

COUPLING SOME CONDITIONED LÉVY TREES WITH THE KESTEN TREE

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ABSTRACT. We consider locally compact Lévy trees conditioned to be large, with respect to different criterion: its height, its maximal “size” vertex and its total “mass”. In the critical case, we provide a coupling with a truncated Kesten tree which then allows to directly prove the local convergence in distribution of the conditioned Lévy tree to be large towards the Kesten tree. We also consider the sub-critical and super-critical cases. In the former case the results can be partial, due to a possible condensation phenomenon which is outside the mathematical framework used in this paper.

1. INTRODUCTION

Local limits (in distribution) of Bienaymé-Galton-Watson (BGW) trees conditioned to be large have been extensively studied in the recent years, and various conditioning have been considered: on the height [27], on the total population size [13], on the total number of leaves [28], see also [7] and references therein for other examples. When the BGW tree is critical (the mean value of the offspring distribution is 1), all these conditionings converge locally to the same limit: the size-biased tree also called Kesten tree. (Of course, not all conditioning of critical BGW tree converges locally to the Kesten tree, see [1].)

Scaling limits of BGW trees are given by random real trees called (continuum) Lévy trees, see [19], and here denoted by \mathcal{T} ; and analogous results also hold for conditioned critical Lévy trees [30, 18] and the limiting tree is the continuum analogue of the discrete Kesten tree. For convenience, we will still call this random real tree a Kesten tree in this paper and denote it by \mathcal{K} ; it is composed of one semi-infinite branch called the spine on which are grafted random real trees (whose distribution is closely related to the initial Lévy tree, see Section 2.8). This tree appears first in the Brownian tree context in [12] and is called the self-similar continuum random tree, and is constructed for a general branching mechanism ψ using exploration processes in [18]. Let us mention that the Lévy tree are given under the so-called excursion measure \mathbb{N}^ψ , and that the Kesten tree is a random variable defined under the probability measure \mathbb{P}^ψ , see Sections 2.6-2.8.

The goal of this paper is to give a coupling between the Lévy trees conditioned to be large and a truncation/transformation of the Kesten tree, so that we can read the convergence in distribution of the critical Lévy trees conditioned to be large towards the Kesten tree as an a.s. (local) convergence of the truncated Kesten tree to the (untruncated) Kesten tree. We shall consider three kind of conditioning for the Lévy tree \mathcal{T} to be large, that is, to have:

- (i) its height $\mathbf{H}(\mathcal{T})$ equal to h ,
- (ii) a maximal vertex with “size” $\Delta(\mathcal{T})$ equal to δ ,
- (iii) its total “mass” $\sigma = \sigma(\mathcal{T})$ equal to r ,

and then let this parameter goes to infinity.

A Lévy tree (resp. Kesten tree) represents the genealogy of a continuous state branching (CB) process (resp. CB with immigration process), so that the previous coupling could also have been stated for conditioned CB process and truncated CB process with immigration; however the picture is much more intuitive when using the genealogical structure given by the tree. To state precisely the results, we consider Lévy trees as random variables taking values in the Polish metric space $(\mathbb{T}, d_{\text{LGH}})$ of (Gromov-Hausdorff isometric classes of) rooted complete locally compact real trees. To do so requires some usual hypothesis

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on the branching mechanism ψ associated with the Lévy tree or the corresponding CB process such as conservativeness (that is, non explosion in finite time) and Grey condition (that is, local compactness of the Lévy trees); we also assume the CB process has infinite variation and thus the total length of the Lévy tree is infinite (see Section 3 in [32] when this last hypothesis is not fulfilled).

Some of the results for the critical case ($\psi'(0) = 0$) have extension to the sub-critical ($\psi'(0) > 0$) or super-critical ($\psi'(0) < 0$) cases. Let us mention that a super-critical Lévy tree \mathcal{T} conditioned to be finite is distributed as a sub-critical Lévy tree, see the Girsanov transformation (28) with ψ the branching mechanism associated to \mathcal{T} and θ replaced by θ_0 the largest root of $\psi = 0$. Let us mention that there are other possible limits than the Kesten tree for local limits (in distribution) of Lévy trees conditioned to be large, see [8].

1.1. Lévy trees conditioned to have large height. Let us first consider the tree conditioned on having height h . To construct this tree from the Kesten tree \mathcal{K} , we cut the spine of \mathcal{K} at level h (and forget everything that is above this cut-point), and also remove all the sub-trees grafted on the remaining spine that reach level h . This gives a tree $\mathcal{K}_h^{\text{height}}$, whose construction is illustrated in Fig. 1. We denote by h the vertex on the spine at distance h from the root ρ , and see it as a distinguished vertex of $\mathcal{K}_h^{\text{height}}$.

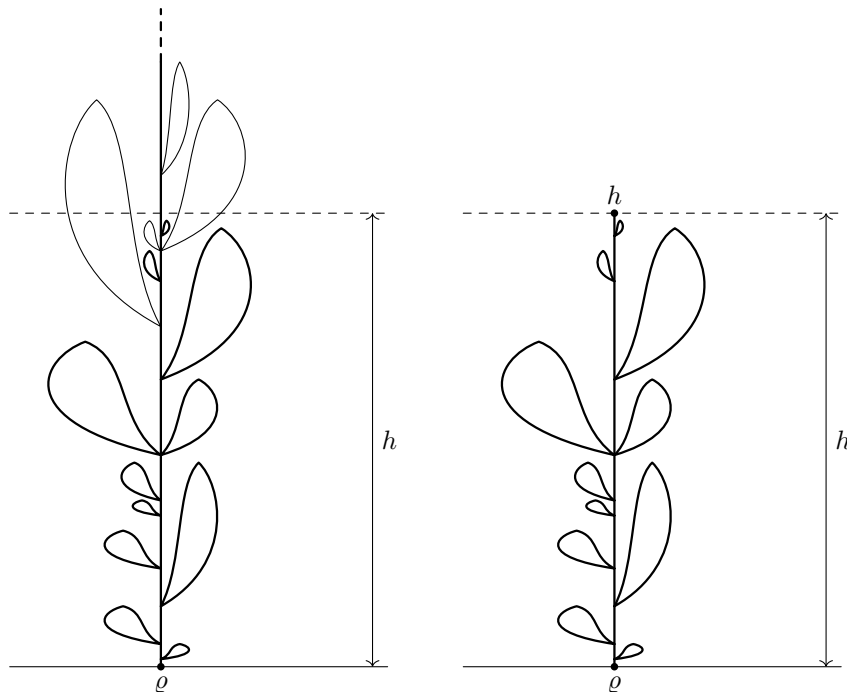


FIGURE 1. Left: the Kesten tree (a spine given by an semi-infinite branch on which are grafted Lévy sub-trees); Right: the truncated Kesten tree $\mathcal{K}_h^{\text{height}}$ with the distinguished vertex h on the spine being the only vertex at distance h from the root ρ .

We summarize Theorem 3.1 on the coupling, and the local convergence of the truncated Kesten trees from Proposition 3.3 and the direct consequence of the local convergence in distribution of the conditioned Lévy tree to have height h , say $\mathcal{T}_h^{\text{height}}$, towards the Kesten tree as h goes to infinity, see Theorem 3.4.

Theorem (Conditioning w.r.t. the height). *We assume the infinite variation setting and the Grey condition. The (sub-)critical Lévy tree \mathcal{T} conditioned under the excursion measure to have height h , say $\mathcal{T}_h^{\text{height}}$, with its only vertex X_h at distance h from the root ρ , is distributed as $(\mathcal{K}_h^{\text{height}}, h)$:*

$$(\mathcal{T}_h^{\text{height}}, X_h) \stackrel{(d)}{=} (\mathcal{K}_h^{\text{height}}, h).$$

Furthermore the “increasing” sequence $(\mathcal{K}_h^{\text{height}})_{h \geq 0}$ converges to the Kesten tree \mathcal{K} in $(\mathbb{T}, d_{\text{LGH}})$. In particular, we have the following local convergence in distribution:

$$(1) \quad \mathcal{T}_h^{\text{height}} \xrightarrow[h \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in } (\mathbb{T}, d_{\text{LGH}}).$$

The convergence (1) was already proven in [2], and the coupling was implicit. We refer to Remark 3.5 for the super-critical case.

1.2. Lévy trees conditioned to have large maximal vertex “size”. According to [19], the Lévy tree \mathcal{T} can be constructed using a coding by the so-called height process which is a functional of a spectrally positive Lévy process \mathbf{V} with Laplace exponent ψ . Then, each jump of the process \mathbf{V} corresponds to the “size” of a vertex of infinite degree in \mathcal{T} . We refer to (25) in Section 2.5 (vi) for a more intrinsic way to define the “size” of a vertex as the limit, when ε goes to 0, of the number of sub-trees attached to this point with height larger than ε with a correct renormalization.

The local limit in distribution of the Lévy tree conditioned of having maximal vertex “size” δ , say $\mathcal{T}_\delta^{\text{node}}$, is obtained in [11] in the critical and sub-critical cases under further assumption on the Lévy measure π of the branching mechanism. To simplify, in the introduction, we shall consider the critical case and assume that π has no atom and support $(0, +\infty)$ in $(0, +\infty)$. Then the tree $\mathcal{T}_\delta^{\text{node}}$ has exactly one node of “size” δ , say X_δ , and we denote by $\mathcal{T}_\delta^{\text{node},*}$ the tree $\mathcal{T}_\delta^{\text{node}}$ when one remove the sub-tree above X_δ . We now provide a representation of $\mathcal{T}_\delta^{\text{node},*}$ as a truncation of the Kesten tree. Let H_δ be the vertex on the spine of the Kesten tree \mathcal{K} being the lowest branching point of the spine on which is grafted a tree with a vertex of “size” $\geq \delta$ (which is possibly its root which belongs to the spine). Then we cut the spine of \mathcal{K} at H_δ , and remove the sub-tree grafted at H_δ to get the tree $\mathcal{K}_\delta^{\text{node},*}$. Then, we built the tree $\mathcal{K}_\delta^{\text{node}}$ by grafting at H_δ in $\mathcal{K}_\delta^{\text{node},*}$ a sub-tree whose root is a branching point with “size” δ and which is distributed as the Lévy tree conditioned to have no other vertices of “size” $\geq \delta$. See Fig. 2 for a representation of \mathcal{K} , $\mathcal{K}_\delta^{\text{node}}$ and $\mathcal{K}_\delta^{\text{node},*}$.

We summarize Theorem 4.1 on the coupling, and the local convergence of the truncated Kesten trees from Proposition 4.3 and the direct consequence of the local convergence in distribution of the conditioned Lévy tree to have maximal vertex “size” equal to δ , say $\mathcal{T}_\delta^{\text{node}}$, towards the Kesten tree as δ goes to infinity, see Theorem 4.4.

Theorem (Conditioning w.r.t. the maximal vertex “size”). *We assume the infinite variation setting and the Grey condition, as well as the Lévy measure π has no atom and full support in $(0, +\infty)$. The (sub-)critical Lévy tree \mathcal{T} conditioned under the excursion measure to have maximal vertex “size” δ , say $\mathcal{T}_\delta^{\text{node}}$, with its only vertex X_δ of “size” δ is distributed as $(\mathcal{K}_\delta^{\text{node}}, H_\delta)$:*

$$(2) \quad (\mathcal{T}_\delta^{\text{node}}, X_\delta) \stackrel{(d)}{=} (\mathcal{K}_\delta^{\text{node}}, H_\delta) \quad \text{and} \quad \mathcal{T}_\delta^{\text{node},*} \stackrel{(d)}{=} \mathcal{K}_\delta^{\text{node},*}.$$

Furthermore in the critical case, the “increasing” sequence $(\mathcal{K}_\delta^{\text{node},*})_{\delta \geq 0}$ converges to the Kesten tree \mathcal{K} in $(\mathbb{T}, d_{\text{LGH}})$, and we have the following local convergence in distribution:

$$(3) \quad \mathcal{T}_\delta^{\text{node}} \xrightarrow[\delta \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in } (\mathbb{T}, d_{\text{LGH}}).$$

The convergence (3) was already proven in [11] and the sub-critical case is also considered therein. The coupling (2) is also valid in the sub-critical case with a slightly different definition of the cutting vertex H_δ , see Theorem 4.1; and in this case the local limit, when δ goes to infinity, of $\mathcal{T}_\delta^{\text{node},*}$ is the truncated Kesten tree $\widehat{\mathcal{K}}_E$ whose spine is cut at vertex E at distance from the root ϱ given by an independent exponential random variable with mean $1/\psi'(0)$. Intuitively, the limit, when δ goes to infinity, of $\mathcal{T}_\delta^{\text{node}}$ in this case, would consist of the compact tree $\widehat{\mathcal{K}}_E$ on which is grafted at E a (non locally compact) Lévy sub-tree whose root is a branching point with infinite “size” (which implies the non local compactness). This is the so-called condensation phenomenon. The super-critical case can be deduce from the sub-critical case by the Girsanov transformation (28) (with θ the positive root of $\psi = 0$), as on the non-extinction event the

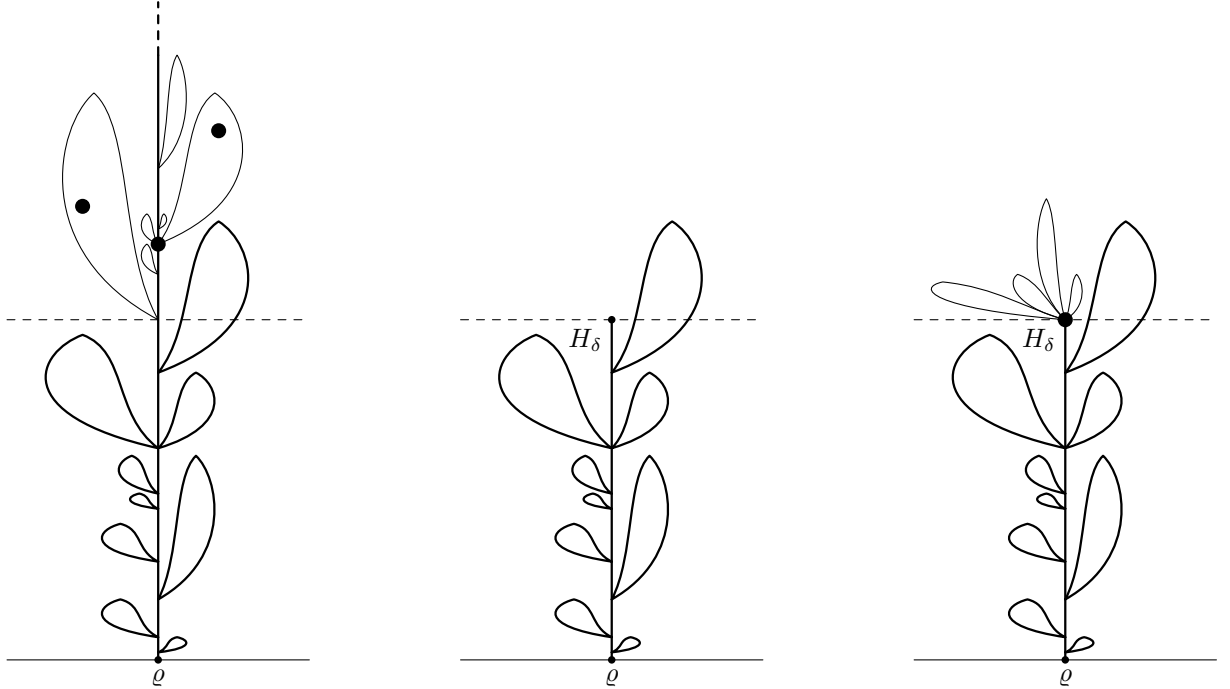


FIGURE 2. Left: the Kesten tree with vertices of “size” $\geq \delta$; Center: the truncated Kesten tree $\mathcal{K}_\delta^{\text{node},*}$ with the distinguished vertex H_δ on the spine; Right: the tree $\mathcal{K}_\delta^{\text{node}}$, with an independent sub-tree grafted at H_δ whose root is of “size” $= \delta$ and which does not have any vertex of “size” $\geq \delta$.

maximal vertex “size” is $+\infty$ as the support of the Lévy measure π is unbounded, and thus conditioning on the maximal vertex “size” to be δ implies the extinction of the super-critical Lévy tree.

1.3. Lévy trees conditioned to have large total mass. Eventually, let us recall that a Lévy tree is naturally endowed with a mass measure supported by its leaves, see Section 2.5 (iv). Let us denote by σ the total mass of this measure, which can also be seen as the life-time of the excursion of the height process that codes the Lévy tree.

To simplify, we consider the critical case ($\psi'(0) = 0$). In this case, the coupling is less straightforward. Consider the Kesten tree \mathcal{K} , and denote by $(h_i, \sigma_i, \mathcal{T}_i)_{i \in I}$ the height h_i at which the sub-tree \mathcal{T}_i is grafted on the spine, and σ_i denote its mass, see Fig.3. The process $\mathbf{S}^\mathcal{K}$ defined $S_h^\mathcal{K} = \sum_{i \in I} \sigma_i \mathbf{1}_{\{h_i \leq h\}}$ is then a subordinator started at 0 with Laplace exponent $\varphi = \psi' \circ \psi^{-1}$, see Section 5.4. For simplicity, we write \mathbf{S} for $\mathbf{S}^\mathcal{K}$.

The idea, is to consider the subordinator \mathbf{S} started at 0 and conditioned to die continuously at level r , which we denote by \mathbf{S}^r and write ζ^r for its lifetime (in particular $S_{\zeta^r-}^r = r$ a.s.). Following [29], provided the potential of \mathbf{S} has a continuous density, say u , w.r.t. the Lebesgue measure on $(0, +\infty)$, that is, $\int_{\mathbb{R}^+} \mathbb{P}(S_t \leq x) dt = \int_{(0,x]} u(y) dy$ for all $x > 0$ and u is continuous, then the process \mathbf{S}^r is well defined and with \mathbb{P}^φ and $\mathbb{P}^{\varphi,r}$ denoting the distribution of \mathbf{S} and \mathbf{S}^r , we get for all $h \geq 0$ and F a non-negative measurable functional:

$$\mathbb{E}^{\varphi,r} \left[\mathbf{1}_{\{h < \zeta^r\}} F(S_{[0,h]}^r) \right] = \mathbb{E}^\varphi \left[\frac{u(r - S_h)}{u(r)} \mathbf{1}_{\{S_h < r\}} F(S_{[0,h]}) \right].$$

Let us mention that $u(r)/r$ is also the density w.r.t. the Lebesgue measure of the total mass of the Lévy tree under the excursion measure \mathbb{N}^ψ , see (70).

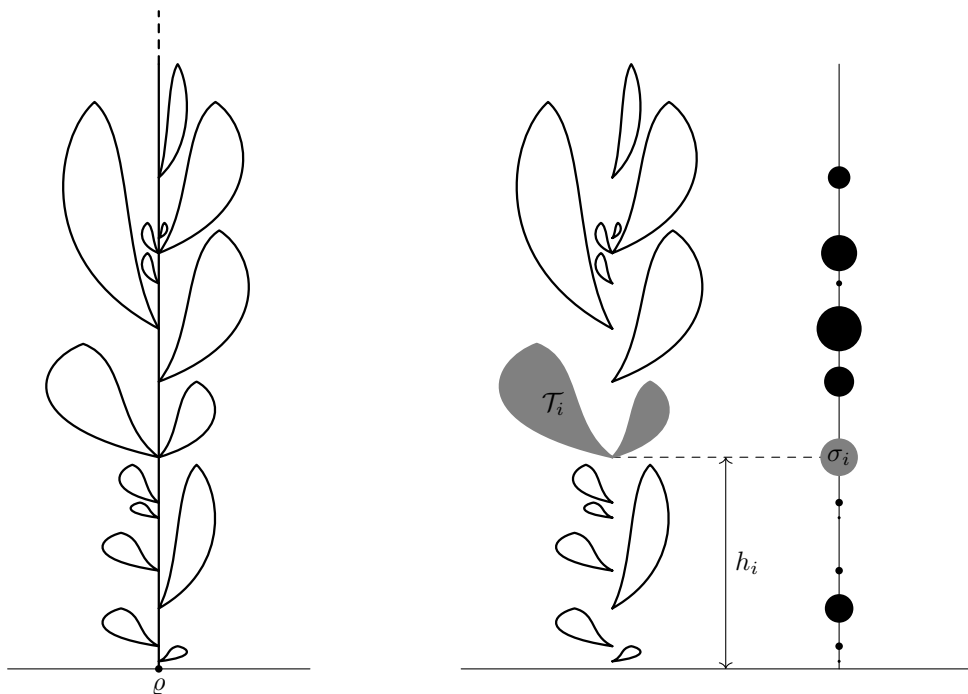


FIGURE 3. Left: the Kesten tree; Center: the sub-trees grafted on the spine, with in gray an instance of a tree \mathcal{T}_i (with the root being a branching point) grafted at height h_i with mass σ_i ; Right: mass of the grafted trees.

From the process \mathbf{S}^r , we build a modified truncated Kesten tree, say $\mathcal{K}_r^{\text{mass}}$, by considering a spine of length ζ_r on which we graft at height h_j an independent Lévy tree \mathcal{T}_j with total mass σ_j , where $(h_j, \sigma_j)_{j \in J}$ are the jumping times and jumps of the process \mathbf{S}^r . In particular the total mass of $\mathcal{K}_r^{\text{mass}}$ is given by $\sum_{j \in J} \sigma_j = S_{\zeta_r}^r = r$. (We stress the distribution of \mathcal{T}_j conditionally on $\sigma_j = r$ is the conditional distribution of a grafted tree $\mathbb{N}^\psi[d\mathcal{T} | \sigma = r]$ with the grafting measure \mathbf{N} defined in (30).) The tree $\mathcal{K}_r^{\text{mass}}$ is not a functional of the Kesten tree. However, in the stable case $\psi(\lambda) = \lambda^a$ with $a \in (1, 2]$, one could use a scaling argument to recover the jumps $(\sigma_j)_{j \in J}$ from the jumps $(\sigma_i)_{i \in I}$, see [15], and then use that conditionally on σ_j the tree \mathcal{T}_j is distributed as a scaled normalized random tree \mathcal{T}' with total mass 1. We also recall that in the stable case one can get a regular version of $\mathcal{T}_r^{\text{mass}}$ the Lévy tree conditioned to have total mass $r > 0$ by scaling, see Section 3.3 in [19].

We get from Corollary 5.8 that $\mathcal{T}_r^{\text{mass}}$, the Lévy tree conditioned to have total mass $r > 0$, is distributed as $\mathcal{K}_r^{\text{mass}}$ (in fact the distribution $\mathcal{K}_r^{\text{mass}}$ provide a regular version of the distribution of $\mathcal{T}_r^{\text{mass}}$), and from Proposition 5.11 (see also Theorem 5.12) we get the local convergence in distribution of $\mathcal{T}_r^{\text{mass}}$ towards the Kesten tree as r goes to infinity when the potential density is non-increasing (this relies on the Fatou lemma and the fact that $\liminf_{r \rightarrow \infty} \frac{u(r - S_h)}{u(r)} \mathbf{1}_{\{S_h < r\}} \geq 1$ which is a direct consequence of u being non-increasing).

Theorem (Conditioning w.r.t. the total mass). *We assume the infinite variation setting and the Grey condition, as well as the existence of continuous density on $(0, +\infty)$, say f_σ , of the total mass under the excursion measure \mathbb{N}^p si such that $r \mapsto r f_\sigma(r)$ is non-increasing. The critical Lévy tree \mathcal{T} conditioned under the excursion measure to have total mass r , say $\mathcal{T}_r^{\text{mass}}$, is distributed as $\mathcal{K}_r^{\text{mass}}$:*

$$(4) \quad \mathcal{T}_r^{\text{mass}} \stackrel{(d)}{=} \mathcal{K}_r^{\text{mass}},$$

and the sequence $(\mathcal{T}_r^{\text{mass}})_{r>0}$ converges to the Kesten tree \mathcal{K} in $(\mathbb{T}, d_{\text{LGH}})$, and we have the following local convergence in distribution:

$$(5) \quad \mathcal{T}_r^{\text{mass}} \xrightarrow[r \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in} \quad (\mathbb{T}, d_{\text{LGH}}).$$

Using a Girsanov transformation, we have a similar result for the super-critical case, see Corollary 5.14. The sub-critical case is more delicate: in the generic case (where one can extend the function ψ to negative values such that there exists a root θ^* of $\psi' = 0$), then one can get a similar result, see Corollary 5.14. In the non-generic case, one expect to have a condensation phenomenon similar to what is observed in the discrete case, see [25, 23, 5]. But in the continuous setting the formalism is more delicate (as the conjectured limiting tree is no more locally compact) and the proof of the condensation is still an open question.

2. NOTATIONS

2.1. Real trees. We refer to [22, 31, 24] for a general presentation of random real trees. Informally, real trees are metric spaces without loops, locally isometric to the real line. More precisely, a (non-empty) metric space (\mathbf{t}, d) is a real tree if the following properties are satisfied:

- (1) For every $x, y \in \mathbf{t}$, there is a unique isometric map $f_{x,y}$ from $[0, d(x, y)]$ to \mathbf{t} such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x, y)) = y$.
- (2) For every $x, y \in \mathbf{t}$, if q is a continuous injective map from $[0, 1]$ to \mathbf{t} such that $q(0) = x$ and $q(1) = y$, then $q([0, 1]) = f_{x,y}([0, d(x, y)])$.

If $x, y \in \mathbf{t}$, we will note $\llbracket x, y \rrbracket$ the range of the isometric map $f_{x,y}$ described above and $\llbracket x, y \rrbracket \setminus \{y\}$. The degree $\kappa_x(\mathbf{t})$ of $x \in \mathbf{t}$ is the number of connected components of $\mathbf{t} \setminus \{x\}$, and we simply write κ_x for $\kappa_x(\mathbf{t})$ when there is no ambiguity on the tree \mathbf{t} . We shall consider the set of leaves $\text{Lf}(\mathbf{t}) = \{x \in \mathbf{t}, \kappa_x = 1\}$ (with the convention that $\text{Lf}(\mathbf{t}) = \mathbf{t}$ if \mathbf{t} is reduced to one element), the set of branching vertices $\text{Br}(\mathbf{t}) = \{x \in \mathbf{t}, \kappa_x \geq 3\}$ and the set of infinite branching vertices $\text{Br}_\infty(\mathbf{t}) = \{x \in \mathbf{t}, \kappa_x = \infty\}$. The skeleton of \mathbf{t} is the set of vertices in the tree that aren't leaves: $\text{Sk}(\mathbf{t}) = \mathbf{t} \setminus \text{Lf}(\mathbf{t})$. A real tree \mathbf{t} is discrete if the set of leaves and branching points is locally finite, that is, $\{y \in \text{Lf}(\mathbf{t}) \cup \text{Br}(\mathbf{t}); d(x, y) \leq a\}$ is finite for all $a \geq 0$ and $x \in \mathbf{t}$. If \mathbf{t} is separable, then the set of leaves $\text{Lf}(\mathbf{t})$ is a Borel subset of \mathbf{t} , and there exists a unique measure $L^{\mathbf{t}}$ on \mathbf{t} (endowed with the Borel σ -field), called the length measure, such that:

$$L^{\mathbf{t}}(\text{Lf}(\mathbf{t})) = 0 \quad \text{and} \quad L^{\mathbf{t}}(\llbracket x, y \rrbracket) = d(x, y) \quad \text{for all } x, y \in \mathbf{t};$$

and furthermore the length measure is σ -finite. If the real tree (\mathbf{t}, d) is complete and locally compact, then any bounded closed set is compact by the Hopf-Rinow theorem, and thus the real tree \mathbf{t} is a Polish metric space and thus separable.

We say that (\mathbf{t}, d, ϱ) is a rooted real tree with root $\varrho \in \mathbf{t}$ if (\mathbf{t}, d) is a real tree and ϱ is a distinguished vertex. Let (\mathbf{t}, d, ϱ) be a rooted real tree. We define its height by:

$$\mathbf{H}(\mathbf{t}) = \sup_{x \in \mathbf{t}} d(\varrho, x).$$

For every $x \in \mathbf{t}$, $\llbracket \varrho, x \rrbracket$ is interpreted as the ancestral line of the vertex x in the tree. We define a partial order on \mathbf{t} by setting $x \preceq y$ (x is an ancestor of y) if $x \in \llbracket \varrho, y \rrbracket$. For a non empty subset $\mathbf{s} \subset \mathbf{t}$, there exists a unique $z \in \mathbf{t}$, called the Most Recent Common Ancestor (MRCA) of \mathbf{s} such that $\llbracket \varrho, z \rrbracket = \bigcap_{x \in \mathbf{s}} \llbracket \varrho, x \rrbracket$. We simply write $x \wedge y$ for the MRCA of x and y (that is, of $\mathbf{s} = \{x, y\}$).

We shall consider the infinite branch, denoted $\llbracket 0, \infty \rrbracket$, as the rooted real tree $(\mathbb{R}_+, d, \varrho = 0)$, where d is the usual Euclidean distance.

We say $(\mathbf{t}, d, \varrho, x)$, is a pointed rooted real trees if (\mathbf{t}, d, ϱ) is a rooted real tree and $x \in \mathbf{t}$ is a distinguished vertex. We shall consider the particular case of the tree reduced to a branch with one extremity of the branch being the root and the other the pointed element; more precisely, for $h \geq 0$, we denote by $\llbracket 0, h \rrbracket$ the pointed rooted real tree $([0, h], d, \varrho = 0, h)$, where d is the usual Euclidean distance.

For simplicity, we shall write \mathbf{t} for a rooted real tree, and denote by $d^{\mathbf{t}}$ its distance and by $\varrho^{\mathbf{t}}$ its root.

2.2. Grafting procedure. We recall the grafting procedure where we add (graft) rooted real trees on an existing rooted real trees. More precisely, let \mathbf{t} be a rooted real tree and let $(x_i)_{i \in I}$ and $(\mathbf{t}_i, \varrho^{\mathbf{t}_i}, d^{\mathbf{t}_i})_{i \in I}$ be a family respectively of vertices of \mathbf{t} and of rooted real tree. We set $T = \mathbf{t} \sqcup (\bigsqcup_{i \in I} \mathbf{t}_i \setminus \{\varrho^{\mathbf{t}_i}\})$ where the symbol \sqcup means that we consider the disjoint union of the sets $(\mathbf{t}_i)_{i \in I}$ and \mathbf{t} . We set $\varrho^T = \varrho^{\mathbf{t}}$. The set T is endowed with the following metric d^T : if $s, t \in T$,

$$d^T(s, t) = \begin{cases} d^{\mathbf{t}}(s, t) & \text{if } s, t \in \mathbf{t}, \\ d^{\mathbf{t}}(s, x_i) + d^{\mathbf{t}_i}(\varrho^{\mathbf{t}_i}, t) & \text{if } s \in \mathbf{t}, t \in \mathbf{t}_i \setminus \{\varrho^{\mathbf{t}_i}\}, \\ d^{\mathbf{t}_i}(s, t) & \text{if } s, t \in \mathbf{t}_i \setminus \{\varrho^{\mathbf{t}_i}\}, \\ d^{\mathbf{t}}(x_i, x_j) + d^{\mathbf{t}_j}(\varrho^{\mathbf{t}_j}, s) + d^{\mathbf{t}_i}(\varrho^{\mathbf{t}_i}, t) & \text{if } i \neq j \text{ and } s \in \mathbf{t}_j \setminus \{\varrho^{\mathbf{t}_j}\}, t \in \mathbf{t}_i \setminus \{\varrho^{\mathbf{t}_i}\}. \end{cases}$$

It is clear that (T, d^T, ϱ^T) is a rooted real tree, and we denote it by T for simplicity and use the following notation for the grafted tree:

$$(6) \quad T = \mathbf{t} \otimes_{i \in I} (\mathbf{t}_i, x_i).$$

This construction can be easily extended to the case where \mathbf{t} is a pointed rooted real tree with distinguished vertex $x \in \mathbf{t}$, which is also seen as a distinguished vertex of T .

2.3. Sub-trees above/below a given level. Let (\mathbf{t}, d, ϱ) be a rooted real tree. Let $h > 0$. We define the restriction map $R_h(\mathbf{t})$ which is the sub-tree below level h :

$$(7) \quad R_h(\mathbf{t}) = \{x \in \mathbf{t}, d(\varrho, x) \leq h\},$$

and still denote by $R_h(\mathbf{t})$ the corresponding rooted real tree with root ϱ and distance given by the restriction of d to $R_h(\mathbf{t})$. Let $(\mathbf{t}_i^\circ)_{i \in I}$ be the connected components of $\mathbf{t} \setminus R_h(\mathbf{t})$; let us denote by ϱ_i the MRCA (in \mathbf{t}) of \mathbf{t}_i° and set $\mathbf{t}_i = \mathbf{t}_i^\circ \cup \{\varrho_i\}$ which we consider as a rooted real tree with root ϱ_i . By construction, we have:

$$\mathbf{t} = R_h(\mathbf{t}) \otimes_{i \in I} (\mathbf{t}_i, \varrho_i).$$

We will also consider the point measure:

$$(8) \quad \mathcal{N}_h^{\mathbf{t}} = \sum_{i \in I} \delta_{(\varrho_i, \mathbf{t}_i)}.$$

The support of the measure $\sum_{i \in I} \delta_{\varrho_i}$ is a subset of:

$$(9) \quad \mathbf{t}(h) = \{x \in \mathbf{t} : d(\varrho, x) = h\}.$$

We shall also cut a tree by removing the sub-tree above one of its vertices. For $x \in \mathbf{t}$, we define:

$$(10) \quad \widehat{R}(\mathbf{t}, x) = \{x\} \cup \{y \in \mathbf{t}, x \notin \llbracket \varrho, y \rrbracket\}.$$

Notice that $R_{d(\varrho, x)}(\mathbf{t}) \subset \widehat{R}(\mathbf{t}, x)$. As for $R_h(\mathbf{t})$, we still denote by $\widehat{R}(\mathbf{t}, x)$ the corresponding rooted real tree with root ϱ and distance given by the restriction of d to $\widehat{R}(\mathbf{t}, x)$; according to the context, we might see $x \in \widehat{R}(\mathbf{t}, x)$ as a distinguished vertex.

2.4. Polish spaces of rooted real trees. By the Hopf-Rinow theorem, if (\mathbf{t}, d) is a complete and locally compact metric real tree, then every closed bounded subset of \mathbf{t} is compact. According to [8], one can define a Gromov-Hausdorff metric d_{LGH} on the space \mathbb{T} of (GH-isometric classes of) rooted complete locally compact real trees. Furthermore, following [9, Theorem 2.9], we get that the space $(\mathbb{T}, d_{\text{LGH}})$ is a Polish metric space. When there is no possible confusion, we shall also simply write \mathbf{t} for its equivalence class in \mathbb{T} .

One can define a restriction map R_h on \mathbb{T} which is consistent with (7); then the map $(h, \mathbf{t}) \mapsto R_h(\mathbf{t})$ is continuous from $\mathbb{R}_+ \times \mathbb{T}$ to \mathbb{T} , see [8, Lemma 5.4]. As a trivial application from the definition of the GH-distance d_{LGH} in [8], let us mention that for $h \geq 0$ and $\mathbf{t}, \mathbf{t}' \in \mathbb{T}$:

$$(11) \quad R_h(\mathbf{t}) = R_h(\mathbf{t}') \implies d_{\text{LGH}}(\mathbf{t}, \mathbf{t}') \leq e^{-h}.$$

Similarly, one can also define a Gromov-Hausdorff metric $d_{\text{LGH}}^{(1)}$ on the space \mathbb{T}_1 of (GH-isometric classes of) pointed rooted complete locally compact real trees so that it is a Polish metric space; we refer again

to [8, Section 5.3] for the precise definition of the restriction map which is again continuous. One can define a restriction map $\widehat{\mathbf{R}}$ on \mathbb{T}_1 which is consistent with (10) and, following [8, Lemma 6.22], get that the map $(\mathbf{t}, x) \mapsto \widehat{\mathbf{R}}(\mathbf{t}, x)$ is measurable from \mathbb{T}_1 to \mathbb{T} (or to \mathbb{T}_1 if one consider x as a distinguished vertex of $\widehat{\mathbf{R}}(\mathbf{t}, x)$). The grafting map $((\mathbf{t}, x), \mathbf{t}') \mapsto T = \mathbf{t} \circledast (\mathbf{t}', x)$ is a continuous map from $\mathbb{T}_1 \times \mathbb{T}$ to \mathbb{T} (or to \mathbb{T}_1 if one consider the pointed tree (T, x)), see [8, Lemma 5.13].

Grafting a countable family of trees is more delicate, and we shall not consider it in full generality. Write $\{\varrho\}$ for the tree reduced to its root. Based on [8, Lemma 5.31], we get in particular that if $\mathcal{M} = \sum_{i \in I} \delta_{(h_i, T_i)}$ is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}$ with intensity $dh \nu(dT)$, where ν is a σ -finite measure on \mathbb{T} such that $\nu(\mathbf{H}(T) = 0) = 0$ and $\nu(\mathbf{H}(T) > \varepsilon)$ is finite for all $\varepsilon > 0$, then the random trees:

$$\tau = \llbracket 0, \infty \rrbracket \circledast_{i \in I} (T_i, h_i), \quad \llbracket 0, h \rrbracket \circledast_{i \in I_h} (T_i, h_i) \quad \text{and} \quad \{\varrho\} \circledast_{i \in I_h} (T_i, \varrho),$$

where $h \geq 0$ and $I_h = \{i \in I : h_i \leq h\}$, are well defined \mathbb{T} -valued random variables. The tree $\llbracket 0, h \rrbracket \circledast_{i \in I_h} (T_i, h_i)$ with distinguished vertex h is also a well defined \mathbb{T}_1 -valued random variable.

Consider a non-decreasing sequence $(A_r)_{r \geq 0}$ of measurable subsets of $\mathbb{R}_+ \times \mathbb{T}$ and the random tree:

$$\tau_r = \llbracket 0, \infty \rrbracket \circledast_{i \in I_{A_r}} (T_i, h_i),$$

with $I_{A_r} = \{i \in I : (h_i, T_i) \in A_r\}$, which we call the restriction of the tree τ to A_r .

In this setting, we shall rewrite the monotonicity of the restrictions as:

$$(12) \quad \tau_r \subset \tau_{r'} \quad \text{a.s. for } 0 \leq r < r'.$$

If $\nu(A^c) = 0$ for $A = \lim_{r \rightarrow \infty} A_r$, then we get from [8, Lemma 5.31] that a.s. $\lim_{r \rightarrow \infty} d_{\text{LGH}}(\tau_r, \tau) = 0$, which we shall write as:

$$(13) \quad \lim_{r \rightarrow \infty} \tau_r \stackrel{\text{a.s.}}{=} \tau \quad \text{in } (\mathbb{T}, d_{\text{LGH}}).$$

2.5. Branching process. We consider the branching mechanism ψ :

$$(14) \quad \psi(\lambda) = \alpha' \lambda + \beta \lambda^2 + \int_{(0, +\infty)} (e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r \leq 1\}}) \pi(dr) \quad \text{for } \lambda \geq 0,$$

where α' and the quadratic parameter β are real numbers, the Lévy measure π on $(0, +\infty)$ is Borel, and:

$$(H0) \quad \boxed{\alpha' \in \mathbb{R}, \quad \beta \geq 0 \quad \text{and} \quad \int_{(0, +\infty)} (1 \wedge r^2) \pi(dr) < +\infty.}$$

We shall consider for $r \geq 0$:

$$(15) \quad \bar{\pi}(r) = \pi((r, +\infty)).$$

The corresponding Lévy process $\mathbf{V} = (V_t)_{t \geq 0}$ is a process on \mathbb{R} with independent and stationary increments characterized by $V_0 = 0$ a.s. and:

$$\mathbb{E}^\psi [e^{-\lambda V_t}] = e^{-t\psi(\lambda)} \quad \text{for all } \lambda \geq 0 \text{ and } t \geq 0.$$

The corresponding continuous state branching (CB) process $\mathbf{Y} = (Y_t)_{t \geq 0}$ is an homogeneous Markov process on $[0, +\infty]$ such that, for $x \in \mathbb{R}_+$, \mathbb{P}_x^ψ -a.s. $Y_0 = x$ and:

$$(16) \quad \mathbb{E}_x^\psi [e^{-\lambda Y_t}] = e^{-xv(\lambda, t)} \quad \text{for all } \lambda > 0 \text{ and } t \geq 0,$$

where, for $\lambda > 0$, the function $t \mapsto v(\lambda, t)$ is the unique positive solution of the equation:

$$(17) \quad v(\lambda, t) + \int_0^t \psi(v(\lambda, s)) ds = \lambda.$$

The branching mechanism ψ is convex, of class \mathcal{C}^∞ at least on $(0, \infty)$, and $\psi(0) = 0$. We simply write $\psi'(0)$ for $\lim_{\lambda \rightarrow 0+} \psi'(\lambda) = \alpha' - \int_{(1, +\infty)} r \pi(dr) \in [-\infty, +\infty)$. The CB process \mathbf{Y} and the branching mechanism ψ are critical (resp. sub-critical, resp. super-critical) if $\psi'(0) = 0$ (resp. $\psi'(0) > 0$, resp. $\psi'(0) < 0$). We shall mainly consider the (sub-)critical case; in this case the integral $\int_{(0, +\infty)} (r \wedge r^2) \pi(dr)$

is finite. When this latter integral is finite, we can rewrite (14) as follows and then define (α, β, π) as characteristic of the branching mechanism ψ :

$$(18) \quad \psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr) \quad \text{with} \quad \alpha = \psi'(0) \in \mathbb{R}.$$

We assume (H0) and consider also the following assumptions.

- Infinite variation:

$$(H1) \quad \boxed{\beta > 0 \quad \text{or} \quad \int_{(0,1)} r\pi(dr) = +\infty.}$$

In particular, we then have $\psi \neq 0$. Under (H0), this condition is equivalent to the CB process \mathbf{Y} being of infinite variation. Notice that (H0) and (H1) imply that ψ is strictly convex and a bijection from $[\theta_0, \infty)$ to $[0, \infty)$, with $\theta_0 \in [0, +\infty)$ the largest root of $\psi = 0$; and that ψ is (sub-)critical if and only if $\theta_0 = 0$. So, under (H1), we denote for $\lambda > 0$ by $\psi^{-1}(\lambda)$ the only root in $(0, \infty)$ of $\psi = \lambda$.

- The process \mathbf{Y} and the branching mechanism are conservative (that is, $\mathbb{P}_x^\psi(Y_t < +\infty) = 1$ for all $t \geq 0$ and $x \in \mathbb{R}_+$):

$$(H2) \quad \boxed{\int_{0+} \frac{d\lambda}{|\psi(\lambda)|} = +\infty.}$$

Under (H0), if $\int_{[1, \infty)} r \pi(dr)$ is finite, then the branching mechanism ψ is conservative. Under (H2), the lifetime of \mathbf{Y} is defined by $\zeta = \inf\{t : Y_t = 0\}$ with the convention that $\inf \emptyset = +\infty$.

- The Grey condition: (which we shall consider with (H0) and (H1)):

$$(H3) \quad \boxed{\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty.}$$

Under (H0)-(H2), the Grey condition is equivalent to a.s. either the lifetime ζ is finite or $\lim_{t \rightarrow +\infty} Y_t = +\infty$.

- (Sub-)critical branching:

$$(H4) \quad \boxed{\psi'(0) \geq 0.}$$

Under (H0), this condition is equivalent to $\lim_{t \rightarrow +\infty} Y_t = 0$ a.s., and it implies (H2).

We shall most of the time assume that (H0)-(H3) hold, and add some comments when (H4) does not hold, that is, for the super-critical case.

Under (H0)-(H3), we define the function c on $(0, +\infty)$ as the unique nonnegative solution of the equation:

$$(19) \quad \int_{c(h)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = h \quad \text{for } h > 0.$$

We have $c = \lim_{\lambda \rightarrow +\infty} v(\lambda, \cdot)$ and the distribution of the lifetime ζ of \mathbf{Y} is given by, for all $x \geq 0$:

$$\mathbb{P}_x^\psi(\zeta \leq h) = e^{-xc(h)} \quad \text{for all } h \geq 0.$$

2.6. The Lévy tree. The genealogy of the CB process \mathbf{Y} (under the canonical measure) can be described by a random tree, the so called Lévy tree, see [19]. The next description is from [20] using the coding of compact real trees by height function (the considered (sub-)critical case can be extended to the super-critical case using a Girsanov transformation, see [10]). Under (H0)-(H3), there exists a σ -finite measure $\mathbb{N}^\psi[d\mathcal{T}]$ on \mathbb{T} , or excursion measure of a Lévy tree (with distance d and root ϱ), satisfying the following properties.

- (i) **Height $\mathbf{H}(\mathcal{T})$.** The distribution of the height $\mathbf{H}(\mathcal{T})$ is given by:

$$(20) \quad \mathbb{N}^\psi[\mathbf{H}(\mathcal{T}) = 0] = 0 \quad \text{and} \quad \mathbb{N}^\psi[\mathbf{H}(\mathcal{T}) > h] = c(h) \quad \text{for all } h > 0.$$

(ii) **Local time.** There exists a \mathcal{T} -measure valued process $(\ell^a)_{a \geq 0}$ càdlàg for the weak topology on finite measures on \mathcal{T} such that:

- $\ell^0 = 0$ and, for every $a > 0$, $\{\ell^a \neq 0\} = \{\mathbf{H}(\mathcal{T}) > a\}$, $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e..
- For every $a > 0$, ℓ^a is supported on $\mathcal{T}(a) = \{x \in \mathcal{T} : d(\rho, x) = a\}$ (see (9)) and $\ell^a(\mathcal{T} \setminus \text{Lf}(\mathcal{T})) = 0$, $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e..
- For every $a > 0$, we have $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e. for every bounded continuous function f on \mathcal{T} and with the measure $\mathcal{N}_a^\mathcal{T}$ defined in (8):

$$\begin{aligned} \langle \ell^a, f \rangle &= \lim_{\varepsilon \downarrow 0} \frac{1}{c(\varepsilon)} \int f(x) \mathbf{1}_{\mathbf{H}(\mathcal{T}') \geq \varepsilon} \mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}') \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{c(\varepsilon)} \int f(x) \mathbf{1}_{\{\mathbf{H}(\mathcal{T}') \geq \varepsilon\}} \mathcal{N}_{a-\varepsilon}^\mathcal{T}(dx, d\mathcal{T}'), \text{ if } a > 0. \end{aligned}$$

- The process $(\langle \ell^a, 1 \rangle)_{a \geq 0}$ is distributed as the CB process \mathbf{Y} under its canonical measure with branching mechanism ψ . For simplicity, we shall identify the two processes: $Y_a = \langle \ell^a, 1 \rangle$. In particular, we have for $\lambda > 0$ (or $\lambda \geq 0$ if $\psi'(0) > -\infty$), $a \geq 0$ and v defined by (17) that:

$$(21) \quad \mathbb{N}^\psi [1 - e^{-\lambda Y_a}] = v(\lambda, a),$$

and with θ_0 the largest nonnegative root of $\psi = 0$, and thus $v(\theta_0, a) = \theta_0$, that:

$$(22) \quad \mathbb{N}^\psi [1 - e^{-\theta_0 Y_a}] = \theta_0.$$

(iii) **Branching property.** For every $a > 0$, the conditional distribution of $\mathcal{N}_a(dx, d\mathcal{T}')$ under $\mathbb{N}^\psi[\cdot | \mathbf{H}(\mathcal{T}) > a]$ given $R_a(\mathcal{T})$ is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx) \mathbb{N}^\psi[d\mathcal{T}']$.

(iv) **Mass measure and total mass σ .** The mass measure \mathbf{m} defined by:

$$(23) \quad \mathbf{m}(dx) = \int_0^\infty \ell^a(dx) da,$$

is supported by the leaves: $\mathbf{m}(\mathcal{T} \setminus \text{Lf}(\mathcal{T})) = 0$. The distribution of the total mass $\sigma = \mathbf{m}(\mathcal{T})$ is given by:

$$(24) \quad \mathbb{N}^\psi[\sigma = 0] = 0 \quad \text{and} \quad \mathbb{N}^\psi[1 - e^{-\lambda \sigma}] = \psi^{-1}(\lambda) \quad \text{for } \lambda > 0.$$

In particular, we have (with θ_0 the largest root of $\psi = 0$) that:

$$\mathbb{N}^\psi[\sigma = +\infty] = \psi^{-1}(0+) = \theta_0.$$

If ψ is (sub-)critical, then the Lévy tree \mathcal{T} is compact and σ is finite \mathbb{N}^ψ -a.e. and:

$$\mathbb{N}^\psi[\sigma] = \frac{1}{\psi'(0)} \in (0, +\infty].$$

(v) **Branching points.**

- $\mathbb{N}^\psi[d\mathcal{T}]$ -a.e., for all $x \in \mathcal{T}$, we have $\kappa_x \in \{1, 2, 3, \infty\}$ and $\kappa_\rho = 1$; in particular the branching points of \mathcal{T} have degree 3 (corresponding to 2 children) or infinity.
- \mathbb{N}^ψ -a.e. the set $\{x \in \mathcal{T} : \kappa_x = 3\}$ is empty if $\beta = 0$, or a countable dense subset of \mathcal{T} if $\beta > 0$.
- The set $\text{Br}_\infty(\mathcal{T})$ of infinite branching points is nonempty with \mathbb{N}^ψ -positive measure if and only if $\pi \neq 0$. If $\langle \pi, 1 \rangle = +\infty$, the set $\text{Br}_\infty(\mathcal{T})$ is \mathbb{N}^ψ -a.e. a countable dense subset of \mathcal{T} . If $\langle \pi, 1 \rangle < +\infty$, the set $\text{Br}_\infty(\mathcal{T})$ is \mathbb{N}^ψ -a.e. a finite (possibly empty) subset of \mathcal{T} .

(vi) **“Size” of the vertices.** The set $\{d(\rho, x), x \in \text{Br}_\infty(\mathcal{T})\}$ coincides \mathbb{N}^ψ -a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, \mathbb{N}^ψ -a.e., for every such discontinuity time b of the map $a \mapsto \ell^a$, there is a unique vertex $x_b \in \text{Br}_\infty(\mathcal{T}) \cap \mathcal{T}(b)$ and $\Delta_b > 0$, such that:

$$\ell^b = \ell^{b-} + \Delta_b \delta_{x_b},$$

where the “size” Δ_b of the vertex x_b is positive and can be obtained by the approximation:

$$(25) \quad \Delta_b = \lim_{\varepsilon \rightarrow 0} \frac{1}{c(\varepsilon)} n(x_b, \varepsilon),$$

with $n(x_b, \varepsilon) = \int \mathbf{1}_{\{x_b\}}(x) \mathbf{1}_{\{\mathbf{H}(\mathcal{T}') > \varepsilon\}} \mathcal{N}_b^{\mathcal{T}'}(dx, d\mathcal{T}')$ is the number of sub-trees above level b with root x_b and height larger than ε .

In order to stress the dependence in \mathcal{T} , we may write $\ell^{a, \mathcal{T}}$ for the local time ℓ^a , $\mathbf{m}^{\mathcal{T}}$ for the mass measure and $\sigma^{\mathcal{T}}$ for the total mass.

Remark 2.1 (On Assumptions (H0)-(H4) for the Lévy tree). The Lévy trees are defined under (H0)-(H3), but they could be introduced without some of those assumptions. If Assumption (H1) does not hold, then the Lévy tree is discrete, and the machinery developed here is not adapted. Assuming (H0)-(H2), the Grey condition (H3) implies that the Lévy tree belongs to the Polish space \mathbb{T} of (GH-isometric classes) of locally compact rooted tree; using the mass erasure procedure approach from Duquesne and Winkel [21], it seems possible to get rid of Assumption (H3) by considering a larger space of trees with nice topological properties. Under (H0)-(H3), the condition (H4) is then equivalent to the Lévy tree being \mathbb{N}^ψ -a.e. compact, that is, having a finite height.

2.7. Girsanov transformation and related measure on Lévy trees. Assume (H0)-(H3). We define a probability measure on \mathbb{T} as follow. Let $r > 0$ and $\sum_{i \in \mathcal{I}} \delta_{(h_i, \mathcal{T}_i)}(dh, d\mathcal{T})$ be a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}$ with intensity $dh \mathbb{N}^\psi[d\mathcal{T}]$. Consider $\{\varrho\}$ as the trivial measured rooted real tree reduced to the root. We define for $r > 0$:

$$\mathcal{T}_{[r]} = \{\varrho\} \otimes_{i \in \mathcal{I}_r} (\mathcal{T}_i, \varrho) \quad \text{with} \quad \mathcal{I}_r = \{i \in \mathcal{I} : h_i \leq r\}.$$

Since $\mathbb{N}^\psi[H(\mathcal{T}) = 0] = 0$ and $\mathbb{N}^\psi[H(\mathcal{T}) > \varepsilon]$ is finite for all $\varepsilon > 0$, we deduce from Section 2.4 that $\mathcal{T}_{[r]}$ is a \mathbb{T} -valued random variable. We denote by \mathbb{P}_r^ψ its probability distribution. Notice that \mathbb{P}_r^ψ -a.s., the root is an infinite branching vertex and its size Δ_ϱ defined by (25) (with $b = 0$) is exactly r . The corresponding local time is defined by $\ell_{[r]}^a = \sum_{i \in \mathcal{I}_r} \ell^{a, \mathcal{T}_i}$ for $a > 0$, and we set $\ell_{[r]}^0 = r \delta_\varrho$. Thanks to Property (ii) on the local time, the process $(\langle \ell_{[r]}^a, 1 \rangle_a \geq 0)$ is distributed as the CB process \mathbf{Y} under \mathbb{P}_r^ψ .

We now recall the Girsanov transformation from [10, 4]; this transformation allows in particular to define the Lévy trees in the super-critical regime. Assume (H0). We consider the interval $\Theta' = \{\theta \in \mathbb{R} : \int_{(1, +\infty)} e^{-\theta r} \pi(dr) < +\infty\}$, and notice that $\mathbb{R}_+ \subset \Theta$. The function ψ given by (14) is in fact well defined on Θ' . For $\theta \in \Theta'$, we define the function ψ_θ by:

$$(26) \quad \psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta) \quad \text{for } \lambda \geq 0;$$

it is a branching mechanism whose quadratic parameter is the same $\beta_\theta = \beta$, and whose Lévy measure is given by $\pi_\theta(dr) = e^{-r\theta} \pi(dr)$. For $\theta \in \Theta'$, the branching mechanism ψ_θ satisfies also (H0). Eventually, we consider the interval $\Theta^\psi = \{\theta \in \Theta' : \psi_\theta \text{ is conservative}\}$, and it is elementary to check that $\Theta^\psi \subset \Theta' \subset \Theta^\psi \cup \{\theta_\infty\}$, with $\theta_\infty = \inf \Theta' \in [-\infty, 0]$. If furthermore the branching mechanism ψ satisfies (H1) (resp. (H3)), then the branching mechanism ψ_θ satisfies also (H1) (resp. (H3)).

For $h \geq 0$, we set \mathcal{F}_h the σ -field on \mathbb{T} generated by the truncation map R_h , and under $\mathbb{N}^\psi[d\mathcal{T}]$, we denote $\sigma_h = \mathbf{m}^{\mathcal{T}}(\mathcal{T}_h) = \int_0^h Y_a da$ the total mass of the truncated Lévy tree $\mathcal{T}_h = R_h(\mathcal{T})$, with $Y_a = \langle \ell^a, 1 \rangle$ the total local time at level a of \mathcal{T} . It is elementary to deduce from [4, Theorem 2.22] (see also [3, Lemma 3.8 and Corollary 4.4]) that for $\theta \in \Theta^\psi$ and $h \geq 0$:

$$(27) \quad \frac{d\mathbb{N}^{\psi_\theta}}{d\mathbb{N}^\psi} \Big|_{\mathcal{F}_h} = e^{-\psi(\theta)\sigma_h - \theta Y_h},$$

and furthermore, with $\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T})$ the total mass of \mathcal{T} :

$$(28) \quad \mathbf{1}_{\{\sigma < +\infty\}} \mathbb{N}^{\psi_\theta}[d\mathcal{T}] = \mathbf{1}_{\{\sigma < +\infty\}} e^{-\psi(\theta)\sigma} \mathbb{N}^\psi[d\mathcal{T}].$$

Recall that \mathbb{N}^{ψ_θ} -a.e. σ is finite if and only if ψ_θ is sub-critical or critical. We deduce that the distribution of the total mass σ under \mathbb{N}^ψ on $(0, +\infty)$ has a density w.r.t. the Lebesgue measure, say f_σ , if and only if the distribution of σ under \mathbb{N}^{ψ_θ} on $(0, +\infty)$ has a density w.r.t. the Lebesgue measure, say f_σ^θ , and that in this case:

$$(29) \quad f_\sigma^\theta(r) = e^{-\psi(\theta)r} f_\sigma(r) \quad \text{for } r \in (0, +\infty).$$

Let $\theta \geq 0$. We will consider the following measure $\mathcal{N}_\theta^\psi[d\mathcal{T}] = 2\beta\theta\mathbb{N}^\psi[d\mathcal{T}] + \int_{(0,+\infty)} \pi(dr)(1 - e^{-\theta r})\mathbb{P}_r^\psi(d\mathcal{T})$ on \mathbb{T} and its formal derivative $\partial_\theta \mathcal{N}_\theta^\psi[d\mathcal{T}]$ at $\theta = 0$, that is:

$$(30) \quad \mathbf{N}^\psi[d\mathcal{T}] = 2\beta\mathbb{N}^\psi[d\mathcal{T}] + \int_{(0,+\infty)} r\pi(dr) \mathbb{P}_r^\psi(d\mathcal{T}).$$

Elementary computations yield for $q > 0$ such that $\psi(q) > 0$:

$$(31) \quad \mathbf{N}^\psi \left[1 - e^{-\psi(q)\sigma} \right] = 2\beta q + \int_{(0,+\infty)} (1 - e^{-qr}) r\pi(dr) = \psi'(q) - \psi'(0).$$

2.8. The Kesten tree \mathcal{K} . Recall the infinite branch $\llbracket 0, \infty \rangle$ defined at the end of Section 2.4. Its length measure, denoted by dh , is just the Lebesgue measure on \mathbb{R}_+ .

Definition 2.2 (Kesten tree). *Let ψ be a branching mechanism satisfying (H0)-(H3). Let $(h_i, \mathcal{T}_i)_{i \in I}$ be the atoms of a Poisson point measure on $\llbracket 0, \infty \rangle \times \mathbb{T}$ with intensity $dh \mathbf{N}^\psi[d\mathcal{T}]$ (and \mathbf{N}^ψ given in (30)). The Kesten tree \mathcal{K} , whose distribution is denoted \mathbb{P}^ψ , is a \mathbb{T} -valued random variable defined as the infinite branch (or spine) $\llbracket 0, \infty \rangle$, on which the \mathcal{T}_i are grafted at height h_i , that is:*

$$\mathcal{K} = \llbracket 0, \infty \rangle \otimes_{i \in I} (\mathcal{T}_i, h_i).$$

The left part of Fig. 1 give an illustration of the Kesten tree. Notice the Kesten tree is indeed a \mathbb{T} -valued random variable by [8, Lemma 5.31]. We see $h \geq 0$ as an element of the spine $\llbracket 0, \infty \rangle$ at distance h from the root $\varrho = 0$ and as a distinguished vertex of \mathcal{K} . In particular the Kesten tree whose spine is truncated at level $h \geq 0$, that is, $\widehat{\mathcal{K}}_h = \widehat{\mathbf{R}}(\mathcal{K}, h)$, can be described as:

$$(32) \quad \widehat{\mathcal{K}}_h = \llbracket 0, h \rangle \otimes_{i \in I_h} (\mathcal{T}_i, h_i) \quad \text{with} \quad I_h = \{i \in I : h_i \leq h\},$$

and $(\widehat{\mathcal{K}}_h, h)$ is a \mathbb{T}_1 -valued random variable. The local time of the Kesten tree \mathcal{K} at level $h \geq 0$ is defined by $\ell^{h, \mathcal{K}} = \sum_{i \in I_h} \ell^{h-h_i, \mathcal{T}_i}$.

We can recast Theorem 4.5 from [20] in our setting for the Lévy tree with a marked leaf (see [8, Corollary 5.9] for the supercritical quadratic, that is, when $\pi = 0$).

Proposition 2.3 (Lévy tree with a marked leaf at a given level). *Assume (H0)-(H3) hold and $\psi'(0) > -\infty$. Let G be a nonnegative measurable function defined on \mathbb{T}_1 . We have for $h > 0$:*

$$(33) \quad \mathbb{N}^\psi \left[\int_{\mathcal{T}} \ell^h(dx) G(\mathcal{T}, x) \right] = e^{-h\psi'(0)} \mathbb{E}^\psi [G(\widehat{\mathcal{K}}_h, h)].$$

Proof. For the (sub-)critical case, see [20, Theorem 4.5]. We now assume $\psi'(0) \in (-\infty, 0)$ and we shall use a Girsanov argument to deduce the result from the sub-critical case. Let $h > 0$.

For a pointed tree $(T, d, \varrho, x) \in \mathbb{T}_1$, where $\varrho, x \in T$, we uniquely decompose the tree T according to the branch $\llbracket \varrho, x \rangle$ and the sub-trees grafted on this branch:

$$T = \llbracket \varrho, x \rangle \otimes_{i \in I} (T_i, x_i),$$

where x_i belongs to the branch $\llbracket \varrho, x \rangle$ and T_i is the union of all the connected components of $T \setminus \{x_i\}$ not containing the root nor x , with x_i added as a root. Notice this decomposition is unique (and measurable using an adaptation of [8, Proposition 5.32]). We shall consider a particular choice of function G defined by $G(T, x) = \exp(-\sum_{i \in I} G_0(d(\varrho, x_i), T_i))$, with G_0 a nonnegative measurable function defined on $\mathbb{R}_+ \times \mathbb{T}$ such that $G_0(T, x) = 0$ if $\mathbf{H}(T) > a - h$ for some a large.

Let $\theta_0 > 0$ be the only root of $\psi = 0$. We deduce from the Girsanov transformation (27) that:

$$\begin{aligned} \mathbb{N}^\psi \left[\int_{\mathcal{T}} \ell^h(dx) G(\mathcal{T}, x) \right] &= \mathbb{N}^{\psi_{\theta_0}} \left[\int_{\mathcal{T}} \ell^h(dx) G(\mathcal{T}, x) e^{\theta_0 Y_a} \right] \\ &= e^{-h\psi'(\theta)} \mathbb{E}^{\psi_{\theta}} \left[G(\widehat{\mathcal{K}}_h, h) e^{\theta_0 \ell^{h, \mathcal{K}}} \right] \\ &= e^{-h\psi'(\theta)} \exp \left(- \int_0^h ds \mathbf{N}^{\psi_{\theta_0}} [1 - e^{-G_0(s, \mathcal{T}) + \theta_0 Y_{a-s}}] \right). \end{aligned}$$

Using (27) and then (22), we get that:

$$\mathbb{N}^{\psi_{\theta_0}} [1 - e^{-G_0(s, \mathcal{T}) + \theta_0 Y_{a-s}}] = \mathbb{N}^{\psi} [1 - e^{-G_0(s, \mathcal{T})}] - \mathbb{N}^{\psi} [1 - e^{-\theta_0 Y_{a-s}}] = \mathbb{N}^{\psi} [1 - e^{-G_0(s, \mathcal{T})}] - \theta_0.$$

Using that the Lévy measure associated to ψ_{θ_0} is $e^{-\theta_0 r} \pi(dr)$, we then deduce that:

$$\begin{aligned} \int_{(0, +\infty)} r e^{-\theta_0 r} \pi(dr) \mathbb{P}_r^{\psi_{\theta_0}} \left(1 - e^{-G_0(s, \mathcal{T}) + \theta_0 Y_{a-s}} \right) \\ = \int_{(0, +\infty)} r \pi(dr) \mathbb{P}_r^{\psi} \left(1 - e^{-G_0(s, \mathcal{T})} \right) - \int_{(0, +\infty)} (1 - e^{-\theta_0 r}) r \pi(dr). \end{aligned}$$

Thus, we obtain that:

$$\mathbb{N}^{\psi_{\theta_0}} [1 - e^{-G_0(s, \mathcal{T}) + \theta_0 Y_{a-s}}] = \mathbb{N}^{\psi} [1 - e^{-G_0(s, \mathcal{T})}] - \psi'(\theta_0) + \psi'(0),$$

and then that:

$$\mathbb{N}^{\psi} \left[\int_{\mathcal{T}} \ell^h(dx) G(\mathcal{T}, x) \right] = e^{-h\psi'(0)} \mathbb{E}^{\psi} \left[G(\widehat{\mathcal{K}}_h, h) \right].$$

By dominated convergence, we can remove the condition that $G_0(T, x) = 0$ if $\mathbf{H}(T) > a - h$ for some a large, and then by the monotone class theorem, we deduce the previous equality holds for any nonnegative measurable function G defined on \mathbb{T}_1 . \square

Taking $G \equiv 1$ in (33), if $\psi'(0) > -\infty$, we get that for $h > 0$:

$$(34) \quad \mathbb{N}^{\psi} [\langle \ell^h, 1 \rangle] = \mathbb{N}^{\psi} [Y_h] = e^{-h\psi'(0)}.$$

For the sub-critical case, integrating (33) over h and using (23), we also obtain the following corollary.

Corollary 2.4 (Sub-critical Lévy tree with a marked leaf). *Assume (H0)-(H3) hold and $\psi'(0) > 0$. Let G be a nonnegative measurable function defined on \mathbb{T}_1 . We have:*

$$\frac{1}{\mathbb{N}^{\psi}[\sigma]} \mathbb{N}^{\psi} \left[\int_{\mathcal{T}} \mathbf{m}(dx) G(\mathcal{T}, x) \right] = \mathbb{E}^{\psi} [G(\widehat{\mathcal{K}}_E, E)],$$

where E is under \mathbb{P}^{ψ} an exponential random variable with mean $\mathbb{N}^{\psi}[\sigma] = 1/\psi'(0)$ independent of \mathcal{K} .

Provided $\psi'(0) > -\infty$, the total local time process of the Kesten tree, $(\langle \ell^{a, \mathcal{K}}, 1 \rangle)_{a \geq 0}$, is a CB process $\mathbf{Z} = (Z_a)_{a \geq 0}$ with immigration (CBI process) defined as the CB process with branching mechanism ψ and immigration $\psi' - \psi'(0)$. As a consequence of (33) and (34), if $\psi'(0) > -\infty$ the one-dimensional marginal Z_a is distributed as the size biased distribution of $Y_a = \langle \ell^a, 1 \rangle$ under the excursion measure, that is, for $a > 0$ and g a measurable nonnegative function defined on \mathbb{R}_+ :

$$(35) \quad \mathbb{E}^{\psi} [g(Z_a)] = e^{a\psi'(0)} \mathbb{N}^{\psi} [Y_a g(Y_a)] = \frac{\mathbb{N}^{\psi} [Y_a g(Y_a)]}{\mathbb{N}^{\psi} [Y_a]}.$$

3. LÉVY TREE WITH A GIVEN HEIGHT

3.1. Coupling and convergence for the conditioning by the height. Assume (H0)-(H4) so that $\psi'(0) = \alpha \geq 0$ and the branching is (sub-)critical. We refer to Remark 3.5 for the super-critical case where (H4) is not satisfied. According to (19) and Item (i) in Section 2.6, we get that under $\mathbb{N}^{\psi}[d\mathcal{T}]$ the distribution of the height $\mathbf{H}(\mathcal{T})$ of the Lévy tree \mathcal{T} has the density $|c'|$ w.r.t. the Lebesgue measure. By [26, Corollary 1.29], there exists a regular version of the conditional distribution of \mathcal{T} given its height, $\mathbb{N}^{\psi}[d\mathcal{T} | \mathbf{H}(\mathcal{T}) = h]$ for $h > 0$. See [2, Theorem 3.3] for a nice representation of the compact rooted random tree $\mathcal{T}_h^{\text{height}}$ distributed as \mathcal{T} under \mathbb{N}^{ψ} conditionally on $\{\mathbf{H}(\mathcal{T}) = h\}$ for any given $h > 0$:

$$\mathcal{L}(\mathcal{T}_h^{\text{height}}) = \mathbb{N}^{\psi}[d\mathcal{T} | \mathbf{H}(\mathcal{T}) = h].$$

From this representation, there exists a unique vertex X_h of $\mathcal{T}_h^{\text{height}}$ which is at distance h of the root.

Recall the Kesten tree \mathcal{K} from Definition 2.2. We consider the tree $\mathcal{K}_h^{\text{height}}$ defined as the Kesten tree whose spine is cut at level h and where the trees \mathcal{T}_i , grafted at height h_i (less than h) and with height larger than $h - h_i$ are removed, that is:

$$\mathcal{K}_h^{\text{height}} = \llbracket 0, h \rrbracket \otimes_{i \in I_h^{\text{height}}} (\mathcal{T}_i, h_i) \quad \text{with} \quad I_h^{\text{height}} = \{i \in I : H(\mathcal{T}_i) + h_i \leq h\},$$

and the branch $\llbracket 0, h \rrbracket$ can be seen as the restriction of the semi-infinite spine $\llbracket 0, \infty \rrbracket$ of the Kesten tree up to level h . We shall also see h as a distinguished vertex of $\mathcal{K}_h^{\text{height}}$; it is the only one at distance h from the root. (Let us stress that \mathcal{T}_i is the only tree grafted at height h_i and that its root is an infinite branching point if $\kappa_{h_i}(\mathcal{K}) = +\infty$.) Let us mention the following relation between the different truncations of the Kesten tree: $\mathcal{K}_h^{\text{height}} \subset R_h(\mathcal{K}) \subset \widehat{\mathcal{K}} = \widehat{\mathbf{R}}(\mathcal{K}, h)$.

We have the following coupling of the Lévy tree conditioned on its height and the Kesten tree whose proof is given in Section 3.2.

Theorem 3.1 (Coupling for the height conditioning). *Assume (H0)-(H4) hold. Let $h \in (0, +\infty)$. Then, the Lévy tree under \mathbb{N}^ψ conditioned to have height h , denoted $\mathcal{T}_h^{\text{height}}$, is distributed as the truncated Kesten tree $\mathcal{K}_h^{\text{height}}$, more precisely (in \mathbb{T}_1):*

$$(36) \quad (\mathcal{T}_h^{\text{height}}, X_h) \stackrel{(d)}{=} (\mathcal{K}_h^{\text{height}}, h).$$

Remark 3.2 (Reconstruction of the Lévy tree). If ψ is (sub-)critical and H is distributed as $\mathbf{H}(\mathcal{T})$ under $\mathbb{N}^\psi[d\mathcal{T}]$ (that is, with distribution $|c'(h)| dh$ on $(0, +\infty)$) and independent of the Kesten tree \mathcal{K} under \mathbb{P}^ψ , we get that $\mathcal{K}_H^{\text{height}}$ is distributed as the unconditioned Lévy tree \mathcal{T} under $\mathbb{N}^\psi[d\mathcal{T}]$.

By construction, we have the following result on the coupled Kesten sub-trees. Recall the convention (12) and (13) for the inclusion and limit of trees.

Proposition 3.3 (Monotony and local convergence for the height coupling). *Assume (H0)-(H4) hold. We have that for $h' \geq h > 0$:*

$$\mathcal{K}_h^{\text{height}} \subset \mathcal{K}_{h'}^{\text{height}} \quad \text{a.s.},$$

and the following convergence holds:

$$\lim_{h \rightarrow +\infty} \mathcal{K}_h^{\text{height}} \stackrel{\text{a.s.}}{=} \mathcal{K} \quad \text{in} \quad (\mathbb{T}, d_{\text{LGH}}).$$

Proof. The inclusion is clear. Notice that $R_a(\mathcal{K}_h^{\text{height}}) = R_a(\mathcal{K})$ for all a strictly less than $A_h = \inf\{h_i : i \in I \text{ and } h_i + \mathbf{H}(\mathcal{T}_i) > h\}$. Since ψ is (sub-)critical by (H4), we have $\mathbb{N}^\psi[\mathbf{H}(\mathcal{T}) \geq r]$ finite for all $r > 0$. We deduce that the map $h \mapsto A_h$ which is non-decreasing satisfies a.s. $\lim_{h \rightarrow \infty} A_h = +\infty$. This and (11) readily imply the a.s. convergence in $(\mathbb{T}, d_{\text{LGH}})$. \square

We immediately deduce from Theorem 3.1 and Proposition 3.3 the following well known result.

Theorem 3.4 (Local convergence for the height conditioning). *Assume (H0)-(H4) hold. The following local convergence holds in distribution:*

$$(37) \quad \mathcal{T}_h^{\text{height}} \xrightarrow[h \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in} \quad (\mathbb{T}, d_{\text{LGH}}).$$

Remark 3.5 (The super-critical case). Assume (H0)-(H3) hold and that ψ is super-critical. Then, according to (28), the Lévy tree \mathcal{T} conditioned to have a finite mass under \mathbb{N}^ψ , or equivalently a finite height thanks to the Grey condition, is distributed as the Lévy tree under $\mathbb{N}^{\psi_{\theta_0}}$, where θ_0 is the only positive root of $\psi = 0$ and ψ_{θ_0} is defined by (26). Furthermore, it is elementary to check that the branching mechanism ψ_{θ_0} defined by (26) is sub-critical. Thus, under \mathbb{N}^ψ , the conditioned Lévy tree $\mathcal{T}_h^{\text{height}}$, which is still well defined, is distributed as the conditioned Lévy tree $\mathcal{T}_h^{\text{height}}$ under $\mathbb{N}^{\psi_{\theta_0}}[d\mathcal{T}]$. This gives that (36) and (37) hold if ψ is super-critical under (H0)-(H3), but with \mathcal{K} the Kesten tree associated to the sub-critical branching mechanism ψ_{θ_0} .

3.2. Proof of Theorem 3.1. Let $h > 0$ be fixed. Recall the function c defined in (19) so that $c(h) = \mathbb{N}^\psi[\mathbf{H}(\mathcal{T}) > h]$. According to [2, Theorem 3.3], the distribution of $\mathcal{T}_h^{\text{height}}$ and X_h is given by:

$$\llbracket 0, h \rrbracket \otimes_{i \in I'_h} (\mathcal{T}'_i, h'_i) \quad \text{and} \quad h,$$

where $(h'_i, \mathcal{T}'_i)_{i \in I}$ are the atoms of a Poisson point measure on $[0, h] \times \mathbb{T}$ with intensity $dh' \mathbf{N}_{h'}^\psi[d\mathcal{T}]$ with:

$$\mathbf{N}_{h'}^\psi[d\mathcal{T}] = 2\beta \mathbb{N}^\psi[d\mathcal{T}, \mathbf{H}(\mathcal{T}) \leq h - h'] + \int_{(0, +\infty)} r e^{-rc(h-h')} \pi(dr) \mathbb{P}_{r, h'}^\psi(d\mathcal{T}),$$

and $\mathbb{P}_{r, h'}^\psi(d\mathcal{T})$ is the distribution of $\{\varrho\} \otimes_{j \in \mathcal{J}} (\mathcal{T}''_j, \varrho)$ where $(\mathcal{T}''_j)_{j \in \mathcal{J}}$ are the atoms of a Poisson point measure on \mathbb{T} with intensity $r \mathbb{N}^\psi[d\mathcal{T}, \mathbf{H}(\mathcal{T}) \leq h - h']$. Notice that X_h in $\mathcal{T}_h^{\text{height}}$ and $h \in \mathcal{K}_h^{\text{height}}$ are the only element at distance h from the root of their tree.

So, the proof is complete once we get that $(h_i, \mathcal{T}_i)_{i \in I_h^{\text{height}}}$ and $(h'_i, \mathcal{T}'_i)_{i \in I'}$ have the same distribution. Let G be a nonnegative measurable function defined on $\mathbb{R}_+ \times \mathbb{T}$. We have:

$$\mathbb{E} \left[e^{-\sum_{i \in I'} G(h'_i, \mathcal{T}'_i)} \right] = \exp \left\{ - \int_0^h dh' \mathbf{N}_{h'}^\psi \left[1 - e^{-G(h', \mathcal{T})} \right] \right\}$$

and:

$$\begin{aligned} \mathbf{N}_{h'}^\psi \left[1 - e^{-G(h', \mathcal{T})} \right] &= 2\beta \mathbb{N}^\psi \left[(1 - e^{-G(h', \mathcal{T})}) \mathbf{1}_{\{\mathbf{H}(\mathcal{T}) + h' \leq h\}} \right] \\ &\quad + \int_{(0, +\infty)} r e^{-c(h-h')} \pi(dr) \mathbb{E}_{r, h'}^\psi \left[1 - e^{-G(h', \mathcal{T})} \right]. \end{aligned}$$

Now considering first a function G such that $G(h', \{\varrho\} \otimes_{j \in \mathcal{J}'} (\mathcal{T}^j, \varrho)) = \sum_{j \in \mathcal{J}} G_0(h', \mathcal{T}^j)$ with G_0 a nonnegative measurable function defined on $\mathbb{R}_+ \times \mathbb{T}$, we get:

$$\begin{aligned} (38) \quad \mathbb{E}_{r, h'}^\psi \left[e^{-G(h', \mathcal{T})} \right] &= \exp \left\{ -r \mathbb{N}^\psi \left[(1 - e^{-G_0(h', \mathcal{T})}) \mathbf{1}_{\{\mathbf{H}(\mathcal{T}) + h' \leq h\}} \right] \right\} \\ &= \exp \left\{ -r \mathbb{N}^\psi \left[1 - e^{-G_0(h', \mathcal{T})} \mathbf{1}_{\{\mathbf{H}(\mathcal{T}) + h' \leq h\}} \right] + r \mathbb{N}^\psi[\mathbf{H}(\mathcal{T}) > h - h'] \right\} \\ &= \mathbb{E}_r^\psi \left[e^{-G(h', \mathcal{T})} \mathbf{1}_{\{\mathbf{H}(\mathcal{T}) + h' \leq h\}} \right] e^{rc(h-h')}. \end{aligned}$$

By the monotone class theorem, we deduce that for all nonnegative measurable function G :

$$(39) \quad e^{-rc(h-h')} \mathbb{E}_{r, h'}^\psi \left[e^{-G(h', \mathcal{T})} \right] = \mathbb{E}_r^\psi \left[e^{-G(h', \mathcal{T})} \mathbf{1}_{\{\mathbf{H}(\mathcal{T}) + h' \leq h\}} \right]$$

and thus:

$$\begin{aligned} \mathbf{N}_{h'}^\psi[d\mathcal{T}] &= 2\beta \mathbb{N}^\psi[d\mathcal{T}, \mathbf{H}(\mathcal{T}) + h' \leq h] + \int_{(0, +\infty)} r \pi(dr) \mathbb{P}_r^\psi(d\mathcal{T}, \mathbf{H}(\mathcal{T}) + h' \leq h) \\ &= \mathbf{N}^\psi[d\mathcal{T}, \mathbf{H}(\mathcal{T}) + h' \leq h]. \end{aligned}$$

By construction the left hand-side of the above equality times dh' is the intensity of the Poisson point measure $\sum_{i \in I_h^{\text{height}}} \delta_{(h_i, \mathcal{T}_i)}$. We have thus obtained that $(h_i, \mathcal{T}_i)_{i \in I_h^{\text{height}}}$ and $(h'_i, \mathcal{T}'_i)_{i \in I'}$ have the same distribution, which completes the proof.

4. LÉVY TREE WITH A GIVEN MAXIMAL VERTEX ‘‘SIZE’’

4.1. Coupling and convergence for the conditioning by the maximal vertex size. We assume (H0)-(H4) and that the Lévy measure π is non trivial and denote by $\text{supp}(\pi)$ its closed support (in $(0, +\infty)$). In particular $\psi'(0) \geq 0$. For $x \in \text{Br}_\infty(\mathcal{T})$, we define the ‘‘size’’ of the vertex at x by Δ_x given as the right hand-side of (25). Recall that $\text{Br}_\infty(\mathcal{T})$ is \mathbb{N}^ψ -a.e. a countable dense (resp. finite) subset of \mathcal{T} if $\langle \pi, 1 \rangle = \infty$ (resp. $\langle \pi, 1 \rangle < \infty$), and that the set of vertices with size larger than $\varepsilon > 0$ is finite for any $\varepsilon > 0$. We consider the maximal ‘‘size’’ of the vertices in the Lévy tree \mathcal{T} under \mathbb{N}^ψ :

$$\Delta = \max\{\Delta_x : x \in \text{Br}_\infty(\mathcal{T})\},$$

with the convention $\max \emptyset = 0$. If necessary, we shall write $\Delta(\mathcal{T})$ to stress the dependence in the tree \mathcal{T} . According to [11, Proposition 3.1] (where the size of a vertex is called mass therein), the distribution of Δ under \mathbb{N}^ψ is given by, for $\delta > 0$:

$$(40) \quad \mathbb{N}^\psi[\Delta > \delta] = \psi_{\{\delta\}}^{-1}(\bar{\pi}(\delta)),$$

where $\bar{\pi}(\delta)$ is defined in (15) and for $\lambda \geq 0$:

$$(41) \quad \psi_{\{\delta\}}(\lambda) = \psi(\lambda) + \int_{(\delta, \infty)} (1 - e^{-\lambda r}) \pi(dr).$$

Intuitively, the function ψ_δ , with characteristic $(\alpha_{\{\delta\}}, \beta_{\{\delta\}}, \pi_{\{\delta\}})$, is the branching mechanism of the CB process with branching mechanism ψ after removing all the jumps of size strictly larger than δ (and killed when it reaches 0); notice the quadratic parameter is the same $\beta_{\{\delta\}} = \beta$, the Lévy measure is given by $\pi_{\{\delta\}}(dr) = \mathbf{1}_{\{r \leq \delta\}} \pi(dr)$ and the drift is given by $\psi'_{\{\delta\}}(0) = \alpha_{\{\delta\}} = \psi'_\theta(0) + \int_{(\delta, \infty)} r \pi(dr)$. Thanks to [11, Corollary 3.2], we get that $\mathbb{N}^\psi[\Delta \in \cdot]$ and π have the same support in $(0, \infty)$, and the same atoms in $(0, \infty)$, and thus for $\delta > 0$:

$$(42) \quad \pi(\{\delta\}) = 0 \iff \mathbb{N}^\psi[\Delta \geq \delta; \cdot] = \mathbb{N}^\psi[\Delta > \delta; \cdot].$$

We follow [11] (notice (H3) is not assumed therein, but one need them here for \mathcal{T} to be locally compact). Let $\delta > 0$ and consider a random marked tree $(\mathcal{T}_\delta^{\text{node},*}, X_\delta) \in \mathbb{T}_1$ whose distribution is characterized by, for any nonnegative measurable function G defined on \mathbb{T}_1 :

$$(43) \quad \mathbb{E} \left[G \left(\mathcal{T}_\delta^{\text{node},*}, X_\delta \right) \right] = \frac{1}{\mathbb{N}^\psi[\sigma \mathbf{1}_{\{\Delta \leq \delta\}}]} \mathbb{N}^\psi \left[\mathbf{1}_{\{\Delta \leq \delta\}} \int_{\mathcal{T}} \mathbf{m}(dx) G(\mathcal{T}, x) \right].$$

Then define the random marked tree $(\mathcal{T}_\delta^{\text{node}}, X_\delta) \in \mathbb{T}_1$ by:

$$(44) \quad \mathcal{T}_\delta^{\text{node}} = \mathcal{T}_\delta^{\text{node},*} \otimes (\mathcal{T}'_{\{\delta\}}, X_\delta),$$

where $\mathcal{T}'_{\{\delta\}}$ is independent of $(\mathcal{T}_\delta^{\text{node},*}, X_\delta)$ and distributed according to $\mathbb{P}_\delta^\psi(\cdot | \Delta \leq \delta)$. By construction, if δ is not an atom of π , we get that X_δ is the only vertex of $\mathcal{T}_\delta^{\text{node}}$ with size δ and that $\Delta = \delta$. We thus get $\widehat{\mathbb{R}}(\mathcal{T}_\delta^{\text{node}}, X_\delta) = \mathcal{T}_\delta^{\text{node},*}$, with the restriction map $\widehat{\mathbb{R}}$ defined in (10). We denote the probability distribution of $\mathcal{T}_\delta^{\text{node}}$ by $\mathbb{N}^\psi[d\mathcal{T} | \Delta = \delta]$; it is defined even for δ not in the support of the distribution of Δ . By the measurability of the grafting procedure, see [8], we deduce that the map $\delta \mapsto \mathbb{N}^\psi[d\mathcal{T} | \Delta = \delta]$ (from \mathbb{R}_+^* to the set of probability measures on the Polish space \mathbb{T}) is measurable. According to [11, Theorem 5.7], if $\pi \neq 0$ and π **has no atom**, we have:

$$\int_{\delta > 0} \mathbb{N}^\psi[d\mathcal{T} | \Delta = \delta] \mathbb{N}^\psi[\Delta = d\delta] = \mathbb{N}^\psi[d\mathcal{T}].$$

For this reason, if $\pi \neq 0$ has no atom, we shall say that $\mathcal{T}_\delta^{\text{node}}$ **is distributed as the Lévy tree \mathcal{T} conditioned on having one vertex of maximal size δ** .

Now we consider a Kesten tree \mathcal{K} under \mathbb{P}^ψ . We define the tree $\mathcal{K}_\delta^{\text{node},*}$ as follows: let H'_δ be the height of the lower vertex on the spine on which is grafted a tree with a vertex (possibly its root) of size larger or equal than δ . Notice that H'_δ is a.s. finite as $\pi([\delta, \infty)) > 0$. Let E be under \mathbb{P}^ψ an independent exponential random variable with mean $1/\psi'(0)$, with the convention that $E = +\infty$ in the critical case, and set $H_\delta = \min(H'_\delta, E)$. We also identify H_δ with the vertex on the infinite spine of \mathcal{K} at distance H_δ from the root. We define $\mathcal{K}_\delta^{\text{node},*}$ as the closure of the connected component containing the root of the Kesten tree \mathcal{K} when the vertex H_δ has been removed. More formally, we have:

$$(45) \quad \mathcal{K}_\delta^{\text{node},*} = \llbracket 0, H_\delta \rrbracket \otimes_{i \in I_\delta^{\text{node}}} (\mathcal{T}_i, h_i) \quad \text{with} \quad I_\delta^{\text{node}} = \{i \in I : h_i < H_\delta\},$$

Let $(h''_j, \mathcal{T}''_j)_{j \in J}$ be the atoms of a Poisson point process on $\mathbb{R}_+ \times \mathbb{T}$ with intensity $dh \mathbb{N}^\psi[d\mathcal{T}]$ and independent of the Kesten tree \mathcal{K} . Recall $\{\varrho\}$ denote the trivial rooted real tree reduced to the root. We set for $\delta > 0$:

$$(46) \quad \mathcal{T}_{[\delta]}^{\text{node}} = \{\varrho\} \otimes_{j \in J_\delta^{\text{node}}} (\mathcal{T}''_j, \varrho) \quad \text{with} \quad J_\delta^{\text{node}} = \{j \in J : h''_j \leq \delta \quad \text{and} \quad \Delta(\mathcal{T}''_j) \leq \delta\},$$

so that $\mathcal{T}_{[\delta]}^{\text{node}}$ is independent of \mathcal{K} and distributed according to $\mathbb{P}_\delta^\psi(\cdot | \Delta \leq \delta)$, with \mathbb{P}_δ^ψ defined in Section 2.7. We then denote by $\mathcal{K}_\delta^{\text{node}}$ the tree $\mathcal{K}_\delta^{\text{node},*}$ on which the tree $\mathcal{T}_{[\delta]}^{\text{node}}$ is grafted at the top H_δ of the truncated spine $\llbracket 0, H_\delta \rrbracket$:

$$(47) \quad \mathcal{K}_\delta^{\text{node}} = \mathcal{K}_\delta^{\text{node},*} \otimes (\mathcal{T}_{[\delta]}^{\text{node}}, H_\delta).$$

Notice that $\widehat{\mathbb{R}}(\mathcal{K}_\delta^{\text{node}}, H_\delta) = \mathcal{K}_\delta^{\text{node},*}$. The next theorem is proved in Section 4.2.

Theorem 4.1 (Coupling for the maximal size vertex conditioning). *Assume (H0)-(H4) hold and that $\pi \neq 0$. Let $\delta > 0$ such that $\pi(\{\delta\}) = 0$ and $\bar{\pi}(\delta) > 0$. Then, the Lévy tree under \mathbb{N}^ψ conditioned to have maximal vertex size δ , $\mathcal{T}_\delta^{\text{node}}$, is distributed as the truncated Kesten tree $\mathcal{K}_\delta^{\text{node}}$ under \mathbb{P}^ψ , more precisely, we have on \mathbb{T}_1 :*

$$(48) \quad (\mathcal{T}_\delta^{\text{node}}, X_\delta) \stackrel{(d)}{=} (\mathcal{K}_\delta^{\text{node}}, H_\delta) \quad \text{and} \quad (\mathcal{T}_\delta^{\text{node},*}, X_\delta) \stackrel{(d)}{=} (\mathcal{K}_\delta^{\text{node},*}, H_\delta).$$

The first equality in (48) is a consequence of the second one and of (44) and (47).

Remark 4.2 (Reconstruction of the Lévy tree). Let \mathcal{K} be an independent Kesten tree. Assume that π has no atom and $\pi(\mathbb{R}_+^*) = +\infty$ (that is, $\mathbb{N}^\psi[\Delta = 0] = 0$). If Δ is distributed as $\mathbb{N}^\psi[d\Delta(\mathcal{T})]$ and independent of \mathcal{K} and of the atoms $(h_j'', \mathcal{T}_j'')_{j \in J}$ of the independent Poisson point process which appears in (46), we deduce from Theorem 4.1 that $\mathcal{K}_\Delta^{\text{node}}$, defined as in (47) with δ replaced by Δ , is distributed as the unconditioned Lévy tree \mathcal{T} under $\mathbb{N}^\psi[d\mathcal{T}]$. The condition $\pi(\mathbb{R}_+^*) = +\infty$ can be removed at the cost of considering the event $\{\Delta(\mathcal{T}) = 0\}$ which has then a positive measure under $\mathbb{N}^\psi[d\mathcal{T}]$.

By construction, we have the following result on the coupled Kesten sub-trees. Notice the increasing limit of $\mathcal{T}_{[\delta]}^{\text{node}}$ as δ goes to infinity is not locally compact as there is a condensation phenomenon at the root; thus the sequence $(\mathcal{T}_{[\delta]}^{\text{node}})_{\delta > 0}$ does not converge in \mathbb{T} . One would need to enlarge the set \mathbb{T} and change the topology for this sequence to converge, see [5] in the discrete setting of Bienaymé-Galton-Watson trees. For this reason, we can only consider local limit when $\lim_{\delta \rightarrow \infty} d^{\mathcal{K}}(0, H_\delta) = +\infty$, that is when the branching is critical.

Proposition 4.3 (Monotony and local convergence for the maximal size vertex coupling). *Assume (H0)-(H4) hold and $\pi \neq 0$. If $\delta' \geq \delta > 0$ are not atoms of π and $\bar{\pi}(\delta') > 0$, then we have a.s. that:*

$$(49) \quad \mathcal{K}_\delta^{\text{node},*} \subset \mathcal{K}_{\delta'}^{\text{node},*}, \quad H_\delta \in \llbracket \varrho, H_{\delta'} \rrbracket \quad \text{and} \quad \mathcal{T}_{[\delta]}^{\text{node}} \subset \mathcal{T}_{[\delta']}^{\text{node}}.$$

Furthermore, in the critical case, we have $\lim_{\delta \rightarrow \infty} d^{\mathcal{K}}(0, H_\delta) = +\infty$ and, if the support of π is unbounded and π has no atom, the following convergences hold in $(\mathbb{T}, d_{\text{LGH}})$:

$$(50) \quad \mathcal{K}_\delta^{\text{node},*} \xrightarrow[\delta \rightarrow +\infty]{\text{a.s.}} \mathcal{K} \quad \text{and} \quad \mathcal{K}_\delta^{\text{node}} \xrightarrow[\delta \rightarrow +\infty]{\text{a.s.}} \mathcal{K}.$$

Proof. Equation (49) is obvious as by construction $H_\delta' \leq H_{\delta'}$, and thus $H_\delta \leq H_{\delta'}$.

We now consider the critical case. Notice that $R_a(\mathcal{K}_\delta^{\text{node}}) = R_a(\mathcal{K})$ for all a strictly less than H_δ . Then use (11) and that a.s. $\lim_{\delta \rightarrow +\infty} H_\delta = +\infty$ to get the convergences in (50). \square

We deduce from Theorem 4.1 and Proposition 4.3 the following well known result from [11].

Theorem 4.4 (Local convergence for the maximal size vertex conditioning). *Assume (H0)-(H3) hold, the regime is critical (that is $\psi'(0) = 0$), the Lévy measure π has no atom and its support is unbounded. The following local convergence holds in distribution:*

$$\mathcal{T}_\delta^{\text{node}} \xrightarrow[\delta \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in} \quad (\mathbb{T}, d_{\text{LGH}}).$$

Remark 4.5 (The sub-critical case). Notice that (50) no longer holds if ψ is sub-critical. Assume the support of π is unbounded and π has no atom. According to [11, Theorem 1.6] the limit of $\mathcal{T}_\delta^{\text{node}}$ is informally a tree with a condensation vertex at level given by an exponential random variable with mean $1/\psi'(0)$. From Theorem 4.1, we deduce that $\lim_{\delta \rightarrow +\infty} H_\delta = E$, and thus $\mathcal{K}_\delta^{\text{node},*} \xrightarrow[\delta \rightarrow +\infty]{\text{a.s.}} \widehat{\mathcal{K}}_E$, where

$\widehat{\mathcal{K}}_E = \widehat{\mathbf{R}}(\mathcal{K}, E)$ is the Kesten tree whose spine has been cut at level E . Now informally the increasing random sequence $(\mathcal{T}_{[\delta]}^{\text{node}})_{\delta > 0}$ converges to a (non locally compact) tree with condensation at the root; and it should then be possible to give sense to the convergence of $\mathcal{K}_\delta^{\text{node}}$ towards a tree with condensation at vertex E .

4.2. Proof of Theorem 4.1. We assume (H0)-(H4) and let $\delta > 0$ be such that $\pi(\{\delta\}) = 0$ and $\bar{\pi}(\delta) > 0$, see (15). Notice the function $\psi_{\{\delta\}}$ from (41) satisfies also (H0)-(H4) and it is then a bijection of \mathbb{R}_+ . We define ϕ as ψ_θ in (26) with ψ replaced by $\psi_{\{\delta\}}$ and θ by:

$$(51) \quad \psi_{\{\delta\}}^{-1}(\bar{\pi}(\delta)) = \mathbb{N}^\psi[\Delta > \delta] = \mathbb{N}^\psi[\Delta \geq \delta] > 0,$$

(where we used that δ is not an atom of π for the second equality) that is, for $\lambda \geq 0$:

$$(52) \quad \phi(\lambda) = \psi_{\{\delta\}}(\lambda + \psi_{\{\delta\}}^{-1}(\bar{\pi}(\delta))) - \bar{\pi}(\delta).$$

The characteristic $(\alpha_\phi, \beta_\phi, \pi_\phi)$ of ϕ are given by:

$$\alpha_\phi = \phi'(0) = \psi'_{\{\delta\}}(\mathbb{N}^\psi[\Delta \geq \delta]) > 0, \quad \beta_\phi = \beta \quad \text{and} \quad \pi_\phi(dr) = \mathbf{1}_{\{r < \delta\}} e^{-r\mathbb{N}^\psi[\Delta \geq \delta]} \pi(dr).$$

In particular the branching mechanism ϕ is sub-critical. We deduce from (28) that:

$$(53) \quad \mathbb{N}^{\psi_{\{\delta\}}} \left[e^{-\bar{\pi}(\delta)\sigma} F(\mathcal{T}) \right] = \mathbb{N}^\phi [F(\mathcal{T})].$$

Recall we only need to prove the second equality in (48). Let F be a measurable nonnegative function defined on \mathbb{T} . Using [11, Eq. (4.2)] for the first equality and (53) for the second, we have:

$$(54) \quad \mathbb{N}^\psi \left[\mathbf{1}_{\{\Delta \leq \delta\}} F(\mathcal{T}) \right] = \mathbb{N}^{\psi_{\{\delta\}}} \left[e^{-\bar{\pi}(\delta)\sigma} F(\mathcal{T}) \right] = \mathbb{N}^\phi [F(\mathcal{T})].$$

The distribution of the marked random tree $(\mathcal{T}_\delta^{\text{node},*}, X_\delta) \in \mathbb{T}_1$, with X_δ the only vertex of size δ in $\mathcal{T}_\delta^{\text{node}}$ is characterized by (43). We have:

$$\mathbb{N}^\psi \left[\mathbf{1}_{\{\Delta \leq \delta\}} \int_{\mathcal{T}} \mathbf{m}(dx) G(\mathcal{T}, x) \right] = \mathbb{N}^\phi \left[\int_{\mathcal{T}} \mathbf{m}(dx) G(\mathcal{T}, x) \right] = \mathbb{N}^\phi[\sigma] \mathbb{E}^\phi [G(\mathcal{K}_E, E)],$$

where we used (54) for the first equality, and Corollary 2.4 for the second with, under \mathbb{P}^ϕ , E an exponential random variable with mean $\mathbb{N}^\phi[\sigma] = 1/\phi'(0)$ independent of the Kesten tree \mathcal{K} .

Recall the function ϕ depends on δ . Taking $G = 1$ in the previous equality gives:

$$\mathbb{N}^\psi \left[\sigma \mathbf{1}_{\{\Delta \leq \delta\}} \right] = \mathbb{N}^\phi[\sigma].$$

So we deduce from (43) that the proof of the second equality in (48) is complete, once we prove that:

$$(55) \quad \mathbb{E}^\psi \left[G(\mathcal{K}_\delta^{\text{node},*}, H_\delta) \right] = \mathbb{E}^\phi [G(\mathcal{K}_E, E)].$$

Consider the Kesten tree \mathcal{K} under \mathbb{E}^ψ . The intensity of grafting a tree having a vertex with size larger than δ is $dh \mathbb{N}^\psi[\Delta \geq \delta]$ with \mathbb{N}^ψ given in (30), so the height H'_δ at which is grafted a tree with maximal vertex size larger than δ is an exponential random variable with parameter:

$$\begin{aligned} \mathbb{N}^\psi[\Delta \geq \delta] &= 2\beta \mathbb{N}^\psi[\Delta \geq \delta] + \int_0^{+\infty} r\pi(dr) \mathbb{P}_r^\psi(\Delta \geq \delta) \\ &= 2\beta \mathbb{N}^\psi[\Delta \geq \delta] + \int_{(0, \delta)} r\pi(dr) \left(1 - e^{-r\mathbb{N}^\psi[\Delta \geq \delta]} \right) + \int_{[\delta, \infty)} r\pi(dr) \\ &= \psi'_\delta(\mathbb{N}^\psi[\Delta \geq \delta]) - \psi'(0) \\ &= \phi'(0) - \psi'(0), \end{aligned}$$

where we used (41) for the third equality with $\pi(\{\delta\}) = 0$, and (52) for the last. We deduce that $\min(H'_\delta, E)$ with E independent of \mathcal{K} and distributed under \mathbb{P}^ψ as an exponential random variable with mean $1/\psi'(0) \in (0, \infty]$ is an exponential random variable with mean $1/\phi'(0) \in (0, \infty)$, which is the distribution of E under \mathbb{P}^ϕ .

Since $\bar{\pi}(\delta) > 0$, the random tree $\mathcal{K}_\delta^{\text{node},*}$ conditionally on H_δ is distributed under \mathbb{P}^ψ as the cut spine $\llbracket 0, H_\delta \rrbracket$ on which are grafted trees:

$$\mathcal{K}_\delta^{\text{node},*} \stackrel{(d)}{=} \llbracket 0, H_\delta \rrbracket \otimes_{i \in I'} (\mathcal{T}'_i, h'_i),$$

where $(h'_i, \mathcal{T}'_i)_{i \in I'}$ are the atoms of a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}$ with intensity $dh \mathbf{N}^\psi[d\mathcal{T}; \Delta < \delta]$. So to prove the second equality in distribution of (48), we deduce from (55) that it is enough to check that the grafting intensity on the spine $\llbracket 0, H_\delta \rrbracket$ for $\mathcal{K}_\delta^{\text{node},*}$, that is $\mathbf{N}^\psi[d\mathcal{T}; \Delta < \delta]$, coincides with the grafting intensity on the spine $\llbracket 0, E \rrbracket$ for \mathcal{K} under \mathbb{P}^ϕ , that is, with $\mathbf{N}^\phi[d\mathcal{T}]$.

Recall \mathbb{P}_r^ϕ (with ψ instead of ϕ) defined in Section 2.7 denotes the distribution of the random tree $\mathcal{T}_{[r]} = \{\varrho\} \otimes_{i \in I''} (\mathcal{T}''_i, \varrho)$, where $(h''_i, \mathcal{T}''_i)_{i \in I''}$ are the atoms of a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}$ with intensity $dh \mathbf{N}^\phi[d\mathcal{T}]$ and $I''_r = \{i \in I'' : h''_i \leq r\}$. We first notice that for G a measurable nonnegative function on $\mathbb{R}_+ \times \mathbb{T}$ and $r > 0$:

$$\begin{aligned} \mathbb{E}_r^\phi \left[e^{-\sum_i G(h_i, \mathcal{T}_i)} \right] &= \exp \left\{ - \int_0^r ds \mathbf{N}^\phi[1 - e^{-G(s, \mathcal{T})}] \right\} \\ &= \exp \left\{ - \int_0^r ds \mathbf{N}^\psi[(1 - e^{-G(s, \mathcal{T})}) \mathbf{1}_{\{\Delta < \delta\}}] \right\} \\ &= \exp \left\{ - \int_0^r ds \mathbf{N}^\psi[1 - e^{-G(s, \mathcal{T})} \mathbf{1}_{\{\Delta < \delta\}}] + r \mathbf{N}^\psi[\Delta \geq \delta] \right\} \\ &= e^{r \mathbf{N}^\psi[\Delta \geq \delta]} \mathbb{E}_r^\psi \left[e^{-\sum_i G(h_i, \mathcal{T}_i) \mathbf{1}_{\{\Delta(\mathcal{T}_i) < \delta\}}} \right], \end{aligned}$$

where we used (54) for the second equality. Arguing similarly as in the proof of (39) given by the computations in (38), we deduce that for $r \in (0, \delta)$:

$$\mathbb{P}_r^\psi(d\mathcal{T}; \Delta < \delta) = e^{-r \mathbf{N}^\psi[\Delta \geq \delta]} \mathbb{P}_r^\phi(d\mathcal{T}).$$

Then, using (54), we obtain:

$$\begin{aligned} \mathbf{N}^\psi[d\mathcal{T}; \Delta < \delta] &= 2\beta \mathbf{N}^\psi[d\mathcal{T}; \Delta < \delta] + \int_{(0, +\infty)} r \pi(dr) \mathbb{P}_r^\psi(d\mathcal{T}; \Delta < \delta) \\ &= 2\beta \mathbf{N}^\phi[d\mathcal{T}] + \int_{(0, \delta)} r \pi(dr) e^{-r \mathbf{N}^\psi[\Delta \geq \delta]} \mathbb{P}_r^\phi(d\mathcal{T}) \\ &= 2\beta \mathbf{N}^\phi[d\mathcal{T}] + \int_{(0, \delta)} r \pi_\phi(dr) \mathbb{P}_r^\phi(d\mathcal{T}) \\ &= \mathbf{N}^\phi[d\mathcal{T}]. \end{aligned}$$

This concludes the proof of the second equality in (48). (We already noticed that the first equality in (48) is a direct consequence of the second one.)

5. LÉVY TREE WITH A GIVEN MASS

5.1. Subordinator and Bernstein function. Let \mathbb{P}_x^φ denote the distribution of a subordinator $\mathbf{S} = (S_t)_{t \geq 0}$ starting at $x \geq 0$ at time 0, that is $S_0 = x$ a.e., with general Laplace exponent φ :

$$(56) \quad \varphi(\lambda) = \kappa \lambda + \int_{(0, +\infty)} (1 - e^{-\lambda r}) \nu(dr) \quad \text{for } \lambda \geq 0,$$

where the drift κ belongs to \mathbb{R}_+ and the Lévy measure ν is a measure on $(0, +\infty)$ such that $\int_{(0, +\infty)} (1 \wedge r) \nu(dr)$ is finite. In particular, we have for $\lambda, t \geq 0$ that:

$$\mathbb{E}_x^\varphi[e^{-\lambda S_t}] = e^{-t\varphi(\lambda) - \lambda x}.$$

We recall that:

$$(57) \quad \lim_{\lambda \rightarrow \infty} \frac{\varphi(\lambda)}{\lambda} = \kappa.$$

The corresponding α -potential measure $U^{(\alpha)}$ on \mathbb{R}_+ , for $\alpha \geq 0$, is defined by $U^{(\alpha)}(A) = \int_0^\infty dt e^{-\alpha t} \mathbb{P}(S_t \in A)$ for all Borel sets $A \subset \mathbb{R}_+$. Its Laplace transform is given by:

$$(58) \quad \int_{[0, \infty)} e^{-\lambda r} U^{(\alpha)}(dr) = \frac{1}{\alpha + \varphi(\lambda)} \quad \text{for all } \lambda \geq 0.$$

We simply write U for the 0-potential $U^{(0)}$.

Remark 5.1. We recall that a Bernstein function, say φ_0 , is the Laplace exponent of a killed subordinator (that is, of the form $\varphi_0 = \varphi + c_0$, with $c_0 \geq 0$) and that φ_0 is a special Bernstein function if the function ϕ defined by $\phi(\lambda) = \lambda/\varphi_0(\lambda)$ is also a Bernstein function, see [33, Sec. 11].

Provided that $\mathbf{S} \neq 0$, the Laplace exponent φ of the subordinator \mathbf{S} is a special Bernstein function if and only if the 0-potential measure $U(dr)$ of φ can be written as $c\delta_0(dr) + u(r)dr$, with $c \geq 0$ (and δ_0 the Dirac mass at 0) and the nonnegative density function u defined on \mathbb{R}_+^* is non-increasing. Thus we have that $u(\infty) = 1/(\kappa + \int_{(0, \infty)} r\nu(dr))$, and also, if $\kappa > 0$ or $\nu(\mathbb{R}_+^*) = +\infty$, that $c = 0$, $\lim_{r \rightarrow 0^+} ru(r) = 0$ and $\lim_{r \rightarrow 0^+} \int_0^r u(s) ds = 0$.

5.2. The subordinator conditioned to die at a given level. In this section, we follow [29]. Let $\mathbf{S} = (S_t)_{t \geq 0}$ be a subordinator with Laplace exponent given by (56). We assume that $\varphi(+\infty) = +\infty$, that is $\kappa > 0$ or $\nu((0, 1)) = +\infty$. Notice that the 0-potential $U = U^{(0)}$ is well defined as for any $\lambda > 0$:

$$U([0, x]) = \int_0^\infty dt \mathbb{P}^\varphi(S_t \leq x) \leq \int_0^\infty dt \mathbb{E}^\varphi[e^{\lambda(x-S_t)}] = \frac{1}{\varphi(\lambda)} e^{\lambda x} < +\infty.$$

Notice also that $\varphi(+\infty) = +\infty$ implies that $\mathbb{P}^\varphi(S_t = 0) = 0$ for all $t > 0$ and thus $\mathbb{P}^\varphi(S_t = a) = 0$ for all $a > 0$, see [14] p. 30. For $\alpha \geq 0$, we deduce that the potential measure $U^{(\alpha)}$ has no atom. We recall that if $U^{(\alpha)}$ is absolutely continuous w.r.t. Lebesgue measure on \mathbb{R}_+ for some $\alpha \geq 0$, then it is absolutely continuous w.r.t. Lebesgue measure on \mathbb{R}_+ for any $\alpha \geq 0$.

The results from [29] are stated for $\alpha = 0$, but they can be extended to the case $\alpha \geq 0$ with minor modifications of the proofs which we shall omit. We now fix $\alpha \geq 0$ and assume that the potential $U^{(\alpha)}$ has a continuous density $u^{(\alpha)}$ w.r.t. the Lebesgue measure on $(0, \infty)$, see [29, Hyp. (DA)] when $\alpha = 0$.

Let $r > 0$. Following Lemma 2.4 therein, the process $(M_t^r)_{t \geq 0}$ defined by:

$$(59) \quad M_t^r = e^{-\alpha t} u^{(\alpha)}(r - S_t) \mathbf{1}_{\{S_t < r\}}$$

is a super-martingale. Using that the support of the distribution of S_t is \mathbb{R}_+ for $t > 0$, see [34], it is then easy to check that the density $u^{(\alpha)}$ is positive on $(0, +\infty)$. So, we can define the following h-transform of the Lévy process \mathbf{S} with lifetime ζ^r and starting at $x \in [0, r)$ under $\mathbb{P}_x^{\varphi, r}$ defined by, for all $t \in [0, \infty)$, $A \in \mathcal{F}_t = \sigma((S_s)_{s \in [0, t]})$:

$$(60) \quad \mathbb{P}_x^{\varphi, r}(A, t < \zeta^r) = e^{-\alpha t} \mathbb{E}_x^\varphi \left[\frac{u^{(\alpha)}(r - S_t)}{u^{(\alpha)}(r - x)} \mathbf{1}_{\{S_t < r\}} \mathbf{1}_A \right].$$

Since \mathbb{P}_x^φ -a.s. $\lim_{t \rightarrow \infty} S_t = +\infty$, we get that $\lim_{t \rightarrow \infty} \mathbb{P}_x^{\varphi, r}(t < \zeta^r) = 0$, and thus the lifetime ζ^r is a.s. finite. Under $\mathbb{P}_x^{\varphi, r}$, the process \mathbf{S} is started at x , killed at time ζ^r and continuously absorbed at r , that is $\mathbb{P}_x^{\varphi, r}$ a.s. $S_{\zeta^r-} = r$. Indeed, for $y \geq 0$, set $\tau_y = \inf\{t \geq 0 : S_t > y\}$. Extending Theorem 2.5 in [29] given for $\alpha = 0$ to any $\alpha \geq 0$, we also get that for $0 \leq x < b < r$, $t \geq 0$ and $A \in \mathcal{F}_t$:

$$(61) \quad \mathbb{P}_x^{\varphi, r}(A, t < \tau_b) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_x^\varphi [e^{-\alpha \tau_r}; A, t < \tau_b, S_{\tau_r-} \geq r - \varepsilon]}{\mathbb{E}_x^\varphi [e^{-\alpha \tau_r}; S_{\tau_r-} \geq r - \varepsilon]}.$$

For $\alpha = 0$, see [29], this reduces to:

$$\mathbb{P}_x^{\varphi, r}(A, t < \tau_b) = \lim_{\varepsilon \downarrow 0} \mathbb{P}_x^\varphi(A, t < \tau_b \mid S_{\tau_r-} \geq r - \varepsilon).$$

Eventually, we simply write $\mathbb{P}_x^{\varphi, r}$ for \mathbb{P}_x^φ when $x = 0$.

5.3. On the density of σ under the excursion measure. We assume (H0)-(H4) and thus $\psi'(0) \geq 0$, which corresponds to the (sub-)critical case. We shall consider the associated critical branching mechanism ψ_{crit} defined by:

$$\psi_{\text{crit}}(\lambda) = \psi(\lambda) - \psi'(0)\lambda.$$

Notice that ψ' and ψ'_{crit} are Bernstein functions as for $\lambda \geq 0$:

$$\psi'(\lambda) = \psi'(0) + \psi'_{\text{crit}}(\lambda) \quad \text{and} \quad \psi'_{\text{crit}}(\lambda) = 2\beta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda r}) r\pi(dr).$$

Let $\mathbf{V} = (V_t)_{t \geq 0}$ be a subordinator with Laplace exponent ψ'_{crit} . Let $\mathbf{T} = (T_t)_{t \geq 0}$ be a subordinator independent of \mathbf{V} with no killing term, no drift and Lévy measure $\mathbb{N}[d\sigma]$, that is, with Laplace exponent ψ^{-1} by (24). The process $\mathbf{S} = (S_t(\omega) = T_{V_t(\omega)}(\omega))_{t \geq 0}$ is thus a subordinator with Laplace exponent:

$$(62) \quad \varphi = \psi'_{\text{crit}} \circ \psi^{-1}.$$

Remark 5.2. Since φ is the Laplace exponent of the subordinator \mathbf{S} , it can be written as in (56). Notice that (H0)-(H4) imply that $\psi(+\infty) = \psi'_{\text{crit}}(+\infty) = \varphi(+\infty) = +\infty$. Using that $x(1 - e^{-x}) \geq e^{-x} - 1 + x \geq x(1 - e^{-x})/2$ for $x \geq 0$, we get that for $\lambda \geq 0$:

$$(63) \quad \psi(\lambda) \leq \lambda\psi'(\lambda) \leq 2\psi(\lambda).$$

This implies that:

$$(64) \quad \lim_{\lambda \rightarrow \infty} \frac{\varphi(\lambda)}{\lambda} = \lim_{\lambda \rightarrow \infty} \frac{\psi'(\lambda)}{\psi(\lambda)} = 0,$$

and thus $\kappa = 0$ by (57). We deduce that $\nu((0, 1)) = \infty$ and from [14] p. 30 that $\mathbb{P}(S_t = 0) = 0$ for all $t > 0$ and thus $\mathbb{P}(S_t = a) = 0$ for all $a > 0$. In particular, we get that the corresponding α -potential has no atom on \mathbb{R}_+ .

We denote by $U^{(\alpha)}$ the α -potential associated to \mathbf{S} .

Lemma 5.3. *Under (H0)-(H4), with $\alpha = \psi'(0) \geq 0$, we have for any Borel set $A \subset \mathbb{R}_+$ that:*

$$U^{(\alpha)}(A) = \mathbb{N}^\psi[\sigma \mathbf{1}_A(\sigma)].$$

In particular the measure $\mathbb{N}^\psi[d\sigma]$ has no atom on $[0, \infty)$.

Proof. Using that $\mathbb{N}^\psi[1 - e^{-\lambda\sigma}] = \psi^{-1}(\lambda)$, we get that for $\lambda > 0$:

$$(65) \quad \int_{[0,\infty)} e^{-\lambda r} U^{(\alpha)}(dr) = \int_0^\infty e^{-\alpha t} \mathbb{E}[e^{-\lambda S_t}] dt = \frac{1}{\psi' \circ \psi^{-1}(\lambda)} = \partial_\lambda \mathbb{N}^\psi[1 - e^{-\lambda\sigma}] = \mathbb{N}^\psi[\sigma e^{-\lambda\sigma}],$$

and thus $U^{(\alpha)}(A) = \mathbb{N}^\psi[\sigma \mathbf{1}_A(\sigma)]$ for any Borel set $A \subset \mathbb{R}_+$. By Remark 5.2, the measure $U^{(\alpha)}$ has no atom on \mathbb{R}_+ , and thus, as $\mathbb{N}^\psi[\sigma = 0] = 0$ by (24), the excursion measure $\mathbb{N}^\psi[d\sigma]$ has no atom on $[0, \infty)$. \square

Notice from (65) that σ under \mathbb{N}^ψ has a density w.r.t. the Lebesgue measure on $[0, \infty)$ if and only if the α -potential $U^{(\alpha)}$ of \mathbf{S} has a density w.r.t. the Lebesgue measure on $(0, \infty)$.

Remark 5.4 (The critical stable case). We consider the critical stable case $\psi(\lambda) = \lambda^a$ with $a \in (1, 2]$ (and thus $\psi'(0) = \alpha = 0$). From (65), we deduce that, with $u = u^{(0)}$ the density of $U^{(0)}$ on $(0, +\infty)$:

$$f_\sigma(r) = \frac{1}{a\Gamma\left(\frac{a-1}{a}\right) r^{(a+1)/a}} \quad \text{and} \quad \frac{u(r-x)}{u(r)} = \left(\frac{r}{r-x}\right)^{1/a} \quad \text{for } r > x > 0.$$

5.4. The Lévy tree conditioned by the total mass. We assume (H0)-(H4), and we refer to Section 5.6 for the super-critical case. For simplicity we write σ for the total mass $\mathbf{m}(\mathcal{T})$ of the Lévy tree \mathcal{T} (see Section 2.6), and the tree \mathcal{T} which is implicitly considered in σ shall be clear from the context. By [26, Corollary 1.29], there exists a regular version of the conditional distribution of \mathcal{T} given its total mass, $\mathbb{N}^\psi[d\mathcal{T} | \sigma = r]$ for $r > 0$, and let $\mathcal{T}_r^{\text{mass}}$ denote a \mathbb{T} -valued random variable with this distribution. In what follows we shall give a nice representation of $\mathcal{T}_r^{\text{mass}}$ and of $\mathbb{N}^\psi[d\mathcal{T} | \sigma = r]$.

Remark 5.5 (The stable case). Assume that $\psi(\lambda) = \lambda^a$ with $a \in (1, 2]$. It is possible to give an explicit construction of $\mathcal{T}_r^{\text{mass}}$, see [17, Sec. 3.2]. In particular, we have the following scaling property for $r > 0$:

$$(66) \quad (\mathcal{T}_r^{\text{mass}}, d, \varrho) \stackrel{(d)}{=} (\mathcal{T}_1^{\text{mass}}, r^{1-1/a}d, \varrho).$$

We can also represent the Kesten tree as:

$$(67) \quad \mathcal{K} = \llbracket 0, \infty \rrbracket \otimes_{i \in I} (\mathcal{T}_{\sigma_i}^{\text{mass}}, h_i),$$

where $(h_i, \sigma_i, \mathcal{T}_{\sigma_i}^{\text{mass}})_{i \in I}$ are the atoms of a Poisson point measure on \mathbb{R}_+^2 with intensity $dh \mathbf{N}^\psi[\sigma \in dr] \mathbf{N}^\psi[d\mathcal{T} | \sigma = r]$. The process $\mathbf{S}^\mathcal{K} = (S_h^\mathcal{K})_{h>0}$ defined by:

$$(68) \quad S_h^\mathcal{K} = \sum_{h_i \leq h} \sigma_i,$$

is, thanks to (31), a subordinator with Laplace exponent:

$$(69) \quad \varphi = \psi'_{\text{crit}} \circ \psi^{-1}.$$

When there is no ambiguity, we shall simply write \mathbf{S} for $\mathbf{S}^\mathcal{K}$. Notice that $\varphi(0) = 0$.

Notice that the distribution of σ under \mathbb{N}^ψ has no atom by Lemma 5.3 and $\mathbb{N}[\sigma = +\infty] = 0$ as the branching mechanism is (sub-)critical. We further assume in this section that the distribution of the total mass σ on $(0, +\infty)$ has a continuous density, say f_σ , w.r.t. the Lebesgue measure:

$$(H5) \quad \boxed{\mathbb{N}^\psi[\sigma \in dr] = f_\sigma(r) dr \quad \text{and} \quad f_\sigma \text{ is continuous on } \mathbb{R}_+^*}.$$

In particular, Assumption (H5) (and also (H6) below) is satisfied in the critical stable case $\psi(\lambda) = \lambda^a$ with $a \in (1, 2]$, see Remark 5.4.

Under (H5), we deduce from Remark 5.2 and Lemma 5.3 that the α -potential $U^{(\alpha)}$ of \mathbf{S} has no Dirac mass at 0 and has a continuous density, say $u^{(\alpha)}$, w.r.t. the Lebesgue measure on $(0, \infty)$; thus we have $U^{(\alpha)}(dr) = u^{(\alpha)}(r) dr$ and from Lemma 5.3 that:

$$(70) \quad r f_\sigma(r) = u^{(\alpha)}(r) \quad \text{for all } r \in (0, +\infty).$$

We define a modified Kesten tree $\mathcal{K}_r^{\text{mass}}$ as follows. Let $r > 0$ and let \mathbf{S}^r be with distribution $\mathbb{P}_0^{\varphi, r}$ from Section 5.2, that is, when $\alpha = 0$ defined as \mathbf{S} be continuously absorbed at r . Denote by $(h'_i, \sigma'_i)_{i \in I'}$ the jumping times and the jumps of \mathbf{S}^r , and ζ^r the finite lifetime of \mathbf{S}^r . We consider the \mathbb{T} -valued random variable defined by:

$$(71) \quad \mathcal{K}_r^{\text{mass}} = \llbracket 0, \zeta^r \rrbracket \otimes_{i \in I'} (\mathcal{T}_{\sigma'_i}^{\text{mass}}, h'_i),$$

where the compact rooted random trees $(\mathcal{T}_{\sigma'_i}^{\text{mass}})_{i \in I'}$ are conditionally on \mathbf{S}^r independent with $\mathcal{T}_{\sigma'_i}^{\text{mass}}$ distributed according to $\mathbb{N}^\psi[d\mathcal{T} | \sigma = \sigma'_i]$ for all $i \in I'$. In particular, we see ζ^r as a leaf of $\mathcal{K}_r^{\text{mass}}$, and $(\mathcal{K}_r^{\text{mass}}, \zeta^r)$ as an element of \mathbb{T}_1 . We shall denote by $\mathbb{P}^{\varphi, r}$ the distribution of $(\mathcal{K}_r^{\text{mass}}, \zeta^r)$.

For $h < \zeta^r$, we shall see h as the vertex of the branch $\llbracket \varrho, \zeta^r \rrbracket$ at distance h from the root, as well as the vertex of the semi-infinite spine of the Kesten tree \mathcal{K} at distance h from the root. From the representation (67) of the subordinator $\mathbf{S}^\mathcal{K}$ from the Kesten tree \mathcal{K} , notice that $S_h^\mathcal{K}$ is the total mass of the random tree $\widehat{\mathbf{R}}(\mathcal{K}, h)$, the Kesten tree with the spine truncated at level h , see (10); it is a.s. larger than the total mass of $\mathbf{R}_h(\mathcal{K})$, the Kesten tree truncated at level h ; the difference comes from the fact that $S_h^\mathcal{K}$ counts also the mass above h of the sub-trees grafted on the semi-infinite spine below level h .

Remark 5.6 (On the distributions of \mathcal{K} and $\mathcal{K}_r^{\text{mass}}$). We recall that for $(\mathbf{t}, x) \in \mathbb{T}_1$, a tree \mathbf{t} with a distinguished vertex $x \in \mathbf{t}$, one can also see x as a distinguished vertex of the truncated tree $\widehat{\mathbf{R}}(\mathbf{t}, x)$, which is then considered as an element of \mathbb{T}_1 . Using (60), for any nonnegative measurable function G defined on the set \mathbb{T}_1 , we have, with $\mathbf{S}^{\mathcal{K}}$ given by (68) and the representation (67) of the Kesten tree, that, with $\alpha = \psi'(0) \geq 0$, for all $h > 0$:

$$(72) \quad \mathbb{E}^{\varphi, r} \left[\mathbf{1}_{\{\zeta^r > h\}} G(\widehat{\mathbf{R}}(\mathcal{K}_r^{\text{mass}}, h)) \right] = e^{-\alpha h} \mathbb{E}^{\psi} \left[\frac{u^{(\alpha)}(r - S_h^{\mathcal{K}})}{u^{(\alpha)}(r)} \mathbf{1}_{\{S_h^{\mathcal{K}} < r\}} G(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right].$$

Notice that the right hand-side is a measurable function of $r \in (0, +\infty)$ which is lower semi-continuous as the non-decreasing limits as $n \rightarrow +\infty$ of the continuous functions $r \mapsto e^{-\alpha h} \mathbb{E}^{\psi} \left[\frac{n \wedge u^{(\alpha)}(r - S_h^{\mathcal{K}})}{u^{(\alpha)}(r)} \mathbf{1}_{\{S_h^{\mathcal{K}} < r\}} G(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right]$ for $n \in \mathbb{N}$ (the continuity of the latter functions is a consequence of the continuity of $u^{(\alpha)}$ and the fact that \mathbb{P}^{ψ} -a.s. $S_h^{\mathcal{K}} \neq r$).

The proof of the next result is given in Section 5.7.

Theorem 5.7 (A first representation for the mass conditioning). *Assume (H0)-(H5) hold. For $r \in (0, +\infty)$, let $\mathcal{T}_r^{\text{mass}}$ be the Lévy tree under \mathbb{N}^{ψ} conditioned to have total mass r , U^r a leaf of $\mathcal{T}_r^{\text{mass}}$ chosen uniformly (w.r.t. the probability measure $r^{-1} \mathbf{m}^{\mathcal{T}_r^{\text{mass}}}(dx)$) and on $\{\mathbf{H}(U^r) \geq h\}$ denote by U_h^r the vertex of $[[\varrho, U^r]]$ at distance h from the root of $\mathcal{T}_r^{\text{mass}}$. For any nonnegative measurable function G defined on the set \mathbb{T}_1 , we have, with $\mathbf{S}^{\mathcal{K}}$ given by (68) and the representation (67) of the Kesten tree, that dr-a.e., for all $h > 0$:*

$$(73) \quad \mathbb{N}_r^{\psi} \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] = \mathbb{E}^{\varphi, r} \left[\mathbf{1}_{\{\zeta^r > h\}} G(\widehat{\mathbf{R}}(\mathcal{K}_r^{\text{mass}}, h)) \right].$$

For an other local absolute continuity representation, we also refer to [16, Proposition 3] in the critical quadratic case ($\psi(\lambda) = \beta\lambda^2$) using the contour process of the Lévy tree (called in this quadratic case the continuum Brownian tree) given by the Brownian excursion and the contour process of the corresponding Kesten tree given by two independent 3-dimensional Bessel processes, and to [15, Theorem 4] in the stable non-quadratic case ($\psi(\lambda) = \lambda^a$ with $a \in (1, 2)$) with a representation of the subordinator killed continuously at level r (that is, under $\mathbb{P}_0^{\varphi, r}$) using scaling of bridges.

We deduce the following corollary using first the regularity of $r \mapsto \mathbb{E}^{\varphi, r} \left[\mathbf{1}_{\{\zeta^r > h\}} G(\widehat{\mathbf{R}}(\mathcal{K}_r^{\text{mass}}, h)) \right]$, see Remark 5.6 and that the distribution of $(\mathcal{T}_r^{\text{mass}}, U^r)$ under \mathbb{N}_r^{ψ} is completely determined by the functionals $\mathbb{N}_r^{\psi} \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right]$ when h ranges over \mathbb{R}_+ and G ranges over the set of nonnegative bounded measurable functions defined on \mathbb{T}_1 .

Corollary 5.8 (Representation for the mass conditioning). *Assume (H0)-(H5) hold. Let $r \in (0, +\infty)$. The distribution of $(\mathcal{K}_r^{\text{mass}}, \zeta^r)$ under $\mathbb{P}^{\varphi, r}$ is a regular version of the distribution of $(\mathcal{T}_r^{\text{mass}}, U^r)$, where the random tree $\mathcal{T}_r^{\text{mass}}$ is distributed as the Lévy tree under \mathbb{N}^{ψ} conditioned to have mass r , and U^r is a leaf of $\mathcal{T}_r^{\text{mass}}$ chosen uniformly (that is, w.r.t. the probability measure $r^{-1} \mathbf{m}^{\mathcal{T}_r^{\text{mass}}}(dx)$).*

Remark 5.9 (Scaling properties of $\mathcal{K}_r^{\text{mass}}$). In the stable critical case $\psi(\lambda) = \lambda^a$ with $a \in (1, 2]$, a regular version of \mathbb{N}_r^{ψ} can be obtained by scaling using (66), see Remark 5.5. Let us check it also coincides with the one given by (73).

First, we have $\varphi(\lambda) = a\lambda^{1-1/a}$. Thus, the subordinator \mathbf{S} enjoy the scaling property $r^{-1} \mathbf{S}_{\bullet} \stackrel{(d)}{=} \mathbf{S}_{(r^{-1+1/a}\bullet)}$. By (60) with $\alpha = 0$, the process \mathbf{S}^r started from 0 enjoy a scaling property, see the formula for the potential densities ratio in Remark 5.4: $(S_t^r, t \in [0, \zeta_r)) \stackrel{d}{=} (r S_{r^{-1+1/a}t}^1, t \in [0, r^{1-1/a}\zeta_1))$. In particular ζ^r and $r^{(1-1/a)}\zeta^1$ have the same distribution. Furthermore, using (66), the grafted trees in $\mathcal{K}_r^{\text{mass}}$ enjoy also a similar scaling property (notice that the masses σ'_j in $\mathcal{K}_r^{\text{mass}}$ are distributed as the masses $r\sigma'_j$ in $\mathcal{K}_1^{\text{mass}}$). Then using the definition of $\mathcal{K}_r^{\text{mass}}$ from (71), we deduce that $(\mathcal{K}_r^{\text{mass}}, d, \varrho)$ is distributed as $(\mathcal{K}_1^{\text{mass}}, r^{1-1/a}d, \varrho)$, which is in agreement with (66). So the right-hand side of (73) enjoys the same scaling property in r as the left hand-side. This implies that the regular version of the conditional distribution of $\mathcal{T}_r^{\text{mass}}$ given by scaling and the one given by (73) coincide.

5.5. Convergence for the conditioning by the total mass in the critical case. We assume in this section that (H0)-(H3) and (H5) hold, that the regime is critical (that is, $\alpha = \psi'(0) = 0$) as well as a regularity condition on the density f_σ of σ under \mathbb{N}^ψ :

$$(H6) \quad \boxed{\psi'(0) = 0 \quad \text{and the function } r \mapsto r f_\sigma(r) \text{ is non-increasing on } (0, \infty).}$$

We simply write u for $u^{(0)}$, the density of the potential of the subordinator with Laplace exponent $\varphi = \psi' \circ \psi^{-1}$.

Remark 5.10 (On Assumption (H6)). Recall we assume (H0)-(H3) and (H5). Notice that Assumption (H6) is equivalent, by (70), to the function u being continuous and non-increasing on $(0, \infty)$. By Remark 5.1, the function u (or equivalently the function $r \mapsto r f_\sigma(r)$) is then non-increasing if and only if φ is a special Bernstein function, that is $\lambda/\varphi(\lambda)$ is the Laplace exponent of a subordinator. (Notice this subordinator is not killed as $\lim_{\lambda \rightarrow 0^+} \lambda/\varphi(\lambda) = \lim_{\lambda \rightarrow 0^+} \psi(\lambda)/\psi'(\lambda) = 0$ by (63).)

Assumption (H6) holds in particular in the critical stable case, see also Remark 5.4.

We now state the main result of this section.

Proposition 5.11 (Strong local convergence for the mass conditioning). *Assume (H0)-(H6) hold (in particular the branching mechanism ψ is critical, that is $\psi'(0) = 0$). Let F be bounded nonnegative measurable function on \mathbb{T} . We have:*

$$(74) \quad \lim_{r \rightarrow \infty} \mathbb{N}_r^\psi [F(\mathbf{R}_h(\mathcal{T}_r^{\text{mass}}))] = \lim_{r \rightarrow \infty} \mathbb{E}^{\varphi, r} [F(\mathbf{R}_h(\mathcal{K}_r^{\text{mass}}))] = \mathbb{E}^\psi [F(\mathbf{R}_h(\mathcal{K}))].$$

Proof. By (H6), we get that the 0-potential density u exists and is non-increasing, so we get:

$$(75) \quad \frac{u(r - S_h^\mathcal{K})}{u(r)} \mathbf{1}_{\{S_h^\mathcal{K} < r\}} \geq \mathbf{1}_{\{S_h^\mathcal{K} < r\}} \quad \text{and thus} \quad \liminf_{r \rightarrow \infty} \frac{u(r - S_h^\mathcal{K})}{u(r)} \mathbf{1}_{\{S_h^\mathcal{K} < r\}} \geq 1.$$

Let G be a nonnegative measurable function on \mathbb{T}_1 . By Fatou lemma, we deduce from (72), (73) and (75) that:

$$\liminf_{r \rightarrow \infty} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] \geq \mathbb{E}^\psi \left[G(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right].$$

Since $\mathbb{N}_r^\psi[\mathbf{H}(U^r) > h] \leq 1$ and, by the previous inequality with $G = 1$, $\liminf_{r \rightarrow \infty} \mathbb{N}_r^\psi[\mathbf{H}(U^r) > h] \geq 1$, we get that:

$$(76) \quad \lim_{r \rightarrow \infty} \mathbb{N}_r^\psi[\mathbf{H}(U^r) > h] = 1.$$

Now assume that G is also bounded by 1 and set $G^* = 1 - G$. We get:

$$\begin{aligned} \limsup_{r \rightarrow \infty} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] &= 1 - \liminf_{r \rightarrow \infty} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G^*(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] \\ &\leq 1 - \mathbb{E}^\psi \left[G^*(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right] \\ &= \mathbb{E}^\psi \left[G(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right]. \end{aligned}$$

This implies that for any bounded nonnegative measurable function G on \mathbb{T}_1 :

$$\lim_{r \rightarrow \infty} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] = \mathbb{E}^\psi \left[G(\widehat{\mathbf{R}}(\mathcal{K}, h)) \right],$$

and thus for any $h \geq 0$ and any bounded nonnegative measurable function F on \mathbb{T} such that $F(\mathbf{t}) = 0$ on $\mathbf{H}(\mathbf{t}) < h$:

$$\lim_{r \rightarrow \infty} \mathbb{N}_r^\psi [F(\mathbf{R}_h(\mathcal{T}_r^{\text{mass}}))] = \mathbb{E}^\psi [F(\mathbf{R}_h(\mathcal{K}))].$$

Then use Corollary 5.8 to get $\mathbb{N}_r^\psi [F(\mathbf{R}_h(\mathcal{T}_r^{\text{mass}}))] = \mathbb{E}^{\varphi, r} [F(\mathbf{R}_h(\mathcal{K}_r^{\text{mass}}))]$ so that (74) holds for F such that $F(\mathbf{t}) \mathbf{1}_{\{\mathbf{H}(\mathbf{t}) < h\}} = 0$. To conclude use Corollary 5.8 and then (76) to get that $\lim_{r \rightarrow \infty} \mathbb{E}^{\varphi, r} [\mathbf{H}(\mathcal{K}_r^{\text{mass}}) < h] = \lim_{r \rightarrow \infty} \mathbb{N}_r^\psi [\mathbf{H}(\mathcal{T}_r^{\text{mass}}) < h] = 0$. \square

We get in particular the following (weaker) result.

Theorem 5.12 (Local convergence for the mass conditioning in the critical case). *Assume (H0)-(H6) hold (in particular the branching mechanism ψ is critical, that is $\psi'(0) = 0$). The following local convergence holds in distribution:*

$$\mathcal{T}_r^{\text{mass}} \xrightarrow[r \rightarrow +\infty]{(d)} \mathcal{K} \quad \text{in } (\mathbb{T}, d_{\text{LGH}}).$$

As it shall be used in a forthcoming work, we also mention the next result, whose proof is immediate thanks to (72), (73) and (75) (with the choice of the regular version of the conditional distribution \mathbb{N}_r^ψ from Corollary 5.8).

Corollary 5.13 (Lower bound for the mass conditioning). *Assume (H0)-(H6) hold (in particular the branching mechanism ψ is critical). Let $r \in (0, +\infty)$, $\mathcal{T}_r^{\text{mass}}$ be the Lévy tree under \mathbb{N}^ψ conditioned to have total mass r and U^r a leaf of $\mathcal{T}_r^{\text{mass}}$ chosen uniformly. For any nonnegative measurable functions F and G defined respectively on \mathbb{T} and \mathbb{T}_1 , we have that for all $h > 0$:*

$$\begin{aligned} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} F(\mathbf{R}_h(\mathcal{T}_r^{\text{mass}})) \right] &\geq \mathbb{E}^\psi \left[\mathbf{1}_{\{S_h^\kappa < r\}} F(\mathbf{R}_h(\mathcal{K})) \right], \\ \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} G(\widehat{\mathbf{R}}_h(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] &\geq \mathbb{E}^\psi \left[\mathbf{1}_{\{S_h^\kappa < r\}} G(\widehat{\mathbf{R}}_h(\mathcal{K}, h)) \right]. \end{aligned}$$

5.6. Convergence for the conditioning by the total mass in super-critical cases and some sub-critical cases. Intuitively, in the critical case $\alpha = 0$, $u^{(0)}(r+a)/u^{(0)}(r)$ “converges” to 1 as r goes to infinity (this is indeed the case in the stable case); but this is no more expected in the sub-critical case (in the sub-critical quadratic case, one has that $\log(u^{(\alpha)}(r+a)/u^{(\alpha)}(r))$ converges to a non trivial constant times a). We shall use the same trick as in the discrete case for Bienaymé-Galton-Watson (BGW) tree, see [6] or [7] and the references therein, where one can exhibit a parameterized family of offspring distributions such that the corresponding conditioned BGW trees have the same distribution. To do so, we consider the Girsanov transformation stated in Section 2.7. Recall the set Θ^ψ defined therein.

Notice that under (H0)-(H1) the function ψ is strictly convex, thus there exists at most one root of $\psi' = 0$ in Θ^ψ ; such value when it exists will be denoted θ^* , and then the branching mechanism ψ_{θ^*} and the density $f_\sigma^{\theta^*}$ (which exists if and only if the distribution of σ on $(0, +\infty)$ under \mathbb{N}^ψ has a density, see (29)), simply denoted ψ_* and f_σ^* , is critical. In this section, we shall consider the following assumption:

(H7)

$$\boxed{\text{There exists } \theta^* \in \Theta^\psi \text{ such that } \psi'(\theta^*) = 0.}$$

Under (H0), Assumption (H7) is trivially satisfied in the critical case (with $\theta^* = 0$) and in the super-critical case (where $\psi'(0) < 0$) under the sufficient condition (H1). In the sub-critical case, Assumption (H7) is not satisfied in general. For example in the sub-critical case $\psi(\lambda) = \alpha\lambda + \lambda^a$, with $\alpha > 0$ and $a \in (1, 2)$, then $\pi(dr) \propto r^{-a-1} dr$ so that $\Theta^\psi = [0, \infty)$ and thus (H7) is not satisfied. Notice (H7) is satisfied for $a = 2$ and that $\Theta^\psi = \mathbb{R}$ in this case.

Under (H0), (H1) and (H7), let \mathcal{K}^* be the Kesten tree associated with the critical branching mechanism ψ_* . The following result is a direct consequence of Theorem 5.12 (with ψ replaced by the critical branching mechanism ψ_*) and the Girsanov transformation which implies, thanks to (28), that:

$$\mathbb{N}_r^\psi[d\mathcal{T}] = \mathbb{N}_r^{\psi_{\theta^*}}[d\mathcal{T}] \quad \text{for all } \theta \in \Theta^\psi \text{ and } r \in (0, +\infty).$$

Corollary 5.14 (Local convergence for the mass conditioning). *Assume (H0)-(H3) and (H7) hold (recall that if ψ is (super-)critical, then (H7) holds), and the function $r \mapsto r e^{-\psi(\theta^*)r} f_\sigma(r)$ is non increasing on $(0, +\infty)$ (that is, (H6) holds with f_σ replaced by f_σ^*). The following local convergence holds in distribution:*

$$\mathcal{T}_r^{\text{mass}} \xrightarrow[r \rightarrow +\infty]{(d)} \mathcal{K}^* \quad \text{in } (\mathbb{T}, d_{\text{LGH}}).$$

Remark 5.15 (The non generic case). The so-called non generic case corresponds to the case where (H7) is not satisfied; this can only happen when ψ is sub-critical. In the discrete setting for BGW tree, see [25, 23] or [5], the local limit of sub-critical BGW trees conditioned to have a large number of vertices is a tree with a vertex at random finite distance from the root with infinitely many children. We expect also to observe a condensation at finite height for the local limit of Lévy trees conditioned to have a very large

mass when (H7) is not satisfied (possibly under other hypothesis than (H0)-(H5)): however the limit would not be locally compact and thus the convergence would not be in \mathbb{T} , see also the discussion from Remark 4.5 where we could use therein a trick to represent the “local limit” with condensation at random finite height E .

5.7. Proof of Theorem 5.7. Let \mathcal{T} be a Lévy tree with distribution $\mathbb{N}^\psi[d\mathcal{T}]$ and U a leaf of \mathcal{T} chosen according to the mass measure \mathbf{m} . Recall that σ is the total mass of the measure \mathbf{m} . Let G be a nonnegative measurable function defined on the set \mathbb{T}_1 of pointed locally compact rooted trees. By construction, we have with U^r as in Theorem 5.7:

$$\mathbb{N}^\psi [G(\mathcal{T}, U)] = \int_0^\infty \mathbb{N}_r^\psi [G(\mathcal{T}_r^{\text{mass}}, U^r)] r f_\sigma(r) dr.$$

We deduce that for $\gamma \geq 0$, $h > 0$, U_h^r defined in Theorem 5.7, and U_h defined similarly as the ancestor of U at level h on the event $\{\mathbf{H}(U) > h\}$, that:

$$(77) \quad \mathbb{N}^\psi \left[\mathbf{1}_{\{\mathbf{H}(U) > h\}} e^{-\gamma \mathbf{H}(U)} G(\mathcal{T}, U_h) \right] = \int_0^\infty \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} e^{-\gamma \mathbf{H}(U^r)} G(\mathcal{T}_r^{\text{mass}}, U_h^r) \right] r f_\sigma(r) dr.$$

For a pointed tree $(T, d, \varrho, x) \in \mathbb{T}_1$, where $\varrho, x \in T$, we uniquely decompose the tree T according to the branch $[[\varrho, x]]$ and the sub-trees grafted on this branch:

$$T = [[\varrho, x]] \otimes_{i \in I} (T_i, x_i),$$

where x_i belongs to the branch $[[\varrho, x]]$ and T_i is the union of all the connected components of $T \setminus \{x_i\}$ not containing the root nor x , with x_i added as a root. Notice this decomposition is unique (and measurable using an adaptation of [8, Proposition 5.32]). We shall consider a particular choice of function G defined by:

$$(78) \quad G(T, x) = \exp \left(- \sum_{i \in I} G_0(d(\varrho, x_i), T_i) \right),$$

with G_0 a nonnegative measurable function defined on $\mathbb{R}_+ \times \mathbb{T}$.

Set $\alpha = \psi'(0) \geq 0$ and for $\lambda, \gamma \geq 0$ and $h > 0$:

$$\begin{aligned} A(\gamma) &= \int_0^\infty e^{-\lambda r} \mathbb{N}_r^\psi \left[\mathbf{1}_{\{\mathbf{H}(U^r) > h\}} e^{-\gamma \mathbf{H}(U^r)} G(\widehat{\mathbf{R}}(\mathcal{T}_r^{\text{mass}}, U_h^r)) \right] r f_\sigma(r) dr, \\ B(\gamma) &= \int_{[0, +\infty)} e^{-\lambda r} \mathbf{U}^{(\alpha)}(dr) \mathbb{E}^{\varphi, r} \left[\mathbf{1}_{\{\zeta^r > h\}} e^{-\gamma \zeta^r} G(\widehat{\mathbf{R}}(\mathcal{K}_r^{\text{mass}}, h)) \right]. \end{aligned}$$

To prove that (73) holds dr -a.e., it is enough in view of (70) to prove that $A(0) = B(0)$ for all $\lambda > 0$, $h \geq 0$ and functions G given by (78) with G_0 any nonnegative measurable function such that $G_0(s, \cdot) = 0$ for $s \geq h$.

Set:

$$F(h) = \alpha h + \int_0^h ds \mathbb{N}^\psi [1 - e^{-\lambda \sigma - G_0(s, \mathcal{T})}].$$

On the one hand, using (77) and the decomposition of the Lévy tree along an ancestral line which is a consequence of (33) at a given level a and the occupation formula (23) for the local times, we obtain:

$$\begin{aligned}
A(\gamma) &= \mathbb{N}^\psi \left[e^{-\gamma \mathbf{H}(U)} \mathbf{1}_{\{\mathbf{H}(U) > h\}} e^{-\lambda \sigma - \sum_{i \in I_U} G_0(h_i^U, \mathcal{T}_i^U)} \mathbf{1}_{\{h_i^U \leq h\}} \right] \\
&= \int_h^\infty da e^{-\gamma a} \mathbb{N}^\psi \left[\int_{\mathcal{T}} \ell^a(dx) e^{-\sum_{i \in I} (\lambda \sigma_i^x + G_0(h_i^x, \mathcal{T}_i^x)} \mathbf{1}_{\{h_i^x \leq h\}} \right] \\
&= \int_h^\infty da e^{-\gamma a} e^{-\alpha a - \int_0^a ds} \mathbb{N}^\psi \left[1 - e^{-\lambda \sigma - G_0(s, \mathcal{T})} \mathbf{1}_{\{s \leq h\}} \right] \\
&= \int_h^\infty da e^{-(\gamma + \alpha)a - \int_0^h ds} \mathbb{N}^\psi \left[1 - e^{-\lambda \sigma - G_0(s, \mathcal{T})} \right] - \int_h^a ds \mathbb{N}^\psi \left[1 - e^{-\lambda \sigma} \right] \\
&= \int_h^\infty da e^{-\gamma a - F(h) - (a-h)(\alpha + \varphi(\lambda))} \\
&= \frac{e^{-\gamma h - F(h)}}{\gamma + \psi' \circ \psi^{-1}(\lambda)},
\end{aligned}$$

where for the fifth equality we used (31) to get $\int_h^a ds \mathbb{N}^\psi \left[1 - e^{-\lambda \sigma} \right] = (h - a)\varphi(\lambda)$ with $\varphi = \psi'_{\text{crit}} \circ \psi^{-1}$ and that $\alpha + \varphi = \psi' \circ \psi^{-1}$.

On the other hand, set $\mathbf{S}_{[0, h]} = (S_t)_{t \in [0, h]}$ and $Q_h = \mathbb{E}^{\varphi, r} \left[G(\widehat{\mathcal{R}}(\mathcal{K}_r^{\text{mass}}, h)) \mid \mathbf{S}_{[0, h]} \right]$. We obtain:

$$\begin{aligned}
B(\gamma) &= \int_{(0, +\infty)} e^{-\lambda r} U^{(\alpha)}(dr) \mathbb{E}^{\varphi, r} \left[\mathbf{1}_{\{\zeta^r > h\}} e^{-\gamma \zeta^r} Q_h \right] \\
&= e^{-\gamma h} \int_{(0, +\infty)} e^{-\lambda r} dr \int_0^\infty \gamma e^{-\gamma s} ds u^{(\alpha)}(r) \mathbb{E}^{\varphi, r} \left[(\mathbf{1}_{\{\zeta^r > h\}} - \mathbf{1}_{\{\zeta^r > s+h\}}) Q_h \right] \\
&= e^{-\gamma h} \int_{(0, +\infty)} e^{-\lambda r} dr \int_0^\infty \gamma e^{-\gamma s} ds \\
&\quad \mathbb{E}^\varphi \left[\left(e^{-\alpha h} u^{(\alpha)}(r - S_h) \mathbf{1}_{\{S_h < r\}} - e^{-\alpha(s+h)} u^{(\alpha)}(r - S_{s+h}) \mathbf{1}_{\{S_{s+h} < r\}} \right) Q_h \right] \\
&= \frac{e^{-(\gamma + \alpha)h}}{\psi' \circ \psi^{-1}(\lambda)} \int_0^\infty \gamma e^{-\gamma s} ds \mathbb{E}^\varphi \left[(e^{-\lambda S_h} - e^{-\lambda S_{s+h} - \alpha s}) Q_h \right] \\
&= \frac{e^{-(\gamma + \alpha)h}}{\psi' \circ \psi^{-1}(\lambda)} \int_0^\infty \gamma e^{-\gamma s} ds \mathbb{E}^\varphi \left[e^{-\lambda S_h} \left(1 - e^{-s(\alpha + \varphi(\lambda))} \right) Q_h \right] \\
&= \frac{e^{-(\gamma + \alpha)h}}{\gamma + \psi' \circ \psi^{-1}(\lambda)} \mathbb{E}^\varphi \left[e^{-\lambda S_h} Q_h \right],
\end{aligned}$$

where we used $e^{-\gamma z} \mathbf{1}_{\{z > h\}} = e^{-\gamma h} \int_0^\infty \gamma e^{-\gamma s} (\mathbf{1}_{\{z > h\}} - \mathbf{1}_{\{z > h+s\}}) ds$ and the density $u^{(\alpha)}$ of the potential $U^{(\alpha)}$ for the second equality, (60) for the third as Q_h is \mathcal{F}_h -measurable with $\mathbb{P}^{\varphi, r} = \mathbb{P}_0^{\varphi, r}$ (see also (72)), the equalities:

$$\int_z^\infty dr e^{-\lambda r} u^{(\alpha)}(r - z) = e^{-\lambda z} \int_{[0, \infty)} e^{-\lambda r} U^{(\alpha)}(dr) = \frac{e^{-\lambda z}}{\alpha + \varphi(\lambda)} = \frac{e^{-\lambda z}}{\psi' \circ \psi^{-1}(\lambda)}$$

for the fourth, and the Markov property of \mathbf{S} at time h for the fifth. To conclude, we use the representation of the tree $\mathcal{K}_r^{\text{mass}}$, see (71), to get:

$$\mathbb{E}^\varphi \left[e^{-\lambda S_h} Q_h \right] = \mathbb{E}^\psi \left[e^{-\sum_{h_i \leq h} \lambda \sigma_i + G_0(h_i, \mathcal{T}_i)} \right] = e^{-\int_0^h ds} \mathbb{N}^\psi \left[1 - e^{-\lambda \sigma - G_0(s, \mathcal{T})} \right] = e^{-F(h) + \alpha h},$$

so we obtain:

$$B(\gamma) = \frac{e^{-\gamma h - F(h)}}{\gamma + \psi' \circ \psi^{-1}(\lambda)}.$$

We deduce that $A(\gamma) = B(\gamma)$ for $\gamma \geq 0$ and thus $A(0) = B(0)$. So the proof is complete.

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Index of notation

Trees and pointed trees

- $\mathbf{t}, T, \tau, \mathcal{T}, \mathcal{K}$: generic notations for trees (or class of equiv. trees).
- (\mathbf{t}, x) : a (or a class of equiv. of) tree with distinguished vertex $x \in \mathbf{t}$.
- d : generic distance on a tree.
- ϱ : generic notation for the root of a tree.
- $\mathbf{H}(x)$ height of vertex x (or distance from x to ϱ).
- $R_h(\mathbf{t})$: the tree \mathbf{t} truncated at level h .
- $\widehat{\mathbf{R}}(\mathbf{t}, x)$: the tree \mathbf{t} without the subtree above the vertex x .
- $\mathbf{H}(\mathbf{t})$: height of the tree \mathbf{t} .
- $\llbracket x, y \rrbracket$: the branch joining the vertices x to y .
- $\llbracket 0, h \rrbracket$: the segment $[0, h]$ seen as a tree with root $\varrho = 0$, and h as a distinguished vertex.
- $\llbracket 0, \infty \rrbracket$: the infinite spine tree with root $\varrho = 0$.
- \mathbb{T} : Polish space of (equiv. class of) rooted loc. compact closed trees.
- \mathbb{T}_1 : Polish space of (equiv. class of) rooted loc. compact closed pointed trees.

Functions and random variables

- ψ branching mechanism with Lévy measure π and quadratic parameter β .
- $\bar{\pi}(r) = \pi((r, \infty))$.
- \mathbf{Y} CB starting at x with branching mechanism ψ under \mathbb{P}_x^ψ .

- $\psi_\theta(\cdot) = \psi(\cdot + \theta) - \psi(\theta)$.
- $\psi_{\text{crit}}(\lambda) = \psi(\lambda) - \psi'(0)\lambda$.
- Laplace exponent $\varphi = \psi'_{\text{crit}} \circ \psi^{-1}$ from Section 5.3.
- $U^{(\alpha)}$: α -potential of subordinator with Laplace exponent φ .
- $u^{(\alpha)}$ density of $U^{(\alpha)}$.

Lévy tree and Kesten tree

- \mathcal{T} : a Lévy tree.
- $\mathcal{T}_{[r]}$: a Lévy tree with root with “size” r .
- \mathcal{K} : a Kesten tree, with infinite spine $\llbracket 0, \infty \rrbracket$.
- $\widehat{\mathcal{K}}_h = \widehat{\mathbf{R}}(\mathcal{K}, h)$: the Kesten tree whose infinite spine is truncated at level h .
- $\ell^a(dx)$: the local time at level a on a Lévy tree.
- $\ell^{a, \mathcal{K}}(dx)$: the local time at level a on the Kesten tree \mathcal{K} .
- $S_h^{\mathcal{K}}$ total mass of the trees grafted on the spine of the Kesten tree up to level h .
- $\sigma = \sigma^{\mathcal{T}}$ total mass of the Lévy tree \mathcal{T} .
- f_σ “density” of σ under \mathbb{N}^ψ .
- $\mathcal{T}_h^{\text{height}}$: Lévy tree cond. to have height h .
- $\mathcal{T}_\delta^{\text{node}}$: Lévy tree cond. to have one vertex of maximal size δ .
- $\mathcal{T}_r^{\text{mass}}$: Lévy tree cond. to have total mass r .

Probability and excursion measures

- $\mathbb{N}^\psi[d\mathcal{T}]$: excursion measure for Lévy trees.
- \mathbb{P}_r^ψ : distribution of $\mathcal{T}_{[r]}$.
- \mathbf{N}^ψ : intensity of the grafting on the infinite spine of the Kesten tree.
- \mathbb{P}^ψ : distribution of the Kesten tree.

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