Approximation of OT problems with marginal moments contraints

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Summary

Kantorovich problem with two marginal laws

For any (open or compact) subset $\mathcal{X} \subset \mathbb{R}^d$ ($d \in \mathbb{N}^*$), let us denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$.

- Let $d_1, d_2 \in \mathbb{N}^*$, $\mathcal{X}_1 \subset \mathbb{R}^{d_1}$ and $\mathcal{X}_2 \subset \mathbb{R}^{d_2}$ be open or compact subsets.
- For $\nu_1 \in \mathcal{P}(\mathcal{X}_1)$ and $\nu_2 \in \mathcal{P}(\mathcal{X}_2)$, let

$$\Pi(\nu_1, \nu_2) = \left\{ \begin{array}{c} \gamma \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2), \\ \int_{\mathcal{X}_2} \mathrm{d}\gamma(x_1, x_2) = \mathrm{d}\nu_1(x_1), \int_{\mathcal{X}_1} \mathrm{d}\gamma(x_1, x_2) = \mathrm{d}\nu_2(x_2) \end{array} \right\}$$

- Let $c : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}_+ \cup \{+\infty\}$ be lower semi-continuous (l.s.c) cost function.

The Kantorovich optimal transport problem reads:

$$\boxed{\inf_{\gamma \in \Pi(\nu_1, \nu_2)} \int_{\mathcal{X}_1, \mathcal{X}_2} c(x_1, x_2) \mathrm{d}\gamma(x_1, x_2).}$$

Multi-marginal Kantorovich problem

- Let $N \in \mathbb{N}^*$, and for all $1 \leq i \leq N$, let $d_i \in \mathbb{N}^*$, $\mathcal{X}_i \subset \mathbb{R}^{d_i}$ be an open or compact subset.
- For all $1 \leq i \leq N$, let $\nu_i \in \mathcal{P}(\mathcal{X}_i)$, and let

$$\Pi^N((\nu_i)_{1\leq i\leq N}) := \left\{\gamma \in \mathcal{P}(\mathcal{X}_1, ..., \mathcal{X}_N), \mathrm{d}\mu^i_\gamma(x_i) = \mathrm{d}\nu_i(x_i), \forall 1 \leq i \leq N\right\},$$

where $\mu^i_\gamma \in \mathcal{P}(\mathcal{X}_i)$ denotes the i^{th} marginal law of γ, defined by

$$\mathrm{d}\mu^i_\gamma(x_i) := \int_{\mathcal{X}_1 \times \ldots \mathcal{X}_{i-1} \times \mathcal{X}_{i+1} \times \ldots \times \mathcal{X}_N} \mathrm{d}\gamma(x_1, ..., x_N).$$

- Let $c : \mathcal{X}_1 \times ... \times \mathcal{X}_N \to \mathbb{R}_+ \cup \{+\infty\}$ be lower semi-continuous (l.s.c.) cost function.

The multi-marginal Kantorovich optimal transport problem reads:

$$\boxed{I = \inf_{\gamma \in \Pi^N((\nu_i)_{1\leq i\leq N})} \int_{\mathcal{X}_1 \times \ldots \times \mathcal{X}_N} c\,\mathrm{d}\gamma.}$$

discretization

Let $M \in \mathbb{N}^*$, we discretize the measure $\nu_i \in \mathcal{P}(\mathcal{X}_i)$ on a fixed discretization grid of $\mathcal{X}_i$, $x_i^1, \ldots, x_i^M \in \mathcal{X}_i$.

$$\mathrm{d}\nu_i(x) \approx \sum_{j=1}^{M} \bar{\nu}_i^j \delta_{x_i^j},$$

for some $\bar{\nu}_i^j \in \mathbb{R}_+$ such that $\sum_{j=1}^{M} \bar{\nu}_i^j = 1$.

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for some $\bar{\nu}_i^j \in \mathbb{R}_+$ such that $\sum_{j=1}^M \bar{\nu}_i^j = 1$.
In the two marginal laws case, introduce

$$\Gamma := \left\{ \begin{array}{c} (\bar{\gamma}^{j_1,j_2}) \in \mathbb{R}_+^{M^2} \\ \forall 1 \leq j \leq M, \sum_{j_1=1}^M \bar{\gamma}^{j_1,j} = \bar{\nu}_2^j, \sum_{j_2=1}^M \bar{\gamma}^{j,j_2} = \bar{\nu}_1^j \end{array} \right\}$$

so that

$$\gamma \approx \sum_{j_1,j_2} \bar{\gamma}^{j_1,j_2} \delta_{x_1^{j_1},x_2^{j_2}}$$

and we solve the **linear problem under linear constraints in** $\mathbb{R}^{M^2}$

$$\inf_{(\bar{\gamma}_{j_1,j_2}) \in \Gamma} \sum_{j_1,j_2}^{M} c(x_1^{j_1}, x_2^{j_2}) \bar{\gamma}^{j_1,j_2}.$$

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In the multi-marginal case with N marginal laws, introduce

$$\Gamma := \left\{ \begin{array}{c} (\bar{\gamma}^{j_1,\cdots,j_N}) \in \mathbb{R}_+^{M^N} \\ \displaystyle\sum_{1 \leq j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_N \leq M} \bar{\gamma}^{j_1,\cdots,j_{i-1},j,j_{i+1},\cdots,j_N} = \bar{\nu}_i^j, \ \forall i,j \end{array} \right\}$$

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Curse of dimensionality!

Summary

Alternative discretization: Moments Constrained Optimal Transport Problem

Let $M \in \mathbb{N}^*$ and $\phi_1^i, ..., \phi_M^i \in \mathcal{C}_b(\mathcal{X}_i)$ be some continuous bounded functions on $\mathcal{X}_i$. They will be called hereafter **test functions**.

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For all $1 \leq i \leq N$, the marginal constraint

$$\mathrm{d}\mu_\gamma^i(x_i) = \mathrm{d}\nu_i(x_i)$$

is then approximated by the M moment constraints: for all $1 \leq j \leq M$,

$$\int_{\mathcal{X}_i} \phi_j^i(x_i)\mathrm{d}\mu_\gamma^i(x_i) = \int_{\mathcal{X}_1\times...\times\mathcal{X}_N} \phi_j^i(x_i)\mathrm{d}\gamma(x_1, ..., x_N) = \int_{\mathcal{X}_i} \phi_j^i(x_i)\mathrm{d}\nu_i(x_i) =: \bar{\nu}_i^j$$

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and we consider the following approximate problem (**MCOT problem**)

$$I^M = \inf_{\substack{\gamma\in\mathcal{P}(\mathcal{X}_1\times\ldots\times\mathcal{X}_N) \\ \forall 1\leq i\leq N, \forall 1\leq j\leq M, \\ \int_{\mathcal{X}_1\times\ldots\times\mathcal{X}_N} \phi_j^i(x_i)\mathrm{d}\gamma(x_1,\ldots,x_N)=\bar{\nu}_i^j \\ \int_{\mathcal{X}_1\times\ldots\times\mathcal{X}_N} \sum_{i=1}^N \theta_{\nu_i}(|x_i|)\mathrm{d}\gamma(x_1,\ldots,x_N)\leq A_0}} \int_{\mathcal{X}_1\times\ldots\times\mathcal{X}_N} c\mathrm{d}\gamma. \tag{1}$$

Alternative discretization: Moments Constrained Optimal Transport Problem

Remark

$$I^M \leq I$$

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Under appropriate conditions on the test functions,

Theorem ([Alfonsi, C., Ehrlacher, Lombardi, 2019])

For all $M \in \mathbb{N}^*$, it holds that $I^M \leq I$ and $I^M \xrightarrow[M \to +\infty]{} I$.
Besides, there exists an integer $1 \leq K \leq MN + 2$, and for all $1 \leq k \leq K$, points $z^k = (x_1^k, ..., x_N^k) \in \mathcal{X}_1 \times ... \times \mathcal{X}_N$ and weights $w_k > 0$ such that

$$\gamma^M = \sum_{k=1}^{K} w_k \delta_{x_1^k, \ldots, x_N^k}$$

is a minimizer of the MCOT Problem (1).

$MN + 2$ bound on the number of charged points

Theorem (Tchakaloff [Bayer & Teichmann, 2006])

Let $d \in \mathbb{N}^*$ and let γ be a measure on $\mathbb{R}^d$ concentrated on a Borel set $A \in \mathcal{F}$, i.e. $\gamma(\mathbb{R}^d \setminus A) = 0$. Let $M_0 \in \mathbb{N}^*$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}^{M_0}$ a Borel measurable map. Assume that the first moments of $\Lambda \# \gamma$ exist, i.e.

$$\int_{\mathbb{R}^{M_0}} \|u\| \mathrm{d}\Lambda\#\gamma(u) = \int_{\mathbb{R}^d} \|\Lambda(z)\| \mathrm{d}\gamma(z) < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^{M_0}$. Then, there exist an integer $1 \leq K \leq M_0$, points $z_1, \ldots, z_K \in A$ and weights $w_1, \ldots, w_K > 0$ such that

$$\forall 1 \leq j \leq M_0, \quad \int_{\mathbb{R}^d} \Lambda_j(z) \mathrm{d}\gamma(z) = \sum_{k=1}^{K} w_k \Lambda_j(z_k),$$

where Λ_j denotes the j-th component of Λ.

Numerical Interest

This suggests to consider the following problem (**particle problem**) as a numerical scheme

$$\inf_{\substack{w_1,\dots,w_{MN+2}\geq 0 \\ \sum_{k=1}^{MN+2} w_k = 1 \\ (x_1^k,\dots,x_{MN+2}^k)_{1\leq k\leq MN+2}\in(\mathcal{X}_1\times\dots\times\mathcal{X}_N)^{MN+2} \\ \forall 1\leq i\leq N,\ \forall 1\leq j\leq M, \sum_{k=1}^{MN+2} w_k\phi_j^i(x_i^k)=\bar{\nu}_i^j}} \sum_{k=1}^{MN+2} w_k c(x_1^k,\dots,x_N^k). \tag{2}$$

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Remark

The number of particles grows linearly with the number of marginal laws, as well as the number of coordinates of those particles.

Summary

Martingale Optimal Transport problem

For the sake of simplicity, we will only consider two marginal laws $\nu_1 \in \mathcal{P}(\mathbb{R}^d)$ and $\nu_2 \in \mathcal{P}(\mathbb{R}^d)$, $d \in \mathbb{N}^*$.
We assume that there exist a martingale coupling between ν_1 and ν_2:

$$\exists \gamma \in \Pi(\nu_1, \nu_2),\ \forall x_1 \in \mathbb{R}^d,\ \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1, x_2) = x_1,$$

and a l.s.c. cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$.
The Martingale Optimal Transport Problem reads as

$$I = \inf_{\substack{\gamma \in \Pi(\nu_1, \nu_2) \\ \forall x_1 \in \mathbb{R}^d,\ \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1, x_2) = x_1}} \int_{\mathbb{R}^{2d}} c(x_1, x_2) \mathrm{d}\gamma(x_1, x_2).$$

In finance, ν_1 and ν_2 are known distribution of prices and $\int_{\mathbb{R}^{2d}} c(x_1, x_2)\mathrm{d}\gamma(x_1, x_2)$ is the payoff of an option.

Martingale MCOT problems

Let us introduce the following MCOT Problems, for all $M \in \mathbb{N}^*$, with the same constraints on the test functions as in the non compact case,

$$I_{A_0}^{M,\mathrm{mg}} = \inf_{\substack{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \\ \forall i=1,2,\, \forall 1 \leq j \leq M, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_j^i(x_i) \mathrm{d}\gamma(x_1,x_2) = \bar{\nu}_i^j \\ \forall x_1 \in \mathbb{R}^d,\, \int_{\mathbb{R}^d} x_2 \mathrm{d}\gamma(x_1,x_2) = x_1, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^2 \theta_{\nu_i}(|x_i|) \mathrm{d}\gamma(\mathbb{R}^d \times \mathbb{R}^d) \leq A_0}} \int_{\mathbb{R}^d \times \mathbb{R}^d} c \mathrm{d}\gamma. \tag{3}$$

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and

$$I_{A_0}^{M,M'} = \inf_{\substack{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \\ \forall i=1,2,\ \forall 1 \le j \le M, \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_j^i(x_i)\mathrm{d}\gamma(x_1,x_2) = \bar{\nu}_i^j \\ \forall 1 \le l \le M',\ \int_{\mathbb{R}^d} x_2 \chi_l(x_1)\mathrm{d}\gamma(x_1,x_2) = \int_{\mathbb{R}^d} x_1 \chi_l(x_1)\mathrm{d}\nu_1(x_1), \\ \int_{\mathbb{R}^d \times \mathbb{R}^d} \sum_{i=1}^2 \theta_{\nu_i}(|x_i|)\mathrm{d}\gamma(\mathbb{R}^d \times \mathbb{R}^d) \le A_0}} \int_{\mathbb{R}^d \times \mathbb{R}^d} c\mathrm{d}\gamma. \tag{4}$$

Martingale MCOT results

With appropriate constraints on the additional test functions χ_I, one can prove that

- The problem (4) admits a finite discrete minimizer (with $2M + M'$ charged points)
- $I_{A_0}^{M,M'} \xrightarrow[M' \to \infty]{} I_{A_0}^{M,\mathrm{mg}} < +\infty$
- $I_{A_0}^{M,\mathrm{mg}} \xrightarrow[M \to \infty]{} I$

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Remark

- One can also define a multi-marginal martingale optimal transport problem and thus, for example, match price at several time steps.
- If we use as test functions the functions $(\cdot - K)^+$ and $(K - \cdot)^+$ for various values of K, the Martingale MCOT problem consists in knowing only option prices at various strikes and maturity with **no hypothesis** on the underlying asset price.
- However, our numerical scheme can only approximate the martingale constraints on a set of test functions (with $NM + M'$ charged points).
- Other numerical methods for Martingale Optimal Transport use sampling techniques [Alfonsi, Corbetta, Jourdain, '19] or entropic regularization [De March, '18] [Guo, Obloj, '17].

Summary

Conclusion and perspectives

Conclusions

- Alternative way of discretizing multi-marginal optimal transport problems, in particular for application in finance using test functions: MCOT problems;
- Convergence of MCOT problem towards the OT problem;
- Some minimizers of MCOT problems can be written as discrete measures charging a low number of points
- suggest the use of a particle numerical scheme for the resolution of the MCOT problem
- Preliminary results on simple OT problems on the rate of convergence of the MCOT problem towards the OT problem for piecewise constant and piecewise affine functions

Perspectives

- find an algorithm which can efficiently solve the particle problem
- prove more general convergence rates of the MCOT problem to the exact OT problem.

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Thank you for your attention.

Summary

Sets of test functions

Let $M \in \mathbb{N}^*$ and let us define the intervals

$$T_1^M := \left[0, \frac{1}{M}\right], T_j^M := \left(\frac{j-1}{M}, \quad \frac{j}{M}\right], \forall 2 \leq j \leq M.$$

We consider three different sets of test functions:

- Piecewise constant ($\mathbb{P}_0$) test functions:

$$\forall 1 \leq j \leq M, \quad \phi_j^M := 1_{T_j^M}$$

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- Continuous piecewise affine (continuous $\mathbb{P}_1$) test functions: for $2 \leq j \leq M$

$$\phi_j^M(x) := \left| \begin{array}{ll} M\left(x - \frac{j-2}{M}\right) & \text{if } x \in T_{j-1}^M, \\ M\left(\frac{j}{M} - x\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{array} \right. \quad \phi_1^M(x) := \left| \begin{array}{ll} 1 - Mx & \text{if } x \in T_1^M, \\ 0 & \text{otherwise,} \end{array} \right.$$

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- Discontinuous piecewise affine (discontinuous $\mathbb{P}_1$) test functions: for $1 \leq j \leq M$

$$\phi_{j,1}^M(x) := \begin{cases} M\left(\frac{j}{M} - x\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{cases} \quad \phi_{j,2}^M(x) := \begin{cases} M\left(x - \frac{j-1}{M}\right) & \text{if } x \in T_j^M, \\ 0 & \text{otherwise,} \end{cases}$$

Rates of convergence

Theorem

- $\mathbb{P}_0$ *case:* *if c is Lipschitz with constant C, then for all $M \in \mathbb{N}^*$,*

$$|I - I^M| \leq \frac{C}{M}.$$

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- $\mathbb{P}_0$ *case:* *if c is Lipschitz with constant C, then for all $M \in \mathbb{N}^*$,*

$$|I - I^M| \leq \frac{C}{M}.$$

- *continuous $\mathbb{P}_1$ case and $c(x_1, x_2) = |x_1 - x_2|$: Let us assume that $\mathrm{d}\nu_1(x) = \rho_1(x)\mathrm{d}x$ and $\mathrm{d}\nu_2(x) = \rho_2(x)\mathrm{d}x$. Let us denote by F_1 and F_2 the cumulative distribution functions of ν_1 and ν_2 and let us assume that the function $F_1 - F_2$ changes sign at most $Q \in \mathbb{N}$ times on $[0, 1]$ and that $\rho_1 - \rho_2 \in L^\infty([0, 1], \mathrm{d}x, \mathbb{R})$. Then, for all $M \in \mathbb{N}^*$,*

$$|I - I^M| = |W_1(\nu_1, \nu_2) - I^M| \leq \frac{2Q\|\rho_1 - \rho_2\|_\infty}{M^2}.$$

Rates of convergence

Theorem

- $\mathbb{P}_0$ *case: if c is Lipschitz with constant C, then for all $M \in \mathbb{N}^*$,*

$$|I - I^M| \leq \frac{C}{M}.$$

- *continuous $\mathbb{P}_1$ case and $c(x_1, x_2) = |x_1 - x_2|$: Let us assume that $\mathrm{d}\nu_1(x) = \rho_1(x)\mathrm{d}x$ and $\mathrm{d}\nu_2(x) = \rho_2(x)\mathrm{d}x$. Let us denote by F_1 and F_2 the cumulative distribution functions of ν_1 and ν_2 and let us assume that the function $F_1 - F_2$ changes sign at most $Q \in \mathbb{N}$ times on $[0,1]$ and that $\rho_1 - \rho_2 \in L^\infty([0,1], \mathrm{d}x, \mathbb{R})$. Then, for all $M \in \mathbb{N}^*$,*

$$|I - I^M| = |W_1(\nu_1, \nu_2) - I^M| \leq \frac{2Q\|\rho_1 - \rho_2\|_\infty}{M^2}.$$

- *continuous $\mathbb{P}_1$ case and $c(x_1, x_2) = |x_1 - x_2|^2$: Let us assume that $\mathrm{d}\nu_1(x) = \rho_1(x)\mathrm{d}x$ and $\mathrm{d}\nu_2(x) = \rho_2(x)\mathrm{d}x$ for some $\rho_1, \rho_2 \in L^\infty([0,1], \mathrm{d}x; \mathbb{R}_+)$. Let us denote by F_1 and F_2 the cumulative distribution functions of ν_1 and ν_2. Then, for all $M \in \mathbb{N}^*$,*

$$|I - I^M| = |W_2^2(\nu_1, \nu_2) - I^M| \leq \frac{7}{3}\frac{\|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}}{M^2}.$$