MAXIMUM ENTROPY COPULA WITH GIVEN DIAGONAL SECTION

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Abstract. We consider copulas with a given diagonal section and compute the explicit density of the unique optimal copula which maximizes the entropy. In this sense, this copula is the least informative among the copulas with a given diagonal section. We give an explicit criterion on the diagonal section for the existence of the optimal copula and give a closed formula for its entropy. We also provide examples for some diagonal sections of usual bivariate copulas and illustrate the differences between them and the maximum entropy copula with the same diagonal section.

1. Introduction

Dependence of random variables can be described by copula distributions. A copula is the cumulative distribution function of a random vector $U = (U_1, \ldots, U_d)$ with $U_i$ uniformly distributed on $I = [0, 1]$. For an exhaustive overview on copulas, we refer to Nelsen [16]. The diagonal section $\delta$ of a $d$-dimensional copula $C$, defined on $I$ as $\delta(t) = C(t, \ldots, t)$ is the cumulative distribution function of $\max_{1 \leq i \leq d} U_i$. The function $\delta$ is non-decreasing, $d$-Lipschitz, and verifies $\delta(t) \leq t$ for all $t \in I$ with $\delta(0) = 0$ and $\delta(1) = 1$. It was shown that if a function $\delta$ satisfies these properties, then there exists a copula with $\delta$ as diagonal section (see Bertino [2] or Fredricks and Nelsen [12] for $d = 2$ and Cuculescu and Theodorescu [6] for $d \geq 2$).

Copulas with a given diagonal section have been studied in different papers, as the diagonal sections are considered in various fields of application. Beyond the fact that $\delta$ is the cumulative distribution function of the maximum of the marginals, it also characterizes the tail dependence of the copula (see Joe [14] p.33. and references in Nelsen et al. [18], Durante and Jaworski [8], Jaworski [13]) as well as the generator for Archimedean copulas (Sungur and Yang [25]). For $d = 2$, Bertino in [2] introduces the so-called Bertino copula $B_\delta$ given by $B_\delta(u, v) = u \wedge v - \min_{u \wedge v \leq t \leq u \vee v} (t - \delta(t))$ for $u, v \in I$. Fredricks and Nelsen in [12] give the example called diagonal copula defined by $K_\delta(u, v) = \min(u, v, (\delta(u) + \delta(v))/2)$ for $u, v \in I$. In Nelsen et al. [17, 18] lower and upper bounds related to the pointwise partial ordering are given for copulas with a given diagonal section. They showed that if $C$ is a symmetric copula with diagonal section $\delta$, then for every $u, v \in I$, we have:

$$B_\delta(u, v) \leq C(u, v) \leq K_\delta(u, v).$$

Durante et al. [10] provide another construction of copulas for a certain class of diagonal sections, called MT-copulas named after Mayor and Torrens and defined as $D_\delta(u, v) = \max(0, \delta(x \vee y) - |x - y|)$. Bivariate copulas with given sub-diagonal sections $\delta_{x_0} : [0, 1 - x_0] \to [0, 1 - x_0], \delta_{x_0}(t) = C(x_0 + t, t)$ are constructed from copulas with given diagonal sections in
QUESADA-MOLINA ET AL. [22]. DURANTE ET AL. [9] or [18] introduce the technique of diagonal splicing to create new copulas with a given diagonal section based on other such copulas. According to [8] for \( d = 2 \) and JAWORSKI [13] for \( d \geq 2 \), there exists an absolutely continuous copula with diagonal section \( \delta \) if and only if the set \( \Sigma_\delta = \{ t \in I; \delta(t) = t \} \) has zero Lebesgue measure. DE AMO ET AL. [7] is an extension of [8] for given sub-diagonal sections. Further construction of possibly asymmetric absolutely continuous bidimensional copulas with a given diagonal section is provided in ERDELY AND GONZÁLEZ [11].

Our aim is to find the most uninformative copula with a given diagonal section \( \delta \). We choose here to maximize the relative entropy to the uniform distribution on \( I^d \), among the copulas with given diagonal section. This is equivalent to minimizing the Kullback-Leibler divergence with respect to the independent copula. The Kullback-Leibler divergence is finite only for absolutely continuous copulas. The previously introduced bivariate copulas \( B_\delta, K_\delta \) and \( D_\delta \) are not absolutely continuous, therefore their Kullback-Leibler divergence is infinite. Possible other entropy criteria, such as Rényi, Tsallis, etc. are considered for example in POUGAZA AND MOHAMMAD-DJAFARI [21]. We recall that the entropy of a \( d \)-dimensional absolutely continuous random vector \( X = (X_1, \ldots, X_d) \) can be decomposed as the sum of the entropy of the marginals and the entropy of the corresponding copula (see ZHAO AND LIN [26]):

\[
H(X) = \sum_{i=1}^{d} H(X_i) + H(U),
\]

where \( H(Z) = -\int f_Z(z) \log f_Z(z) dz \) is the entropy of the random variable \( Z \) with density \( f_Z \), and \( U = (U_1, \ldots, U_d) \) is a random vector with \( U_i \) uniformly distributed on \( I \), such that \( U \) has the same copula as \( X \); namely \( U \) is distributed as \((F_1^{-1}(X_1), \ldots, F_d^{-1}(X_d))\) with \( F_i \) the cumulative distribution function of \( X_i \). Maximizing the entropy of \( X \) with given marginals therefore corresponds to maximizing the entropy of its copula. The maximum relative entropy approach for copulas has an extensive literature. Existence results for an optimal solution on convex closed subsets of copulas for the total variation distance can be derived from Csiszár [5]. A general discussion on abstract entropy maximization is given by BORWEIN ET AL. [3]. This theory was applied for copulas and a finite number of expectation constraints in BEDFORD AND WILSON [1]. Some applications for various moment-based constraints include rank correlation (MEEUWISSEN AND BEDFORD [15], CHU [4], PIANTADOSI ET AL. [20]) and marginal moments (PASHA AND MANSOURY [19]).

We shall apply the theory developed in [3] to compute the density of the maximum entropy copula with a given diagonal section. We show that there exists a copula with diagonal section \( \delta \) and finite entropy if and only if \( \delta \) satisfies: \( \int_I (t - \delta(t)) dt < +\infty \). Notice that this condition is stronger than the condition of \( \Sigma_\delta \) having zero Lebesgue measure which is required for the existence of an absolutely continuous copula with diagonal section \( \delta \). Under this condition, and in the case of \( \Sigma_\delta = \{0,1\} \), the optimal copula’s density \( c_\delta \) turns out to be of the form, for \( x = (x_1, \ldots, x_d) \in I^d \):

\[
c_\delta(x) = b(\max(x)) \prod_{x_i \neq \max(x)} a(x_i),
\]

with the notation \( \max(x) = \max_{1 \leq i \leq d} x_i \), see Theorem 2.3. The optimal copula’s density in the general case is given in Theorem 2.4. Notice that \( c_\delta \) is symmetric, that is it is invariant under the permutation of the variables. This provides a new family of absolutely continuous symmetric copulas with given diagonal section enriching previous work on this subject that
we discussed, see [2],[8],[9],[10],[11],[12],[18]. We also calculate the maximum entropy copula for diagonal sections that arise from well-known families of bivariate copulas.

The rest of the paper is organised as follows. Section 2 introduces the definitions and notations used later on, and gives the main theorems of the paper. In Section 3 we study the properties of the feasible solution \( c_\delta \) of the problem for a special class of diagonal sections with \( \Sigma_\delta = \{0, 1\} \). In Section 4, we formulate our problem as a linear optimization problem in order to apply the theory established in [3]. Then in Section 5 we give the proof for our main theorem showing that \( c_\delta \) is indeed the optimal solution when \( \Sigma_\delta = \{0, 1\} \). In Section 6 we extend our results for the general case when \( \Sigma_\delta \) has zero Lebesgue measure. We give in Section 7 several examples with diagonals of popular bivariate copula families such as the Gaussian, Gumbel or Farlie-Gumbel-Morgenstern copulas among others.

2. Main results

Let \( d \geq 2 \) be fixed. We recall a function \( C \) defined on \( I^d \), with \( I = [0, 1] \), is a \( d \)-dimensional copula if there exists a random vector \( U = (U_1, \ldots, U_d) \) such that \( U_i \) are uniform on \( I \) and \( C(u) = \mathbb{P}(U \leq u) \) for \( u \in I^d \), with the convention that \( x \leq y \) for \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) elements of \( \mathbb{R}^d \) if and only if \( x_i \leq y_i \) for all \( 1 \leq i \leq d \). We shall say that \( C \) is the copula of \( U \). We refer to [16] for a monograph on copulas. The copula \( C \) is said absolutely continuous if the random variable \( U \) has a density, which we shall denote by \( c_C \). In this case, we have that a.e. \( c_C(u) = \partial_{u_1, \ldots, u_d} C(u) \) for \( u \in I^d \). When there is no confusion, we shall write \( c \) for the density \( c_C \) associated to the copula \( C \). We denote by \( \mathcal{C} \) the set of \( d \)-dimensional copulas and by \( \mathcal{C}_0 \) the subset of the \( d \)-dimensional absolutely continuous copulas.

The diagonal section \( \delta_C \) of a copula \( C \) is defined by: \( \delta_C(t) = C(t, \ldots, t) \). Let us note, for \( u \in \mathbb{R}^d \), \( \max(u) = \max_{1 \leq i \leq d} u_i \). Notice that if \( C \) is the copula of \( U \), then \( \delta_C \) is the cumulative distribution function of \( \max(U) \) as \( \delta_C(t) = \mathbb{P}(\max(U) \leq t) \) for \( t \in I \). We denote by \( \mathcal{D} = \{\delta_C, C \in \mathcal{C}\} \) the set of diagonal sections of \( d \)-dimensional copulas and by \( \mathcal{D}_0 = \{\delta_C; C \in \mathcal{C}_0\} \) the set of diagonal sections of absolutely continuous copulas. According to [12], a function \( \delta \) defined on \( I \) belongs to \( \mathcal{D} \) if and only if:

1. \( \delta \) is a cumulative function on \([0, 1]\); \( \delta(0) = 0, \delta(1) = 1 \) and \( \delta \) is non decreasing;
2. \( \delta(t) \leq t \) for \( t \in I \) and \( \delta \) is \( d \)-Lipschitz: \( |\delta(s) - \delta(t)| \leq d|s - t| \) for \( s, t \in I \).

For \( \delta \in \mathcal{D} \), we shall consider the set \( \mathcal{C}^\delta = \{C \in \mathcal{C}; \delta_C = \delta\} \) of copulas with diagonal section \( \delta \), and the subset \( \mathcal{C}_0^\delta = \mathcal{C}^\delta \cap \mathcal{C}_0 \) of absolutely continuous copulas with section \( \delta \). According to [8] and [13], the set \( \mathcal{C}_0^\delta \) is non empty if and only if the set \( \Sigma_\delta = \{t \in I; \delta(t) = t\} \) has zero Lebesgue measure.

For a non-negative measurable function \( f \) defined on \( I^k \), \( k \in \mathbb{N}^* \), we set:

\[
\mathcal{I}_k(f) = \int_{I^k} f(x) \log(f(x)) \, dx.
\]

Since copulas are cumulative functions of probability measures, we will consider the Kullback-Leibler divergence relative to the uniform distribution as a measure of entropy, see [5]:

\[
\mathcal{I}(C) = \begin{cases} 
\mathcal{I}_d(c) & \text{if } C \in \mathcal{C}_0, \\
+\infty & \text{if } C \notin \mathcal{C}_0,
\end{cases}
\]

with \( c \) the density associated to \( C \) when \( C \in \mathcal{C}_0 \). Notice the Shannon-entropy introduced in [24] of the probability measure \( P \) defined on \( I^d \) with cumulative distribution function \( C \) is defined as \( H(P) = -\mathcal{I}(C) \). Thus minimizing the Kullback-Leibler divergence \( \mathcal{I} \) (w.r.t. the
uniform distribution) is equivalent to maximizing the Shannon-entropy. It is well known that the copula \( \Pi \) with density \( c_\Pi = 1 \), which corresponds to \((U_i, 0 \leq i \leq d)\) being independent, minimizes \( \mathcal{I}(C) \) over \( \mathcal{C} \).

We shall minimize the entropy \( \mathcal{I} \) over the set \( \mathcal{C}^\delta \) or equivalently over \( \mathcal{C}_0^\delta \) of copulas with a given diagonal section \( \delta \in \mathcal{D} \) (in fact for \( \delta \in \mathcal{D}_0 \) as otherwise \( \mathcal{C}_0^\delta \) is empty). If \( C \) minimizes \( \mathcal{I} \) on \( \mathcal{C}^\delta \), it means that \( C \) is the least informative (or the “most random”) copula with given diagonal section \( \delta \).

For \( \delta \in \mathcal{D} \), let us denote:

\[
\mathcal{J}(\delta) = \int_I \| \log(t - \delta(t)) \| \, dt.
\]

Notice that \( \mathcal{J}(\delta) \in [0, +\infty] \) and it is infinite if \( \delta \notin \mathcal{D}_0 \). Since \( \delta \) is \( d \)-Lipschitz, the derivative \( \delta' \) of \( \delta \) exists a.e. and since \( \delta \) is non-decreasing we have a.e. \( 0 \leq \delta' \leq d \). This implies that \( \mathcal{I}_1(\delta') \) and \( \mathcal{I}_1(d - \delta') \) are well defined. Let us denote:

\[
(2) \quad \mathcal{G}(\delta) = \mathcal{I}_1(\delta') + \mathcal{I}_1(d - \delta') - d \log(d) - (d - 1).
\]

We have the rough upper bound:

\[
(3) \quad \sup_{\delta \in \mathcal{D}} |\mathcal{G}(\delta)| \leq d + d \log(d).
\]

The following Proposition gives an absolutely continuous copula whose diagonal section is \( \delta \). The proof of this Proposition can be found in Section 3 and Section 8 is dedicated to the proof of (6).

**Proposition 2.1.** Let \( \delta \in \mathcal{D}_0 \) with \( \Sigma_\delta = \{0, 1\} \). We define, for \( r \in I \):

\[
h(r) = r - \delta(r), \quad F(r) = \frac{d - 1}{d} \int_r^1 \frac{1}{h(s)} \, ds,
\]

\[
a(r) = \frac{d - \delta'(r)}{d} h(r)^{-1 + 1/d} e^{F(r)} \quad \text{and} \quad b(r) = \frac{\delta'(r)}{d} h(r)^{-1 + 1/d} e^{-(d - 1)F(r)}.
\]

Then \( c_\delta \) defined a.e. by

\[
(5) \quad c_\delta(x) = b(\max(x)) \prod_{x_i \neq \max(x)} a(x_i) \quad \text{for } x = (x_1, \ldots, x_d) \in I^d,
\]

is the density of a symmetric copula \( C_\delta \) with diagonal section \( \delta \). Furthermore, we have:

\[
(6) \quad \mathcal{I}(C_\delta) = (d - 1) \mathcal{J}(\delta) + \mathcal{G}(\delta).
\]

This and (3) readily implies the following Remark.

**Remark 2.2.** Let \( \delta \in \mathcal{D}_0 \) such that \( \Sigma_\delta = \{0, 1\} \). We have \( \mathcal{I}(C_\delta) < +\infty \) if and only if \( \mathcal{J}(\delta) < +\infty \).

We can now state our main result in the simpler case \( \Sigma_\delta = \{0, 1\} \). It gives the necessary and sufficient condition for \( C_\delta \) to be the unique optimal solution of the minimization problem. The proof is given in Section 5.

**Theorem 2.3.** Let \( \delta \in \mathcal{D}_0 \) such that \( \Sigma_\delta = \{0, 1\} \).

a) If \( \mathcal{J}(\delta) = +\infty \) then \( \min_{C \in \mathcal{C}^\delta} \mathcal{I}(C) = +\infty \).

b) If \( \mathcal{J}(\delta) < +\infty \) then \( \min_{C \in \mathcal{C}^\delta} \mathcal{I}(C) < +\infty \) and \( C_\delta \) is the unique copula such that \( \mathcal{I}(C_\delta) = \min_{C \in \mathcal{C}^\delta} \mathcal{I}(C) \).
To give the answer in the general case where $\Sigma_\delta$ has zero Lebesgue measure, we need some extra notations. Since $\delta$ is continuous, we get that $I \setminus \Sigma_\delta$ can be written as the union of non-empty open intervals $((\alpha_j, \beta_j), j \in J)$, with $\alpha_j < \beta_j$ and $J$ at most countable. Notice that $\delta(\alpha_j) = \alpha_j$ and $\delta(\beta_j) = \beta_j$. For $\delta \neq \emptyset$ and $j \in J$, we set $\Delta_j = \beta_j - \alpha_j$ and for $t \in I$:

$$
\delta^j(t) = \frac{\delta(\alpha_j + t\Delta_j) - \alpha_j}{\Delta_j}.
$$

It is clear that $\delta^j$ satisfies (i) and (ii) and it belongs to $D_0$ as $\Sigma_\delta^j = \{0, 1\}$. Let $c_\delta^j$ be defined by (5) with $\delta$ replaced by $\delta^j$. For $\delta \in D_0$ such that $\Sigma_\delta \neq \{0, 1\}$, we define the function $c_\delta$ by, for $u \in I^d$:

$$
c_\delta(u) = \sum_{j \in J} \frac{1}{\Delta_j} c_\delta^j \left( \frac{u - \alpha_j}{\Delta_j} \right) 1_{(\alpha_j, \beta_j]}(u),
$$

with $1 = (1, \ldots, 1) \in \mathbb{R}^d$. It is easy to check that $c_\delta$ is a copula density and that is zero outside $[\alpha_j, \beta_j]^d$ for $j \in J$. We state our main result in the general case whose proof is given in Section 6.

**Theorem 2.4.** Let $\delta \in D$.

a) If $J(\delta) = +\infty$ then $\min_{C \in C^\delta} I(C) = +\infty$.

b) If $J(\delta) < +\infty$ then $\min_{C \in C^\delta} I(C) < +\infty$ and there exists a unique copula $C_\delta \in C^\delta$ such that $I(C_\delta) = \min_{C \in C^\delta} I(C)$. Furthermore, we have:

$$
I(C_\delta) = (d - 1)J(\delta) + G(\delta);
$$

the copula $C_\delta$ is absolutely continuous, symmetric; its density $c_\delta$ is given by (5) if $\Sigma_\delta = \{0, 1\}$ or by (8) if $\Sigma_\delta \neq \{0, 1\}$.

**Remark 2.5.** For $\delta \in D$, notice the condition $J(\delta) < +\infty$ implies that $\Sigma_\delta$ has zero Lebesgue measure, and therefore, according to [8] and [13], $\delta \in D_0$. And if $\delta \notin D_0$, then $I(C) = +\infty$ for all $C \in C^\delta$. Therefore, we could replace the condition $\delta \in D$ by $\delta \in D_0$ in Theorem 2.4.

### 3. Proof of Proposition 2.1

We assume that $\delta \in D_0$ and $\Sigma_\delta = \{0, 1\}$. We give the proof of Proposition 2.1, which states that $C_\delta$, with density $c_\delta$ given by (5), is indeed a symmetric copula with diagonal section $\delta$ whose entropy is given by (6).

Recall the definition of $h, F, a, b$ and $c_\delta$ from Theorem 2.3. Notice that by construction $c_\delta$ is non-negative and well defined on $I^d$. In order to prove that $c_\delta$ is the density of a copula, we only have to prove that for all $1 \leq i \leq d, r \in I$:

$$
\int_{I^d} c_\delta(u) 1_{\{u_i \leq r\}} du = r,
$$

or equivalently

$$
\int_{I^d} c_\delta(u) 1_{\{u_i \geq r\}} du = 1 - r.
$$

We define for $r \in I$:

$$
A(r) = \int_0^r a(t) dt.
$$

Elementary computations yield for $r \in (0, 1)$:

$$
A(r) = h^{1/d}(r) e^{F(r)}.
$$
Notice that $F(0) \in [-\infty, 0]$ which implies that $A(0) = 0$. A direct integration gives:

\[ d \int_I A^{d-1}(s)b(s)1_{\{s \geq r\}} = 1 - \delta(r). \]

We also have:

\[ (d - 1) \int_I A^{d-2}(s)b(s) ds 1_{\{s \geq r\}} = \frac{(d - 1)}{d} \int_I h^{-1/d}(s) e^{-F(s)} 1_{\{s \geq r\}} ds \]

\[ = \left[ -h^{-1/d}(s) e^{-F(s)} \right]_{s=r}^{1} \]

\[ = h^{-1/d}(r) e^{-F(r)}, \]

where we used for the last step that $h(1) = 0$ and $F(1) \in [0, \infty]$. We have:

\[ \int_{I^d} c_\delta(u) 1_{\{u_i \geq r\}} du = \int_{I^d} b(\max(u)) \prod_{u_j \neq \max(u)} a(u_j) 1_{\{u_i \geq r\}} du \]

\[ = \int_{I^d} A^{d-1}(s)b(s) 1_{\{s \geq r\}} ds \]

\[ + (d - 1) \int_I A^{d-2}(s)b(s)(A(s) - A(r)) 1_{\{s \geq r\}} ds \]

\[ = d \int_I A^{d-1}(s)b(s) 1_{\{s \geq r\}} ds \]

\[ - (d - 1)A(r) \int_I A^{d-2}(s)b(s) 1_{\{s \geq r\}} ds \]

\[ = 1 - \delta(r) - (r - \delta(r)) \]

\[ = 1 - r, \]

where we first divided the integral according to which $u_i$ was the maximum; then we used (9) for the second equality, finally (11) and (12) for the forth. This implies that $c_\delta$ is indeed the density of a copula. We denote by $C_\delta$ the copula with density $c_\delta$. We check that $\delta$ is the diagonal section of $C_\delta$. Using (11), we get, for $r \in I$:

\[ \int_{I^d} c_\delta(u) 1_{\{\max(u) \leq r\}} du = \int_{I^d} b(\max(u)) \prod_{u_j \neq \max(u)} a(u_j) 1_{\{\max(u) \geq r\}} du \]

\[ = d \int_I A^{d-1}(s)b(s) 1_{\{s \leq r\}} ds \]

\[ = \delta(r). \]

The calculations which show that the entropy of $C_\delta$ is given by (6) can be found in Section 8.

4. The minimization problem

Let $\delta \in \mathcal{D}_0$. As a first step we will show, using [3], that the problem of a maximum entropy copula with a given diagonal section $\delta$ has at most a unique optimal solution. To formulate this problem in the framework of [3], we introduce the continuous linear functional $\mathcal{A} = (\mathcal{A}_i, 1 \leq i \leq d + 1) : L^1(I^d) \to L^1(I)^{d+1}$ defined by, for $1 \leq i \leq d$, $f \in L^1(I^d)$ and $r \in I$,

\[ \mathcal{A}_i(f)(r) = \int_{I^d} f(u) 1_{\{u_i \leq r\}} du, \quad \text{and} \quad \mathcal{A}_{d+1}(f)(r) = \int_{I^d} f(u) 1_{\{\max(u) \leq r\}} du. \]
We also define $b^\delta = (b_i, 1 \leq i \leq d + 1) \in L^1(I)^{d+1}$ with $b_{d+1} = \delta$ and $b_i = \text{id}_I$ for $1 \leq i \leq d$, with $\text{id}_I$ the identity map on $I$. Notice that the conditions $A_i(c) = b_i$, $1 \leq i \leq d$, and $c \geq 0$ a.e. imply that $c$ is the density of a copula $C \in \mathcal{C}_0$. If we assume further that the condition $A_{d+1}(c) = b_{d+1}$ holds then the diagonal section of $C$ is $\delta$ (thus $C \in \mathcal{C}_0^\delta$).

Since $\mathcal{I}$ is infinite outside $\mathcal{C}_0^\delta$ and the density of any copula in $\mathcal{C}_0$ belongs to $L^1(I^d)$, we get that minimizing $\mathcal{I}$ over $\mathcal{C}_0^\delta$ is equivalent to the linear optimization problem $(P^\delta)$ given by:

$$
(P^\delta) \quad \text{minimize } \mathcal{I}_d(c) \text{ subject to } \begin{cases} A(c) = b^\delta, \\
 c \geq 0 \text{ a.e. and } c \in L^1(I^d). \end{cases}
$$

We say that a function $f$ is feasible for $(P^\delta)$ if $f \in L^1(I^d)$, $f \geq 0$ a.e., $A(f) = b^\delta$ and $\mathcal{I}_d(f) < +\infty$. Notice that any feasible $f$ is the density of a copula. We say that $f$ is an optimal solution to $(P^\delta)$ if $f$ is feasible and $\mathcal{I}_d(f) \leq \mathcal{I}_d(g)$ for all $g$ feasible.

**Proposition 4.1.** Let $\delta \in \mathcal{D}$. If there exists a feasible $c$, then there exists a unique optimal solution to $(P^\delta)$ and it is symmetric.

**Proof.** Since $A(f) = b^\delta$ implies $A_1(f)(1) = b_1(1)$ that is $\int_I f(x) \, dx = 1$, we can directly apply Corollary 2.3 of [3] which states that if there exists a feasible $c$, then there exists a unique optimal solution to $(P^\delta)$. Since the constraints are symmetric and the functional $\mathcal{I}_d$ is also symmetric, we deduce that the unique optimal solution is also symmetric. \(\square\)

The next Proposition gives that the set of zeros of any non-negative solution $c$ of $A(c) = b^\delta$ contains:

$$
Z^\delta = \{ u \in I^d; \delta'(\max(u)) = 0 \text{ or } \exists i \text{ such that } u_i < \max(u) \text{ and } \delta'(u_i) = d \}.
$$

**Proposition 4.2.** Let $\delta \in \mathcal{D}$. If $c$ is feasible then $c = 0$ a.e. on $Z^\delta$ (that is $c 1_{Z^\delta} = 0$ a.e.).

**Proof.** Recall that $0 \leq \delta' \leq d$. Since $c \in L^1(I^d)$, the condition $A_{d+1}(c) = b_{d+1}$, that is for all $r \in I$

$$
\int_I c(u) 1_{\{ \max(u) \leq r \}} \, du = \int_0^r \delta'(s) \, ds,
$$

implies, by the monotone class theorem, that for all measurable subset $H$ of $I$, we have:

$$
\int_I c(u) 1_H(\max(u)) \, du = \int_H \delta'(s) \, ds.
$$

Since $c \geq 0$ a.e., we deduce that a.e. $c(u) 1_{\{ \delta'(\max(u)) = 0 \}} = 0$.

Next, notice that for all $r \in I$, $1 \leq i \leq d$, the symmetrical property of $c$ gives:

$$
\int_I c(u) 1_{\{ u_i < \max(u), u_i \leq r \}} \, du = \int_I c(u) 1_{\{ u_i \leq r \}} \, du - \int_I c(u) 1_{\{ u_i = \max(u), u_i \leq r \}} \, du
$$

$$
= r - \frac{\delta(r)}{d}
$$

$$
= \int_0^r \left( 1 - \frac{\delta'(s)}{d} \right) \, ds.
$$

This implies that a.e. $c(u) 1_{\{ \exists i \text{ such that } u_i < \max(u), \delta'(u_i) = d \}} = 0$. This gives the result. \(\square\)

We define $\mu$ to be the Lebesgue measure restricted to $Z^\delta = I^d \setminus Z^\delta$: $\mu(du) = 1_{Z^\delta}(u) \, du$.

We define, for $f \in L^1(I^d, \mu)$:

$$
\mathcal{I}^\mu(f) = \int_{I^d} f(u) \log(f(u)) \, \mu(du).
$$
From Proposition 4.2 we can deduce that if $c$ is feasible then $\mathcal{I}^\mu(c) = \mathcal{I}_d(c)$. Let us also define, for $1 \leq i \leq d$, $r \in I$:

$$A^\mu_i(c)(r) = \int_{I^d} c(u) 1_{\{u_i \leq r\}} \mu(du), \quad \text{and} \quad A^\mu_{d+1}(c)(r) = \int_{I^d} c(u) 1_{\{\max(u) \leq r\}} \mu(du).$$

The corresponding optimization problem $(P^\delta_\mu)$ is given by:

$$(P^\delta_\mu) \quad \text{minimize } \mathcal{I}^\mu(c) \text{ subject to } \begin{cases} \mathcal{A}^\mu(c) = b^\delta, \\ c \geq 0 \text{ } \mu\text{-a.e. and } c \in L^1(I^d, \mu), \end{cases}$$

with $\mathcal{A}^\mu = (A^\mu_i, 1 \leq i \leq d+1)$. For $f \in L^1(I^d, \mu)$, we define:

$$f^\mu = \begin{cases} f \text{ on } Z^c, \\ 0 \text{ on } Z^c. \end{cases}$$

Using Proposition 4.2, we easily get the following Corollary.

**Corollary 4.3.** If $c$ is a solution of $(P^\delta_\mu)$, then $c^\mu$ is a solution of $(P^\delta)$. If $c$ is a solution of $(P^\delta)$, then it is also a solution of $(P^\delta_\mu)$.

5. **Proof of Theorem 2.3**

5.1. **Form of the optimal solution.** Let $(A^\mu)^* : L^\infty(I)^{d+1} \to L^\infty(I^d, \mu)$ be the adjoint of $A^\mu$. We will use Theorem 2.9. from [3] on abstract entropy minimization, which we recall here, adapted to the context of $(P^\delta_\mu)$.

**Theorem 5.1** (Borwein, Lewis and Nussbaum). Suppose there exists $c > 0$ $\mu$-a.e. which is feasible for $(P^\delta_\mu)$. Then there exists a unique optimal solution, $c^*$, to $(P^\delta_\mu)$. Furthermore, we have $c^* > 0$ $\mu$-a.e. and there exists a sequence $(\lambda^n, n \in \mathbb{N}^*)$ of elements of $L^\infty(I)^{d+1}$ such that:

$$\int_{I^d} c^\mu(x) |(A^\mu)^*(\lambda^n)(x)| \mu(dx) \xrightarrow{n \to \infty} 0. \quad (14)$$

We first compute $(A^\mu)^*$. For $\lambda = (\lambda_i, 1 \leq i \leq d+1) \in L^\infty(I)^{d+1}$ and $f \in L^\infty(I^d, \mu)$, we have:

$$\langle (A^\mu)^*(\lambda), f \rangle = \langle \lambda, A^\mu(f) \rangle = \sum_{i=1}^d \int_I dr \lambda_i(r) \int_{I^d} f(x) 1_{\{x_i \leq r\}} d\mu(x) + \int_I dr \lambda_{d+1}(r) \int_{I^d} f(x) 1_{\{\max(x) \leq r\}} d\mu(x) = \int_{I^d} d\mu(x) f(x) \left( \sum_{i=1}^d \Lambda_i(x_i) + \Lambda_{d+1}(\max(x)) \right),$$

where we used the definition of the adjoint operator for the first equality, Fubini’s theorem for the second, and the following notation for the third equality:

$$\Lambda_i(x_i) = \int_I \lambda_i(r) 1_{\{r \geq x_i\}} dr, \quad \text{and} \quad \Lambda_{d+1}(t) = \int_I \lambda_{d+1}(r) 1_{\{r \geq t\}} dr.$$

Thus, we can set for $\lambda \in L^\infty(I)^{d+1}$ and $x \in I^d$:

$$\langle (A^\mu)^*(\lambda)(x) = \sum_{i=1}^d \Lambda_i(x_i) + \Lambda_{d+1}(\max(x)). \quad (15)$$
Now we are ready to prove that the optimal solution $c^*$ of \((P^\delta_\mu)\) is the product of measurable univariate functions.

**Lemma 5.2.** Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$. Suppose that there exists $c > 0$ $\mu$-a.e. which is feasible for \((P^\delta_\mu)\). Then there exist $a^*, b^*$ non-negative, measurable functions defined on $I$ such that

$$c^*(u) = b^*(\max(u)) \prod_{u_i \neq \max(u)} a^*(u_i) \quad \mu\text{-a.e.}$$

with $a^*(s) = 0$ if $\delta'(s) = d$ and $b^*(s) = 0$ if $\delta'(s) = 0$.

**Proof.** According to Theorem 5.1, there exists a sequence $(\lambda^n, n \in \mathbb{N}^*)$ of elements of $L^\infty(I)^{d+1}$ such that the optimal solution, say $c^*$, satisfies (14). This implies, thanks to (15), that there exist $d + 1$ sequences $(\Lambda^n_i, n \in \mathbb{N}^*, 1 \leq i \leq d+1)$ of elements of $L^\infty(I)$ such that the following convergence holds in $L^1(I^d, c^*\mu)$:

$$\sum_{i=1}^d \Lambda^n_i(u_i) + \Lambda^n_{d+1}(\max(u)) \xrightarrow{n \to \infty} \log(c^*(u)). \quad (16)$$

Arguing as in Proposition 4.1 and since $Z_\delta^i$ is symmetric, we deduce that $c^*$ is symmetric. Therefore we shall only consider functions supported on the set $\Delta = \{u \in I^d; u_d = \max(u)\}$.

The convergence (16) holds in $L^1(\Delta, c^*\mu)$. For simplicity, we introduce the functions $\Gamma^n_i \in L^\infty(I)$ defined by $\Gamma^n_i = \Lambda^n_i$ for $1 \leq i \leq d-1$, and $\Gamma^n_d = \Lambda^n_d + \Lambda^n_{d+1}$. Then we have in $L^1(\Delta, c^*\mu)$:

$$\sum_{i=1}^d \Gamma^n_i(u_i) \xrightarrow{n \to \infty} \log(c^*(u)). \quad (17)$$

We first assume that there exist $\Gamma$ and $\Gamma_d$ measurable functions defined on $I$ such that $\mu$-a.e. on $\Delta$:

$$\sum_{i=1}^{d-1} \Gamma(u_i) + \Gamma_d(u_d) = \log(c^*(u)). \quad (18)$$

The symmetric property of $c^*(u)$ seen in Proposition 4.1 implies we can choose $\Gamma_i = \Gamma$ for $1 \leq i \leq d-1$ up to adding a constant to $\Gamma_d$. Set $a^* = \exp(\Gamma)$ and $b^* = \exp(\Gamma_d)$ so that $\mu$-a.e. on $\Delta$:

$$c^*(u) = b^*(u_d) \prod_{i=1}^{d-1} a^*(u_i). \quad (19)$$

Recall $\mu(du) = 1_{Z_\delta^i}(u) du$. From the definition (13) of $Z_\delta^i$, we deduce that without loss of generality, we can assume that $a^*(u_i) = 0$ if $\delta'(u_i) = d$ and $b^*(u_d) = 0$ if $\delta'(u_d) = 0$. Use the symmetry of $c^*$ to conclude.

To complete the proof, we now show that (18) holds for $\Gamma$ and $\Gamma_d$ measurable functions. We introduce the notation $u_{(-i)} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d) \in I^{d-1}$. Let us define the probability measure $P(dx) = c^*(x) 1_\Delta(x) \mu(dx)/\int_\Delta c^*(y) \mu(dy)$ on $I^d$. We fix $j, 1 \leq j \leq d-1$. In order to apply Proposition 2 of [23], we first check that $P$ is absolutely continuous with respect to $P^1_1 \otimes P^2_2$, where $P^1_1(du_{(-j)}) = \int_{u_j \in I} P(du_{(-j)} | du_j)$ and $P^2_2(du_j) = \int_{u_{(-j)} \in I^{d-1}} P(du_{(-j)} | du_j)$ are
the marginals of $P$. Notice the following equivalence of measures:

$$P(du) \sim 1_{\Delta}(u) \prod_{i=1}^{d-1} 1_{\{\delta'(u_i) \neq \delta\}} 1_{\{\delta'(u_d) \neq 0\}} du.$$  

(20)

Let $B \subset I^{d-1}$ be measurable. We have:

$$P_1(B) = 0 \iff \int_I 1_{\Delta}(u) \prod_{i=1}^{d-1} 1_{\{\delta'(u_i) \neq \delta\}} 1_{\{\delta'(u_d) \neq 0\}} 1_B(u_{(-j)}) du = 0.$$  

By Fubini’s theorem this last equality is equivalent to:

$$\int_{I^{d-1}} \prod_{i=1, i \neq j}^{d-1} \left(1_{\{\delta'(u_i) \neq \delta\}} 1_{\{u_i \leq u_d\}} \right) 1_{\{\delta'(u_d) \neq 0\}} 1_B(u_{(-j)}) \left(\int_I 1_{\{0 \leq u_j \leq u_d\}} 1_{\{\delta'(u_j) \neq \delta\}} du_j\right) du_{(-j)} = 0.$$  

(21)

Since, for $\varepsilon > 0$, $\delta(\varepsilon) < \varepsilon < d\varepsilon$, we have $\int_I 1_{\{0 \leq u_j \leq s\}} 1_{\{\delta'(u_j) \neq \delta\}} du_j > 0$ for all $s \in I$. Therefore (21) is equivalent to

$$\int_{I^{d-1}} \prod_{i=1, i \neq j}^{d-1} \left(1_{\{\delta'(u_i) \neq \delta\}} 1_{\{u_i \leq u_d\}} \right) 1_{\{\delta'(u_d) \neq 0\}} 1_B(u_{(-j)}) du_{(-j)} = 0.$$  

This implies that there exists $h > 0$ a.e. on $I^{d-1}$ such that

$$P_1^j(du_{(-j)}) = h(u_{(-j)}) \prod_{i=1, i \neq j}^{d-1} \left(1_{\{\delta'(u_i) \neq \delta\}} 1_{\{u_i \leq u_d\}} \right) 1_{\{\delta'(u_d) \neq 0\}} du_{(-j)}.$$  

Similarly we have for $B' \subset I$ that $P_2^j(B') = 0$ if and only if

$$\int_I 1_{\{\delta'(u_j) \neq \delta\}} 1_{B'}(u_j) \left(\int_{I^{d-1}} \prod_{i=1, i \neq j}^{d-1} \left(1_{\{\delta'(u_i) \neq \delta\}} 1_{\{u_i \leq u_d\}} \right) 1_{\{\delta'(u_d) \neq 0\}} 1_{\{u_d \geq u_j\}} du_{(-j)}\right) du_j = 0.$$  

(22)

Since, for $\varepsilon > 0$, $\delta(1) - \delta(1 - \varepsilon) > 1 - (1 - \varepsilon) = \varepsilon > 0$, there exists $g > 0$ a.e. on $I$ such that $P_2^j(du_j) = g(u_j) 1_{\{\delta'(u_j) \neq \delta\}} du_j$. Therefore by (20) we deduce that $P$ is absolutely continuous with respect to $P_1^j \otimes P_2^j$. Then according to Proposition 2 of [23], (17) implies that there exist measurable functions $\Phi_j$ and $\Gamma_j$ defined respectively on $I^{d-1}$ and $I$, such that $c^* \mu$-a.e. on $\Delta$:

$$\log(c^*(u)) = \Phi_j(u_{(-j)}) + \Gamma_j(u_j).$$

As $\mu$-a.e. $c^* > 0$, this equality holds $\mu$-a.e. on $\Delta$. Since we have such a representation for every $1 \leq j \leq d - 1$, we can easily verify that there exists a measurable function $\Gamma_d$ defined on $I$ such that $\log(c^*(u)) = \sum_{i=1}^{d} \Gamma_i(u_i)$ $\mu$-a.e. on $\Delta$.

\[ \square \]

5.2. Calculation of the optimal solution. Now we prove that the optimal solution to $(P^\delta)$, if it exists, is indeed $c_\delta$.

**Proposition 5.3.** Let $\delta \in D_0$ such that $\Sigma_\delta = \{0, 1\}$. If there exists an optimal solution to $(P^\delta)$, then it is $c_\delta$ given by (5).
Proof. In Lemma 5.2 we have already shown that if an optimal solution exists for \((P^\delta)\), then it is of the form \(c^*(u) = b^*(\max(u)) \prod_{u_i \neq \max(u)} a^*(u_i)\). Here we will prove that the constraints of \((P^\delta)\) uniquely determine the functions \(a^*\) and \(b^*\) up to a multiplicative constant, giving \(c^* = c_\delta\). We set for \(r \in I:\)

\[
A^*(r) = \int_0^r a^*(s) \, ds
\]

which take values in \([0, +\infty]\). From \(A_{d+1}(c^*) = b^d_{d+1}\), we have for \(r \in I:\)

\[
\delta(r) = \int_{I^d} c^*(u) 1_{\{\max(u) \leq r\}} \, du = \int_{I^d} b^*(\max(u)) \prod_{u_i \neq \max(u)} a^*(u_i) 1_{\{\max(u) \leq r\}} \, du = d \int_I (A^*(s))^{d-1} b^*(s) 1_{\{s \leq r\}} \, ds.
\]

Taking the derivative with respect to \(r\) gives a.e. on \(I:\)

\[
\delta'(r) = d(A^*(r))^{d-1} b^*(r).
\]

This implies that \(A^*(r)\) is finite for all \(r \in [0, 1)\) and thus \(A^*(0) = 0\). Similarly, using that \(A_1(c^*) = b^0_1\), we get that for \(r \in I:\)

\[
1 - r = \int_{I^d} c^*(u) 1_{\{u_1 \geq r\}} \, du = \int_{I^d} b^*(\max(u)) \prod_{u_i \neq \max(u)} a^*(u_i) 1_{\{u_1 \geq r\}} \, du = \int_{I^d} \prod_{i=2}^d (a^*(u_i) 1_{\{u_i \leq u_1\}}) b^*(u_1) 1_{\{u_1 \geq r\}} \, du + (d-1) \int_{I^d} a^*(u_1) \prod_{i=2}^d (a^*(u_i) 1_{\{u_i \leq u_2\}}) b^*(u_2) 1_{\{u_2 \geq u_1 \geq r\}} \, du
\]

\[
= \int_I (A^*(s))^{d-1} b^*(s) 1_{\{s \geq r\}} \, ds + (d-1) \int_I (A^*(s))^{d-2} b^*(s)(A^*(s) - A^*(r)) 1_{\{s \leq r\}} \, ds = d \int_I (A^*(s))^{d-1} b^*(s) 1_{\{s \geq r\}} \, ds - (d-1) A^*(r) \int_I (A^*(s))^{d-2} b^*(s) 1_{\{s \leq r\}} \, ds.
\]

Using this and (23) we deduce that for \(r \in I:\)

\[
h(r) = (d-1)A^*(r) \int_I (A^*(s))^{d-2} b^*(s) 1_{\{s \leq r\}} \, ds.
\]

Since \(r > \delta(r)\) on \((0, 1)\), we have that \(A^*\) and \(\int_I (A^*(s))^{d-2} b^*(s) 1_{\{s \leq r\}} \, ds\) are positive on \((0, 1)\). Dividing (24) by (25) gives a.e. for \(r \in I:\)

\[
\frac{d-1}{d} \frac{\delta'(r)}{h(r)} = \frac{(A^*(r))^{d-2} b(r)}{\int_I (A^*(s))^{d-2} b^*(s) 1_{\{s \leq r\}} \, ds}.
\]
We integrate both sides to get for \( r \in I \):
\[
\frac{d - 1}{d} \left( \log \left( \frac{h(r)}{h(1/2)} \right) - \int_{1/2}^{r} \frac{1}{h(s)} ds \right) = \log \left( \frac{\int_I (A^*(s))^{d-2} b_*(s) 1_{\{r \leq s \leq 1\}} ds}{\int_I (A^*(s))^{d-2} b_*(s) 1_{\{1/2 \leq s \leq 1\}} ds} \right).
\]
Taking the exponential yields:
\[
\alpha h^{(d-1)/d}(r) e^{-F(r)} = \int_I (A^*(s))^{d-2} b_*(s) 1_{\{r \leq s \leq 1\}} ds,
\]
for some positive constant \( \alpha \). From (25) and (26), we derive:
\[
A^*(r) = \frac{1}{\alpha (d - 1)} h^{1/d}(r) e^{F(r)}.
\]
This proves that the function \( A^* \) is uniquely determined up to a multiplicative constant and so is \( a^* \). With the help of (24) and (27), we can express \( b^* \) as, for \( r \in I \):
\[
b^*(r) = \frac{\delta'(r)(\alpha (d - 1))^{d-1}}{d} e^{-(d-1)F(r)}.
\]
The function \( b^* \) is also uniquely determined up to a multiplicative constant. Therefore (24) implies that there is a unique \( c^* \) of the form (19) which solves \( \mathcal{A}(c) = b^\delta \). (Notice however that the functions \( a^* \) and \( b^* \) are defined up to a multiplicative constant.) Then according to Proposition 2.1 we get that \( c_\delta \) defined by (19) with \( a \) and \( b \) defined by (4) solves \( \mathcal{A}(c) = b^\delta \), implying that \( c^* \) is equal to \( c_\delta \).

5.3. **Proof of Theorem 2.3.** Let \( \delta \in D_0 \) such that \( \Sigma_\delta = \{0, 1\} \). Thanks to Proposition 5.3, we deduce that if there exists an optimal solution to \( (P^\delta) \) then it is \( c_\delta \) given by (19). By construction, we have \( \mu \text{-a.e. } c_\delta > 0 \). According to Corollary 2.2, \( c_\delta \) is feasible for \( (P^\delta) \) if and only if \( \mathcal{J}(\delta) < +\infty \). Therefore if \( \mathcal{J}(\delta) < +\infty \), then \( c_\delta \) is the optimal solution. If \( \mathcal{J}(\delta) = +\infty \) then there is no optimal solution.

6. **Proof of Theorem 2.4**

We first state an elementary Lemma, whose proof if left to the reader. For \( f \) a function defined on \( I^d \) and \( 0 \leq s < t \leq 1 \), we define \( f^s,t \) by, for \( u \in I^d \):
\[
f^s,t(u) = (t - s) f(s 1 + u(t - s)).
\]

**Lemma 6.1.** If \( c \) is the density of a copula \( C \) such that \( \delta_C(s) = s \) and \( \delta_C(t) = t \) for some fixed \( 0 \leq s < t \leq 1 \), then \( c^{s,t} \) is also the density of a copula, and its diagonal section, \( \delta^{s,t} \), is given by, for \( r \in I \):
\[
\delta^{s,t}(r) = \frac{\delta_C(s + r(t - s)) - s}{t - s}.
\]

According to Remark 2.5, it is enough to consider the case \( \delta \in D_0 \), that is \( \Sigma_\delta \) with zero Lebesgue measure. We shall assume that \( \Sigma_\delta \neq \{0, 1\} \). Since \( \delta \) is continuous, we get that \( I \setminus \Sigma_\delta \) can be written as the union of non-empty open intervals \( ((\alpha_j, \beta_j), j \in J) \), with \( \alpha_j < \beta_j \) and \( J \) non-empty and at most countable. Set \( \Delta_j = \beta_j - \alpha_j \). Since \( \Sigma_\delta \) is of zero Lebesgue measure, we have \( \sum_{j \in J} \Delta_j = 1 \). We define also \( S = \bigcup_{j \in J} [\alpha_j, \beta_j]^d \)

For \( s \in \Sigma_\delta \), notice that any feasible function \( c \) of \( (P^\delta) \) satisfies for all \( 1 \leq i \leq d \):
\[
\int_{I^d} c(u) 1_{\{u_i < s\}} 1_{D_i}(u) du = \int_{I^d} c(u) 1_{\{u_i < s\}} du = \int_{I^d} c(u) 1_{\{\max(u_i) < s\}} du = s - \delta(s) = 0,
\]
where \( D_i = \{ u \in I^d \text{ such that } \forall j \neq i : u_j < s \} \). This implies that \( c = 0 \) a.e. on \( I^d \setminus S \). We set \( c^j = c^{\alpha_j, \beta_j} \) for \( j \in J \). We deduce that if \( c \) is feasible for \((P^\delta)\), then we have that a.e.:

\[
(29) \quad c(u) = \sum_{j \in J} \frac{1}{\Delta_j} c^j \left( \frac{u - \alpha_j}{\Delta_j} \right) 1_{(\alpha_j, \beta_j)}(u),
\]

and:

\[
(30) \quad \mathcal{I}_d(c) = \sum_{j \in J} \Delta_j \left( \mathcal{I}_d(c^j) - \log(\Delta_j) \right).
\]

Thanks to Lemma 6.1, the condition \( A(c) = b^\delta \) is equivalent to \( A(c^j) = b^{\delta_j} \) for all \( j \in J \). We deduce that the optimal solution of \((P^\delta)\), if it exists, is given by (29), where the functions \( c^j \) are the optimal solutions of \((P^{\delta_j})\) for \( j \in J \). Notice that by construction \( \Sigma_{\delta_j} = \{0, 1\} \). Thanks to Theorem 2.3, the optimal solution to \((P^{\delta_j})\) exists if and only if we have \( J(\delta_j) < +\infty \); and if it exists it is given by \( c_{\delta_j} \). Therefore, if there exists an optimal solution to \((P^\delta)\), then it is \( c_\delta \) given by (8). To conclude, we have to compute \( \mathcal{I}_d(c_\delta) \). Recall that \( x \log(x) \geq -1/e \) for \( x > 0 \). We have:

\[
\mathcal{I}_d(c_\delta) = \lim_{\varepsilon \downarrow 0} \sum_{j \in J} \Delta_j \left( \mathcal{I}_d(c^j) - \log(\Delta_j) \right) 1_{\{\Delta_j > \varepsilon\}}
= \lim_{\varepsilon \downarrow 0} \sum_{j \in J} \Delta_j \left( (d-1)J(\delta_j) - \log(\Delta_j) \right) 1_{\{\Delta_j > \varepsilon\}} + \sum_{j \in J} \Delta_j G(\delta_j)
= \sum_{j \in J} \Delta_j \left( (d-1)J(\delta_j) - \log(\Delta_j) \right) + \sum_{j \in J} \Delta_j G(\delta_j),
\]

where we used the monotone convergence theorem for the first equality, (6) for the second and the fact that \( G(\delta) \) is uniformly bounded over \( D_0 \) and the monotone convergence theorem for the last. Elementary computations yields:

\[
(d-1)J(\delta) = \sum_{j \in J} \Delta_j \left( (d-1)J(\delta_j) - \log(\Delta_j) \right) \quad \text{and} \quad G(\delta) = \sum_{j \in J} \Delta_j G(\delta_j).
\]

So, we get:

\[
\mathcal{I}_d(c_\delta) = (d-1)J(\delta) + G(\delta).
\]

Since \( G(\delta) \) is uniformly bounded over \( D_0 \), we get that \( \mathcal{I}_d(c_\delta) \) is finite if and only if \( J(\delta) \) is finite. To end the proof, recall the definition of \( \mathcal{I}(C_\delta) \) to conclude that \( \mathcal{I}(C_\delta) = (d-1)J(\delta) + G(\delta) \).

7. Examples for \( d = 2 \)

In this section we compute the density of the maximum entropy copula for various diagonal sections of popular bivariate copula families. In this Section, \( u \) and \( v \) will denote elements of \( I \). The density for \( d = 2 \) is of the form \( c_\delta(u, v) = a(\min(u, v))b(\max(u, v)) \). We illustrate these densities by displaying their isodensity lines or contour plots, and their diagonal cross-section \( \varphi \) defined as \( \varphi(t) = c(t, t), t \in I \).

7.1. Maximum entropy copula for a piecewise linear diagonal section. Let \( \alpha \in (0, 1/2] \). Let us calculate the density of the maximum entropy copula in the case of the following diagonal section:

\[
\delta(r) = (r - \alpha)1_{(\alpha, 1-\alpha)}(r) + (2r - 1)1_{[1-\alpha, 1]}(r).
\]
This example was considered for example in [17]. The limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond to the Fréchet-Hoeffding upper and lower bound copulas, respectively. However, for $\alpha = 0$, $\Sigma_\delta = I$, therefore every copula $C$ with this diagonal section gives $I(C) = +\infty$. (In fact the only copula that has this diagonal section is the Fréchet-Hoeffding upper bound $M$ defined by $M(u, v) = \min(u, v)$, $u, v \in I$.) When $\alpha \in (0, 1/2]$, $J(\delta) < +\infty$ is satisfied, therefore we can apply Theorem 2.3 to compute the density of the maximum entropy copula.

The graph of $\delta$ can be seen in Figure 1 for $\alpha = 0.2$.

\[ F(r) = \begin{cases} \frac{1}{2} \log \left( \frac{r}{\alpha} \right) - \frac{1}{4\alpha} + \frac{1}{2} & \text{if } r \in [0, \alpha) \\ \frac{1}{2} \log \left( \frac{\alpha}{r} \right) + \frac{1}{4\alpha} - \frac{1}{2} & \text{if } t \in [\alpha, 1 - \alpha) \\ \frac{1}{2} \log \left( \frac{\alpha}{t} \right) - \frac{1}{4\alpha} + \frac{1}{2} & \text{if } t \in [1 - \alpha, 1] \end{cases} \]

and:

\[ a(r) = \frac{1}{2\sqrt{\alpha}} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} 1_{[0,a]}(r)} + \frac{1}{2\sqrt{\alpha}} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} 1_{[a,1-a]}(r)} \]

and:

\[ b(r) = \frac{1}{2\sqrt{\alpha}} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} 1_{(a,1-a)}(r)} + \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{4\alpha} \frac{1}{2} 1_{[1-a,1]}(r)} \]

The density $c_\delta(u, v)$ consists of six distinct regions on $\triangle = \{(u, v) \in I^2, u \leq v\}$ as shown in Figure 2a and takes the values:

\[
c_\delta(u, v) = \begin{cases} 0 & \text{in I,} \\ \frac{1}{2\alpha} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} u} & \text{in II,} \\ \frac{1}{4\alpha} e^{-\frac{1}{4\alpha} \frac{1}{2\alpha} u} & \text{in III,} \\ \frac{1}{2\alpha} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} 1 - \alpha} & \text{in IV,} \\ \frac{1}{2\alpha} e^{-\frac{1}{2\alpha} \frac{1}{4\alpha} (1 - a)} & \text{in V,} \\ 0 & \text{in VI.} \end{cases}
\]

Figure 2b shows the isodensity lines of $c_\delta$. In the limiting case of $\alpha = \frac{1}{2}$, the diagonal section is given by $\delta(t) = \max(0, 2t - 1)$, which is the pointwise lower bound for all elements in $D$. Accordingly, it is the diagonal section of the Fréchet-Hoeffding lower bound copula $W$.
with generator function also to the family of Cuadras-Augé copulas. The Gumbel copula with parameter \( \alpha \) is an Archimedean copula defined as, for \( u, v \),

\[
W(u, v) = \frac{1}{\alpha} \log \left( \frac{1}{u} + \frac{1}{v} - 1 \right)
\]

with diagonal section given by

\[
\delta_{\alpha}(u, v) = \frac{1}{\alpha} \left( \frac{1}{u} + \frac{1}{v} - 1 \right)
\]

where \( C_1 \) and \( C_2 \) are copula functions. Recall that the independent copula \( \Pi \) with uniform density \( c_{\Pi} = 1 \) on \( I^2 \) minimizes \( J(C) \) over \( C \). According to (31), the maximum entropy copula with diagonal section \( \delta \) is \( D_{\Pi, \delta} \). This corresponds to choosing the maximum entropy copulas on \([0, 1/2] \times [1/2, 1] \) and \([1/2, 1] \times [0, 1/2] \).

7.2. Maximum entropy copula for \( \delta(t) = t^\alpha \). Let \( \alpha \in (1, 2] \). We consider the family of diagonal sections given by \( \delta(t) = t^\alpha \). This corresponds to the Gumbel family of copulas and also to the family of Cuadras-Augé copulas. The Gumbel copula with parameter \( \theta \in [1, \infty) \) is an Archimedean copula defined as, for \( u, v \in I \):

\[
C^G(u, v) = \varphi_{-1}(\varphi_\theta(u) + \varphi_\theta(v))
\]

with generator function \( \varphi_\theta(t) = (-\log(t))^\theta \). Its diagonal section is given by \( \delta^G(t) = t^{2^\frac{1}{\theta}} = t^\alpha \) with \( \alpha = 2^\frac{1}{\theta} \). The Cuadras-Augé copula with parameter \( \gamma \in (0, 1) \) is defined as, for \( u, v \in I \):

\[
C^{CA}(u, v) = \min(uv^{1-\gamma}, u^{1-\gamma}v)
\]

It is a subclass of the two parameter Marshall-Olkin family of copulas given by \( C^M(u, v) = \min(u^{1-\gamma_1}v, uv^{1-\gamma_2}) \). The diagonal section of \( C^{CA} \) is given by \( \delta(t) = t^{2-\gamma} = t^\alpha \) with \( \alpha = 2 - \gamma \). While the Gumbel copula is absolutely continuous, the Cuadras-Augé copula is not, although it has full support. Since \( J(\delta) < +\infty \), we can apply Theorem 2.3. To give the density of the maximum entropy copula, we have to calculate \( F(v) - F(u) \). Elementary computations
These copulas are absolutely continuous with densities $c$ diagonal section short) are defined as:

$$F(v) - F(u) = \frac{1}{2} \int_u^v \frac{ds}{s - s^\alpha} = \frac{1}{2} \log \left( \frac{v}{u} \right) - \frac{1}{2\alpha - 2} \log \left( \frac{1 - v^{\alpha-1}}{1 - u^{\alpha-1}} \right).$$

The density $c_\delta$ is therefore given by, for $(u, v) \in \Delta$:

$$c_\delta(u, v) = \frac{\alpha}{4} \frac{2 - \alpha u^\alpha-1}{(1 - u^\alpha-1)^\alpha/(2\alpha-2)} v^{\alpha-2} (1 - v^{\alpha-1})^{(2-\alpha)/(2\alpha-2)}.$$

Figure 3 represents the isodensity lines of the Gumbel and the maximum entropy copula $c_\delta$ with common parameter $\alpha = 2\frac{1}{4}$, which corresponds to $\theta = 3$ for the Gumbel copula. We have also added a graph of the diagonal cross-section of the two densities. In the limiting case of $\alpha = 2$, the above formula gives $c_\delta(u, v) = 1$, which is the density of the independent copula $\Pi$, which is also maximizes the entropy on the entire set of copulas.

![Figure 3](image)

**Figure 3.** Isodensity lines and the diagonal cross-section of copulas with diagonal section $\delta(t) = t^\alpha$, $\alpha = 2\frac{1}{4}$.

### 7.3. Maximum entropy copula for the Farlie-Gumbel-Morgenstern diagonal section.

Let $\theta \in [-1, 1]$. The Farlie-Gumbel-Morgenstern family of copulas (FGM copulas for short) are defined as:

$$C(u, v) = uv + \theta uv(1 - u)(1 - v).$$

These copulas are absolutely continuous with densities $c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$. Its diagonal section $\delta_\theta$ is given by:

$$\delta(t) = t^2 + \theta t^2 (1 - t)^2 = \theta t^4 - 2\theta t^3 + (1 + \theta)t^2.$$

Since $\delta_\theta(t) < t$ on $(0, 1)$ and it verifies $\mathcal{J}(\delta) < +\infty$, we can apply Theorem 2.3 to calculate the density of the maximum entropy copula. For $F(r)$, we have:

$$F(r) = \begin{cases} 
\frac{1}{2} \log \left( \frac{r}{1 - r} \right) + \frac{\theta}{\sqrt{4\theta - \theta^2}} \arctan \left( \frac{2\theta r - \theta}{\sqrt{4\theta - \theta^2}} \right) & \text{if } \theta \in (0, 1], \\
\frac{1}{2} \log \left( \frac{r}{1 - r} \right) & \text{if } \theta = 0, \\
\frac{1}{2} \log \left( \frac{r}{1 - r} \right) - \frac{\theta}{\sqrt{\theta^2 - 4\theta}} \arctanh \left( \frac{2\theta r - \theta}{\sqrt{\theta^2 - 4\theta}} \right) & \text{if } \theta \in [-1, 0). 
\end{cases}$$

The density $c_\delta$ is given by, for $\theta \in (0, 1]$ and $(u, v) \in \Delta$:
\[
c_\delta(u, v) = \frac{(1 - 2\theta u^3 + 3\theta u^2 + (1 + \theta)u) (2\theta v^2 + 3\theta v + (1 + \theta))}{(1 - u)\sqrt{\theta u^2 - \theta u + 1} \sqrt{\theta v^2 - \theta v + 1}} \exp\left(-\frac{\theta}{\sqrt{4\theta - \theta^2}} \left(\arctan\left(\frac{2\theta v - \theta}{\sqrt{4\theta - \theta^2}}\right) - \arctan\left(\frac{2\theta u - \theta}{\sqrt{4\theta - \theta^2}}\right)\right)\right)
\]

Figure 4 illustrates the isodensities of the FGM copula and the maximum entropy copula with the same diagonal section for \( \theta = 0.5 \) as well as the diagonal cross-section of their densities.

7.4. Maximum entropy copula for the Gaussian diagonal section. The Gaussian (normal) copula takes the form:

\[
C_\rho(u, v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right),
\]

with \( \Phi_\rho \) the joint cumulative distribution function of a two-dimensional normal random variable with standard normal marginals and correlation parameter \( \rho \in [-1, 1] \), and \( \Phi^{-1} \) the quantile function of the standard normal distribution. The density \( c_\rho \) of \( C_\rho \) can be written as:

\[
c_\rho(u, v) = \varphi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v)),
\]

where \( \varphi \) and \( \varphi_\rho \) stand for respectively the densities of a standard normal distribution and a two-dimensional normal distribution with correlation parameter \( \rho \), respectively. The diagonal section and its derivative are given by:

\[
(32) \quad \delta_\rho(t) = \Phi_\rho\left(\Phi^{-1}(t), \Phi^{-1}(t)\right), \quad \delta'_\rho(t) = 2\Phi\left(\sqrt{\frac{1 - \rho}{1 + \rho}}\Phi^{-1}(t)\right).
\]

Since \( \delta_\rho \) verifies \( \delta_\rho(t) < t \) on \((0, 1)\) and \( J(\delta_\rho) < +\infty \), we can apply Theorem 2.3 to calculate the density of the maximum entropy copula. We have calculated numerically the density of the maximum entropy copula with diagonal section \( \delta_\rho \) for \( \rho = 0.95, 0.5, -0.5 \) and \(-0.95 \). The comparison between these densities and the densities of the corresponding normal copula
can be seen in Figures 5, 6 and 7. We observe a very different behaviour of $c_\rho$ and $c_\delta$ in the case of $\rho < 0$. In the limiting case when $\rho$ goes down to $-1$, we retrieve the diagonal $\delta(t) = \max(0, 2t - 1)$, which we have studied earlier in Section 7.1.

![Figure 5](image)

**Figure 5.** Isodensity lines and the diagonal cross-section of copulas with diagonal section given by (32), with $\rho = 0.5$ and $\rho = 0.95$.

## 8. Appendix - Calculation of the entropy of $C_\delta$

Let us first introduce some notations. Let $\varepsilon \in (0, 1/2)$. Since $x \log(x) \geq -1/\epsilon$ for $x > 0$, we deduce by the monotone convergence theorem that:

(33) \[ I(C_\delta) = \lim_{\varepsilon \downarrow 0} I_\varepsilon(C_\delta), \]

with:

\[ I_\varepsilon(C_\delta) = \int_{[\varepsilon, 1-\varepsilon]^d} c_\delta(x) \log(c_\delta(x)) \, dx. \]

Using $\delta \leq t$ and that $\delta$ is a non-decreasing, $d$-Lipschitz function, we get that for $t \in I$:

(34) \[ 0 \leq h(t) \leq \min(t, (d-1)(1-t)) \leq (d-1) \min(t, 1-t). \]

We set:

(35) \[ w(t) = a(t) e^{-F(t)} = \frac{d - \delta'(r)}{d} h^{-1 + 1/d}(r). \]
Figure 6. Isodensity lines and the diagonal cross-section of copulas with diagonal section given by (32), with \( \rho = -0.5 \) and \( \rho = -0.95 \)

From the symmetric property of \( c_\delta \), we have that

\[
\mathcal{I}_\varepsilon(C_\delta) = J_1(\varepsilon) + J_2(\varepsilon) - J_3(\varepsilon),
\]

with:

\[
J_1(\varepsilon) = d \int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x) 1_{\{\max(x) = x_d\}} \left( \sum_{i=1}^{d-1} \log(w(x_i)) \right) dx,
\]

\[
J_2(\varepsilon) = d \int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x) 1_{\{\max(x) = x_d\}} \log \left( \frac{\delta'(x_d)}{d} h^{-1+1/d}(x_d) \right) dx,
\]

\[
J_3(\varepsilon) = d \int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x) 1_{\{\max(x) = x_d\}} \left( (d - 1) F(x_d) - \sum_{i=1}^{d-1} F(x_i) \right) dx.
\]
We introduce $A_ε(r) = \int_r^\infty a(x) \, dx$. For $J_1(ε)$, we have:

$$J_1(ε) = d(d-1) \int_{[ε,1-ε]} 1\{\text{max}(x) = x_d\} b(x_d) \prod_{j=1}^{d-1} a(x_j) \log(w(x_1)) \, dx$$

$$= d(d-1) \int_{[ε,1-ε]} \left( \int_{[t,1-ε]} A^{d-2}_ε(s)b(s) \, ds \right) a(t) \log(w(t)) \, dt.$$

Notice that using (10) and (12), we have:

$$\int_{[t,1-ε]} A^{d-2}_ε(s)b(s) \, ds = \int_{[t,1]} A^{d-2}(s)b(s) \, ds - \int_{[t,1]} \left( A^{d-2}(s) - A^{d-2}_ε(s) \right) b(s) \, ds$$

$$= \frac{h(t)}{(d-1)A(t)} - \int_{[1-ε,1]} \left( A^{d-2}(s) - A^{d-2}_ε(s) \right) b(s) \, ds$$

By Fubini's theorem, we get:

$$J_1(ε) = J_{1,1}(ε) - J_{1,2}(ε) - J_{1,3}(ε),$$
with:

\[ J_{1,1}(\varepsilon) = \int_{[\varepsilon,1-\varepsilon]} \left( d - \delta'(t) \right) \log (w(t)) \, dt \]

\[ J_{1,2}(\varepsilon) = d(d-1) \left( \int_{[1-\varepsilon,1]} A_{\varepsilon}^{d-2}(s)b(s) \, ds \right) \int_{[\varepsilon,1-\varepsilon]} a(t) \log (w(t)) \, dt \]

\[ J_{1,3}(\varepsilon) = d(d-1) \int_{[\varepsilon,1-\varepsilon]} \left( \int_{t}^{1} \left( A^{d-2}(s) - A_{\varepsilon}^{d-2}(s) \right) b(s) \, ds \right) a(t) \log (w(t)) \, dt. \]

To study \( J_{1,2} \), we first give an upper bound for the term \( \int_{[1-\varepsilon,1]} A_{\varepsilon}^{d-2}(s)a(s)b(s) \, ds \):

\[
\int_{[1-\varepsilon,1]} A_{\varepsilon}^{d-2}(s)b(s) \, ds \leq \int_{[1,1-\varepsilon]} A_{\varepsilon}^{d-2}(s)b(s) \, ds \leq \frac{1}{(d-1)} h^{1-1/d}(1-\varepsilon) e^{-F(1-\varepsilon)} \leq (d-1)^{-1/d} e^{1-1/d},
\]

where we used that \( A_\varepsilon(s) \leq A(s) \) for \( s > \varepsilon \) for the first inequality, (12) for the first equality, and (34) for the last inequality. Since \( t \log(t) \geq -1/e \), we have, using (35):

\[
J_{1,2}(\varepsilon) \geq -\frac{d(d-1)}{e} \left( \int_{[1-\varepsilon,1]} A_{\varepsilon}^{d-2}(s)b(s) \, ds \right) \int_{[\varepsilon,1-\varepsilon]} e^{F(t)} \, dt \\
\geq -\frac{d}{e} h^{1-1/d}(1-\varepsilon) \int_{[\varepsilon,1-\varepsilon]} e^{F(t)-F(1-\varepsilon)} \, dt \\
\geq -\frac{d}{e} ((d-1)\varepsilon)^{1-1/d},
\]

where we used (12) for the second inequality, and that \( F \) is non-decreasing and (37) for the third inequality. On the other hand, we have \( t \log(t) \leq t^{\frac{1}{1-1/d}} \), if \( t \geq 0 \), which gives:

\[
J_{1,2}(\varepsilon) \leq d(d-1) \left( \int_{[1-\varepsilon,1]} A_{\varepsilon}^{d-2}(s)b(s) \, ds \right) \int_{[\varepsilon,1-\varepsilon]} e^{F(t)} \left( \frac{d-\delta'(t)}{d} \right)^{\frac{1}{1-1/d}} h(t) \, dt \\
= dh(1-\varepsilon)^{1-1/d} \int_{[\varepsilon,1-\varepsilon]} \frac{e^{F(t)-F(1-\varepsilon)}}{h(t)} \, dt \\
= dh(1-\varepsilon)^{1-1/d} \left( 1 - e^{F(\varepsilon)-F(1-\varepsilon)} \right) \\
\leq d((d-1)\varepsilon)^{1-1/d},
\]

where we used (37) and \( t^{\frac{1}{1-1/d}} \leq 1 \) for \( t \in I \) for the first inequality, and that \( F \) is non-decreasing for the last. This proves that \( \lim_{\varepsilon \to 0} J_{1,2}(\varepsilon) = 0 \). For \( J_{1,3}(\varepsilon) \), we first observe that for \( s \in [\varepsilon,1-\varepsilon] \) we have \( A_\varepsilon(s) \leq A(s) \) and thus:

\[
(\text{A}^{d-2}(s) - A_{\varepsilon}^{d-2}(s)) = A(\varepsilon) \sum_{i=0}^{d-3} A^i(s)A_{\varepsilon}^{d-3-i}(s) \leq (d-2)A(\varepsilon)A^{d-3}(s).
\]
Using the previous inequality we obtain:

\[
J_{1,3}(\varepsilon) = d(d - 1) \int_{[\varepsilon, 1 - \varepsilon]} \left( \int_t^1 (A^{d-2}(s) - A^{d-2}_\varepsilon(s)) b(s) ds \right) a(t) \log(w(t)) \, dt \\
\geq -\frac{d(d - 1)}{e} \int_{[\varepsilon, 1 - \varepsilon]} \left( \int_t^1 (A^{d-2}(s) - A^{d-2}_\varepsilon(s)) b(s) ds \right) e^{F(t)} \, dt \\
\geq -\frac{d(d - 1)(d - 2)A(\varepsilon)}{e} \int_{[\varepsilon, 1 - \varepsilon]} \left( \int_t^1 A^{d-3}(s)b(s) ds \right) e^{F(t)} \, dt \\
\geq -\frac{d(d - 1)(d - 2)A(\varepsilon)}{e} \int_{[\varepsilon, 1 - \varepsilon]} \frac{h(t)}{A^2(t)} e^{F(t)} \, dt \\
= -\frac{d(d - 2)A(\varepsilon)}{e} \int_{[\varepsilon, 1 - \varepsilon]} h(t)^{1/2} dt \\
\geq -\frac{d(d - 2)(d - 1)^{1/2} \varepsilon^{1/2}}{e}
\]

where we used \( t \log(t) \geq -1/e \) for the first inequality, (38) for the second, (10) and (12) in the following equality, and (34) to conclude. For an upper bound, we have after noticing that \( t \log(t) \leq t^2 \):

\[
J_{1,3}(\varepsilon) = d(d - 1) \int_{[\varepsilon, 1 - \varepsilon]} \left( \int_t^1 (A^{d-2}(s) - A^{d-2}_\varepsilon(s)) b(s) ds \right) a(t) \log(w(t)) \, dt \\
\leq d(d - 1) \int_{[\varepsilon, 1 - \varepsilon]} \left( \int_t^1 (A^{d-2}(s) - A^{d-2}_\varepsilon(s)) b(s) ds \right) e^{F(t)} w^2(t) \, dt \\
\leq d(d - 1)(d - 2)A(\varepsilon) \int_{[\varepsilon, 1 - \varepsilon]} \frac{\int_t^1 A^{d-2}(s)b(s) ds}{A(t)} e^{F(t)} h^{-2+2/d}(t) \, dt \\
= d(d - 2)A(\varepsilon) \int_{[\varepsilon, 1 - \varepsilon]} \frac{e^{-F(t)}}{h(t)} \, dt \\
= d(d - 2)h^{1/d}(\varepsilon)(1 - e^{F(1-\varepsilon)}) \\
\leq d(d - 2)(d - 1)^{1/d} \varepsilon^{1/d},
\]

where we used (38) and \( 0 \leq (d - \delta(t))/d \leq 1 \) for the second inequality; (10) and (12) in the second equality; and (34) to conclude. The results on the two bounds show that
\[ \lim_{\varepsilon \to 0} J_{1,3}(\varepsilon) = 0. \] Similarly, for \( J_2(\varepsilon) \), we get:

\[
J_2(\varepsilon) = \int_{[\varepsilon, 1-\varepsilon]^d} 1\{\max(x) = x_d\} b(x_d) \prod_{j=1}^{d-1} a(x_j) \log \left( \frac{\delta'(x_d)}{d} h^{-1+1/d}(x_d) \right) dx
\]

\[
= d \int_{[\varepsilon, 1-\varepsilon]} A_{\varepsilon}(t) b(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt
\]

\[
= d \int_{[\varepsilon, 1-\varepsilon]} A_{\varepsilon}(t) b(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt
\]

\[
- d \int_{[\varepsilon, 1-\varepsilon]} \left( A_{\varepsilon}(t) - A_{\varepsilon}^{d-1}(t) \right) b(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt
\]

\[
= J_{2,1}(\varepsilon) - J_{2,2}(\varepsilon)
\]

with \( J_{2,1}(\varepsilon) \) and \( J_{2,2}(\varepsilon) \) given by, using (11):

\[
J_{2,1}(\varepsilon) = d \int_{[\varepsilon, 1-\varepsilon]} A_{\varepsilon}^{d-1}(t) b(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt
\]

\[
J_{2,2}(\varepsilon) = d \int_{[\varepsilon, 1-\varepsilon]} \left( A_{\varepsilon}^{d-1}(t) - A_{\varepsilon}^{d-1}(t) \right) b(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt.
\]

By (11), we have:

\[
J_{2,1}(\varepsilon) = \int_{[\varepsilon, 1-\varepsilon]} \delta'(t) \log \left( \frac{\delta'(t)}{d} h^{-1+1/d}(t) \right) dt.
\]

Similarly to \( J_{1,3}(\varepsilon) \) we can show that \( \lim_{\varepsilon \to 0} J_{2,2}(\varepsilon) = 0 \).

Adding up \( J_1(\varepsilon) \) and \( J_2(\varepsilon) \) gives

\[
J_1(\varepsilon) + J_2(\varepsilon) = J_\varepsilon(\delta) + J_4(\varepsilon) - d \log(d)(1 - 2\varepsilon) - J_{1,2}(\varepsilon) - J_{1,3}(\varepsilon) - J_{2,2}(\varepsilon)
\]

with

\[
J_\varepsilon(\delta) = (d - 1) \int_{\varepsilon}^{1-\varepsilon} |\log(h(t))| dt,
\]

\[
J_4(\varepsilon) = \int_{\varepsilon}^{1-\varepsilon} (d - \delta'(t)) \log \left( d - \delta'(t) \right) dt + \int_{\varepsilon}^{1-\varepsilon} \delta'(t) \log \left( \delta'(t) \right) dt.
\]

Notice that \( J_\varepsilon(\delta) \) is non-decreasing in \( \varepsilon > 0 \) and that:

\[
J(\delta) = \lim_{\varepsilon \to 0} J_\varepsilon(\delta).
\]

Since \( \delta'(t) \in [0, d] \), we deduce that \( (d - \delta') \log(d - \delta') \) and \( \delta' \log(\delta') \) are bounded on \( I \) from above by \( d \log(d) \) and from below by \( -1/e \) and therefore integrable on \( I \). This implies:

\[
\lim_{\varepsilon \to 0} J_3(\varepsilon) = I_1(\delta') + I_1(d - \delta').
\]

As for \( J_3(\varepsilon) \), we have by integration by parts:
\begin{align*}
J_3(\varepsilon) &= d \int_{[\varepsilon, 1-\varepsilon]^d} 1_{\{\max(x) = x_d\}} b(x_d) \prod_{i=1}^{d-1} a(x_i) \left( (d-1) F(x_d) - \sum_{i=1}^{d-1} F(x_i) \right) dx \\
&= d(d-1) \int_{[\varepsilon, 1-\varepsilon]} A_\varepsilon^{d-1}(t) b(t) F(t) dt \\
& \quad - d(d-1) \int_{[\varepsilon, 1-\varepsilon]} A_\varepsilon^{d-2}(t) b(t) \left( \int_{\varepsilon}^{t} a(s) F(s) \right) dt \\
&= d(d-1) \int_{[\varepsilon, 1-\varepsilon]} A_\varepsilon^{d-1}(t) b(t) F(t) dt \\
& \quad - d(d-1) \int_{[\varepsilon, 1-\varepsilon]} A_\varepsilon^{d-2}(t) b(t) \left( A_\varepsilon(t) F(t) - \frac{d-1}{d} \int_{\varepsilon}^{t} \frac{A_\varepsilon(s)}{h(s)} ds \right) dt \\
&= (d-1)^2 \int_{[\varepsilon, 1-\varepsilon]} \left( \int_{t}^{1-\varepsilon} A_\varepsilon^{d-2}(s) b(s) \right) \frac{A(t)}{t-\delta(t)} dt.
\end{align*}

By the monotone convergence theorem, (10) and (12) we have:

\[
\lim_{\varepsilon \to 0} J_3(\varepsilon) = (d-1)^2 \int_{I} \left( \int_{t}^{1} A^{d-2}(s) b(s) \right) \frac{A(t)}{t-\delta(t)} dt = d - 1.
\]

Summing up all the terms and taking the limit \(\varepsilon = 0\) give:

\[
\mathcal{I}(C_\delta) = (d-1) \int_{I} |\log(t - \delta(t))| \, dt + \mathcal{I}_1(\delta') + \mathcal{I}_1(d - \delta') - d \log(d) - (d-1) = (d-1) \mathcal{J}(\delta) + \mathcal{G}(\delta).
\]

References


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