leq K$, points $z^k = (x_1^k, ..., x_N^k) \in \mathcal{X}_1 \times ... \times \mathcal{X}_N$ and weights $w_k > 0$ such that*

$$\gamma_{\mathrm{sym}}^M := \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \sum_{k=1}^{K} w_k \delta_{x_{\sigma(1)}^k, \dots, x_{\sigma(N)}^k}$$

*is a minimizer of MCOT Problems* (9) *and* (10)*.*

# Symmetric multi-marginal OT problem with Coulomb Cost

## Theorem

*For all $M \in \mathbb{N}^*$, it holds that $I_{A_0}^M = \tilde{I}_{A_0}^M$.*
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*is a minimizer of MCOT Problems (9) and (10).*

## Remark

To compute a minimizer of the MCOT problem, one only needs to find at most $M+2$ scalars $w_k$ and points $z^k = (x_1^k, ..., x_N^k) \in (\mathbb{R}^d)^N$:
$\mathcal{O}\left(3(M+2)N\right)$.
In [Friesecke, Vögler, 2018], a different result (but in the same spirit) was obtained by the authors in the case when the OT problem is discretized in a finite state space.

# Martingale Optimal Transport problem

For the sake of simplicity, we will only consider two marginal laws $\nu_1 \in \mathcal{P}(\mathbb{R}^d)$ and $\nu_2 \in \mathcal{P}(\mathbb{R}^d)$, $d \in \mathbb{N}^*$.
We assume that there exist a martingale coupling between $\nu_1$ and $\nu_2$:

$$\exists \gamma \in \Pi(\nu_1, \nu_2),\ \forall x_1 \in \mathbb{R}^d,\ \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1, x_2) = x_1,$$

and a l.s.c. cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$.
The Martingale Optimal Transport Problem reads as

$$I = \inf_{\substack{\gamma \in \Pi(\nu_1, \nu_2) \\ \forall x_1 \in \mathbb{R}^d,\ \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1, x_2) = x_1}} \int_{\mathbb{R}^{2d}} c(x_1, x_2) \mathrm{d}\gamma(x_1, x_2).$$

This problem arises in finance where $\nu_1$ and $\nu_2$ are known distribution of prices and $\int_{\mathbb{R}^{2d}} c(x_1, x_2)\mathrm{d}\gamma(x_1, x_2)$ is the payoff of an option for a given model.

# Martingale Optimal Transport problem

Let us introduce the following MCOT Problems, for all $M \in \mathbb{N}^*$, with the same constraints on the test functions as in the non compact case,

$$I_{A_0}^{M,\mathrm{mg}} = \inf_{\substack{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \\ \forall i=1,2,\ \forall 1 \leq j \leq M, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_j^i(x_i) \mathrm{d}\gamma(x_1,x_2) = \bar{\nu}_i^j \\ \forall x_1 \in \mathbb{R}^d,\ \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1,x_2) = x_1, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^2 \theta_{\nu_i}(|x_i|) \mathrm{d}\gamma(\mathbb{R}^d \times \mathbb{R}^d) \leq A_0}} \int_{\mathbb{R}^d \times \mathbb{R}^d} c \mathrm{d}\gamma. \tag{11}$$

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and

$$I_{A_0}^{M,M'} = \inf_{\substack{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \\ \forall i=1,2,\, \forall 1 \le j \le M, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_j^i(x_i) \mathrm{d}\gamma(x_1,x_2) = \bar{\nu}_i^j \\ \forall 1 \le l \le M',\, \int_{\mathbb{R}^d} x_2 \chi_l(x_1) \mathrm{d}\gamma(x_1,x_2) = \int_{\mathbb{R}^d} x_1 \chi_l(x_1) \mathrm{d}\nu_1(x_1), \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^2 \theta_{\nu_i}(|x_i|) \mathrm{d}\gamma(\mathbb{R}^d \times \mathbb{R}^d) \le A_0}} \int_{\mathbb{R}^d \times \mathbb{R}^d} c \mathrm{d}\gamma. \tag{12}$$

# Martingale Optimal Transport problem

With appropriate constraints on the additional test functions $\chi_I$, one can prove that

- The problem (12) admits a finite discrete minimizer.
- $I_{A_0}^{M,M'} \xrightarrow[M' \to \infty]{} I_{A_0}^{M,\mathrm{mg}} < +\infty$
- $I_{A_0}^{M,\mathrm{mg}} \xrightarrow[M \to \infty]{} I$

**Remark**

In the practical application in finance, the marginal laws $\nu_1$ and $\nu_2$ are in general not observed, and market data only provide some moments, as in Problem (11).

# Summary

1 Introduction

2 MCOT problems and applications

3 Rates of convergence

4 Conclusion and perspectives

# Back to the 2-marginal Kantorovich problem

From now on, let $\nu_1, \nu_2, \in \mathcal{P}([0,1])$. Let us consider the following two-marginal optimal transport problem

$$I = \inf_{\substack{\gamma \in \mathcal{P}([0,1]\times[0,1]) \\ \mathrm{d}\mu_\gamma^1 = \mathrm{d}\nu_1 \\ \mathrm{d}\mu_\gamma^2 = \mathrm{d}\nu_2}} \int_{[0,1]\times[0,1]} c(x_1,x_2)\mathrm{d}\gamma(x_1,x_2),$$

with $c(x_1,x_2) = |x_1 - x_2|$ or $c(x_1,x_2) = |x_1 - x_2|^2$.
Let $M \in \mathbb{N}^*$ and let us define the intervals

$$T_1^M := \left[0, \frac{1}{M}\right], T_j^M := \left(\frac{j-1}{M}, \quad \frac{j}{M}\right], \forall 2 \leq j \leq M.$$

# Sets of test functions

We consider three different sets of test functions:

- Piecewise constant ($\mathbb{P}_0$) test functions:

$$\forall 1 \leq j \leq M, \quad \phi_j^M := 1_{T_j^M}$$

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- Continuous piecewise affine (continuous $\mathbb{P}_1$) test functions: for $2 \leq j \leq M$

$$\phi_j^M(x) := \left| \begin{array}{ll} M\left(x - \frac{j-2}{M}\right) & \text{if } x \in T_{j-1}^M, \\ M\left(\frac{j}{M} - x\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{array} \right. \quad \phi_1^M(x) := \left| \begin{array}{ll} 1 - Mx & \text{if } x \in T_1^M, \\ 0 & \text{otherwise,} \end{array} \right.$$

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- Discontinuous piecewise affine (discontinuous $\mathbb{P}_1$) test functions: for $1 \leq j \leq M$

$$\phi_{j,1}^M(x) := \left| \begin{array}{ll} M\left(\frac{j}{M} - x\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{array} \right. \quad \phi_{j,2}^M(x) := \left| \begin{array}{ll} M\left(x - \frac{j-1}{M}\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{array} \right.$$

# MCOT problems

- in the $\mathbb{P}_0$ or continuous $\mathbb{P}_1$ case:

$$I^M := \inf_{\substack{\gamma \in \mathcal{P}([0,1]\times[0,1]) \\ \forall i=1,2,\ \forall 1 \leq j \leq M, \\ \int_{[0,1]\times[0,1]} \phi_j^M(x_i) \mathrm{d}\gamma(x_1,x_2) = \int_{[0,1]} \phi_j^M(x_i) \mathrm{d}\nu_i(x_i)}} \int_{[0,1]\times[0,1]} c \mathrm{d}\gamma.$$

- or in the discontinous $\mathbb{P}_1$ case:

$$I^M := \inf_{\substack{\gamma \in \mathcal{P}([0,1]\times[0,1]) \\ \forall i=1,2,\ \forall 1 \leq j \leq M,\ \forall l=1,2 \\ \int_{[0,1]\times[0,1]} \phi_{j,l}^M(x_i) \mathrm{d}\gamma(x_1,x_2) = \int_{[0,1]} \phi_{j,l}^M(x_i) \mathrm{d}\nu_i(x_i)}} \int_{[0,1]\times[0,1]} c \mathrm{d}\gamma.$$

# Rates of convergence

**Theorem**

- $\mathbb{P}_0$ *case: if c is Lipschitz with constant C, then for all* $M \in \mathbb{N}^*$,

$$|I - I^M| \leq \frac{C}{M}.$$

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**Theorem**

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$$|I - I^M| \leq \frac{C}{M}.$$

- *continuous $\mathbb{P}_1$ case and $c(x_1, x_2) = |x_1 - x_2|$: Let us assume that $\mathrm{d}\nu_1(x) = \rho_1(x)\mathrm{d}x$ and $\mathrm{d}\nu_2(x) = \rho_2(x)\mathrm{d}x$. Let us denote by $F_1$ and $F_2$ the cumulative distribution functions of $\nu_1$ and $\nu_2$ and let us assume that the function $F_1 - F_2$ changes sign at most $Q \in \mathbb{N}$ times on $[0, 1]$ and that $\rho_1 - \rho_2 \in L^\infty([0, 1], \mathrm{d}x, \mathbb{R})$. Then, for all $M \in \mathbb{N}^*$,*

$$|I - I^M| = |W_1(\nu_1, \nu_2) - I^M| \leq \frac{2Q\|\rho_1 - \rho_2\|_\infty}{M^2}.$$

# Idea of proof (continuous $\mathbb{P}_1$ case, $c(x_1, x_2) = |x_1 - x_2|$)

- Remark that a measure $\gamma \in \mathcal{P}([0,1] \times [0,1])$ with marginal laws cumulative distribution functions $\tilde{F}_1$ and $\tilde{F}_2$ and satisfying the $\mathbb{P}_1$ moment constraints is such that

$$\forall i = 1, 2,\ \forall 1 \leq j \leq M, \quad \int_{T_j^M} F_i = \int_{T_j^M} \tilde{F}_i$$

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- For all $1 \leq j \leq M$, if $F_1 - F_2$ does not change of sign on $T_j^M$ then
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- For all $1 \leq j \leq M$, if $F_1 - F_2$ does change of sign on $T_j^M$ then
$$\int_{T_j^M} |F_1 - F_2| \leq \int_{T_j^M} |\tilde{F}_1 - \tilde{F}_2| + 2 \int_{T_j^M} (F_1 - F_2)^-$$
$$\leq \int_{T_j^M} |\tilde{F}_1 - \tilde{F}_2| + 2\|\rho_1 - \rho_2\|_\infty \frac{1}{N^2}$$

# Idea of proof (continuous $\mathbb{P}_1$ case, $c(x_1, x_2) = |x_1 - x_2|$)

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$$\forall i = 1, 2, \ \forall 1 \leq j \leq M, \quad \int_{T_j^M} F_i = \int_{T_j^M} \tilde{F}_i$$
- For all $1 \leq j \leq M$, if $F_1 - F_2$ does not change of sign on $T_j^M$ then
$$\int_{T_j^M} |F_1 - F_2| \leq \int_{T_j^M} |\tilde{F}_1 - \tilde{F}_2|$$
- For all $1 \leq j \leq M$, if $F_1 - F_2$ does change of sign on $T_j^M$ then
$$\begin{aligned} \int_{T_j^M} |F_1 - F_2| &\leq \int_{T_j^M} |\tilde{F}_1 - \tilde{F}_2| + 2 \int_{T_j^M} (F_1 - F_2)^- \\ &\leq \int_{T_j^M} |\tilde{F}_1 - \tilde{F}_2| + 2 \|\rho_1 - \rho_2\|_\infty \frac{1}{N^2} \end{aligned}$$
- by hypothesis, only $Q$ intervals of the last type and if $\gamma$ is a minimizer of the MCOT problem,
$$\int_0^1 |F_1 - F_2| \geq \int_0^1 |\tilde{F}_1 - \tilde{F}_2|.$$

# Rates of convergence

## Theorem

- *continuous* $\mathbb{P}_1$ *case and* $c(x_1, x_2) = |x_1 - x_2|^2$*: Let us assume that* $\mathrm{d}\nu_1(x) = \rho_1(x)\mathrm{d}x$ *and* $\mathrm{d}\nu_2(x) = \rho_2(x)\mathrm{d}x$ *for some* $\rho_1, \rho_2 \in L^\infty([0,1], \mathrm{d}x; \mathbb{R}_+)$*. Let us denote by* $F_1$ *and* $F_2$ *the cumulative distribution functions of* $\nu_1$ *and* $\nu_2$*. Then, for all* $M \in \mathbb{N}^*$*,*

$$|I - I^M| = |W_2^2(\nu_1, \nu_2) - I^M| \leq \frac{7}{3} \frac{\|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}}{M^2}.$$

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# Conclusion and perspectives

**Conclusions**

- Alternative way of discretizing multi-marginal optimal transport problems, in particular for application in quantum chemistry or finance using test functions: MCOT problems;
- Convergence of MCOT problem towards the OT problem;
- Some minimizers of MCOT problems can be written as discrete measures charging a low number of points (and even lower using symmetries in quantum chemistry)
- suggest the use of a particle numerical scheme for the resolution of the MCOT problem
- Preliminary results on simple OT problems on the rate of convergence of the MCOT problem towards the OT problem for piecewise constant and piecewise affine functions

**Perspectives**

- find an algorithm which can efficiently solve the particle problem
- prove more general convergence rates of the MCOT problem to the exact OT problem.

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Thank you for your attention.