

OPTIMAL VACCINATION STRATEGIES FOR A HETEROGENEOUS SIS MODEL*

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Abstract. We study in a general mathematical framework the optimal allocation of vaccine in an heterogeneous population. We cast the problem of optimal vaccination as a biobjective minimization problem $\min(C(\eta), L(\eta))$, where C and L stand, respectively, for the cost and the loss incurred when following the vaccination strategy η , where the function $\eta(x)$ represents the proportion of *nonvaccinated* among individuals of feature x . To measure the loss, we consider either the effective reproduction number, a classical quantity appearing in many models in epidemiology, or the overall proportion of infected individuals after vaccination in the maximal equilibrium, also called the endemic state. We only make few assumptions on the cost $C(\eta)$, which cover the uniform cost, that is, the total number of vaccinated people. The analysis of the biobjective problem is carried out in a general framework, and we check that it is well posed for the SIS model and has Pareto optima, which can be interpreted as the “best” vaccination strategies. We provide properties of the corresponding Pareto frontier given by the outcomes $(C(\eta), L(\eta))$ of Pareto optimal strategies.

Key words. SIS model, vaccination strategy, effective reproduction number, multiobjective optimization, Pareto frontier

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1. Introduction.

1.1. Motivation. Targeted vaccination problems have mainly been studied using two different points of view: eradication of a disease in heterogeneous population, or removing vertices of a network for minimizing its spectral radius.

1.1.1. Optimal vaccination in heterogeneous population. Increasing the prevalence of immunity from contagious disease in a population limits the circulation of the infection among the individuals who lack immunity. This so-called “herd effect” plays a fundamental role in epidemiology as it has had a major impact in the eradication of smallpox and rinderpest or the near eradication of poliomyelitis [20]. Targeted vaccination strategies, based on the heterogeneity of the infection spreading in the population, are designed to increase the level of immunity of the population with a limited quantity of vaccine. These strategies rely on identifying groups of individuals that should be vaccinated in priority in order to slow down or eradicate the disease.

In this article, we establish a theoretical framework to study targeted vaccination strategies for the deterministic infinite-dimensional SIS model (with S = Susceptible and I = Infectious) introduced in [5], which encompasses as particular cases the SIS model on graphs or on stochastic block models. The simplicity of the SIS model allows us to derive strong results on optimal vaccination strategies and existence of Pareto optima under minimal assumptions for general nonhomogeneous populations. On the

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one hand, using this mathematical framework, we can consider in companion papers a series of particular cases and specific examples that complete and illustrate the present work; see section 1.5 for more details. On the other hand, we expect the results obtained here for the SIS model to be generic, in the sense that behaviors exhibited here should be also observed in more realistic and complex models in epidemiology for nonhomogeneous populations; see, for example, the discussion in [7].

1.1.2. Removing vertices of a network in an optimal way. Mathematical epidemiologists usually consider a small number of subpopulations, often with a dense structure of interaction, in the sense that every subpopulation may directly infect all the others; see section 1.3 for further references. Other research communities have looked into a similar problem for graphs. Indeed, given a large, possibly random, graph, with epidemic dynamics on it, and supposing we are able to suppress vertices by vaccinating, one may ask for the best way to choose which vertices to remove.

The importance of the spectral radius of the network, that is, the spectral radius of its adjacency matrix, has been quickly identified as its value determines if the epidemic dies out quickly or survives for a long time [21, 31]. Since Van Mieghem et al. proved in [35] that the problem of minimizing the spectral radius of a graph by removing a given number of vertices is NP-complete (and therefore unfeasible in practice), many computational heuristics have been put forward to give approximate solutions; see, for example, [32] and references therein. The theoretical framework developed below gives the existence of Pareto optima for the continuous relaxation of this problem.

1.2. Herd immunity and targeted vaccination strategies. We first recall classical results in mathematical epidemiology; we refer to Keeling and Rohani's monograph [27] for an extensive introduction to this field, including details on various classical models such as SIS, SIR, and SEIR (with R = Recovered and E = Exposed).

In an homogeneous population, the basic reproduction number of an infection, denoted by R_0 , is defined as the number of secondary cases one individual generates on average over the course of its infectious period, in an otherwise uninfected (susceptible) population. This number plays a fundamental role in epidemiology as it provides a scale to measure how difficult an infectious disease is to control. Intuitively, the disease should die out if $R_0 < 1$ and invade the population if $R_0 > 1$. For many classical mathematical models of epidemiology, such as SIS or S(E)IR, this intuition can be made rigorous: the quantity R_0 may be computed from the parameters of the model, and the threshold phenomenon occurs.

Assuming $R_0 > 1$ in an homogeneous population, suppose now that only a proportion η^{uni} of the population can catch the disease, the rest being immunized. An infected individual will now only generate $\eta^{\text{uni}} R_0$ new cases, since a proportion $(1 - \eta^{\text{uni}})$ of previously successful infections will be prevented. Therefore, the new *effective reproduction number* is equal to $R_e(\eta^{\text{uni}}) = \eta^{\text{uni}} R_0$. This fact led to the recognition by Smith in 1970 [33] and Dietz in 1975 [14] of a simple threshold theorem: the incidence of an infection declines if the proportion of nonimmune individuals is reduced below $\eta_{\text{crit}}^{\text{uni}} = 1/R_0$. This effect is called *herd immunity*.

It is of course unrealistic to depict human populations as homogeneous, and many generalizations of the homogeneous model have been studied; see [27, Chapter 3] for examples and further references. For most of these generalizations, it is still possible to define a meaningful reproduction number R_0 , as the number of secondary cases generated by a *typical* infectious individual when all other individuals are uninfected [13]. After a vaccination campaign, let the vaccination strategy η denote the (not

necessarily homogeneous) proportion of the *nonvaccinated* population, and let the effective reproduction number $R_e(\eta)$ denote the corresponding reproduction number of the nonvaccinated population. The vaccination strategy η is *critical* if $R_e(\eta) = 1$. The possible choices of η naturally raise a question that may be expressed as the following informal optimization problem:

$$(1.1) \quad \begin{cases} \textbf{Minimize:} & \text{the quantity of vaccine to administrate,} \\ \textbf{subject to:} & \text{herd immunity is reached, that is, } R_e \leq 1. \end{cases}$$

If the quantity of available vaccine is limited, then one is also interested in the following:

$$(1.2) \quad \begin{cases} \textbf{Minimize:} & \text{the effective reproduction number } R_e, \\ \textbf{subject to:} & \text{a given quantity of available vaccine.} \end{cases}$$

Interestingly enough, when $R_0 > 1$, the strategy $\eta_{\text{crit}}^{\text{uni}}$, which consists in delivering the vaccine *uniformly* to the population, without taking inhomogeneity into account, leaves a proportion $\eta_{\text{crit}}^{\text{uni}} = 1/R_0$ of the population unprotected and is therefore critical since $R_e(\eta_{\text{crit}}^{\text{uni}}) = 1$. In particular it is admissible for the optimization problem (1.1).

However, herd immunity may be achieved even if the proportion of unprotected people is *greater* than $1/R_0$, by targeting certain group(s) within the population; see Figure 3.3 in [27].

The main goal of this paper is two-fold: formalize the optimization problems (1.1) and (1.2) for a particular infinite-dimensional SIS model, recasting them as a more general biobjective optimization problem; and give existence and properties of solutions to this biobjective problem. We also study a closely related problem, where one wishes to minimize the size of the epidemic rather than the reproduction number.

1.3. Literature on targeted vaccination. Problems (1.1) and (1.2) have been examined in depth for deterministic *metapopulation* models, that is, models in which a heterogeneous population is stratified into a finite number of homogeneous subpopulations (by age group, gender, ...). Such models are specified by choosing the sizes of the subpopulations and quantifying the degree of interactions between them, in terms of various mixing parameters. In this setting, R_0 can often be identified as the spectral radius of a *next-generation matrix* whose coefficients depend on the subpopulation sizes, and the mixing parameters. It turns out that the next-generation matrices take similar forms for many dynamics (SIS, SIR, SEIR, ...); see the discussions in [25, section 10] and [7, section 2]. Vaccination strategies are defined as the levels at which each subpopulation is immunized. After vaccination, the next-generation matrix is changed and its new spectral radius corresponds to the effective reproduction number R_e .

Problem (1.1) has been studied in this setting by Hill and Longini [25]. These authors studied the geometric properties of the so-called threshold hypersurface, that is, the vaccination allocations for which $R_e = 1$. They also computed the vaccination belonging to this surface with minimal cost for an influenza A model. Making structural assumptions on the mixing parameters, Poghotanyan et al. derived in [30] an analytical formula for the solutions of problem (1.2), for populations divided in two groups. Many papers also contain numerical studies of the optimization problems (1.1) and (1.2) on real-world data using gradient techniques or similar methods; see, for example, [22, 19, 15, 18, 36].

Finally, the effective reproduction number is not the only reasonable way of quantifying a population's vulnerability to an infection. For an SIR infection, for example, the proportion of individuals that eventually catch (and recover from) the disease, often referred to as the *attack rate*, is broadly used. We refer to [15, 16] for further discussion on this topic.

1.4. Main results.

1.4.1. Costs and losses in a SIS model. The differential equations governing the epidemic dynamics in metapopulation SIS models were developed by Lajmanovich and Yorke in their pioneer paper [28] and generalized in [5] to a possibly infinite space Ω , where $x \in \Omega$ represents a feature, and the probability measure $\mu(dx)$ represents the fraction of the population with feature x . We describe briefly and informally the key quantities that we would like to minimize, and refer to section 4.5 for details.

A vaccination strategy is represented by a function $\eta : \Omega \rightarrow [0, 1]$, where $\eta(x)$ represents the fraction of nonvaccinated individuals with feature x . In particular, with $\mathbb{1}$ the constant function equal to 1, the strategy $\eta = \mathbb{1}$ corresponds to doing nothing; similarly the constant strategy $\eta = 0$ corresponds to vaccinating everybody. As mentioned before, to any vaccination strategy we associate an epidemic dynamics on the nonvaccinated individuals (that is, vaccinated individuals are completely immunized and do not get infected or infect others). For this effective dynamics, we may compute two interesting quantities.

The first is the effective reproduction number $R_e(\eta)$ of the effective dynamics; a vaccination strategy should try to get this number below 1. If the effective reproduction number $R_e(\eta) > 1$, the generalization of the SIS dynamics we introduced in [5], just like its classical counterparts, converges to a nontrivial equilibrium. In this equilibrium a certain number of people are infected. The size of this equilibrium infection under the effective dynamics defined by η will be denoted by $\mathfrak{I}(\eta)$. In the SIS model, the quantity $\mathfrak{I}$ appears as a natural analogue of the attack rate in the SIR model and is therefore a natural optimization objective.

Following [9], we equip the set of vaccination strategies Δ_{SIS} with the weak-* topology: this choice makes Δ_{SIS} compact and the loss functions R_e and $\mathfrak{I}$ continuous.

To any vaccination strategy $\eta \in \Delta_{\text{SIS}}$ we also associate a *cost* $C(\eta)$, measuring the production and diffusion costs to pay in order to deploy the strategy. Since the vaccination strategy η represents the nonvaccinated population, the cost function C should be nonincreasing or decreasing (roughly speaking $\eta < \eta'$ implies $C(\eta) > C(\eta')$; see Definition 2.1). We shall also assume that C is continuous and that doing nothing costs nothing, that is, $C(\mathbb{1}) = 0$. A simple and natural choice is the uniform cost C_{uni} given by the overall proportion of vaccinated individuals.

1.4.2. Optimizing the protection of the population. Our problem may now be seen as a biobjective minimization problem: we wish to minimize both the loss $L(\eta)$ and the cost $C(\eta)$, subject to $\eta \in \Delta_{\text{SIS}}$, with the loss function L being either R_e or $\mathfrak{I}$. Following classical terminology for multiobjective optimization problems [29], we call a strategy η_* *Pareto optimal* if no other strategy is strictly better:

$$C(\eta) < C(\eta_*) \implies L(\eta) > L(\eta_*) \quad \text{and} \quad L(\eta) < L(\eta_*) \implies C(\eta) > C(\eta_*).$$

The set of Pareto optimal strategies, which might be empty a priori, will be denoted by $\mathcal{P}$. We define the *Pareto frontier* as the set of Pareto optimal outcomes:

$$\mathcal{F} = \{(C(\eta_*), L(\eta_*)) \in \mathbb{R}^2 : \eta_* \in \mathcal{P}\}.$$

For any strategy η , the cost and loss of η vary between the following bounds:

$$\begin{aligned} 0 &= C(\mathbb{1}) \leq C(\eta) \leq C(\emptyset) = c_{\max} = \text{cost of vaccinating the whole population,} \\ 0 &= L(\emptyset) \leq L(\eta) \leq L(\mathbb{1}) = \ell_{\max} = \text{loss incurred in the absence of vaccination.} \end{aligned}$$

Let $L_\star$ be the *optimal loss* function and $C_\star$ the *optimal cost* function defined by

$$\begin{aligned} L_\star(c) &= \inf \{L(\eta) : \eta \in \Delta_{\text{SIS}}, C(\eta) \leq c\} \quad \text{for } c \in [0, c_{\max}], \\ C_\star(\ell) &= \inf \{C(\eta) : \eta \in \Delta_{\text{SIS}}, L(\eta) \leq \ell\} \quad \text{for } \ell \in [0, \ell_{\max}]. \end{aligned}$$

Our main result, illustrated in Section 1.4.3 on a multipartite graph, see Figure 1, states that the Pareto frontier is nonempty and has a continuous parametrization for both losses $L = R_e$ and $L = \mathfrak{I}$. More precisely, using results developed in a more general framework, see section 2, we get the following result, see section 4.8 and Table 1 therein for a more complete picture.

THEOREM 1.1 (the Pareto frontier). *Consider the SIS model with the loss $L \in \{R_e, \mathfrak{I}\}$ and the uniform cost. Under integrability assumptions, see Assumption 6,*

- (i) *The function $C_\star$ is continuous and decreasing on $[0, \ell_{\max}]$.*
- (ii) *The function $L_\star$ is continuous on $[0, c_{\max}]$, decreasing on $[0, C_\star(0)]$, and zero on $[C_\star(0), c_{\max}]$.*
- (iii) *The Pareto frontier is compact and connected, and*

$$\mathcal{F} = \{(c, L_\star(c)) : c \in [0, C_\star(0)]\} = \{(C_\star(\ell), \ell) : \ell \in [0, \ell_{\max}]\}.$$

- (iv) *The set of Pareto optima $\mathcal{P}$ is compact in Δ_{SIS} .*

In this introduction we focus, for readability's sake, on a concrete setting. Let us note that the result stated above is a particular case of the more general statements proved below, which we now briefly discuss. The set of vaccinations, denoted by Δ , may be a more general partially ordered topological space than $\Delta_{\text{SIS}} = \{\eta : 0 \leq \eta \leq \mathbb{1}\}$; in particular we may consider strict subsets of Δ_{SIS} , to model constraints on the strategies that are possible to deploy; see the examples in section 2.1.1 and Remark 4.7. We are able to tackle different, nonuniform, cost functions, and other loss functions provided they have enough regularity.

This generality allows us to give also results on the “worst” strategies, which we call anti-Pareto optimal, as opposed to the “best,” Pareto optimal ones. Surprisingly, proving properties of the anti-Pareto frontier sometimes necessitates stronger assumptions than in the Pareto case. For example, under some *irreducibility* assumption on the kernel developed in section 4.6, we establish the connectedness of the anti-Pareto frontier when the loss is given by $L = R_e$ (see also [3, section 5] in this direction), and a slightly different behavior when the loss is given by $L = \mathfrak{I}$.

We also prove that the set of outcomes or feasible region $\mathbf{F} = \{(C(\eta), L(\eta)), \eta \in \Delta\}$ has no holes in Proposition 3.1, and that the Pareto frontier is convex if C and L are convex in Proposition 3.4.

Remark 1.2 (optimal critical strategies do not depend on the loss). Recall a vaccination strategy $\eta \in \Delta$ is critical if $R_e(\eta) = 1$. Consider the uniform cost C_{uni} and write $\mathcal{P}_L$ for the Pareto optima associated to the loss L . A priori $\mathcal{P}_{\mathfrak{I}}$ and $\mathcal{P}_{R_e}$ are not comparable. Nevertheless, we have the following result on critical Pareto optima:

$$(1.3) \quad \eta \in \mathcal{P}_{R_e} \text{ and } R_e(\eta) = 1 \iff \eta \in \mathcal{P}_{\mathfrak{I}} \text{ and } \mathfrak{I}(\eta) = 0.$$

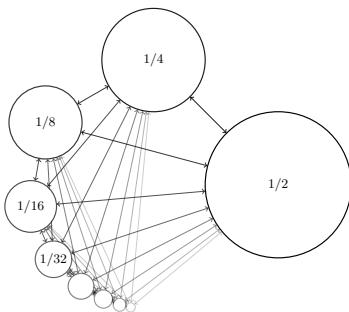
Indeed, in [5], we proved that, for all $\eta \in \Delta$, the equilibrium infection size $\mathfrak{I}(\eta)$ is nonzero if and only if $R_e(\eta) > 1$. This implies that $\mathcal{P}_{\mathfrak{I}}$ is a subset of $\{\eta \in \Delta : R_e(\eta) \geq 1\}$, and (1.3) follows thanks to Theorem 1.1.

1.4.3. An illustrative example: The multipartite graph. Let us illustrate some of our results on an example, which is discussed in detail in [6].

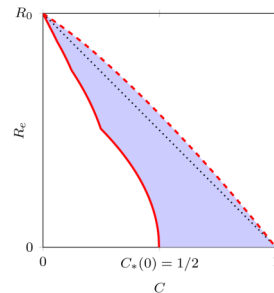
Graphs whose vertices can be colored with ℓ colors, so that the endpoints of every edge are colored differently, are known as ℓ -partite graphs. In a biological setting, this corresponds to a population of ℓ groups such that individuals in a group only contaminate individuals of the other groups. (For example, sexually transmitted disease can be roughly approximated as an epidemic on a bipartite graph; see [23].) Let us generalize and assume there are infinitely many groups, $\ell = \infty$, of respective size $(2^{-n}, n \in \mathbb{N}^*)$, and that there is a constant intergroup positive rate and that there is no intragroup contamination; see Figure 1.

Consider the loss $L = R_e$ and the uniform cost $C = C_{\text{uni}}$ giving the overall proportion of vaccinated individuals. Building on the results of [17, 34], we prove in [6] that vaccinating individuals with the highest number of connections (that is, individuals in the smallest groups) and stopping when all the groups but the first one are fully vaccinated gives a Pareto optimal vaccination. In particular, since the cost of vaccinating the whole population but the first group (whose size is $1/2$ in a total population of size 1) has cost $1/2$, we deduce that $C_*(0) = 1/2$.

In Figure 1, the corresponding Pareto frontier (i.e., the outcome of the “best” vaccination strategies) is drawn as the solid red line; the blue-colored zone corresponds to the feasible region, that is, all the possible values of $(C(\eta), R_e(\eta))$ where η ranges over Δ_{SIS} ; the dotted line corresponds to the outcome of the uniform vaccination strategy, that is, $(C(\eta), R_e(\eta)) = (c, (1-c)R_0)$, where c ranges over $[0, 1]$; and the red dashed curve corresponds to the anti-Pareto frontier (i.e., the outcome of the “worst” vaccination strategies), which for this model correspond to the uniform vaccination of the nodes with the updated lower degree; see [6]. The set of optimal Pareto vaccination strategies given by vaccinating the individuals with the highest number of connections gives a complete parametrization of the Pareto frontier.



(A) Partial multipartite graph.



(B) The Pareto frontier in solid red line compared to the cost and loss of the uniform vaccinations in dotted line and the worst vaccination strategy in red dashed line.

FIG. 1. Example of optimization with $L = R_e$ for a multipartite graph with no intrapopulation transmission.

1.5. On the companion papers. Let us discuss briefly the results that can be found in the companion papers [9, 2] and the related papers [7, 3, 6, 10, 4].

In [9], we prove that the loss functions we consider for the SIS model satisfy the continuity assumptions used here to prove the existence of optima. In [2], we describe how two seemingly different feature spaces may in fact describe essentially the same model; in particular we show how discrete models like the stochastic block models may be encoded in a continuous model, giving the same Pareto and anti-Pareto frontiers.

These core results are then used and extended in various directions. The discussion of gonorrhea vaccination in [24, section 4.5] suggests to prioritize the vaccination of people that have already caught the disease: this leads us to consider in [10] a critical vaccination strategy guided by the equilibrium state. It seems difficult to give conditions for this vaccination strategy to be Pareto or anti-Pareto optimal.

In [7], motivated by a conjecture formulated by Hill and Longini in finite dimension (see [25, Conjecture 8.1]), we investigate the convexity and concavity of the effective reproduction function R_e . In [3], we prove for the loss $L = R_e$ that a disconnecting strategy is better than the worst, i.e., is not anti-Pareto optimal; and we give a graph interpretation of $C_*(0)$ which is the minimal cost to ensure that no infection occurs. We also provide in this setting further results on the anti-Pareto frontier using a decomposition of the kernel on its irreducible components.

In [6], we study the Pareto and anti-Pareto frontier for a number of examples. In some of them, like the multipartite graph of section 1.4.3, the optimal vaccinations target the individuals with the highest number of contacts. When the individuals have the same number of contacts, this heuristic cannot be used, and the situations may be extremely varied: for example, we give models for which the uniform vaccination is Pareto optimal, or anti-Pareto optimal, or not optimal for either problem.

1.6. Structure of the paper. We present the multiobjective optimization problem in section 2 in an abstract setting, under general conditions on the loss and cost functions, and prove the first results on the biobjective problem and its links with the single-objective problems obtained when fixing the value of either the cost or the loss, and optimizing the other. This is completed in section 3 with miscellaneous properties of the Pareto frontier. Section 4 is dedicated to the proper presentation of the SIS setting and the consequences of the general results in this framework.

2. Pareto and anti-Pareto frontiers.

2.1. The setting.

2.1.1. Vaccination strategies. In the SIS model from [5] recalled in section 4, vaccination strategies are encoded by functions from the trait space Ω to $[0, 1]$: we equip $E = L^\infty(\Omega, \mu)$ with the weak-* topology and with the natural partial order $\leq$ on real-valued functions, and we let $\Delta = \Delta_{\text{SIS}} = \{\eta \in E : 0 \leq \eta \leq \mathbb{1}\}$, with 0 (resp., $\mathbb{1}$) the constant function equal to zero (resp., one). We generalize this setting in this section and the next one, keeping in mind the SIS case as a guiding example.

Let $(E, \mathcal{O})$ be a Hausdorff topological vector space, and denote by 0 its zero element. Let $K \subset E$ be a salient pointed convex cone, that is, $\lambda K \subset K$ for all $\lambda \geq 0$, $K + K \subset K$, and $K \cap (-K) = \{0\}$. Notice we don't assume that K is closed. The cone K defines a partial order $\preceq$ on E : for $\eta, \eta' \in E$, we set $\eta' \preceq \eta$ if $\eta - \eta' \in K$. We also write $\eta' \prec \eta$ if $\eta' \preceq \eta$ and $\eta' \neq \eta$. For $\Delta \subset E$, we say that η is the *maximum* of Δ and write $\eta = \max \Delta$ if $\eta \in \Delta$ and $\eta' \preceq \eta$ holds for all $\eta' \in \Delta$; we also say that η is the *minimum* of Δ and write $\eta = \min \Delta$ if $\eta = -\max(-\Delta)$.

From now on, we consider a subset $\Delta \subset E$ with an element $\mathbb{1} \in \Delta$ such that

$$(2.1) \quad \Delta \text{ is convex, compact, } \mathbb{0} = \min \Delta, \text{ and } \mathbb{1} = \max \Delta.$$

In particular, the set Δ is closed (as a compact subset of a Hausdorff space), and the latter two conditions of (2.1) are equivalent to $\mathbb{0}, \mathbb{1} \in \Delta$ and $\Delta \subset K \cap (1 - K)$. Let us stress that writing $\mathbb{1} = \max \Delta$ implies that all elements of Δ are comparable with (and thus less than) $\mathbb{1}$.

Naturally in the SIS model, the set Δ_{SIS} defined above satisfies (2.1). We now discuss three additional choices for Δ , which are all subsets of Δ_{SIS} :

Almost uniform vaccinations. Let Δ_{Osc} denote the set of strategies $\eta \in \Delta_{\text{SIS}}$ that are “almost uniform” in the sense that the oscillation $\sup \eta - \inf \eta \leq \delta$ for some $\delta \in (0, 1)$, with the extrema understood as essential extrema. Restricting strategies to Δ_{Osc} ensures that no subset of the population is too favored.

Blockwise strategies. Let $(\Omega_i)_{i \in I}$ be a measurable partition of Ω into sets of positive measure, and consider the set of corresponding blockwise constant vaccination strategies $\Delta_{\text{Block}} = \{\eta \in \Delta_{\text{SIS}} : \eta \text{ constant on each } \Omega_i\}$. Restricting strategies to Δ_{Block} is natural if one considers public policies focusing on a finite set of identifiable subgroups and treats each subgroup uniformly.

Ordered blockwise strategies. We elaborate on the previous example by adding another constraint on the possible strategies. Assume that the index set I is totally ordered. Consider the subset of the corresponding blockwise constant vaccination strategies $\Delta_{\text{Ord}} = \{\eta \in \Delta_{\text{Block}} : \eta_i \leq \eta_j \text{ for } i \leq j\}$, where η_i is the (constant) value of $\eta \in \Delta_{\text{Ord}}$ on Ω_i . This corresponds to situations where some categories must be at least as vaccinated as others.

The three sets Δ_{Osc} , Δ_{Block} , and Δ_{Ord} all satisfy (2.1). We refer to [8, Remark 2.1] for additional comments on condition (2.1).

2.1.2. Costs and losses. We let E^* denote the topological dual of E , that is, the set of continuous real-valued linear functions defined on E , with the dual pairing $\langle \cdot, \cdot \rangle$ and the dual cone $K^* = \{x^* \in E^* : \langle x^*, \eta \rangle \geq 0 \text{ for all } \eta \in K\}$.

DEFINITION 2.1. We say that a real-valued function H defined on Δ is

- **Nondecreasing:** if for any $\eta_1 \preceq \eta_2$ in Δ , we have $H(\eta_1) \leq H(\eta_2)$.
- **Decreasing:** if for any $\eta_1 \prec \eta_2$ in Δ , we have $H(\eta_1) > H(\eta_2)$.
- **Subhomogeneous:** if $H(\lambda\eta) \leq \lambda H(\eta)$ for all $\eta \in \Delta$ and $\lambda \in [0, 1]$.
- **Affine:** if there exists $c \in \mathbb{R}$ and $x^* \in E^*$ such that $H(\eta) = c - \langle x^*, \eta \rangle$.

The definitions of nonincreasing function and increasing function are similar. We shall study a biobjective optimization problem for a cost function C and loss function L , see (2.2) below, under various regularity conditions. In particular the next assumption will always hold.

Assumption 1 (on the cost function and loss function). The loss function $L : \Delta \rightarrow \mathbb{R}$ is nondecreasing and continuous with $L(\mathbb{0}) = 0$. The cost function $C : \Delta \rightarrow \mathbb{R}$ is nonincreasing and continuous with $C(\mathbb{1}) = 0$. The loss and cost functions are not degenerate, that is,

$$\ell_{\max} := \max_{\Delta} L > 0 \quad \text{and} \quad c_{\max} := \max_{\Delta} C > 0.$$

In particular, the loss and the cost functions are nonnegative and nonzero. Notice that $\ell_{\max} = L(\mathbb{1})$ and $c_{\max} = C(\mathbb{0})$.

Remark 2.2 (affine cost functions). Let $x^* \in K^*$ be such that $\langle x^*, \mathbb{1} \rangle \neq 0$. The affine function defined on E by $\eta \mapsto C(\eta) = \langle x^*, \mathbb{1} - \eta \rangle$ clearly satisfies Assumption 1.

2.2. Optimization problems. We will consider the multiobjective minimization and maximization problems:

$$(2.2) \quad \left\{ \begin{array}{ll} \text{Minimize:} & (C(\eta), L(\eta)), \\ \text{subject to:} & \eta \in \Delta; \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \text{Maximize:} & (C(\eta), L(\eta)), \\ \text{subject to:} & \eta \in \Delta. \end{array} \right.$$

Multiobjective problems are in a sense ill-defined because in most cases, it is impossible to find a single solution that would be optimal to all objectives simultaneously. Hence, we recall the concept of Pareto optimality. Since the minimization problem is crucial for our application to vaccination, we shall define Pareto optimality for the biobjective minimization problem. A strategy $\eta_* \in \Delta$ is said to be *Pareto optimal* for the minimization problem in (2.2) if any improvement of one objective leads to a deterioration of the other, for $\eta \in \Delta$:

$$(2.3) \quad C(\eta) < C(\eta_*) \implies L(\eta) > L(\eta_*) \quad \text{and} \quad L(\eta) < L(\eta_*) \implies C(\eta) > C(\eta_*).$$

The set of Pareto optimal strategies will be denoted by $\mathcal{P}$, and the Pareto frontier is defined as the set of Pareto optimal outcomes:

$$\mathcal{F} = \{(C(\eta), L(\eta)) : \eta \text{ Pareto optimal}\}.$$

Similarly, a strategy $\eta^* \in \Delta$ is *anti-Pareto optimal* if it is Pareto optimal for the biobjective maximization problem in (2.2). Intuitively, for the vaccination application, the “best” vaccination strategies are the Pareto optima and the “worst” vaccination strategies are the anti-Pareto optima. We denote similarly by $\mathcal{P}^{\text{Anti}}$ the set of anti-Pareto optimal strategies, and by $\mathcal{F}^{\text{Anti}}$ its frontier:

$$\mathcal{F}^{\text{Anti}} = \{(C(\eta), L(\eta)) : \eta \text{ anti-Pareto optimal}\}.$$

Finally, we define the *feasible region* as the set of all possible outcomes:

$$\mathbf{F} = \{(C(\eta), L(\eta)), \eta \in \Delta\}.$$

We now consider the classical, single-objective minimization problems related to the “best” strategies, with a fixed loss $\ell \in [0, \ell_{\max}]$ or a fixed cost $c \in [0, c_{\max}]$:

$$(2.4) \quad \text{Minimize: } L(\eta), \text{ subject to: } \eta \in \Delta, C(\eta) \leq c,$$

$$(2.5) \quad \text{Minimize: } C(\eta), \text{ subject to: } \eta \in \Delta, L(\eta) \leq \ell.$$

We denote the values of problems (2.4) and (2.5) by

$$\begin{aligned} L_*(c) &= \inf\{L(\eta) : \eta \in \Delta \text{ and } C(\eta) \leq c\} \quad \text{for } c \in [0, c_{\max}], \\ C_*(\ell) &= \inf\{C(\eta) : \eta \in \Delta \text{ and } L(\eta) \leq \ell\} \quad \text{for } \ell \in [0, \ell_{\max}]. \end{aligned}$$

Similarly, the maximization problem related to the “worst” strategies for a fixed loss $\ell \in [0, \ell_{\max}]$ or a fixed cost $c \in [0, c_{\max}]$ are defined by

$$(2.6) \quad \text{Maximize: } L(\eta), \text{ subject to: } \eta \in \Delta, C(\eta) \geq c,$$

$$(2.7) \quad \text{Maximize: } C(\eta), \text{ subject to: } \eta \in \Delta, L(\eta) \geq \ell.$$

We denote the values of problems (2.6) and (2.7) by

$$\begin{aligned} L^*(c) &= \sup\{L(\eta) : \eta \in \Delta \text{ and } C(\eta) \geq c\} \quad \text{for } c \in [0, c_{\max}], \\ C^*(\ell) &= \sup\{C(\eta) : \eta \in \Delta \text{ and } L(\eta) \geq \ell\} \quad \text{for } \ell \in [0, \ell_{\max}]. \end{aligned}$$

Under Assumption 1, as the loss and the cost functions are continuous on the compact set Δ , the infima in the definitions of the value functions C_* and L_* are minima, and the suprema in the definitions of C^* and L^* are maxima. The minima and maxima are not unique in general; see the examples in [6].

See Figure 2 for a typical representation of the possible aspects of the feasible region $\mathbf{F}$ (in light blue), the value functions and the Pareto and anti-Pareto frontiers under the general Assumption 1, and the connected Pareto and anti-Pareto frontiers under further regularity on the cost and loss functions (see Assumptions 2–5 below) in Figure 2(D). In Figure 1, we have plotted in solid red line the Pareto frontier and in dashed red line the anti-Pareto frontier from section 1.4.3.

2.2.1. Outline of the section. It turns out that the anti-Pareto optimization problem can be recast as a Pareto optimization problem by changing signs and exchanging the cost and loss functions. In order to make use of this property in our application for vaccination in SIS models, we study the Pareto problem under assumptions on the loss L that are general enough to cover the effective reproduction function R_e and the fraction of infected individuals at equilibrium $\mathfrak{I}$, as well as assumptions on the cost that cover the uniform or affine cost functions and $-L$.

The main result of this section states that all the solutions of the optimization problems (2.4) or (2.5) are Pareto optimal, and gives a description of the Pareto frontier $\mathcal{F}$ as a graph in section 2.3, and similarly for the anti-Pareto frontier in section 2.4. Surprisingly, those two problems are not completely symmetric for the loss functions R_e and $\mathfrak{I}$ for SIS models considered in section 4.7; see Proposition 4.5 for an irreducibility condition on the kernels to study the anti-Pareto frontier.

2.3. On the Pareto frontier. The problems (2.4) and (2.5) have solutions.

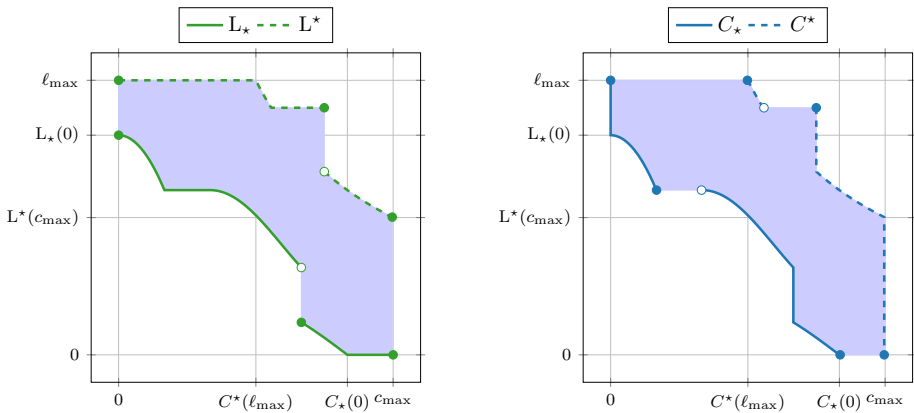
PROPOSITION 2.3 (optimal solutions for fixed cost or fixed loss). *Suppose Assumption 1 holds. For any cost $c \in [0, c_{\max}]$, there exists a minimizer of the loss under the cost constraint $C(\cdot) \leq c$, that is, a solution to problem (2.4). Similarly, for any loss $\ell \in [0, \ell_{\max}]$, there exists a minimizer of the cost under the loss constraint $L(\cdot) \leq \ell$, that is, a solution to problem (2.5).*

Proof. Let $c \in [0, c_{\max}]$. Since $C(\mathbf{1}) = 0$, the set $\{\eta \in \Delta : C(\eta) \leq c\}$ is nonempty. It is also compact as C is continuous on the compact set Δ . Therefore, since the loss function L is continuous, we get that L restricted to this compact set reaches its minimum: problem (2.4) has a solution. The proof is similar for the existence of a solution to problem (2.5). $\square$

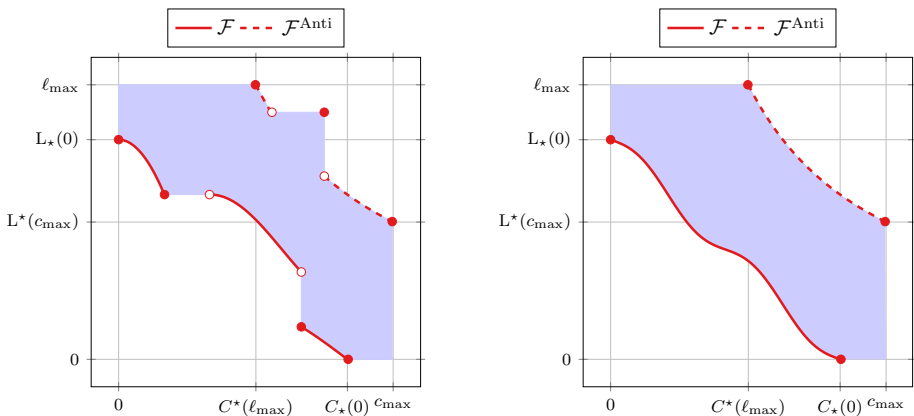
PROPOSITION 2.4 (elementary properties of L_*, L^*, C_*, C^*). *If Assumption 1 holds, then the functions L_* and C_* are right-continuous, while L^* and C^* are left-continuous; the four functions are nonincreasing. We also have*

$$L_*(c_{\max}) = 0, \quad L^*(0) = \ell_{\max}, \quad C_*(\ell_{\max}) = 0, \quad \text{and} \quad C^*(0) = c_{\max}.$$

Proof. We only consider L_* , the three other cases being similar. By definition, L_* is nonincreasing. It remains to check that it is right-continuous. Let c_n be a decreasing sequence converging to c , and let $\ell_n = L_*(c_n) \leq L_*(c)$. By monotonicity,



(A) Value functions for problems (2.4) and (2.6). (B) Value functions for problems (2.5) and (2.7).



(C) Pareto and anti-Pareto frontier. (D) Pareto and anti-Pareto frontier under additional regularity Assumptions 2-5.

FIG. 2. Possible aspects of the feasible region $\mathbf{F}$ (in light blue), the value functions L_* , L^* , C_* , C^* , and the (anti-)Pareto frontier (in red) under Assumption 1.

the sequence ℓ_n is nondecreasing and converges to a limit $\ell_\infty \leq L_*(c)$. To prove the reverse inequality, recalling that $\mathbf{F}$ denotes the set of all possible outcomes, we write

$$L_*(c) = \inf\{L(\eta) : \eta \in \Delta, C(\eta) \leq c\} = \inf\{\ell : (c', \ell) \in \mathbf{F}, c' \leq c\}.$$

Since $\mathbf{F}$ is compact (as the image of the compact set Δ by the continuous map (C, L)), and the condition $c' \leq c$ is closed, the infimum is a minimum. By definition there exist (c'_n, ℓ_n) such that $(c'_n, \ell_n) \in \mathbf{F}$, with $c'_n \leq c_n$. By compactness, along a subsequence we have $(c'_n, \ell_n) \rightarrow (c'_\infty, \ell_\infty) \in \mathbf{F}$, where $c'_\infty \leq c$, so $L_*(c) \leq L_*(c'_\infty) \leq \ell_\infty$. This implies that $L_*(c_n)$ converges to $\ell_\infty = L_*(c)$, so L_* is right-continuous.

Eventually the equality $L_*(c_{\max}) = 0$ is obvious by definition of L_* . $\square$

We start by a general result concerning the links between the three optimization problems (2.2), (2.4), and (2.5).

PROPOSITION 2.5 (single/biobjective problems). *Suppose Assumption 1 holds.*

- (i) *If η_* is Pareto optimal, then η_* is a solution of (2.4) for the cost $c = C(\eta_*)$ and a solution of (2.5) for the loss $\ell = L(\eta_*)$. Conversely, if η_* is a solution to both problems (2.4) and (2.5) for some values c and ℓ , then η_* is Pareto optimal. In particular, we have*

$$(2.8) \quad \mathcal{P} = \{\eta \in \Delta : L(\eta) = L_*(C(\eta))\} \cap \{\eta \in \Delta : C(\eta) = C_*(L(\eta))\}.$$

- (ii) *The Pareto frontier is the intersection of the graphs of C_* and L_* :*

$$\mathcal{F} = \{(c, \ell) \in [0, c_{\max}] \times [0, \ell_{\max}] : c = C_*(\ell) \text{ and } \ell = L_*(c)\}.$$

- (iii) *The points $(0, L_*(0))$ and $(C_*(0), 0)$ both belong to the Pareto frontier, and $C_*(L_*(0)) = L_*(C_*(0)) = 0$. Moreover, we also have $C_*(\ell) = 0$ for $\ell \in [L_*(0), \ell_{\max}]$, and $L_*(c) = 0$ for $c \in [C_*(0), c_{\max}]$.*

Proof. Let us prove (i). If η_* is Pareto optimal, then for any strategy η , if $C(\eta) \leq C(\eta_*)$, then $L(\eta) \geq L(\eta_*)$ by taking the contraposition in (2.3), and η_* is indeed a solution of problem (2.4) with $c = C(\eta_*)$. Similarly η_* is a solution of problem (2.5).

For the converse statement, let η_* be a solution of (2.4) for some c and of (2.5) for some ℓ . It is also a solution of (2.4) with $c = C(\eta_*)$. In particular, we get that for $\eta \in \Delta$, $L(\eta) < L(\eta_*)$ implies that $C(\eta) > c = C(\eta_*)$, which is the second part of (2.3). Similarly, use that η_* is a solution to (2.5) to get that the first part of (2.3) also holds. Thus, the strategy η_* is Pareto optimal.

To prove point (ii), we first prove that $\mathcal{F}$ is a subset of $\{(c, \ell) : c = C_*(\ell) \text{ and } \ell = L_*(c)\}$. A point in $\mathcal{F}$ may be written as $(C(\eta_*), L(\eta_*))$ for some Pareto optimal strategy η_* . By point (i), η_* solves problem (2.4) for the cost $C(\eta_*)$, so $L_*(C(\eta_*)) = L(\eta_*)$. Similarly, we have $C_*(L(\eta_*)) = C(\eta_*)$, as claimed.

We now prove the reverse inclusion. Assume that $c = C_*(\ell)$ and $\ell = L_*(c)$, and consider η a solution of problem (2.5) for the loss ℓ : $L(\eta) \leq \ell$ and $C(\eta) = C_*(\ell) = c$. Then η is admissible for problem (2.4) with cost $c = C_*(\ell)$, so $L(\eta) \geq L_*(C_*(\ell)) = L_*(c) = \ell$. Therefore, we get $L(\eta) = L_*(c)$, and η is also a solution of problem (2.4). By point (i), η is Pareto optimal, so $(C(\eta), L(\eta)) = (c, \ell) \in \mathcal{F}$, and the reverse inclusion is proved.

Finally we prove point (iii). We have $C_*(0) = \min\{C(\eta) : \eta \in \Delta \text{ and } L(\eta) = 0\} \in [0, c_{\max}]$. Let $\eta \in \Delta$ such that $L(\eta) = 0$ and $C(\eta) = C_*(0)$. We deduce that $L_*(C_*(0)) \leq L(\eta) = 0$ and thus $L_*(C_*(0)) = 0$ as L is nonnegative. We deduce from (ii) that $(C_*(0), 0)$ belongs to $\mathcal{F}$. Since L_* is nonincreasing, we also get that $L_* = 0$ on $[C_*(0), c_{\max}]$. The other properties of (iii) are proved similarly. $\square$

The next two hypotheses will help to rule out “flat parts” in the Pareto frontier.

Assumption 2. Any local minimum of the cost function C is global.

Assumption 3. Any local minimum of the loss function L is global.

Remark 2.6 (local minimum and topologies). In the framework of the SIS model of section 2.1.1 with $E = L^\infty(\Omega, \mu)$, consider the function C' defined by $C'(\eta) = 1 - \int_\Omega (2\eta - 1)_+ d\mu$, where $x_+ = \max(0, x)$ for $x \in \mathbb{R}$. This function is nonincreasing and satisfies $C'(1) = 0$ and $C'(0) = 1$. Let $\mathcal{E}$ be the weak-* topology and $\mathcal{E}'$ the topology of the uniform convergence associated to the sup norm, so $\mathcal{E} \subset \mathcal{E}'$. Then, the function C' is $\mathcal{E}'$ -continuous, and for any measurable set A such that $0 < \mu(A) < 1$, the function $\mathbb{1}_A$ is a local minimum of the restriction of C' to $\Delta = \{f \in E : 0 \leq f \leq 1\}$. However, in the $\mathcal{E}$ topology, the restriction of C' to Δ has only one local minimum, given by $\mathbb{1}$, which is also a global minimum.

Therefore, Assumption 2 is a strong hypothesis for stronger topologies but a more reasonable one for weaker topologies. Note by contrast that the continuity condition of Assumption 1 is a stronger condition for the weaker topologies (for example, the function C' discussed above is not $\mathcal{E}$ -continuous).

PROPOSITION 2.7 (reduction to single-objective problems). *Under Assumptions 1 and 2, the biobjective problem may be reduced to cost optimization for a fixed loss; more precisely the following properties hold:*

- (i) *The optimal cost C_* is decreasing on $[0, L_*(0)]$.*
- (ii) *For any $\ell \in [0, L_*(0)]$ there exist η that solve problem (2.5). For any such η , $L(\eta) = \ell$ (that is, the constraint is binding), η is Pareto optimal, and*

$$(2.9) \quad L_*(C_*(\ell)) = \ell.$$

In particular, we have

$$(2.10) \quad \mathcal{P} = \{\eta \in \Delta : C(\eta) = C_*(L(\eta)) \quad \text{and} \quad L(\eta) \leq L_*(0)\}.$$

- (iii) *The Pareto frontier is the graph of C_* :*

$$(2.11) \quad \mathcal{F} = \{(C_*(\ell), \ell) : \ell \in [0, L_*(0)]\}.$$

Similarly, under Assumptions 1 and 3, the biobjective problem may be reduced to loss optimization for a fixed cost, that is,

- (iv) *The optimal loss L_* is decreasing on $[0, C_*(0)]$.*
- (v) *For any $c \in [0, C_*(0)]$ there exist η that solve problem (2.4). For any such η , $C(\eta) = c$. Moreover η is Pareto optimal, and $C_*(L_*(c)) = c$. So, we have*

$$(2.12) \quad \mathcal{P} = \{\eta \in \Delta : L(\eta) = L_*(C(\eta)) \quad \text{and} \quad C(\eta) \leq C_*(0)\}.$$

- (vi) *The Pareto frontier is the graph of L_* :*

$$(2.13) \quad \mathcal{F} = \{(c, L_*(c)) : c \in [0, C_*(0)]\}.$$

Proof. We prove (i). Let $0 \leq \ell < \ell' \leq L_*(0)$, and let η_* be a solution of (2.5):

$$(2.14) \quad C(\eta_*) = C_*(\ell) \quad \text{and} \quad L(\eta_*) \leq \ell.$$

The set $\mathcal{O} = \{\eta : L(\eta) < \ell'\}$ is open and contains η_* . Since $L(\eta_*) < L_*(0)$, we get $C(\eta_*) > 0$, so η_* is not a global minimum for C . By Assumption 2, it cannot be a local minimum for C , so $\mathcal{O}$ contains at least one point η' for which $C(\eta') < C(\eta_*)$. Since $\eta' \in \mathcal{O}$, we get $L(\eta') \leq \ell'$, so that $C_*(\ell') \leq C(\eta') < C(\eta_*) = C_*(\ell)$. Since $\ell < \ell'$ are arbitrary, C_* is decreasing on $[0, L_*(0)]$.

We now prove (ii). If the inequality in (2.14) was strict, that is, $L(\eta_*) < \ell$, then we would get a contradiction as $C(\eta_*) \geq C_*(L(\eta_*)) > C_*(\ell) = C(\eta_*)$. Therefore any solution η_* of (2.5) satisfies $L(\eta_*) = \ell$, and in particular $C_*(L(\eta_*)) = C_*(\ell) = C(\eta_*)$. This implies in turn that η_* also solves (2.4): if η satisfies $L(\eta) < L(\eta_*)$, then using the definition of C_* , the fact that it decreases, and the definition of η_* , we get

$$C(\eta) \geq C_*(L(\eta)) > C_*(L(\eta_*)) = C(\eta_*).$$

By contraposition, we have $L(\eta) \geq L(\eta_*)$ for any η such that $C(\eta) \leq C(\eta_*)$, proving that η_* is also a solution of (2.4) with $c = C(\eta_*)$. By point (i) of Proposition 2.5, η_* is

Pareto optimal. Therefore $(C(\eta_\star), L(\eta_\star)) = (C_\star(\ell), \ell)$ belongs to the Pareto frontier. Using point (ii) of Proposition 2.5, we deduce that $\ell = L_\star(C_\star(\ell))$.

To prove point (iii), note that (2.9) shows that, if $c = C_\star(\ell)$ for $\ell \in [0, L_\star(0)]$, then $\ell = L_\star(c)$. Use point (ii) and (iii) of Proposition 2.5 to get that $\mathcal{F} = \{(c, \ell) : c = C_\star(\ell), \ell \in [0, L_\star(0)]\}$.

The claims (iv), (v), and (vi) are proved similarly, exchanging the roles of L and C . $\square$

Under both hypotheses, the picture becomes much nicer, see Figure 2(D), where the only flat parts of the graphs of $C_\star$ and $L_\star$ occur at zero loss or zero cost.

PROPOSITION 2.8 (when all problems are equivalent). *Under Assumptions 1, 2, and 3, the following properties hold:*

- (i) *The optimal loss $L_\star$ is a continuous decreasing bijection of $[0, C_\star(0)]$ onto $[0, L_\star(0)]$, and $C_\star$ is the inverse bijection.*
- (ii) *The Pareto frontier $\mathcal{F}$ is compact and connected.*
- (iii) *The set of Pareto optimal strategies $\mathcal{P}$ is compact.*

Proof. We first check that $C_\star$ and $L_\star$ are continuous under Assumptions 1, 2, and 3. We deduce from Proposition 2.7 (ii) and Proposition 2.3 that $[0, L_\star(0)]$ is in the range of $L_\star$. Since $L_\star$ is decreasing, thanks to Proposition 2.7 (iv) and $L_\star(C_\star(0)) = 0$, see Proposition 2.5 (iii), the function $L_\star$ is continuous and decreasing on $[0, L_\star(0)]$ and thus one-to-one from $[0, C_\star(0)]$ onto $[0, L_\star(0)]$. Thanks to (2.9), its inverse bijection is the function $C_\star$.

Since the frontier $\mathcal{F}$ is given by (2.13), and $L_\star$ is continuous, the frontier $\mathcal{F}$ is compact and connected.

Since $\mathcal{F}$ is closed and $\mathcal{P} = f^{-1}(\mathcal{F})$, where the function $f = (C, L)$ defined on Δ is continuous, we get that $\mathcal{P}$ is a closed subset of Δ and thus compact as Δ is compact. $\square$

Finally, let us check that Assumptions 2 and 3 hold under very simple assumptions, which are in particular satisfied for the SIS model with vaccination by the cost functions C_{uni} and C_{aff} and the loss functions R_e and $\mathcal{J}$; see section 4.7.

LEMMA 2.9. *Suppose Assumption 1 holds. If the cost function C is convex or decreasing, then Assumption 2 holds, and in the latter case we also have $L_\star(0) = \ell_{\max}$. If the loss function L is subhomogeneous, then Assumption 3 holds.*

Proof. Let $\eta \in \Delta$ be a local minimum of C . For $\varepsilon > 0$ small enough, we get $C(\eta + \varepsilon(\mathbb{1} - \eta)) \geq C(\eta)$ using that $\eta + \varepsilon(\mathbb{1} - \eta) = (1 - \varepsilon)\eta + \varepsilon\mathbb{1}$ belongs to Δ as Δ is convex. If C is convex, using that $C(\mathbb{1}) = 0$, we also get $C(\eta) \geq (1 - \varepsilon)C(\eta) \geq C(\eta + \varepsilon(\mathbb{1} - \eta))$. Therefore all inequalities are equalities, so $C(\eta) = 0$. We thus deduce that η is a global minimum for C . If C is decreasing, using that $\eta \leq \eta + \varepsilon(\mathbb{1} - \eta)$, we get $C(\eta) \geq C(\eta + \varepsilon(\mathbb{1} - \eta))$, with the equality only if $\eta = \mathbb{1}$; we deduce that η is a global minimum of C . So if C is convex or decreasing, then Assumption 2 holds. If C is decreasing, as any local minimum is in fact equal to $\mathbb{1}$, this also gives $L_\star(0) = \ell_{\max}$.

Similarly if L has a local minimum at η , then for $\varepsilon > 0$ small enough, we have $L(\eta) \leq L((1 - \varepsilon)\eta) \leq (1 - \varepsilon)L(\eta)$, so $L(\eta) = 0$, where we used that L is subhomogeneous for the second equality. Thus η is a global minimum of L . $\square$

2.4. On the anti-Pareto frontier. Letting $C'(\eta) = \ell_{\max} - L(\eta)$ and $L'(\eta) = c_{\max} - C(\eta)$, it is easy to see that

$$C'_\star(c) = \ell_{\max} - L_\star(c_{\max} - c) \quad \text{and} \quad L'_\star(\ell) = c_{\max} - C_\star(\ell_{\max} - \ell),$$

so that Propositions 2.3 and 2.5 may be applied to the cost function C' and the loss function L' to yield the following result.

PROPOSITION 2.10 (single-objective and biobjective problems—the anti Pareto case). *Suppose Assumption 1 holds.*

- (0) *For any cost $c \in [0, c_{\max}]$, there exists a maximizer of the loss under the cost constraint $C(\cdot) \geq c$, that is, a solution to problem (2.6). Similarly, for any loss $\ell \in [0, \ell_{\max}]$, there exists a maximizer of the cost under the loss constraint $L(\cdot) \geq \ell$, that is, a solution to problem (2.7).*
- (i) *If η^* is anti-Pareto optimal, then η^* is a solution of (2.6) for the cost $c = C(\eta^*)$ and a solution of (2.7) for the loss $\ell = L(\eta^*)$. Conversely, if η^* is a solution to both problems (2.6) and (2.7) for some values c and ℓ , then η^* is anti-Pareto optimal. In particular, we have*

$$(2.15) \quad \mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : L(\eta) = L^*(C(\eta))\} \cap \{\eta \in \Delta : C(\eta) = C^*(L(\eta))\}.$$

- (ii) *The anti-Pareto frontier is the intersection of the graphs of C^* and L^* :*

$$(2.16) \quad \mathcal{F}^{\text{Anti}} = \{(c, \ell) \in [0, c_{\max}] \times [0, \ell_{\max}] : c = C^*(\ell) \text{ and } \ell = L^*(c)\}.$$

- (iii) *The points $(C^*(\ell_{\max}), \ell_{\max})$ and $(c_{\max}, L^*(c_{\max}))$ both belong to the anti-Pareto frontier, and we have $C^*(L^*(c_{\max})) = c_{\max}$ and $L^*(C^*(\ell_{\max})) = \ell_{\max}$. Moreover, we also have $C^*(\ell) = c_{\max}$ for $\ell \in [0, L^*(c_{\max})]$, and $L^*(c) = \ell_{\max}$ for $c \in [0, C^*(\ell_{\max})]$.*

The following additional hypotheses rule out the occurrence of flat parts in the anti-Pareto frontier.

Assumption 4. Any local maximum of the cost function C is global.

Assumption 5. Any local maximum of the loss function L is global.

The following result is now a consequence of Proposition 2.7 and Proposition 2.8 applied to the loss function L' and cost function C' .

PROPOSITION 2.11 (reduction to single-objective problems—the anti-Pareto case). *Under Assumption 1 and 4 the following properties hold:*

- (i) *The optimal cost C^* is decreasing on $[L^*(c_{\max}), \ell_{\max}]$.*
- (ii) *If η solves problem (2.7) for the loss $\ell \in [L^*(c_{\max}), \ell_{\max}]$, then $L(\eta) = \ell$ (that is, the constraint is binding). Moreover η is anti-Pareto optimal, and $L^*(C^*(\ell)) = \ell$. We have*

$$(2.17) \quad \mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : C(\eta) = C^*(L(\eta)) \text{ and } L(\eta) \in [L^*(c_{\max}), \ell_{\max}]\}.$$

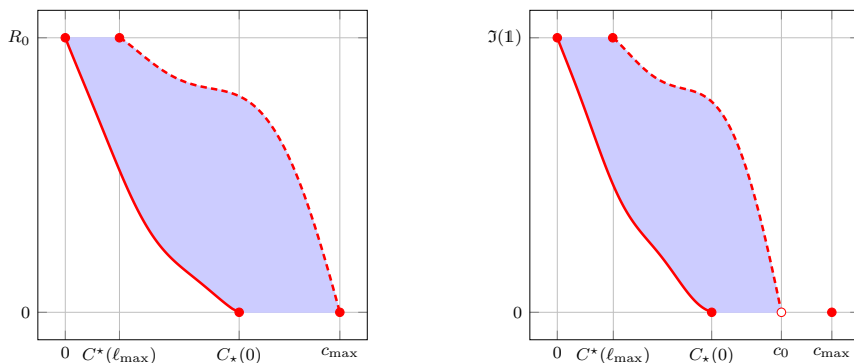
- (iii) *The anti-Pareto frontier is the graph of C^* :*

$$(2.18) \quad \mathcal{F}^{\text{Anti}} = \{(C^*(\ell), \ell) : \ell \in [L^*(c_{\max}), \ell_{\max}]\}.$$

Similarly, under Assumptions 1 and 5, the following properties hold:

- (iv) *The optimal loss L^* is decreasing on $[C^*(\ell_{\max}), c_{\max}]$.*
- (v) *If η solves problem (2.6) for the cost $c \in [C^*(\ell_{\max}), c_{\max}]$, then $C(\eta) = c$. Moreover η is anti-Pareto optimal, and $C^*(L^*(c)) = c$. We have*

$$(2.19) \quad \mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : L(\eta) = L^*(C(\eta)) \text{ and } C(\eta) \in [C^*(\ell_{\max}), c_{\max}]\}.$$



(A) Loss function $L = R_e$ with $R_0 > 0$ (also $\ell_{\max} = R_0$ and $C^*(\ell_{\max}) = C(\mathbb{1}_{\Omega_a})$). (B) Loss function $L = \mathcal{J}$ with $R_0 > 1$ (also $\ell_{\max} = \mathcal{J}_0 = \mathcal{J}(\mathbb{1}) > 0$ and $C^*(\ell_{\max}) = C(\mathbb{1}_{\{g>0\}})$).

FIG. 3. Typical shape of the Pareto frontier $\mathcal{F}$ in solid red line, the anti-Pareto frontier $\mathcal{F}^{\text{Anti}}$ in red dashed line, and the feasible region $\mathbf{F}$ in light blue for the SIS model with a monotonic transmission rate kernel and a continuous decreasing cost function.

(vi) The anti-Pareto frontier is the graph of L^* :

$$(2.20) \quad \mathcal{F}^{\text{Anti}} = \{(c, L^*(c)) : c \in [C^*(\ell_{\max}), c_{\max}]\}.$$

Finally, if the three Assumptions 1, 4, and 5 hold, then L^* is a continuous decreasing bijection from $[C^*(\ell_{\max}), c_{\max}]$ onto $[L^*(c_{\max}), \ell_{\max}]$, C^* is the inverse bijection, the anti-Pareto frontier $\mathcal{F}^{\text{Anti}}$ is compact and connected, and the set of anti-Pareto optimal strategies $\mathcal{P}^{\text{Anti}}$ is compact.

The following result is similar to the first part of Lemma 2.9.

LEMMA 2.12. Suppose the part of Assumption 1 on the cost function holds. If the cost function C is concave or decreasing, then Assumption 4 holds, and in the latter case we also have $L^*(c_{\max}) = 0$.

Proof. Let $\eta \in \Delta$ be a local maximum of C and $\varepsilon \in (0, 1)$ small enough so that $C(\eta) \geq C((1 - \varepsilon)\eta)$. If C is concave, we also get $C((1 - \varepsilon)\eta) \geq (1 - \varepsilon)C(\eta) + \varepsilon C(0) = (1 - \varepsilon)C(\eta) + \varepsilon c_{\max}$. We deduce that $C(\eta) = c_{\max}$, and thus η is a global maximum.

If C is decreasing, we get $C((1 - \varepsilon)\eta) \geq C(\eta)$, with equality if and only if $\eta = 0$. Therefore the only local maximum of C is $\eta = 0$, and it is a global maximum. Since $C(\eta) = c_{\max}$ implies that $\eta = 0$, we also get that $L^*(c_{\max}) = L(0) = 0$. $\square$

In the SIS model, the loss function $L = \mathcal{J}$ does not satisfy Assumption 5 in general because of its zeros, see Remark 4.4, but it satisfies the following weaker condition.

Assumption 5'. If η is a local maximum of L such that $L(\eta) > 0$, then it is a global maximum.

Under this assumption, the function C^* might be discontinuous at 0. We set

$$(2.21) \quad c_0 = C^*(0+) = \lim_{\substack{\ell \rightarrow 0 \\ \ell > 0}} C^*(\ell).$$

Considering Assumption 5' with $c_0 < c_{\max}$ instead of Assumption 5 impacts only items (iv)–(vi) and the conclusion of Proposition 2.11 as follows; we refer to Figure 3 for an illustration and leave the proof to the reader.

PROPOSITION 2.11' (the anti-Pareto case, weaker version). *Suppose Assumptions 1 and 5' hold.*

- (a) *If $c_0 = c_{\max}$, then items (iv)–(vi) and the conclusion of Proposition 2.11 hold.*
- (b) *If $c_0 < c_{\max}$, the following properties hold:*
 - (iv) *The optimal loss L^* is decreasing on $[C^*(\ell_{\max}), c_0]$ and zero on $(c_0, c_{\max}]$.*
 - (v) *If η solves problem (2.6) for the cost $c \in [C^*(\ell_{\max}), c_0] \cup \{c_{\max}\}$, then $C(\eta) = c$, η is anti-Pareto optimal, and $C^*(L^*(c)) = c$. We have*

$$(2.22) \quad \mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : L(\eta) = L^*(C(\eta)) \quad \text{and} \quad C(\eta) \in [C^*(\ell_{\max}), c_0] \cup \{c_{\max}\}\}.$$

- (vi) *The anti-Pareto frontier is also given by*

$$(2.23) \quad \mathcal{F}^{\text{Anti}} = \{(c, L^*(c)) : c \in [C^*(\ell_{\max}), c_0]\} \cup \{(c_{\max}, 0)\}.$$

If furthermore Assumption 4 holds, then $L^(c_0) = 0$, the function L^* is a continuous decreasing bijection of $[C^*(\ell_{\max}), c_0]$ onto $(0, \ell_{\max}]$, C^* is the inverse bijection, and the union of the anti-Pareto frontier with its limit point $\{(c_0, 0)\}$ is compact but not connected.*

Let us give a simple result to help decide which of the cases (a) and (b) holds in the previous proposition.

LEMMA 2.13 (on $c_0 < c_{\max}$). *Suppose Assumption 1 holds. If the cost function is decreasing and if $\{\eta \in \Delta : L(\eta) = 0\}$ contains a neighborhood of $\emptyset$ in Δ , then we have $c_0 < c_{\max}$.*

Proof. Let O be an open neighborhood of $\emptyset$ in Δ such that $L = 0$ on O . Since the cost function is continuous and $\Delta \setminus O$ is compact, it reaches its maximum, say at η . Since $\eta \neq \emptyset$ and the cost is decreasing, we deduce that $C(\eta) < C(\emptyset) = c_{\max}$. This gives also that $c_0 \leq \sup\{C(\eta') : \eta' \in \Delta \setminus O\} = C(\eta) < c_{\max}$. $\square$

It turns out that, under a weak technical assumption that holds for the SIS model, $c_0 = c_{\max}$ together with Assumption 5' imply Assumption 5. We refer to [8, Lemma 2.14 and 4.8, Remarks 2.15–2.16] for an extended discussion of this fact.

3. Miscellaneous properties of the set of outcomes and the Pareto frontier. We prove results concerning the feasible region and its geometry.

3.1. The feasible region. In the following proposition we check a number of topological properties of the set of outcomes $\mathbf{F} = \{(C(\eta), L(\eta)), \eta \in \Delta\}$.

PROPOSITION 3.1 (no hole in the feasible region). *Suppose that Assumption 1 holds. The feasible region $\mathbf{F}$ is compact and path connected, and its complement is connected in $\mathbb{R}^2$. It is the region between the graphs of the value functions:*

$$(3.1) \quad \begin{aligned} \mathbf{F} &= \{(c, \ell) \in \mathbb{R}^2 : 0 \leq c \leq c_{\max}, L_*(c) \leq \ell \leq L^*(c)\} \\ &= \{(c, \ell) \in \mathbb{R}^2 : 0 \leq \ell \leq \ell_{\max}, C_*(\ell) \leq c \leq C^*(\ell)\}. \end{aligned}$$

Proof. Since Δ is convex, it is path connected. The set $\mathbf{F}$ is compact and path connected as the continuous image by (C, L) of the compact, path-connected set Δ .

By symmetry, it is enough to prove that $\mathbf{F}$ is equal to $F_1 = \{(c, \ell) \in \mathbb{R}^2 : 0 \leq c \leq c_{\max}, L_*(c) \leq \ell \leq L^*(c)\}$. Let $(c, \ell) \in \mathbf{F}$ and $\eta \in \Delta$ be such that $(c, \ell) = (C(\eta), L(\eta))$. By definition of L_* and L^* , we have $L_*(c) = L_*(C(\eta)) \leq L(\eta) \leq L^*(C(\eta)) = L^*(c)$. We deduce that $(c, \ell) \in F_1$.

Let us now prove that $F_1 \subset \mathbf{F}$. Let us first consider a point of the form $(c, L_*(c))$, where $0 \leq c \leq c_{\max}$. By definition, there exists η such that $C(\eta) \leq c$ and $L(\eta) = L_*(c)$. Let $\eta_t = t\eta$. The map $t \mapsto C(\eta_t)$ is continuous from $[0, 1]$ to $[C(\eta), c_{\max}]$. As $c \in [C(\eta), c_{\max}]$, there exists s such that $C(\eta_s) = c$. Since L is nondecreasing, $L(\eta_s) \leq L(\eta)$. By definition of $L_*(c)$, $L(\eta_s) \geq L_*(c)$. Therefore $(c, L_*(c)) = (C(\eta_s), L(\eta_s))$ belongs to $\mathbf{F}$. Similarly the graphs of C_* , C^* , and L^* are also included in $\mathbf{F}$.

So, it is enough to check that, if $A = (c, \ell)$ is in F_1 , with $c \in (0, c_{\max})$ and $\ell \in (L_*(c), L^*(c))$, then A belongs to $\mathbf{F}$. We shall assume that $A \notin \mathbf{F}$ and derive a contradiction by building a loop in $\mathbf{F}$ that encloses A and which can be continuously contracted into a point in $\mathbf{F}$.

Since $L_*(c) < \ell < L^*(c)$, there exist η_{SO} and η_{NE} such that

$$C(\eta_{\text{SO}}) \leq c, \quad L(\eta_{\text{SO}}) < \ell, \quad C(\eta_{\text{NE}}) \geq c, \quad \text{and} \quad L(\eta_{\text{NE}}) > \ell.$$

We concatenate the four continuous paths in Δ defined for $u \in [0, 1]$,

$$u \mapsto u\eta_{\text{SO}}, \quad u \mapsto (1-u)\eta_{\text{SO}} + u\mathbb{1}, \quad u \mapsto (1-u)\mathbb{1} + u\eta_{\text{NE}}, \quad \text{and} \quad u \mapsto (1-u)\eta_{\text{NE}},$$

to obtain a continuous loop $(\eta_t, t \in [0, 4])$ from $[0, 4]$ to Δ such that $(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4) = (0, \eta_{\text{SO}}, \mathbb{1}, \eta_{\text{NE}}, 0)$. We now define a continuous family of loops $(\gamma_s, s \in [0, 1])$ in $\mathbb{R}^2$ by

$$\gamma_s(t) = (C(s\eta_t), L(s\eta_t), t \in [0, 4]).$$

By definition, for all $s \in [0, 1]$, γ_s is a continuous loop in $\mathbf{F}$. Since $A = (c, \ell) \notin \mathbf{F}$, the loops γ_s do not contain A , so the winding number $W(\gamma_s, A)$ is well defined (see, for example, [26, Definition 6.1]). As $A \notin \mathbf{F}$, we get that γ_s is a continuous deformation in $\mathbb{R}^2 \setminus \{A\}$ from γ_1 to γ_0 . Thanks to [26, Theorem 6.5], this implies that $W(\gamma_s, A)$ does not depend on $s \in [0, 1]$.

For $s = 0$, the loop degenerates to the single point $(C(0), 0)$ so the winding number is 0. For $s = 1$, let us check that the winding number is 1, which will provide the contradiction. To do this, we compare γ_1 with a simpler loop δ defined by

$$\delta(0) = \delta(4) = (c_{\max}, 0), \quad \delta(1) = (0, 0), \quad \delta(2) = (0, \ell_{\max}), \quad \text{and} \quad \delta(3) = (c_{\max}, \ell_{\max}),$$

and by linear interpolation for other values of t : δ runs around the perimeter of an axis-aligned rectangle with corners $(0, 0)$ and $(c_{\max}, \ell_{\max})$. Clearly $W(\delta, A) = 1$. Let M_t, N_t denote $\gamma_1(t)$ and $\delta(t)$, respectively. For $t \in [0, 1]$, we have $N_t = ((1-t)c_{\max}, 0)$, so the second coordinate of $\overrightarrow{AN_t}$ is nonpositive. On the other hand $L(t\eta_{\text{SO}}) \leq L(\eta_{\text{SO}}) < \ell$, so the second coordinate of $\overrightarrow{AM_t}$ is negative. Therefore the two vectors $\overrightarrow{AN_t}$ and $\overrightarrow{AM_t}$ cannot point in opposite directions. Similar considerations for the other values of $t \in [1, 4]$ show that $\overrightarrow{AN_t}$ and $\overrightarrow{AM_t}$ never point in opposite directions. By [26, Theorem 6.1], the winding numbers $W(\gamma_1, A)$ and $W(\delta, A)$ are equal, and thus $W(\gamma_1, A) = 1$.

This gives that $A \in \mathbf{F}$ by contradiction, and thus $F_1 \subset \mathbf{F}$.

Finally, it is easy to check that F_1 has a connected complement, because F_1 is bounded, and all the points in F_1^c can reach infinity by a straight line: for example, if $\ell > L^*(c)$, then the half-line $\{(c, \ell'), \ell' \geq \ell\}$ is in F_1^c . $\square$

3.2. Geometric properties. When the cost function is affine, then there is a nice geometric property of the Pareto frontier.

LEMMA 3.2. *Suppose that Assumption 1 holds, the cost function C is affine, and the loss function L is subhomogeneous. Then, we have for all $c \in [0, c_{\max}]$ and $\theta \in [0, 1]$,*

$$(3.2) \quad L_*(\theta c + (1-\theta)c_{\max}) \leq \theta L_*(c).$$

Remark 3.3. Geometrically, Lemma 3.2 means that the graph of the loss function $L_\star : [0, c_{\max}] \rightarrow [0, \ell_{\max}]$ is below its chords with end point $(1, L_\star(1)) = (1, 0)$. See Figure 1 for a typical representation of the Pareto frontier (red solid line).

Proof. Let $\theta \in [0, 1]$. For $c \in [C_\star(0), c_{\max}]$, we have $L_\star(c) = 0$, so (3.2) is true.

Let $c \in [0, C_\star(0)]$. Thanks to Lemma 2.9, Assumption 3 holds. Thus, thanks to Propositions 2.3 and 2.7 (v), there exists $\eta \in \mathcal{P}$ with cost $C(\eta) = c$ and thus $L(\eta) = L_\star(c)$. Since C is affine, we have

$$C(\theta\eta) = \theta C(\eta) + (1 - \theta)C(0) = \theta c + (1 - \theta)c_{\max}.$$

Therefore, the strategy $\theta\eta$ is admissible for problem (2.4) with cost constraint $C(\cdot) \leq \theta c + (1 - \theta)c_{\max}$. This implies that $L_\star(\theta c + (1 - \theta)c_{\max}) \leq L(\theta\eta) \leq \theta L(\eta) = \theta L_\star(c)$, thanks to the subhomogeneity of the loss function L . $\square$

In some cases, see, for example, [7, section 4], it is possible to prove that the loss function is a convex function (which in turn implies Assumption 3). In this case, choosing a convex cost function implies that Assumption 2 holds and the Pareto frontier is convex. We leave to the interested reader the proof of the next result.

PROPOSITION 3.4 (convexity/concavity of the Pareto frontier). *Suppose that Assumption 1 holds. If the cost function C and the loss function L are convex, then the functions $C_\star$ and $L_\star$ are convex. If the cost function C and the loss function L are concave, then the functions $C^\star$ and $L^\star$ are concave.*

4. SIS model and vaccination strategies. We follow the presentation of [5] for a heterogeneous SIS model in a constant size population.

4.1. The setting. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $x \in \Omega$ represents a *feature* and the probability measure $\mu(dx)$ represents the fraction of the population with feature x . The parameters of the SIS model are given by a *recovery rate function* γ , which is a positive bounded measurable function defined on Ω , and a *transmission rate kernel* k , which is a nonnegative measurable function defined on Ω^2 .

In accordance with [5], we consider for a kernel k on Ω and $q \in (1, +\infty)$ its norms:

$$\|k\|_{\infty, q} = \sup_{x \in \Omega} \left(\int_{\Omega} k(x, y)^q \mu(dy) \right)^{1/q}.$$

For a kernel k on Ω with $\|k\|_{\infty, q} < \infty$ for some $q \in (1, +\infty)$, we define the integral operator $\mathcal{T}_k$ on the set $\mathcal{L}^\infty$ of bounded measurable real-valued function on Ω by

$$(4.1) \quad \mathcal{T}_k(g)(x) = \int_{\Omega} k(x, y)g(y) \mu(dy) \quad \text{for } g \in \mathcal{L}^\infty \text{ and } x \in \Omega.$$

When k is nonnegative, this operator is also positive, that is, $\mathcal{T}_k(\mathcal{L}_+^\infty) \subset \mathcal{L}_+^\infty$, where $\mathcal{L}_+^\infty = \{f \in \mathcal{L}^\infty : f \geq 0\}$ is the cone of nonnegative bounded measurable functions.

We shall consider the kernel $\mathbf{k} = k/\gamma$ defined by

$$(4.2) \quad \mathbf{k}(x, y) = k(x, y) \gamma(y)^{-1}.$$

Assumption 6 (on the SIS model). Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. The recovery rate function γ defined on Ω is measurable bounded and positive. The transmission rate kernel k on Ω is nonnegative measurable and such that $\|k/\gamma\|_{\infty, q} < +\infty$ for some $q \in (1, +\infty)$.

Under Assumption 6, the integral operator $\mathcal{T}_{\mathbf{k}}$ on $\mathcal{L}^\infty$ is a bounded positive linear operator on $\mathcal{L}^\infty$; it is the so-called *next-generation operator*. Notice also that $\|k\|_{\infty, q}$ is finite (as γ is bounded), and thus the operator $\mathcal{T}_k$ is also a bounded positive linear operator on $\mathcal{L}^\infty$.

4.2. The SIS model. Let $\Delta' = \{f \in \mathcal{L}^\infty : 0 \leq f \leq 1\}$ be the subset of nonnegative functions bounded by 1. The SIS dynamics considered in [5] follows the vector field F defined on Δ' by

$$(4.3) \quad F(g) = (1 - g)\mathcal{T}_k(g) - \gamma g,$$

where the term $(1 - g)$ is the *incidence rate* given by the usual *law of mass action*.

More precisely, let $u_t(x) \in [0, 1]$ model the proportion of individuals with feature $x \in \Omega$ which are infected at time $t \geq 0$. The SIS dynamics considered for $u = (u_t, t \in \mathbb{R})$ is given by the following ODE in $\mathcal{L}^\infty$:

$$(4.4) \quad \partial_t u_t = F(u_t) \quad \text{for } t \in \mathbb{R}_+,$$

with initial condition $u_0 \in \Delta'$. It is proved in [5] that such a solution u exists and is unique.

An *equilibrium* of (4.4) is a function $g \in \Delta'$ such that $F(g) = 0$. Notice that the *disease-free equilibrium* (DFE), which is the zero function, is indeed an equilibrium. According to [5], there exists a maximal equilibrium $\mathbf{g}$, i.e., an equilibrium such that all other equilibria $h \in \Delta'$ are dominated by $\mathbf{g}$: $h \leq \mathbf{g}$. It is toward this maximal equilibrium that the process stabilizes when started from a situation where all the population is infected, that is, $\lim_{t \rightarrow \infty} u_t = \mathbf{g}$ if $u_0(x) = 1$ for all $x \in \Omega$.

For T a bounded operator on $\mathcal{L}^\infty$ endowed with its usual supremum norm, we define by $\|T\|_{\mathcal{L}^\infty}$ its operator norm, and its spectral radius is given by $\rho(T) = \lim_{n \rightarrow \infty} \|T^n\|_{\mathcal{L}^\infty}^{1/n}$. The *reproduction number* R_0 associated to the SIS model defined on (4.4) is the spectral radius of the next-generation operator:

$$(4.5) \quad R_0 = \rho(\mathcal{T}_{\mathbf{k}}).$$

If $R_0 \leq 1$ (subcritical and critical case) the maximal equilibrium is the DFE, and furthermore u_t converges pointwise to zero when $t \rightarrow \infty$. If $R_0 > 1$ (supercritical case), the maximal equilibrium $\mathbf{g}$ is not the DFE, and $\int_\Omega \mathbf{g} \, d\mu > 0$.

We denote by $\mathfrak{I}_0$ the *fraction of infected individuals at the maximal equilibrium*:

$$(4.6) \quad \mathfrak{I}_0 = \int_\Omega \mathbf{g} \, d\mu,$$

where we write $\int_\Omega f \, d\mu$ for $\int_\Omega f(x) \mu(dx)$. We deduce that $\mathfrak{I}_0 > 0 \iff R_0 > 1$.

4.3. Vaccination strategies. We assume the vaccine has *perfect efficiency* and that it is applied once and for all at time $t = 0$. A *vaccination strategy* η is an element of Δ' , where $\eta(x)$ represents the proportion of *nonvaccinated* individuals with feature x . Notice that $\eta \, d\mu$ corresponds in a sense to the effective population. In particular, the “strategy” that consists in vaccinating no one (resp., everybody) corresponds to $\eta(x) = 1$ (resp., $\eta(x) = 0$) for all $x \in \Omega$.

For $\eta \in \Delta'$, the kernel $\mathbf{k}\eta = k\eta/\gamma$, defined by $\mathbf{k}(x, y) = k(x, y)\eta(y)/\gamma(y)$, has finite norm $\|\cdot\|_{\infty, q}$, so we can consider the bounded positive operators $\mathcal{T}_{\mathbf{k}\eta}$ and $\mathcal{T}_{k\eta}$ on $\mathcal{L}^\infty$. According to [5, section 5.3.], the SIS equation with vaccination strategy η is given by $u^\eta = (u_t^\eta, t \geq 0)$ solution to (4.4) with F is replaced by F_η defined by

$$(4.7) \quad F_\eta(g) = (\mathbb{1} - g)\mathcal{T}_{k\eta}(g) - \gamma g.$$

Then the quantity $u_t^\eta(x) = u^\eta(t, x)$ represents the probability for a nonvaccinated individual of feature x to be infected at time t ; so at time t among the population of feature x , a fraction $1 - \eta(x)$ is vaccinated, a fraction $\eta(x)u_t^\eta(x)$ is not vaccinated and infected, and a fraction $\eta(x)(1 - u_t^\eta(x))$ is not vaccinated and not infected.

We define the *effective reproduction number* $R_e(\eta)$ associated to the vaccination strategy η as the spectral radius of the effective next-generation operator $\mathcal{T}_{\mathbf{k}\eta}$:

$$(4.8) \quad R_e(\eta) = \rho(\mathcal{T}_{\mathbf{k}\eta}).$$

We denote by $\mathbf{g}_\eta$ the corresponding maximal equilibrium. We will denote by $\mathfrak{I}(\eta)$ the corresponding fraction of infected individuals at equilibrium. Since the probability for an individual with feature x to be infected in the stationary regime is $\mathbf{g}_\eta(x)\eta(x)$, this fraction is given by the following formula:

$$(4.9) \quad \mathfrak{I}(\eta) = \int_{\Omega} \mathbf{g}_\eta \eta \, d\mu.$$

As $F_\eta(\mathbf{g}_\eta) = 0$, we deduce that $\mathbf{g}_\eta \eta = 0$ μ -almost surely if and only if $\mathbf{g}_\eta = 0$. Applying the results of the previous section to the kernel $k\eta$, we deduce that

$$(4.10) \quad \mathfrak{I}(\eta) > 0 \iff R_e(\eta) > 1.$$

4.4. The set of vaccination strategies Δ . This section is motivated by the continuity results from [9] established for the two loss functions R_e and $\mathfrak{I}$. For $p \in [1, +\infty]$, the space L^p of real-valued measurable functions g defined on Ω such that $\|g\|_p = (\int |g|^p \, d\mu)^{1/p}$ (with the convention that $\|g\|_\infty$ is the μ -essential supremum of $|g|$) is finite, where functions which agree μ -almost surely are identified. We denote by 0 and 1 the elements of L^p which are, respectively, the (class of equivalence of the) constant functions equal to 0 and to 1. We consider the cone $L_+^p = \{f \in L^p : f \geq 0\}$ of L^p ; it is salient pointed convex and corresponds to the usual partial order $\leq$, where $f \leq g$ means that $\mu(f > g) = 0$.

We will mainly be interested in the case $p = +\infty$. With a slight abuse of language we shall identify a function f in $\mathcal{L}^\infty$, with its equivalent class in L^∞ . We consider the set of vaccination strategies:

$$(4.11) \quad \Delta = \Delta_{\text{SIS}} = \{f \in L^\infty : 0 \leq f \leq 1\}.$$

From now on, we only consider this choice for Δ and shall drop the subscript SIS.

We now endow L^∞ with a topology for which Δ is compact. Since the measure μ is finite, the topological dual of L^1 is L^∞ . Let $\mathcal{O}$ denote the weak-* topology on L^∞ , that is, the weakest topology on L^∞ for which all the linear forms $f \mapsto \int_{\Omega} f g \, d\mu$, $g \in L^1$, defined on L^∞ are continuous. Following [1], we recall that $(L^\infty, \mathcal{O})$ is a Hausdorff topological vector space, the topological dual of $(L^\infty, \mathcal{O})$ is L^1 , and a sequence $(g_n, n \in \mathbb{N})$ of elements of L^∞ converges weakly-* to $g \in L^\infty$ if and only if

$$(4.12) \quad \lim_{n \rightarrow \infty} \int_{\Omega} g_n h \, d\mu = \int_{\Omega} g h \, d\mu \quad \text{for all } h \in L^1.$$

According to Alaoglu's theorem, the set Δ is compact for the weak-* topology.

4.5. The cost function and the loss functions for the SIS model. To a vaccination strategy $\eta \in \Delta$, we associate a cost $C(\eta)$ summarizing all production and diffusion costs for the strategy η . It is expected to be a nonincreasing function of η ,

since η encodes the nonvaccinated population. Since doing nothing costs nothing, we also expect $C(\mathbb{1}) = 0$. A simple cost model is the affine cost given by

$$(4.13) \quad C_{\text{aff}}(\eta) = \int_{\Omega} (\mathbb{1} - \eta) c_{\text{aff}} \, d\mu,$$

where $c_{\text{aff}}(x)$ is the cost of vaccinating population of feature $x \in \Omega$, with $c_{\text{aff}} \in L^1_+$ and $c_{\text{aff}} \neq 0$. The particular case $c_{\text{aff}} = \mathbb{1}$ is the uniform cost $C = C_{\text{uni}}$:

$$(4.14) \quad C_{\text{uni}}(\eta) = \int_{\Omega} (1 - \eta) \, d\mu.$$

On the one hand, the real cost of the vaccination may be a complicated function $\psi(C_{\text{aff}}(\eta))$ of the affine cost; but as long as ψ is strictly increasing, this will not affect the optimal strategies. On the other hand, when considering the loss $L = R_e$, it is possible to reduce an affine cost to the uniform cost without affecting the optimal strategies; see [2, Remark 3.9]; this is, however, no longer possible for the loss $L = \mathfrak{I}$.

We now consider the loss $L(\eta)$ which measures the (non)efficiency of the vaccination strategy η . Different choices are possible; we shall concentrate on the effective reproduction number R_e and the asymptotic proportion of infected individuals $\mathfrak{I}$.

We say a function H defined on Δ' is *well defined* on Δ if $H(\eta) = H(\eta')$ for all $\eta, \eta' \in \Delta'$ such that $\eta = \eta'$ μ -almost surely; with a slight abuse of notation, we still denote by H the corresponding function defined on Δ .

PROPOSITION 4.1 (properties of R_e and $\mathfrak{I}$ in [9]). *Suppose Assumption 6 holds.*

- (i) *The functions R_e and $\mathfrak{I}$ are well defined on Δ ; they are also nondecreasing, subhomogeneous, and continuous on Δ (endowed with the weak- $*$ topology).*
- (ii) *$R_e(\mathbb{1}) = R_0$ and $R_e(0) = 0$.*
- (iii) *$\mathfrak{I}(\mathbb{1}) = \mathfrak{I}_0$ and, for $\eta \in \Delta$, we have $\mathfrak{I}(\eta) = 0$ if and only if $R_e(\eta) \leq 1$.*

4.6. Irreducible and monotonic kernels. We follow the presentation in [7, section 5] on the atomic decomposition of positive compact operator and Remark 5.2 therein for the particular case of integral operators; see also the references therein for further results. For $A, B \in \mathcal{F}$, we write $A \subset B$ a.s. if $\mu(B^c \cap A) = 0$ and $A = B$ a.s. if $A \subset B$ a.s. and $B \subset A$ a.s. If $\mathcal{G} \subset \mathcal{F}$ is a σ -field, we recall that $A \in \mathcal{G}$ is an atom of μ in $\mathcal{G}$ if $\mu(A) > 0$ and for any $B \subset A$ we have a.s. either $B = \emptyset$ or $B = A$. Notice that the atoms are defined up to an a.s. equivalence.

Let k be a kernel on Ω such that $\|k\|_{\infty, q} < +\infty$ for some $q \in (1, +\infty)$. A set $A \in \mathcal{F}$ is called *k-invariant*, or simply *invariant* when there is no ambiguity on the kernel k , if $\int_{A^c \times A} k(z, y) \mu(dz) \mu(dy) = 0$. In the epidemiological setting with k the transmission rate kernel, the set A is invariant if the subpopulation A does not infect the subpopulation A^c . The kernel k is *irreducible* (or *connected*) if any invariant set A is equal a.s. either to $\emptyset$ or to Ω . If k is irreducible, then either $\rho(\mathcal{T}_k) > 0$ or $k \equiv 0$, and Ω is an atom of μ in $\mathcal{F}$ (degenerate case). A simple sufficient condition for irreducibility is for the kernel to be a.s. positive.

Let $\mathcal{A}$ be the set of k -invariant sets. Let us stress that $\mathcal{A}$ depends only on the support of the kernel k . Notice that $\mathcal{A}$ is stable by countable unions and countable intersections. Let $\mathcal{F}_{\text{inv}} = \sigma(\mathcal{A})$ be the σ -field generated by $\mathcal{A}$. Usually the σ -field $\mathcal{F}_{\text{inv}}$ is much smaller than the initial σ -field $\mathcal{F}$, and there might exist atoms of μ in $\mathcal{F}_{\text{inv}}$, whereas there might not be atoms of μ in $\mathcal{F}$. (Notice that any atom of μ in $\mathcal{F}$ is a subset of an atom of μ in $\mathcal{F}_{\text{inv}}$.) Then, the kernel k restricted to an atom of μ in $\mathcal{F}_{\text{inv}}$ is irreducible. We shall only consider nondegenerate atoms, and say the atom (of μ in $\mathcal{F}_{\text{inv}}$) is nonzero if the restriction of the kernel k to this atom is nonzero

(and thus the spectral radius of the corresponding integral operator is positive). We say the kernel k is *monatomic* if there exists a unique nonzero atom, say Ω_a , and the kernel is *quasi-irreducible* if it is monatomic and $k \equiv 0$ outside $\Omega_a \times \Omega_a$, where Ω_a is its nonzero atom. Notice that if k is irreducible with $\rho(\mathcal{T}_k) > 0$, then k is monatomic with nonzero atom $\Omega_a = \Omega$ a.s. The quasi-irreducible property is the usual extension of the irreducible property in the setting of symmetric kernels, and the monatomic property is the natural generalization to nonsymmetric kernels; see also [12, 11] for other characterizations of monatomic kernels.

Remark 4.2 (epidemiological interpretation). When the transmission rate kernel k for the SIS model from section 4.2 is monatomic, with nonzero atom Ω_a , then the population with trait in Ω_a can infect itself. It may also infect another part of the population, say with trait in Ω_i , but

- the infection cannot be sustained at all in Ω_i : k is quasi-nilpotent on Ω_i ;
- the population with trait in Ω_i does not infect back the nonzero atom Ω_a .

Recall the reproduction number R_0 defined in (4.5). If furthermore $R_0 > 1$, then the set $\Omega_a \cup \Omega_i$ corresponds to the support of the maximal endemic equilibrium, which is also the unique nonzero equilibrium; see [9].

4.7. Properties of the cost and loss functions. We shall now consider the loss functions R_e and $\mathfrak{J}$, and endowed Δ defined by (4.11) with the weak-* topology.

PROPOSITION 4.3 (loss function properties for the SIS model). *Suppose Assumption 6 on the SIS model holds. Consider the loss function $L \in \{R_e, \mathfrak{J}\}$ on Δ .*

- The loss function is nondecreasing and continuous with $L(0) = 0$. Thus Assumption 1 on the loss holds for the loss $L = R_e$ provided that $R_0 > 0$ (and $\ell_{\max} = R_0 > 0$) and for the loss $L = \mathfrak{J}$ provided that $R_0 > 1$ (and $\ell_{\max} = \mathfrak{J}_0 > 0$).*
- Any local minimum of the loss function L is global; that is, Assumption 3 holds.*
- If the transmission rate kernel is monatomic, then $R_0 > 0$ and any local maximum of the loss function $L = R_e$ is global (that is, Assumption 5 holds for $L = R_e$). If furthermore $R_0 > 1$, then any local maximum of the loss function $L = \mathfrak{J}$, which is not a zero of the loss, is global (that is, Assumption 5' holds for $L = \mathfrak{J}$).*

Proof. We deduce from Proposition 4.1 that the loss function $L = R_e$ (resp., $L = \mathfrak{J}$) satisfies Assumption 1, provided that $R_0 > 0$ (resp., $R_0 > 1$) so that $\ell_{\max} > 0$. This gives point (i). Then use Lemma 2.9 and Proposition 4.1 again to get point (ii), that is, Assumption 3 holds. Point (iii) is in [9, Lemmas 5.4 and 5.5]. $\square$

Remark 4.4 (local maxima of the loss $\mathfrak{J}$). The loss $L = \mathfrak{J}$ does not satisfy Assumption 5 in general even when the kernel k is irreducible with $R_0 > 0$. Indeed, by continuity of R_e , there exists a (weak-*) open neighborhood O of 0 in Δ such that $R_e(\eta) < 1$ for all $\eta \in O$: consequently $\mathfrak{J}$ is identically zero on O , and any $\eta \in O$ is a local maximum of $L = \mathfrak{J}$. However, these maxima are not global if $\mathfrak{J}(1) > 0$.

We give further results on the “worst” vaccination strategy that changes nothing. Those results are stated assuming the transmission rate kernel is monatomic. The general case is more delicate, and it is studied in more details in [3, section 5] for the loss function $L = R_e$; in particular Assumption 5 may not hold and the anti-Pareto frontier may not be connected. Recall that $c_0 = C^*(0+)$; see (2.21).

PROPOSITION 4.5 (on the “worst” vaccination strategy). *Suppose Assumption 6 on the SIS model holds and that the transmission rate kernel is monatomic (and thus*

$R_0 > 0$) with atom Ω_a and maximal equilibrium $\mathfrak{g}$. Suppose also that the cost function satisfies Assumption 1.

- (i) If k is irreducible (with $R_0 > 0$), then we have $C^*(\ell_{\max}) = 0$ for the loss $L = R_e$ and also for the loss $L = \mathfrak{J}$ provided $R_0 > 1$.
- (ii) For the loss $L = R_e$, the strategy $\mathbb{1}_{\Omega_a}$, which vaccinates everyone except the atom, is anti-Pareto optimal, with cost $C(\mathbb{1}_{\Omega_a}) = C^*(\ell_{\max})$.
- (iii) If $R_0 > 1$, for the loss $L = \mathfrak{J}$, the strategy $\mathbb{1}_{\{\mathfrak{g} > 0\}}$, which consists in vaccinating only the types that would not be infected at equilibrium, is anti-Pareto optimal with $C^*(\ell_{\max}) = C(\mathbb{1}_{\{\mathfrak{g} > 0\}})$. Furthermore, if the cost function C is decreasing, then we have $c_0 < c_{\max}$.

According to Remark 4.2, if $R_0 > 1$, we have that $\mathbb{1}_{\Omega_a} \leq \mathbb{1}_{\{\mathfrak{g} > 0\}}$, with an equality if the transmission rate kernel is quasi-irreducible.

Proof. According to [9, Lemma 5.4.(iii)], Assumption 5 holds, $R_0 > 0$, and η is a global maximum if and only if $\eta \geq \mathbb{1}_{\Omega_a}$. This directly implies that $\mathbb{1}_{\Omega_a}$ is a global maximum of R_e . If the cost function satisfies Assumption 1, we also get that $C^*(\ell_{\max}) = C(\mathbb{1}_{\Omega_a})$ and also that $\mathbb{1}_{\Omega_a}$ is anti-Pareto optimal. This gives point (ii).

According to [9, Lemma 5.5.(iii)], if $R_0 > 1$, and thus $\mathfrak{J}(\mathbb{1}) > 0$, then Assumption 5' holds and η is a global maximum if and only if $\eta \geq \mathbb{1}_{\{\mathfrak{g} > 0\}}$. Arguing as above, we get the first part of point (iii). Thanks to Lemmas 2.13 and Remark 4.4, we get that $c_0 < c_{\max}$. This ends the proof of point (iii).

To get point (i), notice that when the transmission rate kernel is irreducible and $R_0 > 0$, then a.s. $\Omega_a = \Omega$ and thus, thanks to Remark 4.2, $\mathbb{1}_{\Omega_a} = \mathbb{1}$. Then use that $C(\mathbb{1}) = 0$ as well as points (ii) and (iii) to conclude for the loss $L = R_e$. The proof for the loss $L = \mathfrak{J}$ is similar using that $\mathbb{1}_{\Omega_a} \leq \mathbb{1}_{\{\mathfrak{g} > 0\}}$ when $R_0 > 1$. $\square$

The following result is an immediate consequence of Lemmas 2.9 and 2.12.

COROLLARY 4.6 (the affine cost function). *The affine cost function C_{aff} defined in (4.13) is continuous on Δ (endowed with the weak-* topology) and thus satisfies Assumptions 1, 2, and 4.*

If furthermore c_{aff} is positive (which is the case for the uniform cost function C_{uni}), then the cost is also decreasing, and thus, under Assumption 6 on the SIS model, $L_(0) = \ell_{\max}$ and $L^*(c_{\max}) = 0$ for the loss $L = R_e$ provided $R_0 > 0$ and for the loss $L = \mathfrak{J}$ provided $R_0 > 1$.*

4.8. Summary of the results for the SIS model. We summarize in Table 1 the results obtained when applying the results of section 2 to the SIS model with

$$L \in \{R_e, \mathfrak{J}\} \quad \text{and} \quad \Delta = \Delta_{\text{SIS}} = \{\eta \in L^\infty : 0 \leq \eta \leq \mathbb{1}\}.$$

Suppose Assumption 6 on the SIS model holds, and consider the loss function $L \in \{R_e, \mathfrak{J}\}$ on Δ . We assume that C is continuous with respect to the weak-* topology on Δ with $C(\mathbb{1}) = 0$. To avoid technicalities, we focus on a decreasing cost function C (this is the case for the affine cost C_{aff} in (4.13) with positive density c_{aff} and thus for the uniform cost C_{uni}). Finally, we assume that $R_0 > 0$ if $L = R_e$ and $R_0 > 1$ if $L = \mathfrak{J}$. We gather in Table 1 the following:

1. The various notation and definitions.
2. Properties of the Pareto frontier and associated functions, in the left column. Most of those follow from Proposition 2.8. Proposition 2.7 is used to show that C_* decreases, and Lemma 2.9 is used to get the value of $L_*(0)$.
3. Properties of the feasible region; see Proposition 3.1.

TABLE 1
Summary for the SIS model with $\Delta = \Delta_{\text{SIS}}$ and $L \in \{R_e, \mathcal{I}\}$. Opt., optimal.

	“Best” vaccinations	“Worst” vaccinations (monatomic k)
Assumptions	Assumption 6 on SIS model, C continuous decreasing.	
Optimal problem	$\min_{\Delta}(C, L)$	$\max_{\Delta}(C, L)$
Range of (cost, loss)	$c_{\max} := \max_{\Delta} C = C(0) > 0$	
	$\ell_{\max} := \max_{\Delta} L = L(\mathbb{1}) > 0$	
	$\mathcal{A} := [0, c_{\max}] \times [0, \ell_{\max}]$	
Optimal cost	$C_{\star}(\ell) := \min_{L \leq \ell} C$	$C^{\star}(\ell) := \max_{L \geq \ell} C$
Optimal loss	$L_{\star}(c) := \min_{C \leq c} L$	$L^{\star}(c) := \max_{C \geq c} L$
Possible outcomes	$\mathbf{F} := (C, L)(\Delta) = \{(c, \ell) \in \mathcal{A} : L_{\star}(c) \leq \ell \leq L^{\star}(c)\}$	
	$= \{(c, \ell) \in \mathcal{A} : C_{\star}(\ell) \leq c \leq C^{\star}(\ell)\}$	
	$\mathbf{F}$ is compact and path connected $\mathbf{F}$ has no hole (that is, $\mathbf{F}^c$ is connected)	
Optimal cost: properties	$C_{\star}$ is decreasing	$C^{\star}$ is decreasing
	$\text{range}(C_{\star}) := C_{\star}([0, \ell_{\max}])$	$\text{range}(C^{\star}) := C^{\star}([0, \ell_{\max}])$
	$C_{\star}(0) \leq c_{\max}$	$C^{\star}(0) = c_{\max}$ and $c_0 := C^{\star}(0+) < c_{\max}$
	$C_{\star}$ is continuous	$C^{\star}$ is continuous on $(0, \ell_{\max}]$
	$C_{\star}(\ell_{\max}) = 0$	$C^{\star}$ is $\begin{cases} \text{continuous at } 0, c_0 = c_{\max} & (L = R_e) \\ \text{discontinuous at } 0, c_0 < c_{\max} & (L = \mathcal{I}) \end{cases}$
Optimal loss: properties	$\text{range}(C_{\star}) = [0, C_{\star}(0)]$	$C^{\star}(\ell_{\max}) = \begin{cases} C(\mathbb{1}_{\Omega_a}) & (L = R_e) \\ C(\mathbb{1}_{\{\mathfrak{g} > 0\}}) & (L = \mathcal{I}) \end{cases}$
	$L_{\star}$ is continuous	$L^{\star}$ is continuous
	$L_{\star}$ is decreasing on $[0, C_{\star}(0)]$	$L^{\star}$ is decreasing on $[C^{\star}(\ell_{\max}), c_0]$
	$L_{\star}$ is constant on $[C_{\star}(0), c_{\max}]$	$L^{\star}$ is constant on $[0, C^{\star}(\ell_{\max})]$ and on $[c_0, c_{\max}]$
	$L_{\star}(0) = \ell_{\max}$ and $L_{\star}(c_{\max}) = 0$	$L^{\star}(0) = \ell_{\max}$ and $L^{\star}(c_{\max}) = 0$
Inverse formula	$L_{\star} \circ C_{\star} = \text{Id}$ on $[0, \ell_{\max}]$	$L^{\star} \circ C^{\star} = \text{Id}$ on $[0, \ell_{\max}]$
	$C_{\star} \circ L_{\star} = \text{Id}$ on $\text{range}(C_{\star})$	$C^{\star} \circ L^{\star} = \text{Id}$ on $\text{range}(C^{\star})$
Optimal strategies	$\mathcal{P} := \{C = C_{\star} \circ L, L = L_{\star} \circ C\}$	$\mathcal{P}^{\text{Anti}} := \{C = C^{\star} \circ L\} \cap \{L = L^{\star} \circ C\}$
	$= \{C = C_{\star} \circ L\}$	$= \{C = C^{\star} \circ L\}$
	$= \{L = L_{\star} \circ C, C \in \text{range}(C_{\star})\}$	$= \{L = L^{\star} \circ C, C \in \text{range}(C^{\star})\}$
	$\mathcal{P}$ is compact	$\mathcal{P}^{\text{Anti}}$ is $\begin{cases} \text{compact} & (L = R_e) \\ \text{not compact} & (L = \mathcal{I}) \end{cases}$
	$\mathbb{1}$ is Pareto opt.	$\begin{matrix} \mathbb{1}_{\Omega_a} & (L = R_e) \\ \mathbb{1}_{\{\mathfrak{g} > 0\}} & (L = \mathcal{I}) \end{matrix}$ is anti-Pareto opt.
Optimal frontier	$\mathcal{F} := (C, L)(\mathcal{P})$	$\mathcal{F}^{\text{Anti}} := (C, L)(\mathcal{P}^{\text{Anti}})$
	$= (C_{\star}, \text{Id})([0, \ell_{\max}])$	$= (C^{\star}, \text{Id})([0, \ell_{\max}])$
	$= (\text{Id}, L_{\star})(\text{range}(C_{\star}))$	$= (\text{Id}, L^{\star})(\text{range}(C^{\star}))$
	$\mathcal{F}$ is connected and compact	$\mathcal{F}^{\text{Anti}}$ is $\begin{cases} \text{connected and compact} & (L = R_e) \\ \text{not connected, not compact} & (L = \mathcal{I}) \end{cases}$

4. Properties of the anti-Pareto frontier and associated functions, in the right column. We additionally assume here that the kernel is monatomic and refer to [3, section 5] when this assumption does not hold. Most results in this column come from Proposition 2.11 (for $L = R_e$) and 2.11' (for $L = \mathcal{I}$). The value of $C^{\star}(\ell_{\max})$ is studied in Proposition 4.5.

We also represent in Figure 3 the typical Pareto and anti-Pareto frontiers for the loss functions R_e and $\mathfrak{I}$, when the transmission rate kernel k is monatomic.

Remark 4.7 (result for other choices of Δ). It is left to the reader to check that the statements given in Table 1 hold also for the vaccinations strategies sets $\Delta_i \subset \Delta_{\text{SIS}}$, with $i \in \{\text{Osc}, \text{Block}, \text{Ord}\}$, described in section 2.1.1, but for the value of $C^*(\ell_{\max})$ and the corresponding anti-Pareto vaccination strategies. Indeed, the strategy $\eta_{\text{SIS}} = \mathbb{1}_{\Omega_a}$ for the loss $L = R_e$ or $\eta_{\text{SIS}} = \mathbb{1}_{\{\mathfrak{g} > 0\}}$ for the loss $L = \mathfrak{I}$ is no longer anti-Pareto optimal, and it has to be replaced by $\eta_i = \min\{\eta \in \Delta_i : \eta \geq \eta_{\text{SIS}}\}$ which is anti-Pareto optimal (in Δ_i) and $C^*(\ell_{\max}) = C(\eta_i)$.

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