

Branching capacity of a random walk in \mathbb{Z}^{5*}

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Abstract

We are interested in the branching capacity of the range of a random walk in \mathbb{Z}^d . Schapira [29] has recently obtained precise asymptotics in the case $d \geq 6$ and has demonstrated a transition at dimension $d = 6$. We study the case $d = 5$ and prove that the renormalized branching capacity converges in law to the Brownian snake capacity of the range of a Brownian motion. The main step in the proof relies on studying the intersection probability between the range of a critical Branching random walk and that of a random walk, which is of independent interest.

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1 Introduction

Let $(\xi_n)_{n \geq 0}$ be a centered, aperiodic and irreducible random walk in \mathbb{Z}^d with finite second moment, whose law is denoted by $P^{(\xi)}$. Denote by $\xi[0, n] := \{\xi_0, \dots, \xi_n\}$ the range of ξ up to time n . Studying the range $\xi[0, n]$ is a classical problem in probability theory.

Concerning the asymptotics of the size of the range $\#\xi[0, n]$, it is well-known (Dvoretzky and Erdős [14]) that there is a transition at dimension $d = 2$: $\#\xi[0, n]$ is of order n when $d \geq 3$ and of order $\frac{n}{\log n}$ when $d = 2$; we refer to Jain and Orey [19], Jain and Pruitt [17], and Le Gall [21, 22] for deep studies on the size of the range.

The capacity of the range depends on its geometry and has recently attracted significant interest. The discrete Newtonian capacity can be defined as follows. Let $d \geq 3$ and $K \subset \mathbb{Z}^d$ be a nonempty finite set. Let S be the range of a simple symmetric random walk on \mathbb{Z}^d starting from x whose law is denoted by P_x . The discrete Newtonian capacity of K , $\text{Cap}(K)$, can be defined for $d \geq 3$ (up to a multiplicative constant) as

$$\text{Cap}(K) := \lim_{x \in \mathbb{Z}^d, x \rightarrow \infty} |x|^{d-2} P_x(S \cap K \neq \emptyset), \quad (1.1)$$

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where $|x|$ denotes the usual Euclidean norm of x and $x \rightarrow \infty$ means that $|x| \rightarrow \infty$. Following the works of Jain and Orey [18], Asselah, Schapira and Sousi [4, 5], and Chang [12], it is known that $\text{Cap}(\xi[0, n])$ is of order n when $d \geq 5$, of order $\frac{n}{\log n}$ when $d = 4$, and of order \sqrt{n} when $d = 3$. This implies a transition at dimension $d = 4$. See Asselah, Schapira, and Sousi [1], Schapira [28] and the references therein for the central limit theorems, and Dembo and Okada [13] for the laws of the iterated logarithm. Additionally, Asselah and Schapira [2] investigated the link between the capacity and the folding phenomenon of a random walk, and Hutchcroft and Sousi [16] explored the capacity of the loop-erased random walk.

Recently, Zhu [31] introduced the concept of branching capacity. The basic idea is to replace the range of a simple random walk S in (1.1) by that of a critical branching random walk. Specifically, let \mathcal{T}_c be a critical Galton-Watson tree with offspring distribution μ , where $\mu = (\mu(i))_{i \geq 0}$ is a probability distribution on \mathbb{N} such that $\sum_{i=0}^{\infty} i\mu(i) = 1$. To avoid triviality, we assume $\mu(1) < 1$. In other words, \mathcal{T}_c is a finite random tree that starts with one particle \emptyset , called the root, and each particle independently produces a random number of offspring according to μ . The critical branching random walk on \mathbb{Z}^d , denoted by V_c , is a random walk indexed by the tree \mathcal{T}_c constructed as follows. Let θ be a probability distribution on \mathbb{Z}^d , representing the common distribution of the displacements of V_c . For each edge e of \mathcal{T}_c , we assign an independent random variable X_e with distribution θ . We set $V_c(\emptyset) := x \in \mathbb{Z}^d$, and for $u \in \mathcal{T}_c \setminus \{\emptyset\}$, $V_c(u) = x + \sum_e X_e$, where the sum is taken over all edges e belonging to the simple path in \mathcal{T}_c connecting u to \emptyset . The range of V_c is denoted by

$$\mathcal{R}_c := \{V_c(u), u \in \mathcal{T}_c\} \subset \mathbb{Z}^d. \tag{1.2}$$

Denote by \mathbf{P}_x the law of V_c and write $\mathbf{P} = \mathbf{P}_0$. Almost surely the random tree \mathcal{T}_c is finite, so is the range \mathcal{R}_c .

For $x \in \mathbb{R}^d$, let $B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}$ be the open ball centered at x with radius $r > 0$. We assume that $d \geq 5$, and for some $q > 4$,

$$\mu \text{ has mean } 1 \text{ and variance } \sigma^2 \in (0, \infty), \tag{1.3}$$

$$\theta \text{ is symmetric, irreducible with covariance matrix } M_\theta, \text{ has } (d - 2)\text{-th moment}, \tag{1.4}$$

$$\text{there exists a finite constant } c \text{ such that for all } r > 0: \theta(B(0, r)^c) \leq cr^{-q}.$$

The last condition in (1.4) is in particular satisfied when θ has a finite q -th moment; and this holds a priori when $d \geq 7$.

Denote by P_x the law of a θ -walk $(S_n)_{n \geq 0}$ on \mathbb{Z}^d started from x , meaning that the random walk S has the step distribution θ . The Green function of (S_n) is given by $g(x, y) := g(x - y)$ for any $x, y \in \mathbb{Z}^d$, and

$$g(x) := \sum_{n=0}^{\infty} P_0(S_n = x) \sim c_g |x|^{2-d} \text{ as } |x| \rightarrow \infty, \tag{1.5}$$

with

$$|x|_\theta := (x^T M_\theta^{-1} x)^{1/2} \text{ and } c_g := \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2} \sqrt{\det M_\theta}},$$

where the equivalence in (1.5) is given by Uchiyama [30, Theorem 2] as θ has finite $(d - 2)$ -th moment. Let $K \subset \mathbb{Z}^d$ be a nonempty finite set. By Zhu [31], when (1.4) holds with $q = d$, the following limit exists and is called the branching capacity of K :

$$\text{Bcap}(K) := \lim_{x \in \mathbb{Z}^d, |x| \rightarrow \infty} \frac{\mathbf{P}_x(\mathcal{R}_c \cap K \neq \emptyset)}{g(x)}. \tag{1.6}$$

Note that $\text{Bcap}(\cdot)$ viewed as a set function is non-decreasing, invariant under translations, and strictly positive. We extend Bcap into a Choquet capacity on \mathbb{R}^d by letting $\text{Bcap}(A) := \text{Bcap}(A \cap \mathbb{Z}^d)$ for any $A \subset \mathbb{R}^d$.

The branching capacity, studied in a series of papers by Zhu [31, 32, 33], has also been the subject of some more recent works. Asselah, Schapira and Sousi [3] have shown that $\text{Bcap}(K)$ can be compared, up to two positive constants, with the discrete Riesz capacity with index $d - 4$, and have revealed a deep relationship between the local time spent in a ball by the branching random walk and the branching capacity of this ball. Moreover, in another work, Asselah, Okada, Schapira and Sousi [6] have demonstrated the comparability of $\text{Bcap}(K)$ with the average limit of the size of the Minkowski sum of K and two independent copies of (S_n) , as well as with the hitting probability of K by this Minkowski sum. See also [7] for further references.

In [7] we also proved the vague convergence of the scaling limit of Bcap towards the capacity related to the Brownian snake, denoted by BScap . More precisely, consider the excursion measure \mathcal{N}_x given by the distribution of a Brownian snake $(W_t)_{t \geq 0}$ started from $x \in \mathbb{R}^d$ whose lifetime process is an Itô excursion. Denote by \mathfrak{R} the range of the Brownian snake, see Le Gall [23, Chapter IV] for the precise definitions. It was shown in [7] that when $d \geq 5$, for any bounded Borel set $A \subset \mathbb{R}^d$, the following limit exists and is finite:

$$\text{BScap}(A) := \lim_{x \rightarrow \infty} |x|^{d-2} \mathcal{N}_x(\mathfrak{R} \cap A \neq \emptyset). \tag{1.7}$$

We call $\text{BScap}(A)$ the Brownian snake capacity of A .

The vague convergence in [7, Theorem 1.4] says that for $d \geq 5$ and for any compact set $K \subset \mathbb{R}^d$ such that

$$\text{BScap}(\overset{\circ}{K}) = \text{BScap}(K), \tag{1.8}$$

where $\overset{\circ}{K}$ denotes the interior of K , we have, under (1.3) and (1.4) with $q = d$,

$$\lim_{n \rightarrow \infty} \frac{\text{Bcap}(\sqrt{n}K)}{n^{(d-4)/2}} = c_\theta \text{BScap}(M_\theta^{-1/2}K), \tag{1.9}$$

with

$$c_\theta := \frac{2}{\sigma^2 c_g} = \frac{4\pi^{d/2} \sqrt{\det M_\theta}}{\sigma^2 \Gamma(\frac{d-2}{2})}. \tag{1.10}$$

We choose a renormalization of $n^{1/2}$ in (1.9) to ensure consistency with the choice of K_n below. The condition (1.8) is in particular satisfied when K is the closure of a bounded open set with Lipschitz boundary, see [7, Proposition 1.3]. It is a natural problem to investigate the branching capacity of (random) compact set K which does not satisfy (1.8). In this paper we consider the case $\sqrt{n}K_n := \xi[0, n]$ of the range of the random walk $(\xi_n)_{n \geq 0}$, which was recently studied by Schapira [29]: when both (S_n) and (ξ_n) are simple symmetric random walks, the following asymptotics hold:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \text{Bcap}(\xi[0, n]) \text{ exists almost surely and is positive when } d \geq 7; \\ \lim_{n \rightarrow \infty} \frac{\log n}{n} \text{Bcap}(\xi[0, n]) = \frac{2\pi^3}{27\sigma^2} \text{ in probability when } d = 6; \\ E^{(\xi)}[\text{Bcap}(\xi[0, n])] \text{ is of order } n^{1/2} \text{ when } d = 5. \end{aligned} \tag{1.11}$$

We aim to give a sharp result in dimension $d = 5$. Assume that

$$\begin{aligned} (\xi_n)_{n \geq 0} \text{ is aperiodic irreducible and } E_0^{(\xi)}[|\xi_1|^3] < \infty, \\ E_0^{(\xi)}[\xi_1] = 0 \text{ and } \xi_1 \text{ has covariance matrix } M_\xi, \end{aligned} \tag{1.12}$$

where $P_x^{(\xi)}$ means that the random walk (ξ_n) starts from $x \in \mathbb{Z}^d$. Let $(\beta_t)_{t \geq 0}$ be a standard Brownian motion in \mathbb{R}^d . For a real $d \times d$ matrix M , we define $M\beta[0, 1] := \{M\beta_t : 0 \leq t \leq 1\}$. Notice that $K_n = n^{-1/2}\xi[0, n]$ converges in law, for the Hausdorff distance, to $M_\xi^{1/2}\beta[0, 1]$.

Theorem 1.1. *Let $d = 5$. Assume (1.3), (1.4) with $q = 5$, and (1.12). We have*

$$\frac{\text{Bcap}(\xi[0, n])}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{(law)} \frac{8\pi^2 \sqrt{\det M_\theta}}{\sigma^2} \text{BScap}(M_\theta^{-1/2} M_\xi^{1/2} \beta[0, 1]).$$

Moreover, $\text{BScap}(M_\theta^{-1/2} M_\xi^{1/2} \beta[0, 1])$ is almost surely positive.

Remark 1.2. (i) In Lemma 2.2, we will show that a.s. $\text{BScap}(M_\theta^{-1/2} M_\xi^{1/2} \beta[0, 1]) > 0$ when $d = 5$, whereas it vanishes when $d \geq 6$.

(ii) In the assumption (1.12), aperiodicity can be easily removed, and the third moment condition is required in Lemma 3.3.

(iii) In the case when both (S_n) and (ξ_n) are simple symmetric random walks, the limiting random variable in Theorem 1.1 becomes

$$\frac{\text{Bcap}(\xi[0, n])}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{(law)} \frac{8\pi^2}{\sigma^2 d^{d/2}} \text{BScap}(\beta[0, 1]).$$

(iv) The condition $q = 5$ is used in the definition of the branching capacity (1.6) as well as in the forthcoming Lemma 2.1.

The key ingredient in the proof of Theorem 1.1 is the following estimate on the intersection probability between (ξ_n) and the branching random walk V_c , which may be of independent interest.

For any $A \subset \mathbb{R}^d$ and $r > 0$, let $A^r := \{x \in \mathbb{R}^d : d(x, A) \leq r\}$ be the closed r -neighborhood, where $d(x, A) := \min_{y \in A} |x - y|$. For any $x \in \mathbb{R}^d$, we denote by $\lfloor x \rfloor$ the point in \mathbb{Z}^d whose coordinates are the integer parts of that of x , thus $|\lfloor x \rfloor - x| \leq \sqrt{d}$. We have the following estimate on the intersection probabilities between \mathcal{R}_c and $\xi[0, n]$.

Proposition 1.3. *Let $d = 5$. Assume (1.3), (1.4) with $q > 4$, and (1.12). For any fixed $x \in \mathbb{R}^5 \setminus \{0\}$ and any $\eta \in (0, 1)$, we have*

$$\limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} n I(\varepsilon, n) = 0, \tag{1.13}$$

where $x_n := \lfloor \sqrt{n}x \rfloor \in \mathbb{Z}^5$ and

$$I(\varepsilon, n) := \mathbf{P}_{x_n} \otimes P_0^{(\xi)} \left(\mathcal{R}_c \cap (\xi[0, n])^{\varepsilon\sqrt{n}} \neq \emptyset, \mathcal{R}_c \cap \xi[0, n] = \emptyset, \xi[0, n] \subset B(0, \eta|x_n|) \right). \tag{1.14}$$

Remark 1.4. (i) The condition $\xi[0, n] \subset B(0, \eta|x_n|)$ in $I(\varepsilon, n)$ guarantees that \mathcal{R}_c must have some growth to reach the $(\varepsilon\sqrt{n})$ -neighborhood of $\xi[0, n]$. Without this condition, Proposition 1.3 is no longer true, see Remark 3.11 for further details.

(ii) We can deduce from (2.1) that at least for small η ,

$$\begin{aligned} n^{-3/2} \text{Bcap}(B(0, \varepsilon n^{1/2})) &\lesssim \mathbf{P}_{x_n} \otimes P_0^{(\xi)} \left(\mathcal{R}_c \cap (\xi[0, n])^{\varepsilon\sqrt{n}} \neq \emptyset, \xi[0, n] \subset B(0, \eta|x_n|) \right) \\ &\lesssim n^{-3/2} \text{Bcap}(B(0, (\eta|x| + \varepsilon)\sqrt{n})). \end{aligned}$$

The left and right terms of the above inequalities are, according to (1.9), of order $\frac{1}{n}$. This explains the factor n in (1.13).

(iii) When $d \geq 6$, (1.13) also holds, under the conditions (1.3), (1.4) with $q = d$, and (1.12). In this case, the proof is elementary, see Lemma 2.3.

(iv) The condition $q > 4$ is required by Lemma 3.3, which will be used in Lemma 3.10.

Let us say a few words on the proofs of Theorem 1.1 and Proposition 1.3. Using Donsker’s invariance principle and the forthcoming (2.1) and (2.4), we can see that the scaling limit of $\text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}})$ can be compared with $\text{BScap}(M_\theta^{-1/2} M_\xi^{1/2} \beta[0, 1])$. Therefore the proof of Theorem 1.1 reduces to showing that

$$\text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) \approx \text{Bcap}(\xi[0, n]),$$

in the sense, see (1.6), that for large x

$$\mathbf{P}_{x_n} \otimes P_0^{(\xi)} \left(\mathcal{R}_c \cap (\xi[0, n]^{\varepsilon n^{1/2}}) \neq \emptyset \right) \approx \mathbf{P}_{x_n} \otimes P_0^{(\xi)} (\mathcal{R}_c \cap \xi[0, n] \neq \emptyset),$$

which is the content of Proposition 1.3 (note that the condition $\xi[0, n] \subset B(0, \eta|x_n|)$ holds with probability $1 - o_{|x|}(1)$ as $|x| \rightarrow \infty$). To prove Proposition 1.3, we will switch the roles of ξ and \mathcal{R}_c in the probability term of Proposition 1.3 and study the probability of the form $\mathbf{P}_0 \otimes P_{x_n}^{(\xi)} (\xi[0, n] \cap \mathcal{R}_c^{\varepsilon n^{1/2}} \neq \emptyset, \xi[0, n] \cap \mathcal{R}_c = \emptyset)$. The latter probability can be estimated by a comparison to $\mathbf{P}_0 \otimes P_{x_n}^{(\xi)} (\xi[0, \infty) \cap \mathcal{R}_{n^2} = \emptyset)$, where \mathcal{R}_{n^2} denotes the range \mathcal{R}_c conditioned on the total population $\#\mathcal{T}_c = n^2$. To achieve such a comparison, we need to introduce some auxiliary trees and their associated branching random walks in Section 3, which will be the most technical part. We refer to Remark 3.9 for more detailed explanations on the proof of Proposition 1.3.

The paper is organized as follows. In Section 2.1, we recall some known results on the branching capacity and the Brownian snake capacity. In Section 2.2, we prove Theorem 1.1 by assuming Proposition 1.3. In Section 3, we introduce auxiliary branching random walks V_+ , V_- , and \widehat{V}_- in Section 3.1, then recall several known results on V_+ and \widehat{V}_- in Section 3.2. After establishing some estimates on the increments of V_+ in Section 3.3, we present the key step in the proof of Proposition 1.3 in Section 3.4, involving a study of intersection probabilities between V_\pm and the random walk ξ . Finally, we provide the proof of Proposition 1.3 in Section 3.5.

For notational convenience, we use the notations C, C', C'' , eventually with some subscripts, to denote some positive constants whose values may vary from one paragraph to another.

2 Brownian snake capacity and proof of Theorem 1.1 by assuming Proposition 1.3

2.1 Branching capacity and Brownian snake capacity

We collect some facts on the Branching and the Brownian snake capacities in this subsection. At first the following result (see [7, Theorem 1.1]) provides a rate of convergence in (1.6) that will also be useful in our proof of Theorem 1.1. Let $d \geq 5$. Assume (1.3), (1.4) with $q = d$. For any $\lambda > 0$, there exist some positive constants α and C such that for any $r \geq 1$, $K \subset B(0, r) \cap \mathbb{Z}^d$, $x \in \mathbb{Z}^d$ with $|x| \geq (1 + \lambda)r$, we have

$$\left| \text{Bcap}(K) - \frac{\mathbf{P}_x(\mathcal{R}_c \cap K \neq \emptyset)}{g(x)} \right| \leq C \left(\frac{r}{|x|} \right)^\alpha \text{Bcap}(K). \tag{2.1}$$

Moreover, there are constants $c_1, c_2 > 0$, such that for any Borel set $A \subset \mathbb{R}^d$,

$$c_1 \text{Cap}_{d-4}(A) \leq \text{BScap}(A) \leq c_2 \text{Cap}_{d-4}(A), \tag{2.2}$$

where for any $\gamma \in (0, d)$,

$$\text{Cap}_\gamma(A) := \left(\inf_\nu \iint |x - y|^{-\gamma} \nu(dx)\nu(dy) \right)^{-1}, \tag{2.3}$$

with the infimum taken over all the probability measures with support in A .

Lemma 2.1. *Let $d \geq 5$. Assume (1.3), (1.4) with $q = d$. For any compact set $K \subset \mathbb{R}^d$, we have*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left| \frac{\text{Bcap}(nK^\varepsilon)}{n^{d-4}} - c_\theta \text{BScap}(M_\theta^{-1/2}K) \right| = 0, \tag{2.4}$$

with c_θ given in (1.10).

Proof. First $\text{BScap}(\cdot)$ is a Choquet capacity relative to the set of compact sets on \mathbb{R}^d . In particular, for every compact set $K \subset \mathbb{R}^d$, we have

$$\lim_{\varepsilon \rightarrow 0+} \text{BScap}(K^\varepsilon) = \text{BScap}(K). \tag{2.5}$$

We check that the condition [7, (1.7)] is satisfied for K^ε : for any $y \in \partial K^\varepsilon$ and n large enough, $K^\varepsilon \cap B(y, 2^{-n})$ contains a ball of radius 2^{-n-1} so $\text{Cap}_{d-2}(K^\varepsilon \cap B(y, 2^{-n})) \geq \text{Cap}_{d-2}(B(0, 2^{-n-1})) = 2^{-(d-2)(n+1)} \text{Cap}_{d-2}(B(0, 1))$. Consequently for any $\varepsilon > 0$, we have $\text{BScap}(K^\varepsilon) = \text{BScap}(K^{\varepsilon-})$, where $K^{\varepsilon-} = \{x \in \mathbb{R}^d : d(x, K) < \varepsilon\}$ the open ε -neighborhood of K . Notice that these results also hold when K is replaced by $M_\theta^{-1/2}K$ and K^ε by $M_\theta^{-1/2}K^\varepsilon$. This and [7, Theorem 1.4] yield that

$$\lim_{n \rightarrow \infty} \frac{\text{Bcap}(nK^\varepsilon)}{n^{d-4}} = c_\theta \text{BScap}(M_\theta^{-1/2}K^\varepsilon). \tag{2.6}$$

Finally, (2.4) follows from (2.6) and (2.5). □

We end this subsection by the following result on the positivity of the Brownian snake capacity of the range of a Brownian motion. Recall that for a real $d \times d$ matrix M , $M\beta[0, 1] = \{M\beta_t : 0 \leq t \leq 1\}$.

Lemma 2.2. *Let $(\beta_t)_{t \geq 0}$ be a Brownian motion in \mathbb{R}^d and M be a symmetric positive definite matrix $d \times d$. Then almost surely,*

$$\text{BScap}(M\beta[0, 1]) \begin{cases} > 0, & \text{if } d = 5, \\ = 0, & \text{if } d \geq 6. \end{cases}$$

Proof. By [25, Theorem 1.1 and formula (9)], a.s., $\text{Cap}_\gamma(\beta[0, 1]) > 0$ if and only if $\gamma < \min(2, d)$. In particular, $\text{Cap}_{d-4}(\beta[0, 1])$ is positive when $d = 5$ and zero when $d \geq 6$. By (2.3), the same result holds when $\beta[0, 1]$ is replaced by $M\beta[0, 1]$, because $\frac{|Mx - My|}{|x - y|}$ is uniformly bounded in $(0, \infty)$ for all $x \neq y$. We conclude by using (2.2). □

2.2 Proof of Theorem 1.1 by assuming Proposition 1.3

By Donsker’s invariance principle and Skorokhod’s representation theorem, on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we may find a version of the random walk (ξ_n) starting from 0 and a standard Brownian motion $(\beta_t)_{t \geq 0}$ in \mathbb{R}^d such that for $\Theta := \{M_\xi^{1/2}\beta_t : 0 \leq t \leq 1\}$, almost surely for every $\varepsilon > 0$, there exists some $n_0 \geq 1$ such that for every $n \geq n_0$,

$$n^{1/2} \cdot \Theta^{\varepsilon/2} \subset \xi[0, n]^{\varepsilon n^{1/2}} \subset n^{1/2} \cdot \Theta^{2\varepsilon}. \tag{2.7}$$

Applying (2.4) to $n^{1/2} \cdot \Theta^{\varepsilon/2}$ and $n^{1/2} \cdot \Theta^{2\varepsilon}$, we deduce that for $d = 5$, \mathbb{P} -almost surely,

$$\limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} \left| \frac{\text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}})}{n^{1/2}} - c_\theta \text{BScap}(M_\theta^{-1/2}\Theta) \right| = 0. \tag{2.8}$$

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 by assuming Proposition 1.3. Let $d = 5$. By (2.8), it suffices to show that

$$\limsup_{\delta \rightarrow 0+} \limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) - \text{Bcap}(\xi[0, n]) > \delta n^{1/2} \right) = 0. \quad (2.9)$$

Let $x \in \mathbb{R}^d \setminus \{0\}$ and let $x_n = \lfloor n^{1/2}x \rfloor$. Let $\eta > 0$ be small whose value will be determined later. For n large enough, on the event

$$E_n = E_n(x, \eta) := \left\{ \xi[0, n] \subset B(0, \eta|x_n|) \right\},$$

by applying twice (2.1) to obtain the following first and third inequalities, we have

$$\begin{aligned} \text{Bcap}(\xi[0, n]) &\geq \frac{\mathbf{P}_{x_n}(\mathcal{R}_c \cap \xi[0, n] \neq \emptyset)}{(1 + C\eta^\alpha)g(x_n)} \\ &= \frac{\mathbf{P}_{x_n}(\mathcal{R}_c \cap \xi[0, n]^{\varepsilon n^{1/2}} \neq \emptyset)}{(1 + C\eta^\alpha)g(x_n)} - \frac{e_n(\varepsilon)}{1 + C\eta^\alpha} \\ &\geq \frac{1 - C(\eta + 2\varepsilon|x|^{-1})^\alpha}{1 + C\eta^\alpha} \text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) - e_n(\varepsilon), \end{aligned}$$

where

$$e_n(\varepsilon) := \frac{1}{g(x_n)} \mathbf{P}_{x_n} \left(\mathcal{R}_c \cap \xi[0, n] = \emptyset, \mathcal{R}_c \cap \xi[0, n]^{\varepsilon n^{1/2}} \neq \emptyset \right).$$

For any $\delta > 0$, we may find and then fix sufficiently small $\eta = \eta(\delta) > 0$ and $\varepsilon_0 := \varepsilon_0(\delta, |x|) > 0$ such that for all $\varepsilon \leq \varepsilon_0$, we have $\frac{1 - C(\eta + 2\varepsilon|x|^{-1})^\alpha}{1 + C\eta^\alpha} \geq 1 - \delta^2$. It follows that

$$\begin{aligned} &\mathbb{P} \left(\text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) - \text{Bcap}(\xi[0, n]) > \delta n^{1/2} \right) \\ &\leq \mathbb{P} \left(\delta^2 \text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) + e_n(\varepsilon) > \delta n^{1/2}, E_n \right) + \mathbb{P}(E_n^c) \\ &\leq \mathbb{P} \left(n^{-1/2} \text{Bcap}(\xi[0, n]^{\varepsilon n^{1/2}}) > \frac{1}{2\delta} \right) + \mathbb{P} \left(e_n(\varepsilon) > \frac{\delta}{2} n^{1/2}, E_n \right) + \mathbb{P}(E_n^c) \\ &=: (2.10)_1 + (2.10)_2 + (2.10)_3. \end{aligned} \quad (2.10)$$

By (2.8),

$$\limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} (2.10)_1 \leq \mathbb{P} \left(\text{BScap}(\Theta) > \frac{1}{4c_\theta \delta} \right)$$

is arbitrarily small as we take $\delta \rightarrow 0+$.

By Markov's inequality, $(2.10)_2 \leq a_n I(\varepsilon, n)$ where $I(\varepsilon, n)$ was defined in (1.14) and $a_n := 2\delta^{-1}n^{-1/2}(g(x_n))^{-1}$ is of order of n by (1.5), as $n \rightarrow \infty$ (recalling that $d = 5$). By Proposition 1.3,

$$\limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} (2.10)_2 \leq C' \limsup_{\varepsilon \rightarrow 0+} \limsup_{n \rightarrow \infty} n I(\varepsilon, n) = 0. \quad (2.11)$$

Finally for $(2.10)_3$, we deduce from the standard random walk fluctuations that (η is fixed)

$$\limsup_{x \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(E_n^c) = 0.$$

Combining the terms above and letting first $n \rightarrow \infty$, then $\varepsilon \rightarrow 0$, $|x| \rightarrow \infty$ and lastly $\delta \rightarrow 0$, we obtain (2.9) and complete the proof of Theorem 1.1. \square

Recall $I(\varepsilon, n)$, defined in (1.14), depends also on x and η . We give an elementary proof of (1.13) when $d \geq 6$.

Lemma 2.3. *Let $d \geq 6$. Assume (1.3), (1.4) with $q = d$. For any fixed $x \in \mathbb{R}^d \setminus \{0\}$ and any $\eta \in (0, 1)$, we have*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} n I(\varepsilon, n) = 0.$$

Proof. Let $\Xi_n := n^{(4-d)/2} \text{Bcap}((\xi[0, n])^{\varepsilon \sqrt{n}} 1_{\{\xi[0, n] \subset B(0, \eta|x_n|\)}})$. By (2.1), there exists some positive constant $C = C(\eta, |x|)$ such that for all n ,

$$n I(\varepsilon, n) \leq C \mathbb{E}(\Xi_n).$$

By [31], there exists some positive constant C' such that for all n , $\text{Bcap}(B(0, \eta|x_n|)) \leq C'|x_n|^{d-4}$. We have $\Xi_n \leq C''$ for some positive constant C'' independent of n .

Now we use (2.7) and (2.6) to see that \mathbb{P} -a.s.

$$\limsup_{n \rightarrow \infty} \Xi_n \leq \limsup_{n \rightarrow \infty} n^{(4-d)/2} \text{Bcap}(n^{1/2} \Theta^{2\varepsilon}) = c_\theta \text{BScap}(M_\theta^{-1/2} \Theta^{2\varepsilon}).$$

Applying Fatou's lemma to $C'' - \Xi_n$, we get that

$$\limsup_{n \rightarrow \infty} \mathbb{E}(\Xi_n) \leq c_\theta \mathbb{E}(\text{BScap}(M_\theta^{-1/2} \Theta^{2\varepsilon})).$$

Finally we remark that $\lim_{\varepsilon \rightarrow 0} \mathbb{E}(\text{BScap}(M_\theta^{-1/2} \Theta^{2\varepsilon})) = \mathbb{E}(\text{BScap}(M_\theta^{-1/2} \Theta)) = 0$ for $d \geq 6$, by Lemma 2.2. This ends the proof. \square

3 Intersection probabilities: Proof of Proposition 1.3

In this section, we shall consider planar tree using the lexicographic order in the Ulam-Harris setting. We set $\mathcal{U}^* = \cup_{n \in \mathbb{N}^*} (\mathbb{N}^*)^n$ and $\mathcal{U} = \mathcal{U}^* \cup \{\emptyset\}$, where \emptyset is called the root. For $u = u_1 \cdots u_n \in \mathcal{U}^*$, the set of ancestors $\text{anc}(u) \subset \mathcal{U}$ of u is given by $\text{anc}(u) = \{u_1 \cdots u_k : k \in \{1, \dots, n\}\} \cup \{\emptyset\}$. We see a rooted planar tree \mathbf{t} as a subset of \mathcal{U} such that: $\emptyset \in \mathbf{t}$; if $u \in \mathbf{t} \cap \mathcal{U}^*$ then $\text{anc}(u) \subset \mathbf{t}$; if $u \in \mathbf{t}$, then there exists $k \geq 0$ such that $u_i \in \mathbf{t}$ if and only if $i \in \{1, \dots, k\}$ (with the convention that $\emptyset_i = i$, and that $k = 0$ implies no children). The tree \mathbf{t} is endowed with the usual lexicographic order \prec with the convention that $\emptyset \prec u$ for all $u \in \mathcal{U}^*$.

We set $\mathcal{U}' = \mathcal{U} \cup \mathcal{U}'_*$, with $\mathcal{U}'_* = \{-u : u \in \mathcal{U}^*\}$, where for the word $u = u_1 \cdots u_n \in \mathcal{U}^*$, we define the word $-u := (-u_1) \cdots (-u_n)$. We extend the lexicographic order \prec on \mathcal{U} to \mathcal{U}' as follows. For $u \in \mathcal{U}'_*$, we have $u \prec v$ if: either $v \in \mathcal{U}$; or $v \in \mathcal{U}'_*$ and $-u \in \text{anc}(-v)$; or $v \in \mathcal{U}'_*$ and $-u \notin \text{anc}(-v)$ and $-v \prec -u$ (in \mathcal{U}^*). For example, we have:

$$(-5) \prec (-1) \prec (-1)(-1)(-1) \prec \emptyset \prec 1 \prec 111 \prec 5.$$

3.1 On the discrete tree models

We will introduce the adjoint tree \mathcal{T}_{adj} and invariant tree \mathcal{T}_∞ , as well as the associated branching random walks.

The random planer tree \mathcal{T}_{adj} is derived from the Galton-Watson tree \mathcal{T}_c , with the only modification being made at the root. In the adjoint tree, the root has an offspring distribution

$$\tilde{\mu}(k) = \sum_{j=k+1}^{\infty} \mu(j), \quad k \geq 0,$$

instead of μ , while all other vertices retain the original offspring distribution μ .

We then construct the invariant tree \mathcal{T}_∞ as follows. Start with an infinite spine from the root \emptyset :

$$\mathcal{X} = \{\emptyset_0 := \emptyset, \emptyset_1, \dots, \emptyset_n, \dots\}.$$

To \emptyset , we graft on the right a planar random tree \mathcal{T}_0^d distributed as the critical Galton-Watson tree \mathcal{T}_c (identifying the root of \mathcal{T}_c with $\emptyset = \emptyset_0$). Then, for any $n \geq 1$, the vertex \emptyset_n has i children on the left and j on the right with probability $\mu(i + j + 1)$, and we graft independent copies of \mathcal{T}_c to each child (identifying the root of the grafted tree with the child). As a remark, the two planar subtrees grafted to the left, \mathcal{T}_n^g , and to the right, \mathcal{T}_n^d , of \emptyset_n (with their root identified with \emptyset_n) are dependent random trees and, for $n \geq 1$, distributed as \mathcal{T}_{adj} . For $n = 0$, the tree \mathcal{T}_0^g is simply reduced to its root.

An element $u \neq \emptyset$ of the planar tree $\mathcal{T}_n^d \subset \mathcal{U}$ is coded by the word nu in \mathcal{T}_∞ , and an element $u \neq \emptyset$ of the planar tree $\mathcal{T}_n^g \subset \mathcal{U}$ is coded by the word $(-n)(-u)$ in \mathcal{T}_∞ , and \emptyset_n is coded by $(-n)$ for $n \geq 1$. One can see the tree \mathcal{T}_∞ as a subset of \mathcal{U}' . Then, we order the vertices of \mathcal{T}_∞ by the lexicographic order \prec on \mathcal{U}' :

$$\dots \prec \mathcal{T}_\infty(-2) \prec \mathcal{T}_\infty(-1) \prec \mathcal{T}_\infty(0) \prec \mathcal{T}_\infty(1) \prec \dots,$$

with $\mathcal{T}_\infty(0) = \emptyset_0 = \emptyset$. Moreover, we denote by $\mathcal{T}_+ := \mathcal{T}_\infty(\mathbb{N}) = \mathcal{T}_\infty \cap \mathcal{U}$ (and $\mathcal{T}_- := \mathcal{T}_\infty(\mathbb{Z}_-) = \mathcal{T}_\infty \cap \mathcal{U}_-$ where $\mathcal{U}_- = \mathcal{U}' \cup \{\emptyset\}$) the subgraph of \mathcal{T}_∞ with non-negative (and non-positive) labels. Notice \mathcal{T}_- is a tree and that the spine \mathcal{X} is a subset of \mathcal{T}_- , but \mathcal{T}_+ is disconnected and thus no longer a tree, see Fig. 1.

The sequence $\mathcal{T}_\infty(0), \mathcal{T}_\infty(-1), \mathcal{T}_\infty(-2), \dots$ is not a depth-first sequence for \mathcal{T}_- . We shall reorganize \mathcal{T}_- in depth-first order by considering the planar rooted tree $\widehat{\mathcal{T}}_- \subset \mathcal{U}$ built as the union of the spine \mathcal{X} and the trees $(\mathcal{T}_n^g)_{n \geq 0}$, where \emptyset_{n+1} is identified as an extra oldest child of the root of \mathcal{T}_n^g (which is still identified with \emptyset_n). We then order the vertices of $\widehat{\mathcal{T}}_-$ using the lexicographic order \prec on \mathcal{U} :

$$\emptyset = \widehat{\mathcal{T}}_-(0) \prec \emptyset_1 = \widehat{\mathcal{T}}_-(1) \prec \widehat{\mathcal{T}}_-(2) \prec \dots.$$

See an illustration in Fig. 1.

For each $\alpha \in \{c, \text{adj}, \infty, +, -\}$, we construct V_α , the branching random walk indexed by \mathcal{T}_α with displacement distribution θ , in the same way as we did for V_c and \mathcal{T}_c . For notational brevity, for any $i \in \mathbb{Z}$, we write $V_\infty(i) := V_\infty(\mathcal{T}_\infty(i))$ the spatial position of the i th vertex of \mathcal{T}_∞ . For $a, b \in \mathbb{Z}$, we denote

$$V_\infty[a, b] = \{V_\infty(i), a \leq i \leq b\},$$

and abbreviate \mathcal{R}_∞ for the range $V_\infty(-\infty, \infty)$. For $\alpha \in \{c, \text{adj}, +\}$, we use similar notations $\mathcal{T}_\alpha(i), V_\alpha(i)$ for $0 \leq i \leq \#\mathcal{T}_\alpha$, i finite, with $\mathcal{T}_\alpha(0)$ the root of \mathcal{T}_α . In particular, $\mathcal{R}_c = V_c[0, \#\mathcal{T}_c]$ is in agreement with (1.2). Finally, the law of a branching random walk started from x is always denoted by \mathbf{P}_x .

Since for all $i \in \mathbb{N}$, there exists a unique $j \in \mathbb{N}$ such that $\widehat{\mathcal{T}}_-(i) = \mathcal{T}_\infty(-j)$, we can define $\widehat{V}_-(i) = V_\infty(-j)$. It is immediate that

$$\widehat{\mathcal{R}}_- := \widehat{V}_-[0, \infty) = V_\infty(-\infty, 0] =: \mathcal{R}_-.$$

The distribution of the random tree \mathcal{T}_∞ is invariant by rerooting at vertex $\mathcal{T}_\infty(n)$ for all $n \in \mathbb{Z}$, see [9, Section 2.2]. Since the distribution θ of the increments of the branching random walk is symmetric, we directly deduce that V_∞ is invariant by translation ([9, Section 2.2] and [24]):

$$(V_\infty(n + i) - V_\infty(n))_{i \in \mathbb{Z}} \stackrel{(\text{law})}{=} (V_\infty(i) - V_\infty(0))_{i \in \mathbb{Z}}, \quad \forall n \in \mathbb{Z}. \tag{3.1}$$

Denote by $(Y_n)_{n \in \mathbb{N}}$ the Lukasiewicz walk associated to \mathcal{T}_c , that is, (Y_n) is a centered random walk on \mathbb{Z} starting from 0 whose step distribution is $\mathbf{P}(Y_1 = i) = \mu(i + 1)$, $i \geq -1$, and such that $Y_{n+1} - Y_n + 1$ is the number of children of $\mathcal{T}_c(n)$ for all $n < \#\mathcal{T}_c$. Similarly,

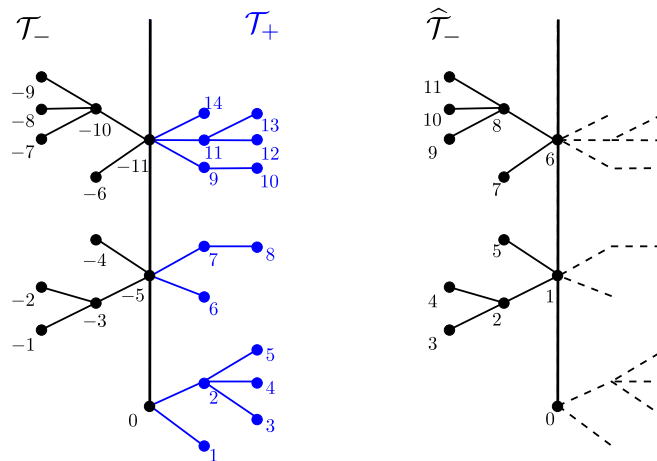


Figure 1: On the left: a sample of $\mathcal{T}_- = \mathcal{T}_\infty(\mathbb{Z}_-)$ in black and $\mathcal{T}_+ = \mathcal{T}_\infty(\mathbb{N})$ in blue (except for the root which is black but belongs also to \mathcal{T}_+) so that $\mathcal{T}_\infty = \mathcal{T}_- \cup \mathcal{T}_+$; notice $\mathcal{T}_\infty(-1)$ is coded by $(-1)(-1)(-1)$, $\mathcal{T}_\infty(-3)$ by $(-1)(-1)$ and $\mathcal{T}_\infty(-5)$ by (-1) . On the right: the tree $\widehat{\mathcal{T}}_-$ which is a different ordering than \mathcal{T}_- of the same graph.

denote by (L_n) the Lukasiewicz walk of $\widehat{\mathcal{T}}_-$ (to be precise, (L_n) is associated to the forest in $\widehat{\mathcal{T}}_-$, where the edges of the infinite spine $(\varnothing_n)_{n \geq 0}$ are removed. In this setting, \varnothing_{n+1} is not seen as child of \varnothing_n). Then its law is as follows: $L_0 = 0, L_1 = -1$; for $n \geq 1$, conditioning on $\sigma\{L_i, i \leq n\}$, $L_{n+1} - L_n + 1$ is distributed as μ if $L_n \neq \min_{0 \leq i \leq n} L_i$, and as $\tilde{\mu}$ if $L_n = \min_{0 \leq i \leq n} L_i$.

By [33, equation after (5.5)], the branching random walk \widehat{V}_- indexed by \mathcal{T}_- can be compared to V_c in terms of (Y_n) and (L_n) : for any $0 \leq k < m$ and nonnegative measurable function F ,

$$\mathbf{E}_x \left(F(V_c[0, k]) \mid \#\mathcal{T}_c = m \right) = \mathbf{E}_x \left(F(\widehat{V}_-[0, k]) \Phi_{m,k}(L_k) \right), \tag{3.2}$$

where

$$\Phi_{m,k}(\ell) := \frac{m \mathbf{P}(Y_{m-k} = -(\ell + 1))}{(m - k) \mathbf{P}(Y_m = -1)}.$$

By the local central limit theorem for the random walk Y , for any $a \in (0, 1)$, there exists some $C_a > 0$ such that, for any $m \geq 1$ and $k \leq am$, we have $\Phi_{m,k}(\ell) \leq C_a$ for all $\ell \in \mathbb{Z}$. Consequently for any nonnegative measurable function F , we have

$$\mathbf{E}_x \left(F(V_c[0, [am]]) \mid \#\mathcal{T}_c = m \right) \leq C_a \mathbf{E}_x \left(F(\widehat{V}_-[0, [am]]) \right). \tag{3.3}$$

Finally, we denote the spatial positions of the spine \mathcal{X} by

$$\mathcal{R}_\mathcal{X} = \{V_\mathcal{X}(0), V_\mathcal{X}(1), \dots\}, \tag{3.4}$$

where for any $i \geq 0$, $V_\mathcal{X}(i) := V_\infty(\varnothing_i) \in \mathbb{Z}^d$ is the position of i th point in the spine (recall that $\varnothing_0 := \varnothing$ is the root). As a matter of fact, the sequence $(V_\mathcal{X}(i))_{i \geq 0}$, forms a θ -walk on \mathbb{Z}^d . We denote by \mathcal{Y} the set of points in $\widehat{\mathcal{T}}_-$ that are not in the spine, with the exception of the root, see Fig. 2. Let

$$\begin{aligned} \mathcal{Y} &:= \{\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \dots\} = \widehat{\mathcal{T}}_- \setminus \{\varnothing_1, \varnothing_2, \dots\}, \\ \mathcal{R}_\mathcal{Y} &:= \{V_\mathcal{Y}(0), V_\mathcal{Y}(1), V_\mathcal{Y}(2), \dots\}, \end{aligned} \tag{3.5}$$

listed in depth-search order, $\mathcal{Y}_0 := \varnothing$ and $V_\mathcal{Y}(0) := V_\infty(\varnothing)$, and for any $i \geq 0$, $V_\mathcal{Y}(i) = V_\infty(\mathcal{Y}_i)$ denotes the spatial position of the i th vertex of \mathcal{Y} . Then we have $\widehat{\mathcal{T}}_- = \mathcal{X} \cup \mathcal{Y}$ and $\mathcal{X} \cap \mathcal{Y} = \{\varnothing\}$.

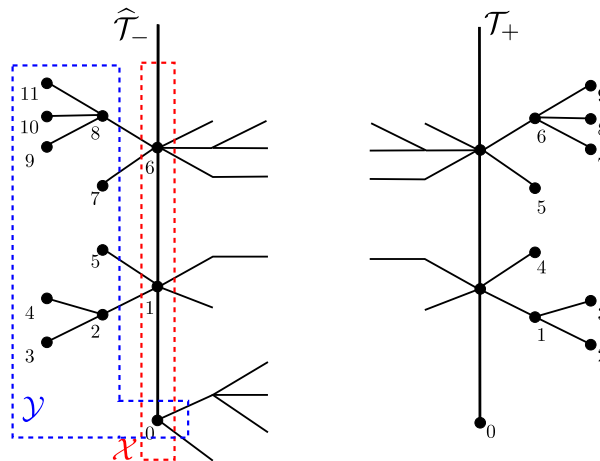


Figure 2: An illustration of $\widehat{\mathcal{T}}_- = \mathcal{X} \cup \mathcal{Y}$, and its comparison to \mathcal{T}_+ when $\mathcal{T}_0^d = \{\emptyset\}$. We have here that $\mathcal{X}(0) = 0, \mathcal{X}(1) = 1, \mathcal{X}(2) = 6$ and $\mathcal{Y}(0) = 0, \mathcal{Y}(1) = \widehat{\mathcal{T}}_-(2), \mathcal{Y}(2) = \widehat{\mathcal{T}}_-(3), \mathcal{Y}(3) = \widehat{\mathcal{T}}_-(4), \mathcal{Y}(4) = \widehat{\mathcal{T}}_-(5)$ and $\mathcal{Y}(5) = \widehat{\mathcal{T}}_-(7)$.

Let $\mathcal{R}_+ := V_+[0, \infty)$ denote the range of V_+ . By construction, the random sets $(\mathcal{R}_\mathcal{X}, \mathcal{R}_\mathcal{Y})$ and $(\mathcal{R}_\mathcal{X}, \mathcal{R}_+)$ conditionally on $\{\mathcal{T}_0^d = \{\emptyset\}\}$ have the same distribution. We deduce that

$$\widehat{\mathcal{R}}_- = \mathcal{R}_\mathcal{X} \cup \mathcal{R}_\mathcal{Y} \stackrel{\text{(law)}}{=} (\mathcal{R}_\mathcal{X} \cup \mathcal{R}_+ \mid \mathcal{T}_0^d = \{\emptyset\}),$$

and that for any nonnegative measurable function F ,

$$\mathbf{E}_x(F(\mathcal{R}_\mathcal{Y}, \mathcal{Y})) = \mathbf{E}_x(F(\mathcal{R}_+, \mathcal{T}_+) \mid \mathcal{T}_0^d = \{\emptyset\}) \leq \frac{1}{\mu(0)} \mathbf{E}_x(F(\mathcal{R}_+, \mathcal{T}_+)). \quad (3.6)$$

This property will be used to compare rare events for \widehat{V}_- and V_+ . As for $V_\alpha[a, b]$ we use the notation: for any $0 \leq i \leq j$,

$$V_\mathcal{Y}[i, j] := \{V_\mathcal{Y}(i), V_\mathcal{Y}(i+1), \dots, V_\mathcal{Y}(j)\}, \quad (3.7)$$

and define for any $m \geq 1$,

$$\mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)} := \mathcal{R}_\mathcal{X} \cap (\widehat{V}_-[0, m]) \quad \text{and} \quad \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} := \mathcal{R}_\mathcal{Y} \cap (\widehat{V}_-[0, m]). \quad (3.8)$$

We end this section by a simple estimate on the number of spine points in $\widehat{\mathcal{T}}_-$.

Lemma 3.1. Assume (1.3). For any $r \in (0, 1)$, there exist some positive constants $C = C_r$ and $c = c_r$ such that for all $n \geq 1$

$$\mathbf{P}(\#\widehat{\mathcal{T}}_-[0, n] \cap \mathcal{X} > rn) \leq C e^{-cn}.$$

Proof. By construction, $\widehat{\mathcal{T}}_-$ consists of i.i.d. adjoint trees in a sequence, and we need to show that it is very unlikely for $\widehat{\mathcal{T}}_-$ to cover $\lceil rn \rceil$ such subtrees. Denote by (Z_i) i.i.d. random variables distributed as the total population of $\#\mathcal{T}_{\text{adj}}$. Then it suffices to show that

$$\mathbf{P}(Z_1 + \dots + Z_{\lceil rn \rceil - 1} \leq n) \leq C e^{-cn},$$

where this probability is equal to 1 if $\lceil rn \rceil = 1$. Let for $s \geq 0$,

$$f(s) := e^{s/r} \mathbf{E}(e^{-sZ_1}).$$

Note that $\mathbf{E}(Z_1) = \infty$. Then $f'(s) = \mathbf{E}[(\frac{1}{r} - Z_1)e^{(\frac{1}{r} - Z_1)s}]$ is continuous on $s \in (0, \infty)$ with $\lim_{s \rightarrow 0^+} f'(s) = -\infty$. Combined with $f(0) = 1$, we know that $f(s_0) < 1$ for some $s_0 > 0$. Therefore,

$$\mathbf{P}(Z_1 + \dots + Z_{\lceil rn \rceil - 1} \leq n) \leq e^{s_0 n} (\mathbf{E}(e^{-s_0 Z_1}))^{\lceil rn \rceil - 1} \leq \mathbf{E}(e^{-s_0 Z_1})^{-1} (f(s_0))^{rn},$$

and we conclude by taking $c = -r \log f(s_0) > 0$ and C large enough. □

3.2 Some known results

We collect here some preliminary estimates on the random walk and the branching random walk.

Lemma 3.2. [20, Proposition 2.1.2, Theorem 2.3.9] *Let $(\xi_n)_{n \geq 0}$ be a centered, aperiodic and irreducible random walk in \mathbb{Z}^d with finite variance. There exists some $C > 0$ such that for every $s > 0$ and $n \geq 1$,*

$$P_0^{(\xi)} \left(\max_{0 \leq i \leq n} |\xi_i| \geq s\sqrt{n} \right) \leq C s^{-2}, \tag{3.9}$$

$$\sup_{x \in \mathbb{Z}^d} P_0^{(\xi)}(\xi_n = x) \leq C n^{-d/2}. \tag{3.10}$$

Indeed, [20, Proposition 2.1.2 (a)], with $k = 1$, gives (3.9), and (3.10) follows from [20, Theorem 2.3.9].

Recall V_+ is the branching random walk in \mathbb{Z}^d indexed by \mathcal{T}_+ defined in Section 3.1.

Lemma 3.3. ([8, Lemma 4.13]) *Let $d = 5$. Assume (1.3), (1.4) with $q > 4$, and (1.12). For any $M > 0$, there exist some $v, C > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\limsup_{n \rightarrow \infty} \mathbf{P}_0 \left(\max_{|x| \leq \varepsilon n^{1/2}, x \in \mathbb{Z}^5} P_x^{(\xi)}(\xi[0, \infty) \cap V_+[0, n^2] = \emptyset) \geq \varepsilon^v \right) \leq C \varepsilon^M. \tag{3.11}$$

Proof. Though [8] only considered simple random walks, a general random walk ξ that is centered with finite variance, aperiodic and irreducible will validate every argument except for [8, Lemma 4.6], which relies on the asymptotics of the Green's function, $g_{(\xi)}(x) = \sum_{n \geq 0} P_0^{(\xi)}(\xi_n = x)$. Given the existence of finite third moment of ξ , we can apply [30, Theorem 2] (with $N = 5, m = 0$) to get $g_{(\xi)}(x) = (C + o(1))|x|^{-3}$. Thus we still get [8, Lemma 4.6] for ξ satisfying (1.12) instead of a simple random walk. □

We end this subsection by some estimates on the branching random walks V_+ and V_c and on the graph distance on \mathcal{T}_+ . We consider \mathcal{T}_+ as a subgraph of \mathcal{T}_∞ and, for all $0 \leq i < j$, we denote by $d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j))$ the graph distance in \mathcal{T}_∞ between the two vertices $\mathcal{T}_+(i)$ and $\mathcal{T}_+(j)$, which is the number of edges in the geodesic path connecting $\mathcal{T}_+(i)$ to $\mathcal{T}_+(j)$ in \mathcal{T}_∞ .

Lemma 3.4. *Assume (1.3) and (1.4) with some $q > 4$.*

(i) *For any $0 < \zeta < \frac{1}{4} - \frac{1}{q}$, there exists some positive constants a' and $C = C_{a'}$ such that all $n \geq 1$ and $0 < \eta < 1$, we have*

$$\mathbf{P}_0 \left(\max_{0 \leq i < j \leq n, 0 \leq j-i \leq \eta n} |V_+(j) - V_+(i)| \geq \eta^\zeta n^{1/4} \right) \leq C \eta^{a'}. \tag{3.12}$$

(ii) *There exists some $C = C_q > 0$ such that for any $n \geq 1$,*

$$\mathbf{E}_0 \left(\max_{z \in \mathcal{R}_c} |z|^q \mid \#\mathcal{T}_c = n \right) \leq C n^{q/4}. \tag{3.13}$$

(iii) *For any $r \geq 1$, there is some positive constant C such that for any $0 \leq i < j$,*

$$\mathbf{E} \left(d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j))^r \right) \leq C (j - i)^{r/2}. \tag{3.14}$$

Proof. The statements (i) and (ii) come from [8, (4.11), (4.24)].

For (iii), we deduce from the invariance \mathcal{T}_∞ by rerooting that for any $j > i$, $d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j))$ is distributed as $d_{\text{gr}}(\mathcal{T}_+(j-i), \emptyset)$. By using [8, (4.7), (4.9), (4.10)], and we note that (4.9) and (4.10) hold for any exponent $r \geq 1$ instead of $q/2$ there, we get (3.14). \square

3.3 Increments of V_+

We first present a general result to estimate the increments of a discrete-time process:

Lemma 3.5. *Let $\alpha \in (0, 1]$, $b > 0$, $p > \max(\frac{1}{\alpha}, \frac{b}{\alpha})$ and $H, K \geq 0$ and $n \geq 1$. Let $\Upsilon_0, \dots, \Upsilon_n$ be real-valued random variables such that for any $0 \leq i \leq j \leq n$,*

$$\mathbb{E}(|\Upsilon_j - \Upsilon_i|^p) \leq H(j-i)^{\alpha p} + K(j-i)^b. \tag{3.15}$$

For any $\max(1, b) < \gamma < \alpha p$, there exists some positive constant $c = c(\alpha, \gamma, p, b)$ only depending on α, γ, p, b such that

$$\mathbb{E}\left(\max_{0 \leq i \leq j \leq n} |\Upsilon_j - \Upsilon_i|^p\right) \leq c(Hn^{\alpha p} + Kn^\gamma). \tag{3.16}$$

In particular, if $\Upsilon_0 = 0$, then

$$\mathbb{E}\left(\max_{0 \leq i \leq n} |\Upsilon_i|^p\right) \leq c(Hn^{\alpha p} + Kn^\gamma). \tag{3.17}$$

When $K = 0$, the above lemma is the discrete-time version of the Garsia-Rodemich-Rumsey lemma. We shall apply Lemma 3.5 to estimate the increments of V_+ and the term $K(j-i)^b$ will appear when we consider a truncated version of V_+ , with K depending on n .

Proof. Consider $(\Upsilon_t^{(n)})_{0 \leq t \leq 1}$ the linear interpolation of the process $(n^{-\alpha} \Upsilon_{\lfloor nt \rfloor})_{0 \leq t \leq 1}$. We are going to apply the Garsia-Rodemich-Rumsey lemma for $\Upsilon^{(n)}$. We claim that there exists some $c_p > 0$ such that for all $0 \leq s < t \leq 1$,

$$\mathbb{E}\left(|\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p\right) \leq c_p(t-s)^\gamma(H + Kn^{\gamma-\alpha p}). \tag{3.18}$$

In fact, if for some $0 \leq i \leq n$, $\frac{i}{n} \leq s < t \leq \frac{i+1}{n}$, then $\Upsilon_t^{(n)} - \Upsilon_s^{(n)} = n^{1-\alpha}(t-s)(\Upsilon_{i+1} - \Upsilon_i)$, hence

$$\mathbb{E}\left(|\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p\right) \leq n^{p(1-\alpha)}(t-s)^p(H + K) \leq (H + K)n^{\gamma-\alpha p}(t-s)^\gamma,$$

proving (3.18) (with $c_p \geq 1$) in this case as $\gamma < \alpha p$. If for some $0 \leq i < j < n$, $\frac{i}{n} \leq s < \frac{i+1}{n}$ and $\frac{j}{n} \leq t < \frac{j+1}{n}$, we distinguish two subcases: either $j = i + 1$, then we use twice the above estimate to the couples $(s, \frac{i+1}{n})$ and $(\frac{i+1}{n}, t)$ and get that

$$\mathbb{E}\left(|\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p\right) \leq 2^p(H + K)n^{\gamma-\alpha p}(t-s)^\gamma;$$

or $j \geq i + 2$, then $t - s \geq \frac{1}{n}$ and

$$\begin{aligned} & \mathbb{E}\left(|\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p\right) \\ & \leq 3^{p-1}\mathbb{E}\left(|\Upsilon_t^{(n)} - \Upsilon_{j/n}^{(n)}|^p\right) + 3^{p-1}\mathbb{E}\left(|\Upsilon_{j/n}^{(n)} - \Upsilon_s^{(n)}|^p\right) + 3^{p-1}\mathbb{E}\left(|\Upsilon_{j/n}^{(n)} - \Upsilon_{i/n}^{(n)}|^p\right) \\ & \leq 2 \cdot 3^{p-1}n^{-\alpha p}(H + K) + 3^{p-1}n^{-\alpha p}\mathbb{E}\left(|\Upsilon_j - \Upsilon_i|^p\right) \\ & \leq c_p H(t-s)^{\alpha p} + c_p K(t-s)^b n^{b-\alpha p} \\ & \leq c_p(t-s)^\gamma(H + Kn^{\gamma-\alpha p}), \end{aligned}$$

where in the last inequality we have used the facts that $b < \gamma < \alpha p$ and $(t - s)^{b-\gamma} \leq n^{\gamma-b}$ in this case. Then the proof of (3.18) is complete.

Now in view of (3.18), we are entitled to apply [10, (3a.3)] and get that for any (arbitrarily) fixed $0 < \delta < \gamma - 1$,

$$\mathbb{E} \left(\sup_{0 \leq s < t \leq 1} \frac{|\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p}{|t - s|^\delta} \right) \leq C (H + Kn^{\gamma-\alpha p}).$$

Hence, we get

$$\mathbb{E} \left(\max_{0 \leq i < j \leq n} |\Upsilon_j - \Upsilon_i|^p \right) = n^{\alpha p} \mathbb{E} \left(\sup_{0 \leq s < t \leq 1} |\Upsilon_t^{(n)} - \Upsilon_s^{(n)}|^p \right) \leq C (Hn^{\alpha p} + Kn^\gamma),$$

proving (3.16). □

We then apply this lemma to get a control on the increments of V_+ .

Lemma 3.6. *Assume (1.3) and (1.4) with some $q > 4$. For any $p \geq \frac{4}{\frac{1}{4} - \frac{1}{q}}$, there exists some $C > 0$ such that for any $\eta \in (0, 1)$ and $r > 0$, we have¹*

$$\limsup_{n \rightarrow \infty} \mathbf{P}_0 \left(\max_{0 \leq k \leq \frac{1}{\eta}} \max_{0 \leq i \leq \eta n} |V_+(i + k\eta n) - V_+(k\eta n)| \geq r n^{1/4} \right) \leq C r^{-p} \eta^{\frac{p}{4}-1}. \quad (3.19)$$

Consequently,

$$\limsup_{n \rightarrow \infty} \mathbf{P}_0 \left(\max_{0 \leq k \leq \frac{1}{\eta}} \max_{0 \leq i \leq \eta n} |V_Y(i + k\eta n) - V_Y(k\eta n)| \geq r n^{1/4} \right) \leq \frac{C}{\mu(0)} r^{-p} \eta^{\frac{p}{4}-1}. \quad (3.20)$$

Typically we will choose $r = \eta^\zeta$ for some $\zeta \in (0, \frac{1}{4})$ and p large enough so that the right-hand-side of (3.19) goes to 0 as $\eta \rightarrow 0$. Comparing such a choice of r in Lemma 3.6 with (3.12), we remark that the condition $\zeta < \frac{1}{4} - \frac{1}{q}$ is relaxed to $\zeta < 1/4$ at the expense of the limit as $n \rightarrow \infty$ in (3.19). This relaxation is crucial for the forthcoming Lemma 3.7, where we must select ζ sufficiently close to $\frac{1}{4}$ (in fact we need $\zeta > \frac{1}{4}$) to control the size of a neighborhood of V_+ or of \widehat{V}_- .

Proof. It is enough to show (3.19), as (3.20) follows from (3.19) by (3.6). To this end, we first use a truncation argument (Step 1) to remove the big jumps in V_+ , then Lemma 3.5 (Step 2) to estimate the increments of the truncated version of V_+ .

Step 1: Set

$$b_n := n^{\frac{1}{8} + \frac{1}{2q}}.$$

Recall that on a branching random walk, X_e denotes the displacement on an edge e ; we denote

$$X_e^{(n)} := X_e \mathbf{1}_{\{|X_e| < b_n\}},$$

and we write $V_{+,n}$ for the corresponding branching random walk where displacements on edges are $(X_e^{(n)})$ instead of (X_e) . Our first step is to show that we may replace $V_+[0, n]$ by $V_{+,n}[0, n]$, i.e.

$$\limsup_{n \rightarrow \infty} \mathbf{P}(V_{+,n}[0, n] = V_+[0, n]) = 1. \quad (3.21)$$

Denote by \mathfrak{t}_n the subtree of \mathcal{T}_∞ spanned by $\mathcal{T}_+[0, n]$ and $\Delta_n := \#(\mathfrak{t}_n \cap \mathcal{X})$ the number of intersections of \mathfrak{t}_n with the spine \mathcal{X} . Then there are $n + \Delta_n$ edges in \mathfrak{t}_n , and we first

¹The term $k\eta n$ is understood as its integer part. Similar remark applies elsewhere without further explanations.

show that Δ_n cannot be too big. Note that the subtree of \mathcal{T}_∞ spanned by $\mathcal{T}_+ \setminus \mathcal{T}_0^d$ and the root \emptyset , in depth-first order, has the same distribution as $\widehat{\mathcal{T}}_-$. Since $\mathcal{T}_+ \setminus \mathcal{T}_0^d$ is independent of \mathcal{T}_0^d , we get that for any $1 \leq j \leq n + 1$ and $k \geq 1$, (see Figure 2 for an illustration with $j = 1, k = 3, n = 9$)

$$\mathbb{P}(\Delta_n = k | \#\mathcal{T}_0^d = j) = \mathbb{P}(\#(\widehat{\mathcal{T}}_-[0, n + k - j] \cap \mathcal{X}) = k) \leq \mathbb{P}(\#(\widehat{\mathcal{T}}_-[0, n + k] \cap \mathcal{X}) \geq k).$$

It follows that $\mathbb{P}(\Delta_n = k) = \sum_{j=1}^{n+1} \mathbb{P}(\Delta_n = k, \#\mathcal{T}_0^d = j) \leq \mathbb{P}(\#(\widehat{\mathcal{T}}_-[0, n + k] \cap \mathcal{X}) \geq k)$, hence

$$\begin{aligned} \mathbb{P}(\Delta_n \geq n) &\leq \sum_{k=n}^{\infty} \mathbb{P}(\#(\widehat{\mathcal{T}}_-[0, n + k] \cap \mathcal{X}) \geq k) \\ &\leq \sum_{k=n}^{\infty} \mathbb{P}\left(\#(\widehat{\mathcal{T}}_-[0, n + k] \cap \mathcal{X}) \geq \frac{n + k}{2}\right) \\ &\leq \sum_{k=n}^{\infty} e^{-c(n+k)} \leq C e^{-cn}, \end{aligned}$$

where the third inequality is due to Lemma 3.1 (with $r = 1/2$ there). This shows that with probability $1 - C e^{-cn}$, \mathfrak{t}_n contains $n + \Delta_n < 2n$ edges.

Then by union bounds and (1.4), we get

$$\mathbb{P}\left(\max_{e \in \mathfrak{t}_n} |X_e| \geq b_n\right) \leq 2n \mathbb{P}(|X| \geq b_n) + C e^{-cn} \leq C n b_n^{-q} + C e^{-cn} \leq C' n^{\frac{1}{2} - \frac{q}{8}}$$

for all large n , where X denotes a random variable with distribution θ . In other words, with probability larger than $1 - C' n^{\frac{1}{2} - \frac{q}{8}}$, there is no edge $e \in \mathfrak{t}_n$ such that $X_e \neq X_e^{(n)}$, hence $V_{+,n}[0, n] = V_+[0, n]$ and we get (3.21). Then it suffices to prove (3.19) for $V_{+,n}[0, n]$ instead of $V_+[0, n]$.

Step 2: For $n \geq j > i \geq 0$, recall that $d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j))$ denotes the graph distance in \mathcal{T}_∞ between the two vertices $\mathcal{T}_+(i)$ and $\mathcal{T}_+(j)$. Conditioning on $\{d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j)) = k\}$, $V_{+,n}(j) - V_{+,n}(i)$ is the sum of k iid copies of $X^{(n)} := X 1_{\{|X| \leq b_n\}}$, where X is distributed as θ . As $X^{(n)}$ is centered (by the symmetry of θ), we use the Rosenthal inequality [26, Theorem 2.9], and get that for any $p > 1$, there exists some $c_p > 0$ such that

$$\mathbf{E}\left(|V_{+,n}(j) - V_{+,n}(i)|^p \mid d_{\text{gr}}(\mathcal{T}_+(i), \mathcal{T}_+(j)) = k\right) \leq c_p (k \mathbf{E}(|X^{(n)}|^p) + k^{p/2} \mathbf{E}(|X^{(n)}|^2)^{p/2}).$$

Note that $\mathbf{E}(|X^{(n)}|^p) \leq b_n^p$ and $\mathbf{E}(|X^{(n)}|^2) \leq \mathbf{E}(|X|^2) < \infty$. Then by (3.14), there is some positive constant C such that for any $0 \leq i < j \leq n$ and $p > 4$,

$$\mathbf{E}\left(|V_{+,n}(j) - V_{+,n}(i)|^p\right) \leq C (b_n^p (j - i)^{1/2} + (j - i)^{p/4}).$$

Applying (3.17) to $\Upsilon_\cdot = V_{+,n}(i + \cdot)$, $\alpha = \frac{1}{4}$, $b = \frac{1}{2}$ and any $\gamma \in (1, p/4)$, we get some positive constant C' such that for any $0 \leq i < \ell \leq n$,

$$\mathbf{E}\left(\max_{i \leq j \leq \ell} |V_{+,n}(j) - V_{+,n}(i)|^p\right) \leq C' ((\ell - i)^{p/4} + b_n^p (\ell - i)^\gamma). \tag{3.22}$$

This, in view of the union bound and the Markov inequality, yields that

$$\begin{aligned} &\mathbf{P}_0\left(\max_{0 \leq k \leq \frac{1}{\eta}} \max_{0 \leq i \leq \eta n} |V_{+,n}(i + k\eta n) - V_{+,n}(k\eta n)| \geq r n^{1/4}\right) \\ &\leq C' \eta^{-1} r^{-p} n^{-p/4} (\eta^{p/4} n^{p/4} + b_n^p \eta^\gamma n^\gamma) \\ &= C' r^{-p} \eta^{p/4 - 1} + C' r^{-p} \eta^{\gamma - 1} b_n^p n^{-p/4 + \gamma}. \end{aligned}$$

Now for any $p \geq \frac{4}{\frac{1}{4} - \frac{1}{q}}$, we fix an arbitrary $\gamma \in (1, 2)$ so that $(\frac{1}{8} + \frac{1}{2q})p - \frac{p}{4} + \gamma < 0$, then $b_n^p n^{-p/4 + \gamma} \rightarrow 0$ as $n \rightarrow \infty$, and we get (3.19) for $V_{+,n}$, hence for V_+ by Step 1. \square

The following lemma says that with high probability the number of points in the neighbourhood of $\widehat{V}_-[0, m]$ is not too big, such an estimate will be important in estimating the forthcoming probability term $p'_{m,n}$ defined in (3.28).

Lemma 3.7. *Let $d \geq 5$. Assume (1.3) and (1.4) with $q > 4$. For any $M > 0$ and $\gamma \in (0, d - 4)$, there exist some positive constants $C = C_d$ and $C' = C_{M,\gamma}$ such that for any $\eta \in (0, 1)$,*

$$\limsup_{m \rightarrow \infty} \mathbf{P}_0 \left(\# \left((\widehat{V}_-[0, m])^{\eta m^{1/4}} \cap \mathbb{Z}^d \right) > C \eta^\gamma m^{d/4} \right) \leq C' \eta^M. \tag{3.23}$$

Proof. Let $\zeta := \frac{1}{d-\gamma} \in (\frac{1}{d}, \frac{1}{4})$. Applying (3.20) (with η replaced by $\eta^{1/\zeta}$ and r by η) to a sufficiently large p , we get that for any $M > 1$, there is some positive constant C such that for all $\eta \in (0, 1)$,

$$\limsup_{m \rightarrow \infty} \mathbf{P}_0 \left(\max_{0 \leq k \leq \eta^{-1/\zeta} m} \max_{0 \leq j \leq \eta^{1/\zeta} m} |V_{\mathcal{Y}}(j + k\eta^{1/\zeta} m) - V_{\mathcal{Y}}(k\eta^{1/\zeta} m)| \geq \eta m^{1/4} \right) \leq C \eta^M. \tag{3.24}$$

Recall the notations $\mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)} = \mathcal{R}_{\mathcal{X}} \cap \widehat{V}_-[0, m]$, $\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} = \mathcal{R}_{\mathcal{Y}} \cap \widehat{V}_-[0, m]$ and $V_{\mathcal{Y}}[i, j] := \{V_{\mathcal{Y}}(i), V_{\mathcal{Y}}(i + 1), \dots, V_{\mathcal{Y}}(j)\}$. Since $\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} \subset V_{\mathcal{Y}}[0, m]$, we deduce from (3.24) that

$$1 - \mathbf{P}_0 \left(\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} \text{ can be covered by } \eta^{-1/\zeta} \text{ balls of radius } \eta m^{1/4} \right) \leq C \eta^M + o_m(1), \tag{3.25}$$

where $o_m(1) \rightarrow 0$ as $m \rightarrow \infty$ and $o_m(1)$ may depend on η .

Let $j \geq 1$. Consider the event $\{d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \geq j\}$ with $x \in \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)}$. There is some i such that $x = V_\infty(\varnothing_i)$. Let $K_i := d_{gr}(\varnothing_i, \mathcal{Y} \cap \widehat{T}_-[0, m])$, where we recall that d_{gr} is the graph distance in \mathcal{T}_∞ . Then $d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)})$ is less than the displacement of a θ -walk at time K_i . Since each spine point has no attached tree with probability $\tilde{\mu}(0) = 1 - \mu(0)$, we deduce from the union bound that for any $j, \ell \geq 1$,

$$\begin{aligned} \mathbf{P}_0 \left(\max_{x \in \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)}} d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \geq j \right) &\leq \mathbf{P}_0 \left(\max_{0 \leq i \leq m-1} K_i \geq \ell \right) + m \mathbf{P}_0(|S_\ell| \geq j) \\ &\leq m(1 - \mu(0))^\ell + m j^{-q'} \mathbf{E}_0(|S_\ell|^{q'}), \end{aligned} \tag{3.26}$$

where we choose (and then fix) an arbitrary constant $q' \in (4, q)$. By [26, Theorem 2.10],

$$\mathbf{E}_0(|S_\ell|^{q'}) \leq C \ell^{q'/2}.$$

Take $\ell = \lfloor \frac{q'}{\mu(0)} \log m \rfloor$ and $j = \eta m^{1/4}$. There exists some $m_0 = m_0(q', \mu(0)) \geq 1$ such that for all $m \geq m_0$,

$$\mathbf{P}_0 \left(\max_{x \in \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)}} d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \geq \eta m^{1/4} \right) \leq C' \eta^{-q'} m^{1-q'/4} (\log m)^{q'/2}. \tag{3.27}$$

Note that on $\{\max_{x \in \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)}} d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) < \eta m^{1/4}\}$, if $\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}$ can be covered by $\eta^{-1/\zeta}$ balls of radius $\eta m^{1/4}$, then $\widehat{V}_-[0, m] = \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)} \cup \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}$ can be covered by $\eta^{-1/\zeta}$ balls of radius $2\eta m^{1/4}$. It follows from (3.25) and (3.27) that

$$\begin{aligned} &1 - \mathbf{P}_0 \left(\widehat{V}_-[0, m] \text{ can be covered by } \eta^{-1/\zeta} \text{ balls of radius } 2\eta m^{1/4} \right) \\ &\leq C \eta^M + C' \eta^{-q'} m^{1-q'/4} (\log m)^{q'/2} + o_m(1) \\ &= C \eta^M + o_m(1). \end{aligned}$$

Finally, when $\widehat{V}_-[0, m]$ is covered by $\eta^{-1/\zeta}$ balls of radius $2\eta m^{1/4}$, its $(\eta m^{1/4})$ -neighborhood is covered by $\eta^{-1/\zeta}$ balls of radius $3\eta m^{1/4}$, hence

$$\#\left(\left(\widehat{V}_-[0, m]\right)^{\eta m^{1/4}} \cap \mathbb{Z}^d\right) \leq \eta^{-1/\zeta} c_d \eta^d m^{d/4} = c_d \eta^\gamma m^{d/4},$$

for some positive constant c_d only depending on d . This proves (3.23). □

3.4 Intersection probabilities between V_\pm and ξ

The proof of Proposition 1.3 can be outlined as follows. By ordering the vertices of \mathcal{T}_c using the depth first search and its reversed sense (see (3.45)), we may reduce the problem of studying $I(\varepsilon, n)$, defined in (1.14), to that of the non-intersection probability

$$\mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(V_c \left[0, \frac{3}{5}m \right] \cap (\xi[0, n])^{\varepsilon n^{1/2}} \neq \emptyset, V_c \left[0, \frac{4}{5}m \right] \cap \xi[0, n] = \emptyset \mid \#\mathcal{T}_c = m \right),$$

where the choice $(\frac{3}{5}, \frac{4}{5})$ can be replaced by any $(c, c + \lambda)$ with $c \in (\frac{1}{2}, 1)$ and $\lambda \in (0, 1 - c)$. Furthermore, the main contribution to $I(\varepsilon, n)$ comes from those m of order n^2 , say $m \in [\delta n^2, \frac{1}{\delta} n^2]$ for small $\delta > 0$. By (3.3), the above probability is dominated, up to a multiplicative constant, by the corresponding probability for \widehat{V}_- instead of V_c conditioned on $\{\#\mathcal{T}_c = m\}$. This latter probability is estimated in the following result.

Proposition 3.8. *Let $d = 5$. Assume (1.3), (1.4) with $q > 4$ and (1.12). Fix $\lambda > 0$. For each $\varepsilon > 0$, $x \in \mathbb{R}^5$ and $x_n = [x\sqrt{n}] \in \mathbb{Z}^5$, we define*

$$p_{m,n} := \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\widehat{V}_-[0, m] \cap (\xi[0, n])^{\varepsilon n^{1/2}} \neq \emptyset, \widehat{V}_-[0, (1 + \lambda)m] \cap \xi[0, n] = \emptyset \right).$$

There is some $c > 0$ such that for any $\delta \in (0, \frac{1}{9})$, there is $C = C_{\delta,x,\lambda} > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\limsup_{n \rightarrow \infty} \max_{m \in [\delta n^2, \frac{1}{\delta} n^2]} p_{m,n} \leq C \varepsilon^c.$$

Remark 3.9. One may compare Proposition 3.8 to Lemma 3.3 (see also [8, Lemma 4.13]). In Lemma 3.3, the non-intersection probability of a random walk starting very close to a branching random walk has been estimated; whereas here, we need the intersection probability when the two random models start relatively far apart, and then consider the joint events that they are close but not intersecting.

This is an essential step due to difference between Cap and Bcap. For Cap, the testing random set (\mathcal{S} in (1.1)) is a random walk with Markov property, so events similar to $p_{m,n}$ can be decomposed into the product of probabilities for two independent events: arrival at a small neighborhood, then escape the target set starting nearby (in the sense of Lemma 3.3). However, for Bcap, the smallest i such that $\widehat{V}_-(i) \in (\xi[0, n])^{\varepsilon n^{1/2}}$ is not a stopping time, the whole random set $\widehat{V}_-[0, i]$ matters.

To address this issue, we develop Proposition 3.8 where the proof (for (3.34) below) is significantly more involved, because both the hitting probability at $(\xi[0, n])^{\varepsilon n^{1/2}}$ and the hitting at $\xi[0, n]$ “conditioned” on arriving at $(\xi[0, n])^{\varepsilon n^{1/2}}$ have to be carefully handled, which might be useful in other contexts involving the study of branching structures.

This subsection is devoted to the proof of Proposition 3.8. Consider $\varepsilon \in (0, 1)$. Let $r > 0$ be some small constant whose value will be determined later. We have

$$p_{m,n} \leq p'_{m,n} + p''_{m,n},$$

with

$$p'_{m,n} := \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\widehat{V}_-[0, m] \cap \xi[(1 - \varepsilon^r)n, n]^{\varepsilon n^{1/2}} \neq \emptyset \right), \tag{3.28}$$

$$p''_{m,n} := \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\widehat{V}_-[0, m] \cap \xi[0, (1 - \varepsilon^r)n]^{\varepsilon n^{1/2}} \neq \emptyset, \right. \\ \left. \widehat{V}_-[0, (1 + \lambda)m] \cap \xi[0, n] = \emptyset \right). \tag{3.29}$$

We outline the steps in estimating $p'_{m,n}$ and $p''_{m,n}$ as follows.

(i) Estimate $p'_{m,n}$. Let $r' \in (0, r)$. According to the random walk estimate (3.9), with high probability as $\varepsilon \rightarrow 0$, $\xi[(1 - \varepsilon^r)n, n]$ is contained in $B(\xi_n, \varepsilon^{r'/2}n^{1/2})$ the ball centered at ξ_n with a radius of $\varepsilon^{r'/2}n^{1/2}$. Therefore, estimating $p'_{m,n}$ boils down to evaluating whether ξ_n is in $(\widehat{V}_-[0, m])^{\eta n^{1/2}}$, where η is some power of ε . This is the motivation for estimating $\#(\widehat{V}_-[0, m])^{\eta n^{1/2}}$, a task addressed in Lemma 3.7.

(ii) Estimate $p''_{m,n}$. We divide \widehat{T}_- into two pieces \mathcal{X} and \mathcal{Y} , as done in (3.4) and (3.5), and give an estimate on their ranges, see Fig. 2. By Lemma 3.1, there are few points in $V_{\mathcal{X}} \cap \widehat{V}_-[0, m]$, then we may replace \widehat{V}_- by $V_{\mathcal{Y}}$ in $p''_{m,n}$. This, in view of (3.6), boils down to estimating the corresponding probability for V_+ . To achieve this, we will cut the range of V_+ into a certain number $\lfloor \varepsilon^{-1/\zeta} \rfloor$ of pieces. An application of the strong Markov property of ξ allows us to reduce the problem to estimating the expectation of the maximum over $\lfloor \varepsilon^{-1/\zeta} \rfloor$ of identically distributed random variables, whose common law is that of $q_{\varepsilon,r}(n)$, where

$$q_{\varepsilon,r}(n) := \max_{|x| \leq 4\varepsilon n^{1/2}, x \in \mathbb{Z}^d} P_x^{(\xi)} (\xi[0, \varepsilon^r n] \cap V_+[0, \varepsilon n^2] = \emptyset) 1_{\{V_+[0, \varepsilon n^2] \subset B(0, \varepsilon \zeta n^{1/2})\}}, \tag{3.30}$$

with $\zeta \in (0, \frac{1}{4} - \frac{1}{q})$ a fixed constant. We mention that $P_x^{(\xi)}$ only computes the probability with respect to the random walk ξ starting from x , so that $q_{\varepsilon,r}(n)$ is a *random variable* depending on V_+ . Finally, $q_{\varepsilon,r}(n)$ is estimated in the following Lemma 3.10.

Lemma 3.10. *Let $d = 5$. Assume (1.3), (1.4) with $q > 4$ and (1.12). Let $\zeta \in (0, \frac{1}{4} - \frac{1}{q})$ and $q_{\varepsilon,r}(n)$ be as in (3.30) and $r \in (0, 2\zeta)$. For any $M > 0$, there exist some $v = v_{r,\zeta} > 0$ and $C = C_{r,\zeta} > 0$ such that for all $\varepsilon \in (0, 1)$,*

$$\limsup_{n \rightarrow \infty} \mathbf{P}_0(q_{\varepsilon,r}(n) \geq \varepsilon^v) \leq C \varepsilon^M. \tag{3.31}$$

Proof. Only small ε needs to be considered. Applying a change of variables $(n', \varepsilon') = (\varepsilon^{1/2}n, 4\varepsilon^{3/4})$ to (3.11) in Lemma 3.3, we deduce that there are some $v', C', C'' > 0$ such that

$$\limsup_{n \rightarrow \infty} \mathbf{P}_0 \left(\max_{|x| \leq 4\varepsilon n^{1/2}, x \in \mathbb{Z}^5} P_x^{(\xi)} (\xi[0, \infty) \cap V_+[0, \varepsilon n^2] = \emptyset) \geq C' \varepsilon^{v'} \right) \leq C'' \varepsilon^M. \tag{3.32}$$

Let $\ell_n = \varepsilon^\zeta n^{1/2}$. On the event $\{V_+[0, \varepsilon n^2] \subset B(0, \ell_n)\}$, we deduce from the Markov property of ξ at $\varepsilon^r n$ that

$$\begin{aligned} & P_x^{(\xi)} (\xi[0, \infty) \cap V_+[0, \varepsilon n^2] = \emptyset) \\ & \geq P_x^{(\xi)} \left(\xi[0, \varepsilon^r n] \cap V_+[0, \varepsilon n^2] = \emptyset, |\xi_{\varepsilon^r n}| > 2\ell_n, \xi[\varepsilon^r n, \infty) \cap B(0, \ell_n) = \emptyset \right) \\ & \geq \inf_{|y| \geq 2\ell_n} P_y^{(\xi)} (\xi[0, \infty) \cap B(0, \ell_n) = \emptyset) P_x^{(\xi)} (\xi[0, \varepsilon^r n] \cap V_+[0, \varepsilon n^2] = \emptyset, |\xi_{\varepsilon^r n}| > 2\ell_n) \\ & \geq c P_x^{(\xi)} (\xi[0, \varepsilon^r n] \cap V_+[0, \varepsilon n^2] = \emptyset, |\xi_{\varepsilon^r n}| > 2\ell_n), \end{aligned}$$

for all n such that $\ell_n \geq R_0$, where $R_0 > 0$ is a large enough constant such that

$$c = \inf_{R \geq R_0} \inf_{|y| > 2R} P_y^{(\xi)}(\xi[0, \infty) \cap B(0, R) = \emptyset) > 0.$$

By the local limit theorem for the random walk (3.10), there is $C''' > 0$ such that

$$\max_{|x| \leq 4\epsilon n^{1/2}} P_x^{(\xi)}(|\xi_{\epsilon r n}| \leq 2\ell_n) \leq P_0^{(\xi)}(|\xi_{\epsilon r n}| \leq 3\ell_n) \leq C''' \epsilon^{\zeta d - rd/2}.$$

It follows that

$$q_{\epsilon, r}(n) \leq \frac{1}{c} \max_{|x| \leq 4\epsilon n^{1/2}} P_x^{(\xi)}(\xi[0, \infty) \cap V_+[0, \epsilon n^2] = \emptyset) + C''' \epsilon^{\zeta d - rd/2}.$$

By (3.32), we obtain Lemma 3.10 by choosing any $v \in (0, \min(v', (\zeta - r/2)d))$. □

We are now ready to present the proof of Proposition 3.8.

Proof of Proposition 3.8. As in Lemma 3.10, let $\zeta \in (0, \frac{1}{4} - \frac{1}{q})$ and fix some constant $r \in (0, 2\zeta)$. Recall (3.28) and (3.29). Let $\delta \in (0, \frac{1}{9})$. It is enough to show that there is some $c > 0$ independent of δ , and some constant $C_\delta > 0$ such that for any small $\epsilon > 0$,

$$\limsup_{n \rightarrow \infty} \max_{m \in [\delta n^2, \frac{1}{3}n^2]} p'_{m, n} \leq C_\delta \epsilon^c. \tag{3.33}$$

$$\limsup_{n \rightarrow \infty} \max_{m \in [\delta n^2, \frac{1}{3}n^2]} p''_{m, n} \leq C_\delta \epsilon^c. \tag{3.34}$$

We mention that the exact values of C_δ, c are not important, they may change from one paragraph to another.

(i) Proof of (3.33). Fix some $0 < r' < r$. Apply (3.9) with $(n, s) = (\epsilon^r n, \epsilon^{(r'-r)/2})$ there, we have

$$P_{x_n}^{(\xi)} \left(\max_{(1-\epsilon^r)n \leq k \leq n} |\xi_k - \xi_n| \geq \epsilon^{r'/2} n^{1/2} \right) \leq C \epsilon^{r-r'}.$$

Since $\epsilon \leq \epsilon^{r'/2}$, we deduce that for all $m \in [\delta n^2, \frac{1}{3}n^2]$,

$$p'_{m, n} \leq C \epsilon^{r-r'} + \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\xi_n \in (\widehat{V}_-[0, m])^{2\epsilon^{r'/2} n^{1/2}} \right). \tag{3.35}$$

To estimate the above probability term, we apply (3.23) to $m = n^2/\delta$, $\eta = 2\delta^{1/4}\epsilon^{r'/2}$ with $d = 5$, $\gamma \in (0, 1)$ and $M = 1$, to see that there are some $C_\delta, c_\delta > 0$ (we may choose $c_\delta := C 2^\gamma \delta^{-(d-\gamma)/4}$ and $C_\delta := 2C' \delta^{1/4}$ with C, C' as in (3.23)) such that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \max_{m \in [\delta n^2, \frac{1}{3}n^2]} \mathbf{P}_0 \left(\#((\widehat{V}_-[0, m])^{2\epsilon^{r'/2} n^{1/2}} \cap \mathbb{Z}^5) > c_\delta \epsilon^{\gamma r'/2} n^{5/2} \right) \\ & \leq \limsup_{n \rightarrow \infty} \mathbf{P}_0 \left(\#((\widehat{V}_-[0, \frac{1}{\delta}n^2])^{2\epsilon^{r'/2} n^{1/2}} \cap \mathbb{Z}^5) > c_\delta \epsilon^{\gamma r'/2} n^{5/2} \right) \\ & \leq C_\delta \epsilon^{r'/2}. \end{aligned} \tag{3.36}$$

Moreover, by (3.10), for any finite set $A \subset \mathbb{Z}^5$,

$$P_{x_n}^{(\xi)}(\xi_n \in A) \leq C n^{-5/2} \#A.$$

It follows that on $\{\#((\widehat{V}_-[0, m])^{2\epsilon^{r'/2} n^{1/2}} \cap \mathbb{Z}^5) \leq c_\delta \epsilon^{\gamma r'/2} n^{5/2}\}$,

$$P_{x_n}^{(\xi)} \left(\xi_n \in (\widehat{V}_-[0, m])^{2\epsilon^{r'/2} n^{1/2}} \right) \leq C c_\delta \epsilon^{\gamma r'/2},$$

which in view of (3.35) and (3.36) yield that for all $m \in [\delta n^2, \frac{1}{8}n^2]$,

$$p'_{m,n} \leq C \varepsilon^{r-r'} + C_\delta \varepsilon^{r'/2} + C c_\delta \varepsilon^{\gamma r'/2} + o_n(1),$$

with $o_n(1)$ independent of m and $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$. This proves (3.33).

(ii) Proof of (3.34). Recall (3.29) for the definition of $p''_{m,n}$. By (3.8), $\widehat{V}_-[0, m] = \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)} \cup \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}$. We are going to apply (3.27) with $\eta = \varepsilon \delta^{1/4}$. Note that for all $m \in [\delta n^2, \frac{1}{8}n^2]$, on the event $\{\max_{x \in \mathcal{R}_{\mathcal{X}, \widehat{V}_-}^{(m)}} d(x, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) < \varepsilon \delta^{1/4} m^{1/4}\}$, we have

$$\widehat{V}_-[0, m] \cap \xi[0, (1 - \varepsilon^r)n]^{\varepsilon n^{1/2}} \neq \emptyset,$$

which implies that $\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} \cap \xi[0, (1 - \varepsilon^r)n]^{2\varepsilon n^{1/2}} \neq \emptyset$. It follows from (3.27) that for all $m \in [\delta n^2, \frac{1}{8}n^2]$ and n large enough,

$$p''_{m,n} \leq p'''_{m,n} + o_n(1), \tag{3.37}$$

where as before, $o_n(1)$ is independent of m and $o_n(1) \rightarrow 1$ as $n \rightarrow \infty$, and

$$p'''_{m,n} := \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} \cap \xi[0, (1 - \varepsilon^r)n]^{2\varepsilon n^{1/2}} \neq \emptyset, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{((1+\lambda)m)} \cap \xi[0, n] = \emptyset \right).$$

Let

$$\tau_{m,n} := \inf\{i \geq 0 : d(\xi_i, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \leq 2\varepsilon n^{1/2}\}.$$

We have

$$\begin{aligned} p'''_{m,n} &= \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\tau_{m,n} \leq (1 - \varepsilon^r)n, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{((1+\lambda)m)} \cap \xi[0, n] = \emptyset \right) \\ &\leq \mathbf{E}_0 \left(\max_{d(y, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \leq 2\varepsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)} (\xi[0, \varepsilon^r n] \cap \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{((1+\lambda)m)} = \emptyset) \right), \end{aligned} \tag{3.38}$$

where the inequality follows from the strong Markov property of ξ at $\tau_{m,n}$. Let

$$A_m := \left\{ \#(\widehat{\mathcal{T}}_-[0, (1 + \lambda)m] \cap \mathcal{X}) < \lambda m / 2 \right\}.$$

On A_m , we have

$$\mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)} \subset V_{\mathcal{Y}}[0, m] \subset V_{\mathcal{Y}}[0, (1 + \frac{\lambda}{2})m] \subset \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{((1+\lambda)m)}. \tag{3.39}$$

By Lemma 3.1, there is some positive constant c_δ such that

$$\max_{m \in [\delta n^2, \frac{1}{8}n^2]} \mathbf{P}_0(A_m^c) \leq e^{-c_\delta n^2},$$

which, in view of (3.38) and (3.39), implies that

$$\begin{aligned} p'''_{m,n} &\leq e^{-c_\delta n^2} + \mathbf{E}_0 \left(1_{A_m} \max_{d(y, \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{(m)}) \leq 2\varepsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)} (\xi[0, \varepsilon^r n] \cap \mathcal{R}_{\mathcal{Y}, \widehat{V}_-}^{((1+\lambda)m)} = \emptyset) \right) \\ &\leq e^{-c_\delta n^2} + \mathbf{E}_0 \left(\max_{d(y, V_{\mathcal{Y}}[0, m]) \leq 2\varepsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)} (\xi[0, \varepsilon^r n] \cap V_{\mathcal{Y}}[0, (1 + \frac{\lambda}{2})m] = \emptyset) \right). \end{aligned}$$

Recall that $\zeta \in (0, \frac{1}{4} - \frac{1}{q})$. Let

$$B_y := \left\{ \max_{0 \leq k \leq \varepsilon^{-1/\zeta}} \max_{0 \leq j \leq \varepsilon^{1/\zeta} m} |V_{\mathcal{Y}}(j + k\varepsilon^{1/\zeta} m) - V_{\mathcal{Y}}(k\varepsilon^{1/\zeta} m)| < \varepsilon n^{1/2} \right\}.$$

On the event B_y , for any $y \in \mathbb{Z}^d$ such that $d(y, V_y[0, m]) \leq 2\epsilon n^{1/2}$, there is some $0 \leq k < \epsilon^{-1/\zeta}$ such that $|y - V_y(k\epsilon^{1/\zeta}m)| \leq 3\epsilon n^{1/2} < 4\epsilon n^{1/2}$, furthermore

$$P_y^{(\xi)}(\xi[0, \epsilon^r n] \cap V_y[0, (1 + \frac{\lambda}{2})m] = \emptyset) \leq P_y^{(\xi)}(\xi[0, \epsilon^r n] \cap V_y[k\epsilon^{1/\zeta}m, k\epsilon^{1/\zeta}m + \lambda m/2] = \emptyset).$$

It follows that

$$\mathbf{E}_0 \left(\max_{d(y, V_y[0, m]) \leq 2\epsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)}(\xi[0, \epsilon^r n] \cap V_y[0, (1 + \frac{\lambda}{2})m] = \emptyset) \right) \leq \mathbf{P}_0(B_y^c) + J_{(3.40)}, \quad (3.40)$$

with

$$J_{(3.40)} := \mathbf{E}_0 \left(\max_{0 \leq k < \epsilon^{-1/\zeta}} \max_{|y - V_y(k\epsilon^{1/\zeta}m)| \leq 4\epsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)} \times (\xi[0, \epsilon^r n] \cap V_y[k\epsilon^{1/\zeta}m, k\epsilon^{1/\zeta}m + \lambda m/2] = \emptyset) \right).$$

Applying (3.20) to $\eta = \epsilon^{1/\zeta}$, $r = \epsilon\delta^{1/4}$ and p sufficiently large such that $p(\frac{1}{4\zeta} - 1) - \frac{1}{\zeta} \geq 1$ and $p \geq \frac{4}{\frac{1}{4} - \frac{1}{\zeta}}$, we get that uniformly in $m \in [\delta n^2, \frac{1}{8}n^2]$,

$$\mathbf{P}_0(B_y^c) \leq C_\delta \epsilon + o_n(1),$$

where $C_\delta := \frac{C}{\mu(0)} \delta^{-p/4}$ in notation of (3.20). Therefore we have shown that uniformly in $m \in [\delta n^2, \frac{1}{8}n^2]$,

$$p'''_{m,n} \leq C_\delta \epsilon + o_n(1) + J_{(3.40)}. \quad (3.41)$$

For $J_{(3.40)}$, we deduce from (3.6) that

$$J_{(3.40)} \leq \frac{1}{\mu(0)} \mathbf{E}_0 \left(\max_{0 \leq k < \epsilon^{-1/\zeta}} U_k \right), \quad (3.42)$$

where for ϵ small enough so that $\epsilon n^2 \leq \lambda m/2$ and for each k ,

$$U_k := \max_{|y - V_+(k\epsilon^{1/\zeta}m)| \leq 4\epsilon n^{1/2}, y \in \mathbb{Z}^5} P_y^{(\xi)}(\xi[0, \epsilon^r n] \cap V_+[k\epsilon^{1/\zeta}m, k\epsilon^{1/\zeta}m + \epsilon n^2] = \emptyset).$$

To estimate $\max_{0 \leq k < \epsilon^{-1/\zeta}} U_k$, we cut V_+ into smaller pieces. Consider the event

$$B_{V_+} = B_{V_+}(m, n) := \left\{ \max_{0 \leq i < j \leq (1 + \frac{\lambda}{2})m, j - i \leq \epsilon n^2} |V_+(j) - V_+(i)| \leq \epsilon^\zeta n^{1/2} \right\}.$$

By (3.12), we deduce from the union bound and the translation invariance for V_+ in (3.1) that for some positive constant a' ,

$$\max_{m \in [\delta n^2, \frac{1}{8}n^2]} \mathbf{P}_0(B_{V_+}^c) \leq C_\delta \epsilon^{a'}. \quad (3.43)$$

It follows that

$$\mathbf{E}_0 \left(\max_{0 \leq k < \epsilon^{-1/\zeta}} U_k \right) \leq C_\delta \epsilon^{a'} + \mathbf{E}_0 \left(\max_{0 \leq k < \epsilon^{-1/\zeta}} U_k 1_{B_{V_+}^c} \right) \leq C_\delta \epsilon^{a'} + \mathbf{E}_0 \left(\max_{0 \leq k < \epsilon^{-1/\zeta}} q_{\epsilon, r}^{(k)}(n) \right), \quad (3.44)$$

where for each $k \geq 0$,

$$q_{\epsilon, r}^{(k)}(n) := U_k 1_{\{\max_{0 \leq i \leq \epsilon n^2} |V_+(k\epsilon^{1/\zeta}m+i) - V_+(k\epsilon^{1/\zeta}m)| \leq \epsilon^\zeta n^{1/2}\}}.$$

By (3.1), $(q_{\epsilon, r}^{(k)}(n))_{0 \leq k < \epsilon^{-1/\zeta}}$ are identically distributed as $q_{\epsilon, r}(n)$, which was defined in (3.30). Applying Lemma 3.10 to an arbitrary $M > \frac{1}{\zeta}$, there exists some ν such that for all large n ,

$$\mathbf{P}_0(q_{\epsilon, r}(n) > \epsilon^\nu) \leq C \epsilon^M + o_n(1).$$

By union bounds,

$$\mathbf{P}_0 \left(\max_{0 \leq k < \varepsilon^{-1/\zeta}} q_{\varepsilon,r}^{(k)}(n) > \varepsilon^v \right) \leq \varepsilon^{-1/\zeta} \mathbf{P}_0(q_{\varepsilon,r}(n) > \varepsilon^v) \leq C \varepsilon^{M-1/\zeta}.$$

Since $q_{\varepsilon,r}^{(k)}(n) \leq 1$, we have

$$\mathbf{E}_0 \left(\max_{0 \leq k < \varepsilon^{-1/\zeta}} q_{\varepsilon,r}^{(k)}(n) \right) \leq \varepsilon^v + C \varepsilon^{M-1/\zeta} + o_n(1).$$

Going back to (3.44), we get that

$$\mathbf{E}_0 \left(\max_{0 \leq k < \varepsilon^{-1/\zeta}} U_k \right) \leq C_\delta \varepsilon^{a'} + \varepsilon^v + C \varepsilon^{M-1/\zeta} + o_n(1),$$

which together with (3.37), (3.41) and (3.42), imply that for all large n and $m \in [\delta n^2, \frac{1}{\delta} n^2]$,

$$p''_{m,n} \leq o_n(1) + C_\delta \varepsilon + C_\delta \varepsilon^{a'} + \varepsilon^v + C \varepsilon^{M-1/\zeta},$$

implying (3.34) and completing the proof of Proposition 3.8. □

3.5 Proof of Proposition 1.3

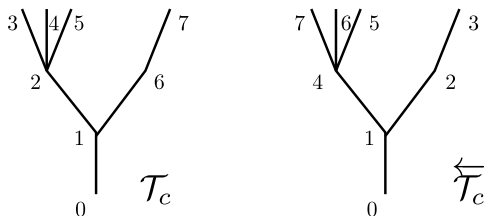


Figure 3: An illustration of \mathcal{T}_c and $\overleftarrow{\mathcal{T}}_c$.

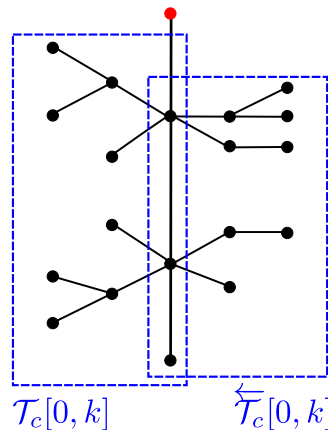


Figure 4: In order for the vertex on the top to not belong to $\mathcal{T}_c[0, k] \cup \overleftarrow{\mathcal{T}}_c[0, k]$, there must be at least $2k + 1 - \#\mathcal{T}_c$ points on the line connecting it to the root. In this figure, $k = 10$.

As shown in Fig. 3, we write $\mathcal{T}_c(0), \mathcal{T}_c(1), \dots$ as \mathcal{T}_c in its depth-first order, and we write $\overleftarrow{\mathcal{T}}_c(0), \overleftarrow{\mathcal{T}}_c(1), \dots$ as the depth-first order in the reversed sense.

Observe that if a vertex does not belong to $\mathcal{T}_c[0, \frac{3}{5}\#\mathcal{T}_c] \cup \overleftarrow{\mathcal{T}}_c[0, \frac{3}{5}\#\mathcal{T}_c]$, then the line connecting it to the root contains at least $\frac{1}{5}\#\mathcal{T}_c$ points, as shown in Fig. 4. Hence there is some positive constant C such that for every $\delta > 0, n \geq 1$,

$$\begin{aligned} & \mathbf{P} \left(\mathcal{T}_c \not\subset \overleftarrow{\mathcal{T}}_c \left[0, \frac{3}{5}\#\mathcal{T}_c \right] \cup \mathcal{T}_c \left[0, \frac{3}{5}\#\mathcal{T}_c \right], \#\mathcal{T}_c \in [\delta n^2, n^2/\delta] \right) \\ & \leq \mathbf{P}(\text{height of } \mathcal{T}_c \text{ is at least } \delta n^2/5) \\ & \leq C \delta^{-1} n^{-2} \end{aligned} \tag{3.45}$$

where the last inequality follows from the classical estimate on the branching process:

$$\mathbf{P}(\text{height of } \mathcal{T}_c \text{ is at least } j) \sim \frac{2}{\sigma^2 j}, \quad j \rightarrow \infty. \tag{3.46}$$

For simplicity we may and will assume that the offspring distribution μ is aperiodic, as one can easily adapt the proof line by line for the periodic case. In particular, given aperiodicity, by Dwass [15] and local central limit theorem (see [20, Theorem 2.3.9]),

$$n^{3/2} \mathbf{P}(\#\mathcal{T}_c = n) \sim \frac{1}{\sigma\sqrt{2\pi}}, \quad n \rightarrow \infty. \tag{3.47}$$

Therefore

$$\mathbf{P}(\#\mathcal{T}_c > j) \sim \frac{2}{\sigma\sqrt{2\pi}j}, \quad j \rightarrow \infty. \tag{3.48}$$

We are now ready to prove Proposition 1.3.

Proof of Proposition 1.3. Let $\delta > 0$ be small, we have

$$\begin{aligned} I(\varepsilon, n) &\leq \mathbf{P}\left(\mathcal{T}_c \not\subseteq \widehat{\mathcal{T}}_c \left[0, \frac{3}{5}\#\mathcal{T}_c\right] \cup \mathcal{T}_c \left[0, \frac{3}{5}\#\mathcal{T}_c\right], \#\mathcal{T}_c \in [\delta n^2, n^2/\delta]\right) \\ &+ 2 \sum_{m=\delta n^2}^{n^2/\delta} \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(V_c \left[0, \frac{3}{5}m\right] \cap (\xi[0, n])^{\varepsilon n^{1/2}} \neq \emptyset, V_c \left[0, \frac{4}{5}m\right] \cap \xi[0, n] = \emptyset, \#\mathcal{T}_c = m \right) \\ &+ \mathbf{P}(\#\mathcal{T}_c > n^2/\delta) + \mathbf{P}_{x_n}(\#\mathcal{T}_c < \delta n^2, \mathcal{R}_c \cap B(0, \eta|x_n| + \varepsilon n^{1/2}) \neq \emptyset) \\ &=: J_0(\delta, n) + 2 \sum_{m=\delta n^2}^{n^2/\delta} J_1(\varepsilon, n, m) + J_2(\delta, n) + J_3(\delta, \varepsilon, n). \end{aligned}$$

We have already bounded J_0 in (3.45). For J_2 , we deduce from (3.48) that for some positive constant C ,

$$J_2(\delta, n) = \mathbf{P}(\#\mathcal{T}_c > n^2/\delta) \leq C \delta^{1/2} n^{-1}.$$

For J_3 , when $\varepsilon < (1 - \eta)|x|/4$, we have for all large n , $\mathcal{R}_c \cap B(0, \eta|x_n| + \varepsilon n^{1/2}) \neq \emptyset$ implies that there is some $z \in \mathcal{R}_c$ such that $|z - x_n| \geq |x_n| - (\eta|x_n| + \varepsilon n^{1/2}) \geq \frac{1}{2}(1 - \eta)|x|n^{1/2}$. It follows that

$$J_3(\delta, \varepsilon, n) \leq \mathbf{P}_0 \left(\max_{z \in \mathcal{R}_c} |z| \geq \frac{1}{2}(1 - \eta)|x|n^{1/2}, \#\mathcal{T}_c < \delta n^2 \right).$$

By (3.13), there exists some $C > 0$ such that for all $m \geq 1$, $\mathbf{E}_0(\max_{z \in \mathcal{R}_c} |z|^q | \#\mathcal{T}_c = m) \leq C m^{q/4}$. Using Markov's inequality and (3.47), we have that for some positive constants $C_{\eta, x}$ and $C'_{\eta, x}$,

$$\begin{aligned} J_3(\delta, \varepsilon, n) &= \sum_{m=1}^{\delta n^2} \mathbf{P}_0(\#\mathcal{T}_c = m) \mathbf{P}_0 \left(\max_{z \in \mathcal{R}_c} |z| \geq \frac{1}{2}(1 - \eta)|x|n^{1/2} \mid \#\mathcal{T}_c = m \right) \\ &\leq C_{\eta, x} n^{-q/2} \sum_{m=1}^{\delta n^2} m^{-3/2} \mathbf{E}_0(\max_{z \in \mathcal{R}_c} |z|^q | \#\mathcal{T}_c = m) \\ &\leq C'_{\eta, x} \delta^{1/2} n^{-1}. \end{aligned}$$

Finally, by taking $k = 4m/5$ in (3.3), we can find some universal constant $C > 0$ such that $J_1(\varepsilon, n, m)$ is bounded above by

$$C \mathbf{P}_0 \otimes P_{x_n}^{(\xi)} \left(\widehat{V}_- [0, 3m/5] \cap (\xi[0, n])^{\varepsilon n^{1/2}} \neq \emptyset, \widehat{V}_- [0, 4m/5] \cap \xi[0, n] = \emptyset \right) \mathbf{P}(\#\mathcal{T}_c = m).$$

Then by Proposition 3.8, we deduce that for some positive constants c , C_δ and C'_δ such that for all $\delta n^2 \leq m \leq n^2/\delta$,

$$J_1(\varepsilon, n, m) \leq (C_\delta \varepsilon^c + o_n(1)) \mathbf{P}(\#\mathcal{T}_c = m),$$

hence by (3.47)

$$\sum_{m=\delta n^2}^{n^2/\delta} J_1(\varepsilon, n, m) \leq (C_\delta \varepsilon^c + o_n(1)) \mathbf{P}(\#\mathcal{T}_c \geq \delta n^2) \leq C'_\delta (\varepsilon^c + o_n(1)) n^{-1}.$$

Combine the above estimates, we see that

$$\limsup_{n \rightarrow \infty} n \left(J_0(\delta, n) + 2 \sum_{m=\delta n^2}^{n^2/\delta} J_1(\varepsilon, n, m) + J_2(\delta, n) + J_3(\delta, \varepsilon, n) \right) \leq C \delta^{1/2} + C'_{\eta, x} \delta^{1/2} + C'_\delta \varepsilon^c,$$

and we conclude Proposition 1.3 by letting $\varepsilon \rightarrow 0$ then $\delta \rightarrow 0$. \square

We end the paper by a remark on the condition $\{\xi[0, n] \subset B(0, \eta|x_n|)\}$ in Proposition 1.3:

Remark 3.11. Without the condition $\xi[0, n] \subset B(0, \eta|x_n|)$, Proposition 1.3 is no longer true. In fact, we have $\mathcal{R}_c = \{x_n\}$ with probability $\mu(0) > 0$, then

$$\begin{aligned} & n \mathbf{P}_{x_n} \otimes P_0^{(\xi)} \left(\mathcal{R}_c \cap (\xi[0, n])^{\varepsilon\sqrt{n}} \neq \emptyset, \mathcal{R}_c \cap \xi[0, n] = \emptyset \right) \\ & \geq \mu(0) n P_0^{(\xi)} \left(\{x_n\} \cap (\xi[0, n])^{\varepsilon\sqrt{n}} \neq \emptyset, \{x_n\} \cap \xi[0, n] = \emptyset \right) \\ & \geq \mu(0) n \left(P_0^{(\xi)}(B(x\sqrt{n}, \varepsilon\sqrt{n}) \cap \xi[0, n] \neq \emptyset) - P_0^{(\xi)}(\{x_n\} \cap \xi[0, n] \neq \emptyset) \right) \\ & \geq \mu(0) c_\varepsilon n \end{aligned}$$

which diverges as $n \rightarrow \infty$, where the last inequality follows from the facts that $c_\varepsilon := \liminf_{n \rightarrow \infty} P_0^{(\xi)}(B(x\sqrt{n}, \varepsilon\sqrt{n}) \cap \xi[0, n] \neq \emptyset) > 0$ and $\lim_{n \rightarrow \infty} P_0^{(\xi)}(\{x_n\} \cap \xi[0, n] \neq \emptyset) = 0$.

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