

TOPOLOGICAL PROPERTIES OF THE EFFECTIVE REPRODUCTION NUMBER IN AN HETEROGENEOUS SIS MODEL

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ABSTRACT. This present results lay the foundations for the study of the optimal allocation of vaccine in the simple epidemiological SIS model where one consider a very general heterogeneous population. In the present setting each individual has a type x belonging to a general space, and a vaccination strategy is a function η where $\eta(x) \in [0, 1]$ represents the proportion of non-vaccinated among individuals of type x . We shall consider two loss functions associated to a vaccination strategy η : either the effective reproduction number, a classical quantity appearing in many models in epidemiology, and which is given here by the spectral radius of a compact operator that depends on η ; or the overall proportion of infected individuals after vaccination in the maximal endemic state.

By considering the weak-* topology on the set Δ of vaccination strategies, so that it is a compact set, we can prove that those two loss functions are continuous using the notion of collective compactness for a family of operators. We also prove their stability with respect to the parameters of the SIS model. Eventually, we consider their monotonicity and related properties in particular when the model is “almost” irreducible.

1. INTRODUCTION

1.1. Motivation. Increasing the prevalence of immunity from contagious disease in a population limits the circulation of the infection among the individuals who lack immunity. This so-called “herd effect” plays a fundamental role in epidemiology as it has had a major impact in the eradication of smallpox and rinderpest or the near eradication of poliomyelitis [19]. It is of course unrealistic to depict human populations as homogeneous, and many generalizations of the homogeneous model have been studied; see [22, Chapter 3] for examples and further references. Targeted vaccination strategies, based on the heterogeneity of the infection spreading in the population, are designed to increase the level of immunity of the population with a limited quantity of vaccine. These strategies rely on identifying groups of individuals that should be vaccinated in priority in order to slow down or eradicate the disease. It is assumed the vaccine is perfect and provide an ever lasting immunity.

In this article, we consider two loss functions to measure the effectiveness of targeted vaccination strategies with perfect vaccine in the deterministic infinite-dimensional SIS model (with S=Susceptible and I=Infectious) introduced in [7], that encompasses as particular cases the SIS model on graphs or on stochastic block models.

The first one is the so-called *effective reproduction number* R_e defined as the number of secondary cases one “typical” infectious individual generates on average over the course of its infectious period, in an otherwise uninfected (susceptible) and non-vaccinated population. When there is no vaccination, this reduces to the *basic reproduction number* denoted by R_0 . This latter number plays a fundamental role in epidemiology as it provides a scale to measure how difficult an infectious disease is to control, see [14]. Intuitively, the disease should die out if $R_0 < 1$ (sub-critical regime) and invade the population if $R_0 > 1$ (super-critical regime). For many classical mathematical

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models of epidemiology, such as SIS or S(E)IR (with R=Recovered and E=Exposed), this intuition can be made rigorous: the quantity R_0 may be computed from the parameters of the model, and the threshold phenomenon occurs.

The second one is the fraction $\mathfrak{I}$ of infected individuals at equilibrium, and set $\mathfrak{I}_0$ when there is no vaccination. (In particular, one get for the SIS model that $\mathfrak{I}_0 = 0$ in the sub-critical regime $R_0 \leq 1$.) For a SIR model, distributing vaccine so as to minimize the attack rate (that is, the proportion of individuals that eventually catch (and recover from) the disease) is at least as natural as trying to minimize the reproduction number; this problem has been studied for example in [15, 16].

The simplicity of the SIS model allows us to study the regularity of the loss functions r_e and $\mathfrak{I}$ under minimal assumptions for general non-homogeneous populations, using theoretical properties on the spectral radius of integral operators and properties of the maximal equilibrium of the SIS infinite dimensional ODE. The mathematical foundation developed here allows us to study Pareto optimal vaccination in SIS model in [8], when taking into account the cost a the vaccination strategy, and illustrate those results in particular cases and specific examples, see references therein. Furthermore, we expect the results obtained for the SIS model to be generic, in the sense that behaviours exhibited here should be also observed in more realistic and complex models in epidemiology for non-homogeneous populations; in this direction, see for example the discussion in [10].

1.2. Main results. The differential equations governing the epidemic dynamics in metapopulation SIS models were developed by Lajmanovich and Yorke in their pioneer paper [24]. In [7], we introduced a natural generalization of their equation, to a possibly infinite space Ω , where $x \in \Omega$ represents a feature and the probability measure $\mu(dx)$ represents the fraction of the population with feature x . Following [7, Section 5], we represent a vaccination strategy by a measurable function $\eta : \Omega \rightarrow [0, 1]$, where $\eta(x)$ represents the fraction of **non-vaccinated** individuals with feature x . In particular, the “strategy” that consists in vaccinating no one (resp. everybody) corresponds to $\eta = \mathbb{1}$, the constant function equal to 1, (resp. $\eta = \mathbb{0}$, the constant function equal to 0). We denote by Δ the set of strategies.

1.2.1. Regularity of the effective reproduction function R_e . We consider the effective reproduction function in a general operator framework which we call the *kernel model*. This model, which will be defined in detail below in Section 2, is characterized by a measured space $(\Omega, \mathcal{F}, \mu)$, with μ a non-zero σ -finite measure, and a measurable non-negative kernel $k : \Omega \times \Omega \rightarrow \mathbb{R}_+$. Considering the kernel model with a general measure μ instead of a probability measure is in particular motivated by [9, 10]. Let T_k be the corresponding integral operator defined on some linear subspace of real-valued measurable functions by:

$$T_k(h) : x \mapsto \int_{\Omega} k(x, y)h(y) \mu(dy).$$

In the setting of [7] (see in particular Equation (11) therein), T_k is the so-called *next generation operator*, where the kernel k is defined in terms of a transmission rate kernel $k(x, y)$ and a recovery rate function γ by the product $k(x, y) = k(x, y)/\gamma(y)$; the *reproduction number* R_0 is then the spectral radius $\rho(T_k)$ of T_k .

The effective reproduction number associated to the vaccination strategy $\eta \in \Delta$ is given by:

$$(1) \quad R_e(\eta) = \rho(T_{k\eta}),$$

where ρ stands for the spectral radius and $k\eta$ stands for the kernel $(k\eta)(x, y) = k(x, y)\eta(y)$. In particular, we have $R_e(\mathbb{1}) = R_0$ (resp. $R_e(\mathbb{0}) = 0$).

Motivated by vaccine allocation optimization, we shall consider a topology on Δ such that it is compact and the function R_e is continuous. It is natural to try and prove this continuity

by writing R_e as the composition of the spectral radius ρ and the map $\eta \mapsto T_{k\eta}$. The spectral radius is indeed continuous at compact operators (and $T_{k\eta}$ is in fact compact under a technical integrability assumption on the kernel k formalized on page 7 as Assumption 1), if we endow the set of bounded operators with the operator norm topology; see [6, 26]. However, this only works if we equip Δ with the uniform topology, for which it is not compact.

We instead consider Δ , the set of functions in Δ where functions which are μ -a.e. equal are identified, endowed with the weak-* topology for which compactness holds; see Lemma 2.3. This forces us to equip the space of bounded operators with the strong topology, for which the spectral radius is in general not continuous [21, p. 431]. However, the family of operators $(T_{k\eta}, \eta \in \Delta)$ is *collectively compact* which enables us to recover continuity, using a series of results obtained by Anselone [4]. After noticing that the function R_e coincide on functions which are μ -a.e. equal, or that R_e is indeed well defined on Δ , this leads to the following statement, proved in Theorem 4.2 below. We recall that Assumption 1, formulated on page 7, provides an integrability condition on the kernel k .

Theorem 1.1 (Continuity of the spectral radius). *Under Assumption 1 on the kernel k , the function $R_e : \Delta \rightarrow \mathbb{R}_+$ is continuous with respect to the weak-* topology on Δ .*

In fact, we also prove the continuity of the spectrum with respect to the Hausdorff distance on the set of compact subsets of $\mathbb{C}$. We shall write $R_e[k]$ to stress the dependence of the function R_e in the kernel k . In Proposition 4.3, we prove the stability of R_e , by giving natural sufficient conditions on a sequence of kernels $(k_n, n \in \mathbb{N})$ converging to k which imply that $R_e[k_n]$ converges uniformly towards $R_e[k]$. This result has both a theoretical and a practical interest: the next-generation operator is unknown in practice, and has to be estimated from data. Thanks to this result, the value of R_e computed from the estimated operator is a converging approximation of the true value.

1.2.2. Regularity of the total proportion of infected population function $\mathfrak{J}$. We consider the *SIS model* from [7]. This model is characterized by a probability space $(\Omega, \mathcal{F}, \mu)$, the transmission kernel $k : \Omega \times \Omega \rightarrow \mathbb{R}_+$ and the recovery rate $\gamma : \Omega \rightarrow \mathbb{R}_+^*$. We suppose in the following that the technical Assumption 2, formulated on page 8, holds, so that the SIS dynamical evolution is well defined.

This evolution is encoded as $u = (u_t, t \in \mathbb{R}_+)$, where $u_t \in \Delta$ for all t and $u_t(x)$ represents the probability of an individual with feature $x \in \Omega$ to be infected at time $t \geq 0$, and follows the equation:

$$(2) \quad \partial_t u_t = F(u_t) \quad \text{for } t \in \mathbb{R}_+, \quad \text{where} \quad F(g) = (\mathbb{1} - g)\mathcal{T}_k(g) - \gamma g \quad \text{for } g \in \Delta,$$

with an initial condition $u_0 \in \Delta$ and with $\mathcal{T}_k$ the integral operator corresponding to the kernel k acting on the set of bounded measurable functions, see (19). It is proved in [7] that such a solution u exists and is unique under Assumption 2. An *equilibrium* of (2) is a function $g \in \Delta$ such that $F(g) = 0$. According to [7], there exists a maximal equilibrium $\mathbf{g}$, *i.e.*, an equilibrium such that all other equilibria $h \in \Delta$ are dominated by $\mathbf{g}$: $h \leq \mathbf{g}$. Furthermore, we have $R_0 \leq 1$ (sub-critical and critical regimes) if and only if $\mathbf{g} = 0$. In the non-trivial connected case (for example if $k > 0$), then 0 and $\mathbf{g}$ are the only equilibria, and $\mathbf{g}$ is the long-time distribution of infected individuals in the population: $\lim_{t \rightarrow +\infty} u_t = \mathbf{g}$ as soon as the initial condition is non-zero; see [7, Theorem 4.14].

According to [7, Section 5.3], the SIS equation with vaccination strategy η is given by (2), where F is replaced by F_η defined by:

$$F_\eta(g) = (\mathbb{1} - g)T_{k\eta}(g) - \gamma g.$$

and u_t now describes the proportion of infected *among the non-vaccinated population*. We denote by $\mathbf{g}_\eta$ the corresponding maximal equilibrium (thus considering $\eta = \mathbb{1}$ gives $\mathbf{g} = \mathbf{g}_1$), so that $F_\eta(\mathbf{g}_\eta) = 0$.

Since the probability for an individual x to be infected in the stationary regime is $\mathfrak{g}_\eta(x) \eta(x)$, the *fraction of infected individuals at equilibrium*, $\mathfrak{I}(\eta)$, is thus given by:

$$(3) \quad \mathfrak{I}(\eta) = \int_{\Omega} \mathfrak{g}_\eta \eta \, d\mu = \int_{\Omega} \mathfrak{g}_\eta(x) \eta(x) \mu(dx).$$

In the SIS model the quantity $\mathfrak{I}$ appears as a natural analogue of the attack rate for SIR models, and is therefore a natural optimization objective.

We obtain results on the functional $\mathfrak{I}$ that are very similar to the ones on R_e . Recall that Assumption 2 on page 8 ensures that the infinite-dimensional SIS model, given by equation (2), is well defined. The next theorem corresponds to Theorem 4.6.

Theorem 1.2 (Continuity of the equilibrium infection size). *Under Assumption 2, the function $\mathfrak{I} : \Delta \rightarrow \mathbb{R}_+$ is continuous with respect to the weak-* topology on Δ .*

We shall write $\mathfrak{I}[k, \gamma]$ to stress the dependence of the function $\mathfrak{I}$ in the kernel k and the function γ . In Proposition 4.7, we prove the stability of $\mathfrak{I}$, by giving natural sufficient conditions on a sequence of kernels and functions $((k_n, \gamma_n), n \in \mathbb{N})$ converging to (k, γ) which imply that $\mathfrak{I}[k_n, \gamma_n]$ converges uniformly towards $\mathfrak{I}[k, \gamma]$.

1.2.3. Other regularity results. We also prove that the loss functions $L = R_e$ and $L = \mathfrak{I}$ are both non-decreasing ($\eta \leq \eta'$ implies $L(\eta) \leq L(\eta')$), and sub-homogeneous ($L(\lambda\eta) \leq \lambda L(\eta)$ for all $\lambda \in [0, 1]$); see Propositions 4.1 and 4.5.

Motivated by the bi-objective minimization problem of the cost and the loss L of vaccination strategies and the description of the corresponding set of Pareto optimal vaccination strategies developed in the companion paper [8], we shall investigate if local extrema of the loss function are in fact global extrema, see Assumptions 3 on pages 12 and 14. It turns out that local minimum are indeed global minimum for the loss functions R_e and $\mathfrak{I}$. However the picture is more involved for the local maximum, and slightly different between R_e and $\mathfrak{I}$. We concentrate in this paper on the case where the model is irreducible and its extension, the so called monatomic case, where intuitively, there is only one maximal irreducible component. Those results are given in Lemmas 5.4 (for R_e) and 5.5 (for $\mathfrak{I}$). We also characterize all the global maxima. Let us mention that the reducible case for the loss R_e is further studied in [9, Section 5].

1.3. Structure of the paper. After recalling a few topological facts in Section 2, we present the vaccination model, the loss functions R_e and $\mathfrak{I}$, and the various assumptions on the parameters in Section 2. We study the regularity properties of R_e and $\mathfrak{I}$ in Section 4. Section 5 is devoted to study of their local extremum. The proofs of a few technical results on $\mathfrak{I}$ are gathered in Section 6.

2. GENERAL SETTING AND NOTATION

2.1. Spaces, operators, spectra. All metric spaces (S, d) are endowed with their Borel σ -field denoted by $\mathcal{B}(S)$. The set $\mathcal{K}$ of compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance d_H is a metric space, and the function rad from $\mathcal{K}$ to $\mathbb{R}_+$ defined by $\text{rad}(K) = \max\{|\lambda|, \lambda \in K\}$ is Lipschitz continuous from $(\mathcal{K}, d_H)$ to $\mathbb{R}$ endowed with its usual Euclidean distance.

Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a σ -finite non-negative measure $\mu \neq 0$. We denote by $\mathcal{L}^\infty$, the Banach spaces of bounded real-valued measurable functions defined on Ω equipped with the sup-norm, $\mathcal{L}_+^\infty$ the subset of $\mathcal{L}^\infty$ of non-negative function, and $\Delta = \{f \in \mathcal{L}^\infty : f(\Omega) \subset [0, 1]\}$ the subset of non-negative functions bounded by 1. For f and g real-valued functions defined on Ω , we may write $\langle f, g \rangle$ or $\int_{\Omega} fg \, d\mu$ for $\int_{\Omega} f(x)g(x) \mu(dx)$ whenever the latter integral is meaningful. For $p \in [1, +\infty]$, we denote by $L^p = L^p(\mu) = L^p(\Omega, \mu)$ the space of real-valued measurable functions g defined Ω such that $\|g\|_p = (\int |g|^p \, d\mu)^{1/p}$ (with the convention that $\|g\|_\infty$ is the μ -essential supremum of $|g|$) is finite, where functions which agree μ -almost surely are identified. We denote by 0 and 1 the elements of $\mathcal{L}^\infty$ which are respectively the (class

of equivalence of the) constant functions equal to 0 and to 1, and with a slight abuse of notation, we also see them as elements of L^∞ . For $f, g \in L^p$, the inequality $f \leq g$ (in L^p) means that $\mu(f > g) = 0$. We consider the Banach lattice $(L^p, \|\cdot\|_p, \leq)$ and its cone $L_+^p = \{f \in L^p : f \geq 0\}$ of non-negative functions from L^p . We shall consider the set $\Delta = \{f \in L^\infty : 0 \leq f \leq 1\}$ corresponding to the set Δ where functions which agree μ -a.e. are identified. For g a measurable function, with a slight abuse of notation, we denote by M_g the multiplication linear map (possibly unbounded) on L^p or on $\mathcal{L}^\infty$ defined by $M_g(h) = gh$.

We now recall some general facts on Banach spaces and Banach lattices. Let $(E, \|\cdot\|)$ be a real or complex Banach space. We denote by $\|\cdot\|_E$ the operator norm on $\mathcal{L}(E)$ the Banach algebra of operators, that is, bounded linear maps. Let $T \in \mathcal{L}(E)$. The spectral radius of T is given by:

$$(4) \quad \rho(T) = \lim_{n \rightarrow \infty} \|T^n\|_E^{1/n}.$$

A sequence $(T_n, n \in \mathbb{N})$ of elements of $\mathcal{L}(E)$ converges strongly to T if $\lim_{n \rightarrow \infty} \|T_n x - Tx\| = 0$ for all $x \in E$. The operator T is compact if the subset $\{Tx : \|x\| \leq 1\}$ of E is relatively compact; and following [4], a set of operators $\mathcal{A} \subset \mathcal{L}(E)$ is *collectively compact* if the subset $\{Sx : S \in \mathcal{A}, \|x\| \leq 1\}$ of E is relatively compact.

If $(E, \|\cdot\|)$ is a complex Banach space, the spectrum $\text{Spec}(T)$ of $T \in \mathcal{L}(E)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda \text{Id}$ does not have a bounded inverse linear map, where Id is the identity operator on E . Recall that $\text{Spec}(T)$ is a compact subset of $\mathbb{C}$, and that the spectral radius of T is also given by:

$$(5) \quad \rho(T) = \text{rad}(\text{Spec}(T)).$$

The element $\lambda \in \text{Spec}(T)$ is an eigenvalue if there exists $x \in E$ such that $Tx = \lambda x$ and $x \neq 0$.

The next result is in [4] (see [10, Lemma 2.1] for details).

Lemma 2.1 (Anselone). *Let $(T_n, n \in \mathbb{N})$ be a collectively compact sequence of $\mathcal{L}(E)$ which converges strongly to $T \in \mathcal{L}(E)$. Then, we have $\lim_{n \rightarrow \infty} \text{Spec}(T_n) = \text{Spec}(T)$ in $(\mathcal{K}, d_H)$, and $\lim_{n \rightarrow \infty} \rho(T_n) = \rho(T)$.*

Let $(E, \|\cdot\|, \leq)$ be a real Banach lattice, that is $(E, \|\cdot\|)$ is a real Banach space with an order relation $\leq$ satisfying some conditions, see [2, Section 9.1]. We denote by $E_+ = \{x \in E : x \geq 0\}$ the positive cone of E . Recall it is a closed set. A linear map T on E is *positive* if $T(E_+) \subset E_+$. According to [2, Theorem 4.3] positive linear maps on Banach lattices are bounded (and thus are operators). If S and T are two operators on E , we write $T \leq S$ if the operator $S - T$ is positive. The next result can be found in [25, Theorem 4.2].

Lemma 2.2. *Let $(E, \|\cdot\|, \leq)$ be a real Banach lattice. Let $S, T \in \mathcal{L}(E)$ be positive operators. If $T \leq S$, then we have:*

$$(6) \quad \rho(T) \leq \rho(S).$$

Any real Banach lattice E and any operator T on E admits a natural complex extension. The spectrum of T will be identified as the spectrum of its complex extension and denoted by $\text{Spec}(T)$, furthermore by [1, Lemma 6.22], the spectral radius of the complex extension of T is also equal to the spectral radius of T . Moreover, by [1, Corollary 3.23], if T is positive (seen as an operator on the real Banach lattice E), then T and its complex extension have the same norm.

2.2. On the weak-* topology on Δ . Let $\mathcal{O}$ denote the *weak-* topology* on L^∞ , that is, the weakest topology on L^∞ for which all the linear forms $f \mapsto \int_\Omega fg d\mu$, $g \in L^1$, defined on L^∞ are continuous. We recall that $(L^\infty, \mathcal{O})$ is an Hausdorff topological vector space, see [5, Proposition 3.11], that the topological dual of $(L^\infty, \mathcal{O})$ is L^1 , see [5, Proposition 3.14], and that a sequence $(f_n, n \in \mathbb{N})$ of elements of L^∞ converges weakly-* to $f \in L^\infty$ if and only if, see [5, Proposition 3.13].

$$(7) \quad \lim_{n \rightarrow \infty} \int_{\Omega} g f_n d\mu = \int_{\Omega} g f d\mu \quad \text{for all } g \in L^1.$$

A set $A \subset L^\infty$ is weak-* sequentially compact if for all sequences of elements of A , there exists a sub-sequence which weakly-* converges to a limit belonging to A . A topological set $(E, \mathcal{O})$ is a sequential space if for any $A \subset E$ which is not closed, there exist $x \in \bar{A} \setminus A$, where $\bar{A}$ is the closure of A , and a sequence in A which converges to x .

Lemma 2.3 (Topological properties of $\Delta \subset L^\infty$). *Let $(\Omega, \mathcal{F}, \mu)$ be a measured space with μ a σ -finite measure, and consider the weak-* topology on L^∞ . The following properties hold.*

- (i) *The set Δ is weak-* compact and weak-* sequentially compact.*
- (ii) *The set Δ endowed with the weak-* topology is a sequential space.*
- (iii) *A function from Δ (endowed with the weak-* topology) to a topological space is continuous if and only if it is sequentially continuous.*

Proof. The Banach-Alaoglu theorem [18, Theorem 3.21] implies that the closed unit ball, say B_{L^∞} , of L^∞ is weak-* compact. According to [18, Example (v), Chapter 11] as μ is σ -finite, the Banach space L^1 is weakly compactly generated (that is, there exists a weakly compact set K whose linear span is dense in L^1). Thus, thanks to the Amir-Lindenstrauss theorem, see Theorem 11.16 or more directly Exercise 11.21 in [18], the unit ball B_{L^∞} is weak-* sequentially compact and in fact weak-* angelic (that is, for all $A \subset B_{L^\infty}$ and all x in the weak-* closure of A , there exists a sequence of elements in A which weak-* converges to x , see [18, Definition 4.48]). In particular, since Δ is the closed ball centered at $2^{-1}\mathbb{1}$ with radius $1/2$ of L^∞ , we get it is weak-* compact, weak-* sequentially compact and weak-* angelic.

Since Δ is weak-* angelic, we deduce that it is a sequential space. Since continuity and sequential continuity coincide for functions defined on a sequential space, we get (iii). $\square$

Remark 2.4 (On the topology on Δ). Assume that the measure μ is finite. Let $p \in (1, +\infty)$. Using that reflexive Banach spaces are weakly compactly generated according to [18, Example (i), Chapter 11], we get, arguing as in the proof of Lemma 2.3, that the set Δ with the trace of the weak-* topology (and thus of the weak topology as the space is reflexive) on L^p is also a sequential space. Furthermore, with $1/p + 1/q = 1$, a sequence $(f_n, n \in \mathbb{N})$ of elements of L^p converges weakly to $f \in L^p$ if and only if:

$$(8) \quad \lim_{n \rightarrow \infty} \int_{\Omega} g f_n d\mu = \int_{\Omega} g f d\mu \quad \text{for all } g \in L^q.$$

Since the topology on a sequential space is characterized by the converging sequences, see [17, Exercises 1.7.20], and since (7) and (8) are equivalent for sequences $(g_n, n \in \mathbb{N})$ of elements of Δ , we deduce that the trace on Δ of the weak-* topology on L^∞ and of the weak topology on L^p coincide. (Let us stress that there exists a topology different from the weak-* topology which has the same converging sequences, see the last proposition in [27].)

We shall consider loss functions L defined on $\Delta \subset \mathcal{L}^\infty$, and see them as function on $\Delta \subset L^\infty$ when they are compatible with the equivalence relation given by the μ -a.e. equality. In this case, with a slight abuse of notation, we also denote the corresponding function on Δ by L .

Definition 2.5. *A loss function L defined on Δ is:*

- (i) *Well defined (on Δ endowed with the weak-* topology) if for all $\eta_1, \eta_2 \in \Delta$:*

$$(9) \quad \eta_1 = \eta_2 \quad \mu\text{-a.e.} \quad \implies \quad L(\eta_1) = L(\eta_2);$$

- (ii) *Non-decreasing on Δ if for all $\eta_1, \eta_2 \in \Delta$:*

$$(10) \quad \eta_1 \leq \eta_2 \quad \mu\text{-a.e.} \quad \implies \quad L(\eta_1) \leq L(\eta_2);$$

(iii) **Sub-homogeneous on Δ** if for all $\eta \in \Delta$ and $\lambda \in [0, 1]$:

$$(11) \quad L(\lambda\eta) \leq \lambda L(\eta).$$

3. THE KERNEL AND SIS MODELS

3.1. Kernel model ($\mu(\Omega) \in (0, +\infty]$). In the kernel model, we assume that the measure μ is σ -finite and non-zero. We define a *kernel* (resp. *signed kernel*) on Ω as a $\mathbb{R}_+$ -valued (resp. $\mathbb{R}$ -valued) measurable function defined on $(\Omega^2, \mathcal{F}^{\otimes 2})$. For f, g two non-negative measurable functions defined on Ω and k a kernel on Ω , we denote by fkg the kernel on Ω defined by:

$$(12) \quad fkg : (x, y) \mapsto f(x) k(x, y) g(y).$$

When γ is a positive measurable function defined on Ω , we write k/γ for $k\gamma^{-1}$, which differs in general from $\gamma^{-1}k$.

For $p \in (1, +\infty)$, we define the double norm of a signed kernel k by:

$$(13) \quad \|k\|_{p,q} = \left(\int_{\Omega} \left(\int_{\Omega} |k(x, y)|^q \mu(dy) \right)^{p/q} \mu(dx) \right)^{1/p} \quad \text{with } q \text{ given by } \frac{1}{p} + \frac{1}{q} = 1.$$

Assumption 1 (On the kernel model $[(\Omega, \mathcal{F}, \mu), k]$). The kernel k , defined on a measured space $(\Omega, \mathcal{F}, \mu)$, with σ -finite non-zero measure μ , has a finite double-norm, that is, $\|k\|_{p,q} < +\infty$ for some $p \in (1, +\infty)$.

To a kernel k such that $\|k\|_{p,q} < +\infty$, we associate the integral operator T_k on L^p defined by:

$$(14) \quad T_k(g)(x) = \int_{\Omega} k(x, y) g(y) \mu(dy) \quad \text{for } g \in L^p \text{ and } x \in \Omega.$$

This operator is positive (in the sense that $T_k(L_+^p) \subset L_+^p$), and compact (see [20, p. 293]). It is well known and easy to check that:

$$(15) \quad \|T_k\|_{L^p} \leq \|k\|_{p,q}.$$

For $\eta \in \Delta$, the kernel $k\eta$ has also a finite double norm on L^p and the operator M_η is bounded, so that the operator $T_{k\eta} = T_k M_\eta$ is compact. We can define the *effective spectrum* function $\text{Spec}[k]$ from Δ to $\mathcal{K}$ by:

$$(16) \quad \text{Spec}[k](\eta) = \text{Spec}(T_{k\eta}),$$

the *effective reproduction number* function $R_e[k] = \text{rad} \circ \text{Spec}[k]$ from Δ to $\mathbb{R}_+$ by:

$$(17) \quad R_e[k](\eta) = \rho(T_{k\eta}),$$

and the corresponding *reproduction number*:

$$(18) \quad R_0[k] = R_e[k](\mathbb{1}) = \rho(T_k).$$

When there is no ambiguity, we simply write R_e for the function $R_e[k]$, and R_0 for the number $R_0[k]$. Motivated by Section 3.3 below, we say a vaccination strategy $\eta \in \Delta$ is *critical* if $R_e(\eta) = 1$.

3.2. SIS model ($\mu(\Omega) = 1$): **dynamics and equilibria.** In the SIS model, we assume that μ is a probability measure, thus following the framework of [7]. For $q \in (1, +\infty)$, we also consider the following norm for the kernel k :

$$\|k\|_{\infty,q} = \sup_{x \in \Omega} \left(\int_{\Omega} k(x, y)^q \mu(dy) \right)^{1/q}.$$

Since μ is finite, if the norm $\|k\|_{\infty,q}$ is finite, then for p such that $1/p + 1/q = 1$, the norm $\|k\|_{p,q}$ is also finite. When $\|k\|_{\infty,q} < +\infty$, the corresponding positive bounded linear integral operator $\mathcal{T}_k$ on $\mathcal{L}^\infty$ is similarly defined by:

$$(19) \quad \mathcal{T}_k(g)(x) = \int_{\Omega} k(x, y)g(y) \mu(dy) \quad \text{for } g \in \mathcal{L}^\infty \text{ and } x \in \Omega.$$

Notice that the integral operators $\mathcal{T}_k$ and T_k corresponds respectively to the operators T_k and $\hat{T}_k$ in [7]. According to [7, Lemma 3.7], the operator $\mathcal{T}_k^2$ on $\mathcal{L}^\infty$ is compact and $\mathcal{T}_k$ has the same spectral radius as T_k :

$$(20) \quad \rho(\mathcal{T}_k) = \rho(T_k).$$

In accordance with [7], we consider the following assumption. Recall that $k/\gamma = k\gamma^{-1}$.

Assumption 2 (On the SIS model $[(\Omega, \mathcal{F}, \mu), k, \gamma]$). *The recovery rate function γ , defined on a probability space $(\Omega, \mathcal{F}, \mu)$, is bounded and positive. The transmission rate kernel k on Ω is such that $\|k/\gamma\|_{\infty,q} < +\infty$ for some $q \in (1, +\infty)$.*

If k and γ satisfy Assumption 2, then $k = k/\gamma$ clearly satisfies Assumption 1 (as μ is finite). Under Assumption 2, we also consider the bounded operators $\mathcal{T}_{k/\gamma}$ on $\mathcal{L}^\infty$, as well as $T_{k/\gamma}$ on L^p , which are the so called *next-generation operator*. The SIS dynamics considered in [7] under Assumption 2 follows the vector field F defined on $\mathcal{L}^\infty$ by:

$$(21) \quad F(g) = (\mathbb{1} - g)\mathcal{T}_k(g) - \gamma g.$$

More precisely, we consider $u = (u_t, t \in \mathbb{R})$, where $u_t \in \Delta$ for all $t \in \mathbb{R}_+$ such that:

$$(22) \quad \partial_t u_t = F(u_t) \quad \text{for } t \in \mathbb{R}_+,$$

with initial condition $u_0 \in \Delta$. The value $u_t(x)$ models the probability that an individual of feature x is infected at time t ; it is proved in [7] that such a solution u exists and is unique.

An *equilibrium* of (22) is a function $g \in \Delta$ such that $F(g) = 0$ (in $\mathcal{L}^\infty$). According to [7], there exists a maximal equilibrium $\mathbf{g}$, i.e., an equilibrium such that all other equilibria $h \in \Delta$ are dominated by $\mathbf{g}$: $h \leq \mathbf{g}$. The *reproduction number* R_0 associated to the SIS model given by (22) is the spectral radius of the next-generation operator, so that using the definition of the effective reproduction number (17), (18) and (20), this amounts to:

$$(23) \quad R_0 = \rho(\mathcal{T}_{k/\gamma}) = R_0[k/\gamma] = R_e[k/\gamma](\mathbb{1}).$$

If $R_0 \leq 1$ (sub-critical and critical regime), then u_t converges pointwise to 0 when $t \rightarrow \infty$. In particular, the maximal equilibrium $\mathbf{g}$ is equal to 0 everywhere. If $R_0 > 1$ (super-critical regime), then the null function is still an equilibrium but different from the maximal equilibrium $\mathbf{g}$, as $\int_{\Omega} \mathbf{g} d\mu > 0$.

3.3. Vaccination strategies in the SIS model. A *vaccination strategy* η of a vaccine with perfect efficiency is an element of Δ , where $\eta(x)$ represents the proportion of **non-vaccinated** individuals with feature x . Notice that $\eta d\mu$ corresponds in a sense to the effective population.

Recall the definition of the kernel fkg from (12). For $\eta \in \Delta$, the kernels $k\eta/\gamma$ and $k\eta$ have finite norm $\|\cdot\|_{\infty,q}$ under Assumption 2, so we can consider the bounded positive operators $\mathcal{T}_{k\eta/\gamma}$ and $\mathcal{T}_{k\eta}$ on $\mathcal{L}^\infty$. According to [7, Section 5.3.], the SIS equation with vaccination strategy η is given by (22), where F is replaced by F_η defined by:

$$(24) \quad F_\eta(g) = (\mathbb{1} - g)\mathcal{T}_{k\eta}(g) - \gamma g.$$

We denote by $u^\eta = (u_t^\eta, t \geq 0)$ the corresponding solution with initial condition $u_0^\eta \in \Delta$. We recall that $u_t^\eta(x)$ represents the probability for an non-vaccinated individual of feature x to be infected at time t . Since the effective reproduction number is the spectral radius of $\mathcal{T}_{k\eta/\gamma}$, we recover (17)

with $k = k/\gamma$ as $\rho(T_{k\eta/\gamma}) = \rho(T_{k\eta/\gamma}) = R_e[k/\gamma](\eta)$. We denote by $\mathfrak{g}_\eta$ the corresponding maximal equilibrium (so that $\mathfrak{g} = \mathfrak{g}_1$). In particular, we have:

$$(25) \quad F_\eta(\mathfrak{g}_\eta) = 0 \quad (\text{in } \mathcal{L}^\infty).$$

We will denote by $\mathfrak{I}$ the *fraction of infected individuals at equilibrium*. Since the probability for an individual with feature x to be infected in the stationary regime is $\mathfrak{g}_\eta(x) \eta(x)$, this fraction is given by the following formula:

$$(26) \quad \mathfrak{I}(\eta) = \int_{\Omega} \mathfrak{g}_\eta \eta \, d\mu = \int_{\Omega} \mathfrak{g}_\eta(x) \eta(x) \mu(dx).$$

We deduce from (24) and (25) that $\mathfrak{g}_\eta \eta = 0$ μ -almost surely is equivalent to $\mathfrak{g}_\eta = 0$. Applying the results of [7] to the kernel $k\eta$, we deduce that:

$$(27) \quad \mathfrak{I}(\eta) > 0 \iff R_e[k/\gamma](\eta) > 1.$$

4. GENERAL PROPERTIES OF THE FUNCTIONS R_e AND $\mathfrak{I}$

As mentioned in the introduction, see [8], we shall see the functions R_e and $\mathfrak{I}$ defined on $\Delta \subset \mathcal{L}^\infty$ and taking values in $\mathbb{R}_+$ as loss functions, and check they are well defined on $\Delta \subset L^\infty$, see Definition 2.5, and then non-decreasing and continuous on Δ .

4.1. The effective reproduction number R_e . We consider the kernel model $[(\Omega, \mathcal{F}, \mu), k]$ under Assumption 1, so that μ is a non-zero σ -finite measure and k is a kernel on Ω with finite double norm. Recall the effective reproduction number function $R_e[k]$ defined on Δ by (17): $R_e[k](\eta) = \rho(T_k M_\eta)$, and the reproduction number $R_0[k] = \rho(T_k)$. When there is no risk of confusion on the kernel k , we simply write R_e and R_0 for $R_e[k]$ and $R_0[k]$.

Proposition 4.1 (Basic properties of R_e). *Suppose Assumption 1 holds. The function $R_e = R_e[k]$ satisfies the following properties:*

- (i) *The function R_e is well defined and non-decreasing on Δ endowed with the weak- $*$ topology.*
- (ii) *$R_e(0) = 0$ and $R_e(1) = R_0$.*
- (iii) *$R_e(\lambda\eta) = \lambda R_e(\eta)$ for all $\eta \in \Delta$ and $\lambda \in [0, 1]$.*

Proof. If $\eta_1 = \eta_2$ μ -almost surely, then we have that $T_{k\eta_1} = T_{k\eta_2}$, and thus $R_e(\eta_1) = R_e(\eta_2)$. If $\eta_1 \leq \eta_2$ μ -almost everywhere, then the operator $T_{k\eta_2} - T_{k\eta_1}$ is positive. According to (6), we get that $\rho(T_{k\eta_1}) \leq \rho(T_{k\eta_2})$. This concludes the proof of Point (i). Point (ii) is a direct consequence of the definition of R_e . Since for any fixed $\lambda \in \mathbb{R}_+$ and any operator T on L^p , the norm of λT is equal to $\lambda \|T\|_{L^p}$, Point (iii) is clear. $\square$

Similarly, we get that the function $\text{Spec}[k]$ defined on Δ is well defined on Δ . We generalize a continuity property on the spectral radius originally stated in [7] by weakening the topology.

Theorem 4.2 (Continuity of $R_e[k]$ and $\text{Spec}[k]$). *Suppose Assumption 1 holds. Then, the functions $\text{Spec}[k]$ and $R_e[k]$ are continuous functions from Δ (endowed with the weak- $*$ topology) respectively to $\mathcal{K}$ (endowed with the Hausdorff distance) and to $\mathbb{R}_+$ (endowed with the usual Euclidean distance).*

Let us remark the proof holds even if k takes negative values.

Proof. Let B denote the unit ball in L^p , with $p \in (1, +\infty)$ from Assumption 1. Since the operator T_k on L^p is compact, the set $T_k(B)$ is relatively compact. For all $\eta \in \Delta$, set $\eta B = \{\eta g : g \in B\}$. As $\eta B \subset B$, we deduce that $T_{k\eta}(B) = T_k(\eta B) \subset T_k(B)$. This implies that the family $\{T_{k\eta} : \eta \in \Delta\}$ is collectively compact.

Let $(\eta_n, n \in \mathbb{N})$ be a sequence in Δ weak-* converging to some $\eta \in \Delta$. Let $g \in L^p$. The weak-* convergence of η_n to η implies that $(T_{k\eta_n}(g), n \in \mathbb{N})$ converges μ -almost surely to $T_{k\eta}(g)$. Consider the function K defined on Ω by:

$$K(x) = \left(\int_{\Omega} k(x, y)^q \mu(dy) \right)^{1/q},$$

which belongs to L^p , thanks to (13). Since for all x ,

$$|T_{k\eta_n}(g)(x)| \leq T_k(|\eta_n g|)(x) \leq K(x) \|\eta_n g\|_p \leq K(x) \|g\|_p,$$

we deduce, by dominated convergence, that the convergence holds also in L^p :

$$(28) \quad \lim_{n \rightarrow \infty} \|T_{k\eta_n}(g) - T_{k\eta}(g)\|_p = 0,$$

so that $T_{k\eta_n}$ converges strongly to $T_{k\eta}$. Using Lemma 2.1 (with $T_n = T_{k\eta_n}$ and $T = T_{k\eta}$) on the continuity of the spectrum, we get that $\lim_{n \rightarrow \infty} \text{Spec}[k](\eta_n) = \text{Spec}[k](\eta)$. The function $\text{Spec}[k]$ is thus weak-* sequentially continuous, and, thanks to Lemma 2.3, it is continuous from Δ endowed with the weak-* topology to the metric space $\mathcal{K}$ endowed with the Hausdorff distance. The continuity of $R_e[k]$ then follows from its definition (5) as the composition of the continuous functions rad and $\text{Spec}[k]$. $\square$

We now give a stability property of the spectrum and spectral radius with respect to the kernel k .

Proposition 4.3 (Stability of $R_e[k]$ and $\text{Spec}[k]$). *Let μ be a σ -finite non-zero measure on the measurable space $(\Omega, \mathcal{F})$. Let $p \in (1, +\infty)$. Let $(k_n, n \in \mathbb{N})$ and k be kernels on Ω with finite double norms on L^p . If $\lim_{n \rightarrow \infty} \|k_n - k\|_{p,q} = 0$, then we have:*

$$(29) \quad \lim_{n \rightarrow \infty} \sup_{\eta \in \Delta} |R_e[k_n](\eta) - R_e[k](\eta)| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{\eta \in \Delta} d_H(\text{Spec}[k_n](\eta), \text{Spec}[k](\eta)) = 0.$$

Proof. Notice the suprema in (29) can also be taken over Δ as R_e and Spec defined on Δ are well defined on Δ . Let us first prove that, if $(\eta_n, n \in \mathbb{N})$ is a sequence in Δ which weak-* converges to $\eta \in \Delta$, then $\text{Spec}[k_n](\eta_n)$ converges to $\text{Spec}[k](\eta)$ in Hausdorff distance.

All the operators in $\mathcal{A} = \{T_k\} \cup \{T_{k_n} : n \in \mathbb{N}\}$ are compact, and we deduce from (15) that:

$$\lim_{n \rightarrow \infty} \|T_{k_n} - T_k\|_{L^p} = 0.$$

Therefore $\mathcal{A}$ is a compact set in $\mathcal{L}(L^p)$. Since the elements of $\mathcal{A}$ are compact operators, we get by [3, Theorem 2.4], that $\mathcal{A}$ is collectively compact. Since $\{M_\eta : \eta \in \Delta\}$ is a bounded set in $\mathcal{L}(L^p)$, we deduce from [4, Proposition 4.2(2)], that the family $\mathcal{A}' = \{T' M_\eta : T' \in \mathcal{A} \text{ and } \eta \in \Delta\}$ is collectively compact. A fortiori the sequence $(T_n = T_{k_n \eta_n} = T_{k_n} M_{\eta_n}, n \in \mathbb{N})$ of elements of $\mathcal{A}'$ is collectively compact, and $T = T_{k\eta} = T_k M_\eta$ is compact.

Let $g \in L^p$. We have:

$$\|T_n(g) - T(g)\|_p \leq \|T_{k_n} - T_k\|_{L^p} \|g\|_p + \|T_{k\eta_n}(g) - T_{k\eta}(g)\|_p.$$

Using $\lim_{n \rightarrow \infty} \|T_{k_n} - T_k\|_{L^p} = 0$ and (28), we get that $\lim_{n \rightarrow \infty} \|T_n(g) - T(g)\|_p = 0$, thus $(T_n, n \in \mathbb{N})$ converges strongly to T . Thanks to Lemma 2.1, we deduce that $\lim_{n \rightarrow \infty} \text{Spec}(T_n) = \text{Spec}(T)$, that is $\lim_{n \rightarrow \infty} \text{Spec}[k_n](\eta_n) = \text{Spec}[k](\eta)$.

Then, as the function $\eta \mapsto d_H(\text{Spec}[k_n](\eta), \text{Spec}[k](\eta))$ is weak-* continuous on the weak-* compact set Δ , thanks to Theorem 4.2, it reaches its maximum say at $\eta_n \in \Delta$ for $n \in \mathbb{N}$. As Δ

is weak-* sequentially compact, consider a sub-sequence which weak-* converges to a limit say η . Since

$$\begin{aligned} \sup_{\eta \in \Delta} d_H \left(\text{Spec}[k_n](\eta), \text{Spec}[k](\eta) \right) \\ = d_H \left(\text{Spec}[k_n](\eta_n), \text{Spec}[k](\eta_n) \right) \\ \leq d_H \left(\text{Spec}[k_n](\eta_n), \text{Spec}[k](\eta) \right) + d_H \left(\text{Spec}[k](\eta_n), \text{Spec}[k](\eta) \right), \end{aligned}$$

using the weak-* continuity of $\text{Spec}[k]$, we deduce that along this sub-sequence the right hand side converges to 0. Since this result holds for any converging sub-sequence, we get the second part of (29). The first part then follows from the definition (5) of R_e as a composition, and the Lipschitz continuity of the function rad . $\square$

4.2. The asymptotic proportion of infected individuals $\mathfrak{I}$. We consider the SIS model $[(\Omega, \mathcal{F}, \mu), k, \gamma]$ under Assumption 2. Recall from (26) that the asymptotic proportion of infected individuals $\mathfrak{I}$ is given on Δ by $\mathfrak{I}(\eta) = \int_{\Omega} \mathfrak{g}_{\eta} \eta \, d\mu$, where $\mathfrak{g}_{\eta}$ is the maximal solution in Δ of the equation $F_{\eta}(h) = 0$. We first recall [11, Lemma 5.3 and Proposition 5.5] on the properties and characterization of the maximal equilibrium $\mathfrak{g} = \mathfrak{g}_{\mathbf{1}}$.

Lemma 4.4 (Properties of the maximal equilibrium). *Suppose Assumption 2 holds.*

- (i) *Let $\eta, g \in \Delta$. If $F_{\eta}(g) \geq 0$, then we have $g \leq \mathfrak{g}_{\eta}$ (in $\mathcal{L}^{\infty}$).*
- (ii) *For any $h \in \Delta$, we have $h = \mathfrak{g}$ (in $\mathcal{L}^{\infty}$) if and only if $F(h) = 0$ (in $\mathcal{L}^{\infty}$) and $R_e(\mathbf{1} - h) \leq 1$.*
- (iii) *If $R_0 > 1$ (or equivalently $\mathfrak{g} \neq 0$ in $\mathcal{L}^{\infty}$), then we have $R_e(\mathbf{1} - \mathfrak{g}) = 1$.*

We may now state the main properties of the function $\mathfrak{I}$.

Proposition 4.5 (Basic properties of $\mathfrak{I}$). *Suppose that Assumption 2 holds and write R_e for $R_e[k/\gamma]$. The function $\mathfrak{I}$ has the following properties:*

- (i) *The function $\mathfrak{I}$ is well defined and non-decreasing on Δ endowed with the weak-* topology.*
- (ii) *For $\eta \in \Delta$, we have $\mathfrak{I}(\eta) = 0$ if and only if $R_e(\eta) \leq 1$.*
- (iii) *$\mathfrak{I}(\lambda\eta) \leq \lambda\mathfrak{I}(\eta)$ for all $\eta \in \Delta$ and $\lambda \in [0, 1]$.*

Proof. If $\eta_1 = \eta_2$ μ -almost surely, then the operators $\mathcal{T}_{k\eta_1}$ and $\mathcal{T}_{k\eta_2}$ are equal. Thus, the equilibria $\mathfrak{g}_{\eta_1}$ and $\mathfrak{g}_{\eta_2}$ are also equal, which in turns implies that $\mathfrak{I}(\eta_1) = \mathfrak{I}(\eta_2)$. To prove the monotonicity, consider $\eta_1, \eta_2 \in \Delta$ such that a.s. $\eta_1 \leq \eta_2$. This gives $\mathcal{T}_{k\eta_1} \leq \mathcal{T}_{k\eta_2}$. We deduce that $F_{\eta_1}(g) \leq F_{\eta_2}(g)$ in $\mathcal{L}^{\infty}$ for all $g \in \Delta \subset \mathcal{L}^{\infty}$. In particular, taking $g = \mathfrak{g}_{\eta_1}$ and using (25), we get $F_{\eta_2}(\mathfrak{g}_{\eta_1}) \geq 0$. By Lemma 4.4 this implies $\mathfrak{g}_{\eta_1} \leq \mathfrak{g}_{\eta_2}$. To sum up, we get:

$$(30) \quad \eta_1 \leq \eta_2 \quad \text{in } L^{\infty} \quad \implies \quad \mathfrak{g}_{\eta_1} \leq \mathfrak{g}_{\eta_2} \quad \text{in } \mathcal{L}^{\infty}.$$

This readily implies that $\mathfrak{I}(\eta_1) = \int_{\Omega} \mathfrak{g}_{\eta_1} \eta_1 \, d\mu \leq \int_{\Omega} \mathfrak{g}_{\eta_2} \eta_2 \, d\mu = \mathfrak{I}(\eta_2)$. This gives Point (i).

Point (ii) is already stated in Equation (27). We now consider Point (iii). Since $\lambda \in [0, 1]$, we deduce from (30) that $\mathfrak{g}_{\lambda\eta} \leq \mathfrak{g}_{\eta}$. This implies that $\mathfrak{I}(\lambda\eta) = \int_{\Omega} \mathfrak{g}_{\lambda\eta} \lambda\eta \, d\mu \leq \lambda \int_{\Omega} \mathfrak{g}_{\eta} \eta \, d\mu = \lambda\mathfrak{I}(\eta)$. $\square$

The proof of the following continuity results are both postponed to Section 6.

Theorem 4.6 (Continuity of $\mathfrak{I}$). *Suppose that Assumption 2 holds. The function $\mathfrak{I}$ defined on Δ is continuous with respect to the weak-* topology.*

We write $\mathfrak{I}[k, \gamma]$ for $\mathfrak{I}$ to stress the dependence on the parameters k, γ of the SIS model.

Proposition 4.7 (Stability of $\mathfrak{I}$). *Let $((k_n, \gamma_n), n \in \mathbb{N})$ and (k, γ) be a sequence of kernels and functions satisfying Assumption 2. Assume furthermore that there exists $p' \in (1, +\infty)$ such that*

$k = \gamma^{-1}k$ and $(k_n = \gamma_n^{-1}k_n, n \in \mathbb{N})$ have finite double norm in $L^{p'}$ and that $\lim_{n \rightarrow \infty} \|k_n - k\|_{p', q'} = 0$. Then we have:

$$(31) \quad \lim_{n \rightarrow \infty} \sup_{\eta \in \Delta} |\mathfrak{J}[k_n, \gamma_n](\eta) - \mathfrak{J}[k, \gamma](\eta)| = 0.$$

Let us stress that Assumption 2 on k and γ implies that $k\gamma^{-1}$ has a finite double norm. In the proposition above, it is also assumed that $\gamma^{-1}k$ has a finite double norm. Notice those two conditions coincide when $\text{ess inf}_\Omega \gamma$ is positive.

5. OTHER PROPERTIES OF THE FUNCTIONS R_e AND $\mathfrak{J}$

In the companion paper [8] we consider the optimization of the protection of the population, which can be written as the bi-objective minimization problem $\min(C(\eta), L(\eta))$, where C and L stand respectively for the cost and the loss incurred when following the vaccination strategy η . In our setting the loss is given either by the effective reproduction number R_e or the fraction of infected individuals at equilibrium $\mathfrak{J}$. (To fix the ideas, a natural cost C , when the measure μ is finite, is the uniform cost $C_{\text{uni}}(\eta) = \int_\Omega (1 - \eta) d\mu$ corresponding intuitively to the number of doses used in the vaccination strategy η , as we recall that $1 - \eta(x)$ is the proportion of the vaccinated population with given feature x . Notice the cost C_{uni} is well defined on $\Delta \subset L^\infty$.) The bi-objective minimization problem is then studied under some of the following hypothesis on the loss. Recall that Δ is endowed with the weak-* topology.

Assumption 3 (On the loss). *Let L be a loss function from Δ to $\mathbb{R}$.*

- (i) **Monotony.** *The function L is non-decreasing continuous with $L(0) = 0$ and $\max_\Delta L > 0$.*
- (ii) **Minima.** *Any local minimum of the function L is a global minimum.*
- (iii) **Maxima.** *Any local maximum of the function L is a global maximum.*

Notice the loss functions R_e and $\mathfrak{J}$ satisfy clearly Assumption 3 (i) provided they are not trivially equal to zero. Thanks to the next lemma, they also satisfy Assumption 3 (ii) as they are sub-homogeneous.

Lemma 5.1. *Let L be a non-negative and non-decreasing loss function defined on Δ . If it is sub-homogeneous, then Assumption 3 (ii) holds.*

Proof. Let $\eta \in \Delta$. If L has a local minimum at η , then for $\varepsilon > 0$ small enough $L(\eta) \leq L((1 - \varepsilon)\eta) \leq (1 - \varepsilon)L(\eta)$, so $L(\eta) = 0$ and η is a global minimum of L . $\square$

We prove in this section that under some irreducibility condition on the kernel that R_e satisfies Assumption 3 (iii). The situation is a bit more complicated for the loss $\mathfrak{J}$, for which Assumption 3 (iii) does not hold. However, $\mathfrak{J}$ satisfies a weakened version, see Assumption 5.3 (iii') below. The reducible case is more delicate and it is studied in more details in [9, Section 5] for the loss function $L = R_e$; in particular Assumption 3 (iii) may not hold in this case.

In Section 5.1 we consider some irreducibility property for a kernel and its implications for the SIS model, see also [12, 13] for further results in this direction. In Section 5.2, we provide some irreducibility conditions in the kernel model so that the loss function R_e satisfies Assumption 3, see Lemma 5.4. Section 5.3 provide similar results for the loss $\mathfrak{J}$ in the SIS model, see Lemma 5.5.

5.1. Irreducible, quasi-irreducible and monatomic kernels. We follow the presentation in [10, Section 5] on the atomic decomposition of positive compact operator and Remark 5.2 therein for the particular case of integral operators, see also the references therein for further results. Let $(\Omega, \mathcal{F}, \mu)$ be a measured space with μ a non-zero σ -finite measure. For $A, B \in \mathcal{F}$, we write $A \subset B$ a.e. if $\mu(B^c \cap A) = 0$ and $A = B$ a.e. if $A \subset B$ a.e. and $B \subset A$ a.e.. Let $\mathcal{G} \subset \mathcal{F}$ be a σ -field. A set A is an *atom* of μ in $\mathcal{G}$ if A belongs to $\mathcal{G}$, and for all $B \subset A$ with $B \in \mathcal{G}$, we have either $B = \emptyset$ a.e. or $B = A$ a.e.. Notice that the atoms are defined up to an a.e. equivalence.

Let k be a kernel on Ω with a finite double norm. For $A, B \in \mathcal{F}$, $x \in \Omega$, we simply write $k(x, A) = \int_A k(x, y) \mu(dy)$, $k(B, x) = \int_B k(z, x) \mu(dz)$ and:

$$k(B, A) = \int_{B \times A} k(z, y) \mu(dz) \mu(dy) \in [0, +\infty].$$

A set $A \in \mathcal{F}$ is called *k-invariant*, or simply *invariant* when there is no ambiguity on the kernel k , if $k(A^c, A) = 0$. In the epidemiological setting, the set A is invariant if the sub-population A does not infect the sub-population A^c . The kernel k is *irreducible* (or *connected*) if any invariant set A is such that $\mu(A) = 0$ or $\mu(A^c) = 0$. If k is irreducible, then either $R_0[k] > 0$ or $k \equiv 0$ and Ω is an atom of μ in $\mathcal{F}$ (degenerate case). A simple sufficient condition for irreducibility is for the kernel to be a.e. positive.

Let $\mathcal{A}$ be the set of k -invariant sets. Let us stress that the set of k -invariant sets depends only on the support of the kernel k . In particular in the SIS model, with $k = k/\gamma$, the k -invariant sets and the k -invariant sets coincide. Notice that $\mathcal{A}$ is stable by countable unions and countable intersections. Let $\mathcal{F}_{\text{inv}} = \sigma(\mathcal{A})$ be the σ -field generated by $\mathcal{A}$. Then, the operator k restricted to an atom of μ in $\mathcal{F}_{\text{inv}}$ is irreducible. We shall only consider non degenerate atoms, and say the atom (of μ in $\mathcal{F}_{\text{inv}}$) is non-zero if the restriction of the kernel k to this atom is non-zero (and thus the spectral radius of the corresponding integral operator is positive). We say the kernel k is *monatomic* if there exists a unique non-zero atom, say Ω_a , and the kernel is *quasi-irreducible* if it is monatomic and $k \equiv 0$ outside $\Omega_a \times \Omega_a$, where Ω_a is its non-zero atom. Notice that: (i) if k is irreducible with $R_0[k] > 0$, then k is monatomic with non-zero atom $\Omega_a = \Omega$; (ii) if k is monatomic, then $R_0[k] > 0$ by definition. The quasi-irreducible property is the usual extension of the irreducible property in the setting of symmetric kernels; and the monatomic property is the natural generalization to non-symmetric kernels.

According to [10, Lemma 5.3], we get that if a kernel k , with finite double norm, is monatomic with non-zero atom Ω_a and $\eta \in \Delta$, then, with $k_a = \mathbb{1}_{\Omega_a} k \mathbb{1}_{\Omega_a}$ and $\mathbf{k}_a$ (resp. $\boldsymbol{\eta}_a$) the restriction of k (resp. $\eta \in \Delta$) to Ω_a :

$$(32) \quad R_e[k](\eta) = R_e[k_a](\eta) = R_e[k_a](\eta \mathbb{1}_{\Omega_a}) = R_e[\mathbf{k}_a](\boldsymbol{\eta}_a).$$

Remark 5.2 (Epidemiological interpretation). When the kernel $k = k/\gamma$ for the SIS model $[(\Omega, \mathcal{F}, \mu), k, \gamma]$ is monatomic, with non-zero atom Ω_a , then the population with trait in Ω_a can infect itself. It may also infect another part of the population, say with trait in Ω_i , but:

- the infection cannot be sustained at all in Ω_i : k is quasi-nilpotent on Ω_i ;
- the population with trait in Ω_i does not infect back the non-zero atom Ω_a .

If furthermore $R_0 > 1$, then the set $\Omega_a \cup \Omega_i$ corresponds to the support of the maximal endemic equilibrium.

In the monatomic case, the non-zero equilibrium, if it exists, is unique. This result is a direct consequence of Lemma 4.1 (iii) and Corollary 4.11 in [13].

Lemma 5.3 (Equilibrium in the monatomic case). *Assume Assumption 2 holds for the SIS model $\text{Param} = [(\Omega, \mathcal{F}, \mu), k, \gamma]$ and that $R_0[k] > 1$ (super-critical regime) with $k = k/\gamma$. If k (and $\mathbf{k}$) is monatomic, with non-zero atom say Ω_a , then there exists a unique non-zero equilibrium, say $\mathbf{g}$, and its support is the smallest invariant set containing Ω_a , that is, the set $\{\mathbf{g} > 0\}$ is invariant and if A is invariant and $\Omega_a \subset A$, then a.s. $\{\mathbf{g} > 0\} \subset A$.*

5.2. The kernel model. We now check Assumptions 3 (ii)-(iii) for the loss $L = R_e$.

Lemma 5.4 (Extrema of R_e). *Consider the kernel model $\text{Param} = [(\Omega, \mathcal{F}, \mu), k]$ under Assumption 1, and simply write R_e for the loss function $L = R_e[k]$.*

- (i) Assumption 3 (i) holds if $R_0 > 0$.
- (ii) Assumption 3 (ii) holds.

(iii) If k is monatomic with atom Ω_a , then $R_0 > 0$ and Assumption 3 (iii) holds. Furthermore, $\eta \in \Delta$ is a global maximum of R_e if and only if $\eta \geq \mathbb{1}_{\Omega_a}$ (in L^∞).

Proof. Since the function R_e is homogeneous, see Proposition 4.1, we deduce from Lemma 5.1 that Assumption 3 (ii) holds. Using Theorem 4.2, for the continuity, Proposition 4.1, for the monotonicity of the function R_e , and the fact that $R_0 > 0$, the hypotheses on the loss in Assumption 3 (i) hold.

We now prove Point (iii). We first assume that the kernel k is irreducible with $R_0 > 0$. In particular, we have a.e. that $k(\Omega, y) > 0$. Let $\eta \in \Delta$ be a local maximum; we want to show that it is also a global maximum.

Suppose first that $\inf \eta > 0$. Then $k\eta$ is irreducible non-zero with finite double norm. According [28, Theorem V.6.6] and since $T_{k\eta} = T_k M_\eta$ is compact, the eigenspace of $T_{k\eta}$ associated to $R_e(\eta)$ is one-dimensional and it is spanned by a vector v_d such that $v_d > 0$ a.e., and the corresponding left eigenvector associated to $R_e(\eta)$, say v_g , can be chosen such that $\langle v_g, v_d \rangle = 1$ and a.e. $v_g > 0$. According to [23, Theorem 2.6], applied to $L_0 = T_{k\eta}$ and $L = T_{k(\eta + \varepsilon(\mathbb{1} - \eta))}$ with $\varepsilon \in (0, 1)$, we have, using that $\|L_0 - L\|_{L^p} = O(\varepsilon)$ thanks to (15):

$$R_e(\eta + \varepsilon(\mathbb{1} - \eta)) = R_e(\eta) + \varepsilon \langle v_g, T_{k(\mathbb{1} - \eta)} v_d \rangle + O(\varepsilon^2).$$

Since R_e has a local maximum at η , the first order term on the right hand side vanishes, so $v_g(x)k(x, y)(\mathbb{1} - \eta(y))v_d(y) = 0$ for μ almost every x and y . Since v_g and v_d are positive a.e. and k is irreducible, we get that $k(\Omega, y)(\mathbb{1} - \eta(y)) = 0$ a.e. and thus a.e. $\eta(y) = 1$. Therefore $\eta = \mathbb{1}$, which is a global maximum for R_e .

Finally, suppose that $\inf \eta = 0$. Let G be an open subset of Δ on which $R_e \leq R_e(\eta)$ and with $\eta \in G$. For $\varepsilon > 0$ small enough, the strategy $\eta_\varepsilon = \eta + \varepsilon(\mathbb{1} - \eta)$ belongs to G and satisfies $R_e(\eta) \leq R_e(\eta_\varepsilon) \leq R_e(\eta)$ (where the first inequality comes from the fact that R_e is non-decreasing). Therefore η_ε is a local maximum with $\inf \eta_\varepsilon \geq \varepsilon$, and thus, thanks to the first part of the proof, $\eta_\varepsilon = \mathbb{1}$. This readily implies that $\eta = \mathbb{1}$. We deduce that if η is a local maximum, then $\eta = \mathbb{1}$ and thus it is a global maximum. This ends the proof for the irreducible case when $R_0 > 0$.

Recall that R_e and R_0 respectively denote $R_e[k]$ and $R_0[k]$. To treat the monatomic case, recall that for any η , we know by (32) that:

$$R_e(\eta) = R_e[\mathbf{k}_a](\boldsymbol{\eta}_a),$$

where $\mathbf{k}_a$ (resp. $\boldsymbol{\eta}_a$) is the restriction of k (resp. η) to the atom Ω_a , and $R_0 = R_0[\mathbf{k}_a] > 0$. Let $\eta \in \Delta$ be a local maximum for R_e . Then $\boldsymbol{\eta}_a$ is a local maximum for $R_e[\mathbf{k}_a]$. We deduce from the first part of the proof applied to the irreducible kernel $\mathbf{k}_a$ that $\boldsymbol{\eta}_a = \mathbb{1}_a$, and thus $\eta \geq \mathbb{1}_{\Omega_a}$ as well as $R_e(\eta) \geq R_e(\mathbb{1}_{\Omega_a}) = R_e(\mathbb{1})$. Thus, the strategy η is a global maximum. This implies that Assumption 3 (iii) holds.

Use that $\mathbb{1}_a$, the unity function defined on Ω_a , is the only global maximum of $R_e[\mathbf{k}_a]$ thanks to the first part of the proof, to deduce that η is a global maximum of R_e if and only if $\eta \geq \mathbb{1}_{\Omega_a}$ (in L^∞). $\square$

5.3. The SIS model. The loss $L = \mathfrak{J}$ does not satisfies Assumption 3 (iii) in general even when the kernel $k = k/\gamma$ is irreducible with $R_0 = R_0[k] > 0$. Indeed, by continuity of R_e , there exists a (weak-*) open neighborhood G of 0 such that $R_e(\eta) < 1$ for all $\eta \in G$: consequently $\mathfrak{J}$ is identically zero on G , and any $\eta \in G$ is a local maximum of $L = \mathfrak{J}$. However, these maxima are not global in the super-critical regime where $\mathfrak{J}(\mathbb{1}) > 0$ (and $R_0 > 1$). For this reason, we shall consider the following variant of Assumption 3 (iii), where one does not consider the zeros of the loss.

Assumption 3 (On the loss). *Let L be a function from Δ endowed with the weak-* topology to $\mathbb{R}$.*

(iii') **Maxima.** *Any local maximum η of the loss function L , such that $L(\eta) > 0$, is a global maximum.*

We are now ready to check that Assumption 5.3 (iii') holds for the loss $L = \mathfrak{J}$ when the kernel k is monatomic. Recall $\mathbf{g} \in \Delta$ is the maximal equilibrium.

Lemma 5.5. *Consider the SIS model $\text{Param} = [(\Omega, \mathcal{F}, \mu), k, \gamma]$ under Assumption 2 with the loss function $L = \mathfrak{J}$, and simply write R_0 for $R_0[k/\gamma]$.*

- (i) *Assumption 3 (i) holds, and thus $\mathfrak{J}(\mathbf{1}) > 0$, if $R_0 > 1$.*
- (ii) *Assumption 3 (ii) holds.*
- (iii) *If k is monatomic and $R_0 > 1$, then Assumption 5.3 (iii') holds. Furthermore, $\eta \in \Delta$ is a global maximum of $\mathfrak{J}$ if and only if $\eta \geq \mathbf{1}_{\{\mathbf{g} > 0\}}$ (in L^∞).*

Proof. Since the loss $\mathfrak{J}$ is sub-homogeneous, see Proposition 4.5, we deduce from Lemma 5.1 that Assumption 3 (ii) holds. Using Theorem 4.6 (for the continuity), Proposition 4.5 (for the monotonicity of the function $\mathfrak{J}$), and the fact that $\mathfrak{J}(\mathbf{1}) > 0$ if $R_0 > 1$, see (27), we obtain that the hypothesis on the loss in Assumption 3 (i) hold if $R_0 > 1$.

We now prove Point (iii). Assume that $R_0 > 1$, that is, $\mathfrak{J}(\mathbf{1}) > 0$, and set $k = k/\gamma$. Let $\mathbf{g}$ be the maximal equilibrium which is non-zero as $R_0 > 1$. Recall that being k' -invariant depends only on the support of the kernel k' . Since the kernels k and k have the same support, and k is monatomic, we deduce that k is monatomic with the same atom Ω_a and same smallest invariant set containing Ω_a given by $\{\mathbf{g} > 0\}$ thanks to Lemma 5.3. Suppose that $\mathfrak{J}$ has a local maximum at some $\eta \in \Delta$ and $\mathfrak{J}(\eta) > 0$. For $\varepsilon \in (0, 1)$, set $\eta_\varepsilon = \eta + \varepsilon(\mathbf{1} - \eta)$. We have that for $\varepsilon > 0$ small enough:

$$(33) \quad \mathfrak{J}(\eta) \geq \mathfrak{J}(\eta_\varepsilon) = \int_{\Omega} \mathbf{g}_{\eta_\varepsilon} \eta_\varepsilon d\mu \geq \int_{\Omega} \mathbf{g}_{\eta_\varepsilon} \eta d\mu \geq \int_{\Omega} \mathbf{g}_\eta \eta d\mu = \mathfrak{J}(\eta),$$

where we used that $\eta \leq \eta_\varepsilon$ and $0 \leq \mathbf{g}_\eta \leq \mathbf{g}_{\eta_\varepsilon}$, see (30). Therefore all these quantities are equal. Since $k\eta_\varepsilon$ and k have the same support, we deduce that $k\eta_\varepsilon$ is monatomic with non-zero atom Ω_a . From Lemma 5.3, we also obtain that $\{\mathbf{g}_{\eta_\varepsilon} > 0\}$ and $\{\mathbf{g} > 0\}$ are equal, being equal to the smallest invariant set containing Ω_a . We deduce from (33), as all the inequalities are equalities, that $\eta_\varepsilon = \eta$ a.s. on $\{\mathbf{g} > 0\}$, and thus $\eta \geq \mathbf{1}_{\{\mathbf{g} > 0\}}$ a.s.. Recall from (30) that $\mathbf{g}_\eta \leq \mathbf{g}$. So $\mathbf{g}_\eta$ is zero outside $\{\mathbf{g} > 0\}$, and we deduce that changing the value of η outside $\{\mathbf{g} > 0\}$ does not affect the value of $\mathfrak{J}(\eta)$. In conclusion, $\eta \in \Delta$ is a local maximum such that $\mathfrak{J}(\eta) > 0$ if and only if $\eta \geq \mathbf{1}_{\{\mathbf{g} > 0\}}$ a.s., and thus is a global maximum. $\square$

6. TECHNICAL PROOFS: PROPERTIES OF $\mathfrak{J}$ AND OF THE MAXIMAL EQUILIBRIUM

In the SIS model, in order to stress, if necessary, the dependence of a quantity H , such as F_η , R_e or $\mathbf{g}_\eta$, in the parameters k and γ (which satisfy Assumption 2) of the model, we shall write $H[k, \gamma]$. Recall that if k and γ satisfy Assumption 2, then the kernel k/γ has a finite double norm on L^p for some $p \in (1, +\infty)$ (as the measure μ is finite). We now consider the continuity property of the maps $\eta \mapsto \mathbf{g}_\eta[k, \gamma]$ and $(k, \gamma, \eta) \mapsto \mathbf{g}_\eta[k, \gamma]$. Notice the former function defined on Δ is well defined on Δ thanks to (30).

Lemma 6.1. *Let $((k_n, \gamma_n), n \in \mathbb{N})$ and (k, γ) be kernels and functions satisfying Assumption 2 and $(\eta_n, n \in \mathbb{N})$ be a sequence of elements of Δ which weak-* converges to η .*

- (i) *We have $\lim_{n \rightarrow \infty} \mathbf{g}_{\eta_n}[k, \gamma] = \mathbf{g}_\eta[k, \gamma]$ μ -almost surely.*
- (ii) *Assume furthermore there exists $p' \in (1, +\infty)$ such that $k = \gamma^{-1}k$ and $(k_n = \gamma_n^{-1}k_n, n \in \mathbb{N})$ have finite double norm on $L^{p'}$ and that $\lim_{n \rightarrow \infty} \|k_n - k\|_{p', q'} = 0$. Then, we have $\lim_{n \rightarrow \infty} \mathbf{g}_{\eta_n}[k_n, \gamma_n] = \mathbf{g}_\eta[k, \gamma]$ μ -almost surely.*

Proof. The proof of (i) and (ii) being rather similar, we only provide the latter and indicate the difference when necessary. To simplify, we write $g_n = \mathbf{g}_{\eta_n}[k_n, \gamma_n]$. We set $h_n = \eta_n g_n \in \Delta$ for $n \in \mathbb{N}$. Since Δ is sequentially weak-* compact, up to extracting a subsequence, we can assume

that h_n weak-* converges to a limit $h \in \Delta$. Since $F_{\eta_n}[k_n, \gamma_n](g_n) = 0$ for all $n \in \mathbb{N}$, see (25), we have:

$$(34) \quad g_n = \frac{\mathcal{T}_{k_n}(\eta_n g_n)}{1 + \mathcal{T}_{k_n}(\eta_n g_n)} = \frac{\mathcal{T}_{k_n}(h_n)}{1 + \mathcal{T}_{k_n}(h_n)}.$$

We set $g = \mathcal{T}_k(h)/(1 + \mathcal{T}_k(h))$. Notice that $\mathcal{T}_{k_n}(h_n) = (\mathcal{T}_{k_n} - \mathcal{T}_k)(h_n) + \mathcal{T}_k(h_n)$. We have $\lim_{n \rightarrow \infty} \mathcal{T}_k(h_n) = \mathcal{T}_k(h)$ pointwise. Since $\|(\mathcal{T}_{k_n} - \mathcal{T}_k)(h_n)\|_{p'} \leq \|k_n - k\|_{p', q'}$, up to taking a sub-sequence, we deduce that $\lim_{n \rightarrow \infty} (\mathcal{T}_{k_n} - \mathcal{T}_k)(h_n) = 0$ almost surely. (Notice the previous step is not used in the proof of (i) as $k_n = k$ and $\lim_{n \rightarrow \infty} \mathcal{T}_k(h_n) = \mathcal{T}_k(h)$ pointwise.) This implies that g_n converges almost surely to g . By the dominated convergence theorem (recall μ is finite), we deduce that g_n converges also in L^p to g . This proves that $h = \eta g$ almost surely. We get $g = \mathcal{T}_k(\eta g)/(1 + \mathcal{T}_k(\eta g))$ and thus $F_\eta[k, \gamma](g) = 0$ in $\mathcal{L}^\infty$: g is an equilibrium for $F_\eta[k, \gamma]$. We recall from [10, Section 3] the functional equality $R_e[k'h] = R_e[hk']$, where k' is a kernel, h a non-negative functions such that the kernels $k'h$ and hk' have finite double norm. We get:

$$\begin{aligned} R_e[k\eta/\gamma](1 - g) &= R_e[k](\eta(1 - g)) = \lim_{n \rightarrow \infty} R_e[k_n](\eta_n(1 - g_n)) \\ &= \lim_{n \rightarrow \infty} R_e[k_n \eta_n / \gamma_n](1 - g_n) \\ &\leq 1, \end{aligned}$$

where we used the weak-* continuity and the stability of R_e from Theorem 4.2 and Proposition 4.3 for the second equality, and Lemma 4.4 (ii) for the inequality. (Only the weak-* continuity of $\eta' \mapsto R_e[k/\gamma](\eta')$ is used in the proof of (i) to get $R_e[k/\gamma](\eta(1 - g)) \leq 1$.) Since g is an equilibrium for $F_\eta[k, \gamma]$, we deduce from Lemma 4.4 (ii), with k replaced by $k\eta$, that g is the maximal equilibrium, that is, $g = \mathfrak{g}_\eta[k, \gamma]$. $\square$

Proofs of Theorem 4.6 and Proposition 4.7. Under the assumptions of Lemma 6.1, taking the pair (k_n, γ_n) equal to (k, γ) in the case (i) therein, we deduce that $(\eta_n \mathfrak{g}_{\eta_n}[k_n, \gamma_n], n \in \mathbb{N})$ weak-* converges to $\eta \mathfrak{g}_\eta[k, \gamma]$. This implies that:

$$\lim_{n \rightarrow \infty} \mathfrak{J}[k_n, \gamma_n](\eta_n) = \lim_{n \rightarrow \infty} \int_{\Omega} \eta_n \mathfrak{g}_{\eta_n}[k_n, \gamma_n] d\mu = \int_{\Omega} \eta \mathfrak{g}_\eta[k, \gamma] d\mu = \mathfrak{J}[k, \gamma](\eta).$$

Taking $(k_n, \gamma_n) = (k, \gamma)$ provides the continuity of $\mathfrak{J}[k, \gamma]$ and thus Theorem 4.6. Then, arguing as in the end of the proof of Proposition 4.3, we get Proposition 4.7. $\square$

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