Hybrid high-order methods

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collaboration and support: CEA

MFET, Mülheim an der Ruhr, 22 August 2023

Outline

Hybrid high-order (HHO) methods ...

- In a nutshell
- Links to other methods
- Wave propagation problems

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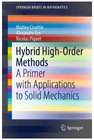
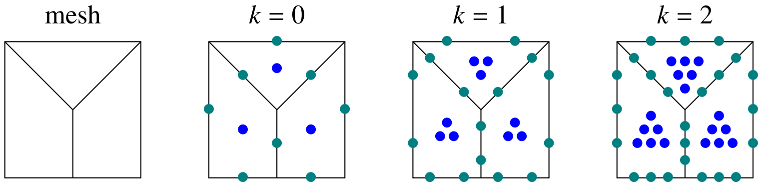
- Links to other methods
- Wave propagation problems
- Seminal references: [Di Pietro, AE, Lemaire 14; Di Pietro, AE 15]
- Two textbooks
  - HHO on polytopal meshes [Di Pietro, Droniou 20]
  - A primer with application to solid mechanics [Cicuttin, AE, Pignet 21]



HHO in a nutshell

Basic ideas

- Degrees of freedom (dofs) located on mesh cells and faces
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces



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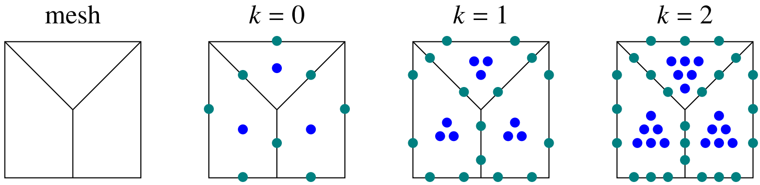

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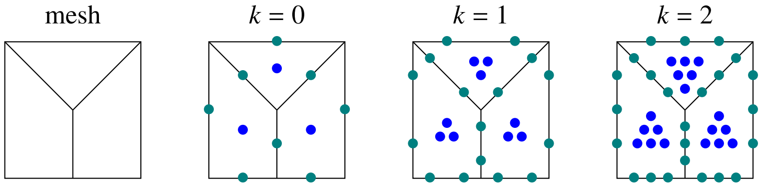

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- One adds a local stabilization to weakly enforce the matching of cell dofs trace with face dofs
- The global problem is assembled cellwise as in FEM
- Generalization to higher order of ideas from Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

Gradient reconstruction and stabilization

- Mesh cell $T \in \mathcal{T}$, cell dofs $u_T \in \mathbb{P}^k(T)$, face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Local potential reconstruction $R_T : \hat{U}_T \to \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla R_T(\hat{u}_T), \nabla q)_T = -(u_T, \Delta q)_T + (u_{\partial T}, \nabla q \cdot \mathbf{n}_T)_{\partial T}, \quad \forall q \in \mathbb{P}^{k+1}(T)/\mathbb{R}$$

together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

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together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

- Local gradient reconstruction $\mathbf{G}_T(\hat{u}_T) := \nabla R_T(\hat{u}_T) \in \nabla \mathbb{P}^{k+1}(T)$
- Local stabilization operator acting on $\delta_{\hat{u}_T} := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\delta_{\hat{u}_T}) := \Pi^k_{\partial T}\Big(\delta_{\hat{u}_T} - \underbrace{\big((I - \Pi^k_T) R_T(0, \delta_{\hat{u}_T})\big)|_{\partial T}}_{\text{high-order correction}}\Big)$$

Taking $S_{\partial T}(\delta_{\hat{u}_T}) := \delta_{\hat{u}_T}$ is suboptimal ...

Local bilinear form

- Local bilinear form for Poisson model problem

$$a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_T(\hat{u}_T), \mathbf{G}_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\delta_{\hat{u}_T}), S_{\partial T}(\delta_{\hat{w}_T}))_{\partial T}$$

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- Stability and boundedness

$$\alpha \|\hat{u}_T\|^2_{\hat{U}_T} \leq a_T(\hat{u}_T, \hat{u}_T) \leq \omega \|\hat{u}_T\|^2_{\hat{U}_T}, \quad \forall \hat{u}_T \in \hat{U}_T$$

with $\|\hat{u}_T\|^2_{\hat{U}_T} := \|\nabla u_T\|^2_T + h_T^{-1}\|\delta_{\hat{u}_T}\|^2_{\partial T}$

Assembly of discrete problem

- Global dofs $\hat{u}_h = (u_{\mathcal{T}}, u_{\mathcal{F}})$ ($\mathcal{T}$:= {mesh cells}, $\mathcal{F}$:= {mesh faces})

$$\hat{U}_h := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F}), \quad \mathbb{P}^k(\mathcal{T}) := \bigtimes_{T\in\mathcal{T}} \mathbb{P}^k(T), \quad \mathbb{P}^k(\mathcal{F}) := \bigtimes_{F\in\mathcal{F}} \mathbb{P}^k(F)$$

- Dirichlet conditions enforced on face boundary dofs

$$\hat{U}_{h0} := \{\hat{v}_h \in \hat{U}_h \mid v_F = 0 \ \forall F \subset \partial\Omega\}$$

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- Dirichlet conditions enforced on face boundary dofs

$$\hat{U}_{h0} := \{\hat{v}_h \in \hat{U}_h \mid v_F = 0 \ \forall F \subset \partial\Omega\}$$

- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \quad \forall \hat{w}_h \in \hat{U}_{h0}$$

(only cell component of test function used on rhs)

Algebraic realization and static condensation

- Algebraic realization

$$\begin{bmatrix} \mathsf{A}_{\mathcal{T}\mathcal{T}} & \mathsf{A}_{\mathcal{T}\mathcal{F}} \\ \mathsf{A}_{\mathcal{F}\mathcal{T}} & \mathsf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{U}_{\mathcal{T}} \\ \mathsf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}} \\ 0 \end{bmatrix}$$

$\Longrightarrow$ submatrix $\mathsf{A}_{\mathcal{T}\mathcal{T}}$ is block-diagonal!

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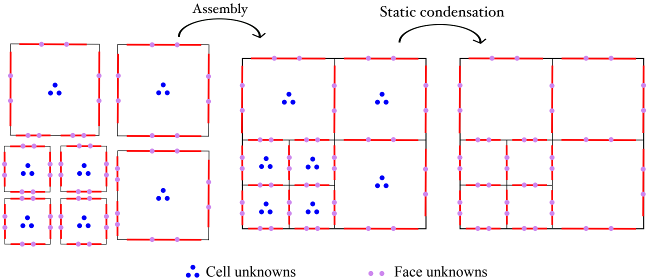
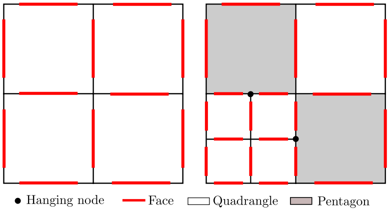
- Cell dofs can be eliminated locally by static condensation
  - global problem couples only face dofs
  - cell dofs recovered by local post-processing

- Summary



Main assets of HHO methods

- General meshes: polytopal cells, hanging nodes



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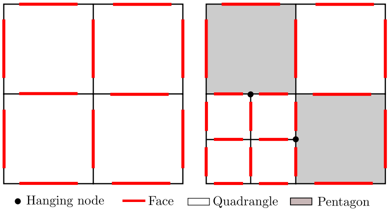

- Local conservation
  - optimally convergent and algebraically balanced fluxes on faces
  - as any face-based method, balance at the cell level
- Attractive computational costs
  - only face dofs are globally coupled
  - compact stencil (slightly less compact than DG though)

Error estimates

- Smooth solutions (in $H^{k+2}(\Omega)$)
  - $O(h^{k+1})$ H^1-error estimate (face dofs of order $k \geq 0$)
  - $O(h^{k+2})$ L^2-error estimate (with full elliptic regularity)

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- Less regularity
  - $O(h^t)$ H^1-error estimate if $u \in H^{1+t}(\Omega)$, $t \in (\frac{1}{2}, k+1]$
  - for $t \in (0, \frac{1}{2})$, see [AE, Guermond 21 (FoCM)]
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  - for $f \in H^{-1}(\Omega)$, see [AE, Zanotti 20 (IMAJNA)]
- Main consistency property: Introduce reduction operator

$$\hat{I}_T : H^1(T) \to \hat{U}_T, \qquad \hat{I}_T(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v|_{\partial T}))$$

  Then we have
  - $h_T^{-1}\|v - R_T(\hat{I}_T(v))\|_T + \|\nabla(v - R_T(\hat{I}_T(v)))\|_T \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$
  - $h_T^{-\frac{1}{2}}\|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$

Variants

- Variant on gradient reconstruction $\mathbf{G}_T : \hat{U}_T \to \mathbb{P}^k(T;\mathbb{R}^d)$ s.t.

$$(\mathbf{G}_T(\hat{u}_T), \mathbf{q})_T = -(u_T, \mathrm{div}\ \mathbf{q})_T + (u_{\partial T}, \mathbf{q}\cdot\mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

  - same scalar mass matrix for each component of $\mathbf{G}_T(\hat{u}_T)$
  - useful for nonlinear problems
    [Di Pietro, Droniou 17; Botti, Di Pietro, Sochala 17; Abbas, AE, Pignet 18]

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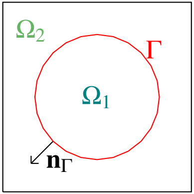
- Variants on cell dofs and stabilization
  - mixed-order setting: $k \geq 0$ for face dofs and $(k+1)$ for cell dofs
  - this variant allows for the simpler Lehrenfeld–Schöberl HDG stabilization

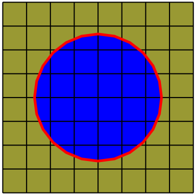
$$S_{\partial T}(\delta_{\hat{u}_T}) := \Pi^k_{\partial T}(\delta_{\hat{u}_T})$$

  - another variant is $k \geq 1$ for face dofs and $(k-1)$ for cell dofs

HHO on unfitted meshes

- Model problem with curved interface/boundary





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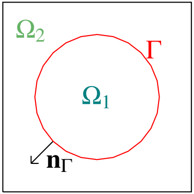

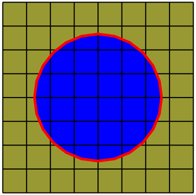

- HHO works optimally on cells with planar faces
- One idea is to use unfitted meshes
  - curved interface can cut arbitrarily through mesh cells
  - numerical method must deal with ill cut cells

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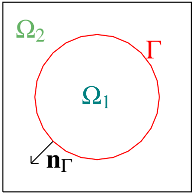

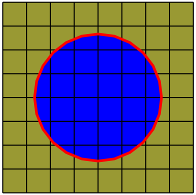

- HHO works optimally on cells with planar faces
- One idea is to use unfitted meshes
  - curved interface can cut arbitrarily through mesh cells
  - numerical method must deal with ill cut cells
- Well developed paradigm for unfitted FEM
  - double unknowns in cut cells and use a consistent Nitsche's penalty technique to enforce jump conditions [Hansbo, Hansbo 02]
  - ghost penalty [Burman 10] to counter ill cuts (gradient jump penalty across faces near curved boundary/interface)

Unfitted HHO

- Main ideas [Burman, AE 18 (SINUM)]
  - double cell and face dofs in cut cells, no dofs on curved boundary/interface
  - mixed-order setting: $k \geq 0$ for face dofs and $(k+1)$ for cell dofs
  - local cell agglomeration as an alternative to ghost penalty
    see [Sollie, Bokhove, van der Vegt 11; Johansson, Larson 13] for dG context

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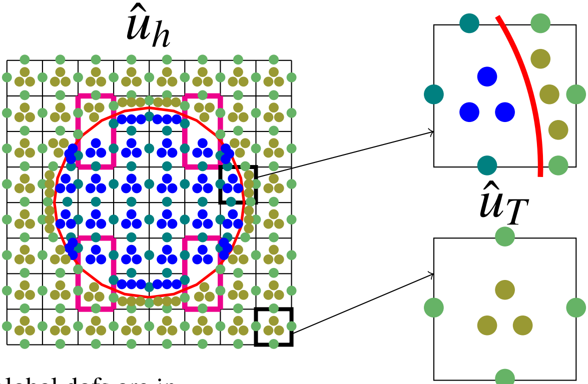
- Improvements in [Burman, Cicuttin, Delay, AE 21 (SISC)]
  - novel gradient reconstruction $\Rightarrow$ $O(1)$ penalty parameter
  - robust cell agglomeration procedure (ensures locality)

- Extensions
  - Stokes interface problems [Burman, Delay, AE 20 (IMANUM)]
  - wave propagation [Burman, Duran, AE 21 (CMAME)]

Global dofs



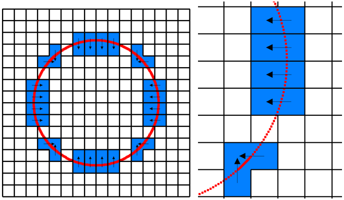
- The global dofs are in

$$\hat{u}_h \in \hat{U}_h := \bigtimes_{T \in \mathcal{T}^1} \mathbb{P}^{k+1}(T_1) \times \bigtimes_{T \in \mathcal{T}^2} \mathbb{P}^{k+1}(T_2) \times \bigtimes_{F \in \mathcal{F}^1} \mathbb{P}^{k}(F_1) \times \bigtimes_{F \in \mathcal{F}^2} \mathbb{P}^{k}(F_2)$$

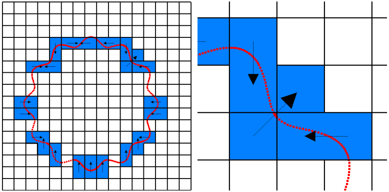
- We set to zero all the face components attached to $\partial\Omega$
- All the cell dofs are eliminated locally by static condensation
- Only the face dofs are globally coupled

Illustration of agglomeration procedure

- Circular interface



- Flower-like interface



Links to other methods

$$\text{HHO} \equiv \text{WG} \equiv \text{HDG} \equiv \text{ncVEM}$$

- [Cockburn, Di Pietro, AE 16 (M2AN)], [Di Pietro, Droniou, Manzini 18 (JCP)], [Cicuttin, AE, Pignet 21 (SpringerBriefs)]
- !! Different devising viewpoints should be mutually enriching !!

Weak Galerkin (WG)

- WG methods devised in [Wang, Ye 13] (vast litterature...)
- Similar devising of HHO and WG
- HHO gradient reconstruction is called weak gradient in WG

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- HHO gradient reconstruction is called weak gradient in WG
- WG often uses plain LS stabilization

$$S_{\partial T}^{\mathrm{WG}}(\delta_{\hat{u}_T}) := \delta_{\hat{u}_T} \quad \text{vs.} \quad S_{\partial T}^{\mathrm{HHO}}(\delta_{\hat{u}_T}) := \begin{cases} \Pi_{\partial T}^{k}\big(\delta_{\hat{u}_T} - \big((I - \Pi_T^k) R_T(0, \delta_{\hat{u}_T})\big)|_{\partial T}\big) & (l = k) \\ \Pi_{\partial T}^{k}(\delta_{\hat{u}_T}) & (l = k+1) \end{cases}$$

- Plain LS stabilization leads to $O(h^k)$ H^1-error bounds (not $O(h^{k+1})$ ...)
  - achieving $O(h^{k+1})$ bounds requires using face polynomials of order $(k+1)$ $\Longrightarrow$ more expensive

Hybridizable DG

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  - the local equation for the dual variable is the grad. rec. formula in HHO!
  - one passes from HDG to HHO formulation by static condensation of dual variable

$$\begin{bmatrix} \mathsf{A}^{\mathrm{HDG}}_{\sigma\sigma} & \mathsf{A}^{\mathrm{HDG}}_{\sigma u} & \mathsf{A}^{\mathrm{HDG}}_{\sigma\lambda} \\ \mathsf{A}^{\mathrm{HDG}}_{u\sigma} & \mathsf{A}^{\mathrm{HDG}}_{uu} & \mathsf{A}^{\mathrm{HDG}}_{u\lambda} \\ \mathsf{A}^{\mathrm{HDG}}_{\lambda\sigma} & \mathsf{A}^{\mathrm{HDG}}_{\lambda u} & \mathsf{A}^{\mathrm{HDG}}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathsf{S}_{\mathcal{T}} \\ \mathsf{U}_{\mathcal{T}} \\ \mathsf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathsf{F}_{\mathcal{T}} \\ 0 \end{bmatrix} \iff \begin{cases} \mathsf{A}^{\mathrm{HDG}}_{\sigma\sigma}\mathsf{S}_{\mathcal{T}} = -\left(\mathsf{A}^{\mathrm{HDG}}_{\sigma u}\mathsf{U}_{\mathcal{T}} + \mathsf{A}^{\mathrm{HDG}}_{\sigma\lambda}\mathsf{U}_{\mathcal{F}}\right) \\ \begin{bmatrix} \mathsf{A}^{\mathrm{HHO}}_{uu} & \mathsf{A}^{\mathrm{HHO}}_{u\lambda} \\ \mathsf{A}^{\mathrm{HHO}}_{\lambda u} & \mathsf{A}^{\mathrm{HHO}}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathsf{U}_{\mathcal{T}} \\ \mathsf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}} \\ 0 \end{bmatrix} \end{cases}$$

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- HHO is an HDG method!
  - this bridge uncovers HHO numerical flux trace

$$\widehat{\mathbf{q}}_{\partial T}(\hat{u}_T) = -\mathbf{G}_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(S^{\star}_{\partial T} \circ S_{\partial T})(\delta_{\hat{u}_T})$$

  - one HHO novelty: use of reconstruction in stabilization (equal-order case)

- Main HHO benefit: simpler analysis based on L^2-projections (avoids special HDG projection!)

Nonconforming virtual elements

- ncVEM devised in [Ayuso, Manzini, Lipnikov 16]
- Virtual space

$$\mathbb{P}^{k+1}(T) \subsetneq \mathcal{V}_T := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^l(T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

(original ncVEM devising with $l = k - 1$, $k \geq 1$)

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- Virtual space

$$\mathbb{P}^{k+1}(T) \subsetneq \mathcal{V}_T := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^l(T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

(original ncVEM devising with $l = k - 1$, $k \geq 1$)

- HHO dof space $\hat{U}_T$ with $l := k - 1$ isomorphic to virtual space $\mathcal{V}_T$
  - virtual reconstruction operator $\mathcal{R}_T : \hat{U}_T \to \mathcal{V}_T$
  - $\hat{\mathcal{J}}_T : \mathcal{V}_T \to \hat{U}_T$: restriction of reduction operator to virtual space
  - then, $\hat{\mathcal{J}}_T \circ \mathcal{R}_T = I_{\hat{U}_T}$ and $\mathcal{R}_T \circ \hat{\mathcal{J}}_T = I_{\mathcal{V}_T}$

Nonconforming virtual elements

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- Stabilization controls energy-norm of noncomputable remainder
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  - purely algebraic stab. from ncVEM could be explored in HHO
- Further link to Multiscale Hybrid Mixed (MHM methods) [Chaumont, AE, Lemaire, Valentin 22]

Wave propagation problems

- Second-order formulation in time: Newmark schemes
- First-order formulation in time: Runge–Kutta (RK) schemes
- [Burman, Duran, AE 22 (CAMC, CMAME)], [Burman, Duran, AE, Steins 21 (JSC)], [Steins, AE, Jamond, Drui 23 (M2AN)]

Acoustic wave equation

- Domain $\Omega \subset \mathbb{R}^d$, time interval $J := (0, T_\mathrm{f})$, $T_\mathrm{f} > 0$
- Acoustic wave equation with wave speed $c := \sqrt{\kappa/\rho}$

$$(\partial_{tt} p(t), w)_{\frac{1}{\kappa};\Omega} + (\nabla p(t), \nabla w)_{\frac{1}{\rho};\Omega} = (f(t), w)_\Omega, \quad \forall w \in H_0^1(\Omega)\, \forall t \in J$$

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$$\mathfrak{E}(t) := \tfrac{1}{2}\|\partial_t p(t)\|^2_{\frac{1}{\kappa};\Omega} + \tfrac{1}{2}\|\nabla p(t)\|^2_{\frac{1}{\rho};\Omega}$$

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- Everything can be extended to elastodynamics

- Local cell dofs in $\mathbb{P}^l(T)$, $l \in \{k, k+1\}$, and local face dofs in $\mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^l(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Local gradient reconstruction $\mathbf{G}_T(\hat{u}_T) \in \mathbb{P}^k(T;\mathbb{R}^d)$ (or in $\nabla\mathbb{P}^{k+1}(T)$)

- Local stabilization acting on $\delta_{\hat{u}_T} := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\delta_{\hat{u}_T}) := \begin{cases} \Pi^k_{\partial T}\big(\delta_{\hat{u}_T} - \big((I - \Pi^k_T)R_T(0,\delta_{\hat{u}_T})\big)|_{\partial T}\big) & \text{if } l = k \\ \Pi^k_{\partial T}(\delta_{\hat{u}_T}) & \text{if } l = k+1 \end{cases}$$

- Local bilinear form (with $\tau_{\partial T} := (\rho_{|T} h_T)^{-1}$)

$$a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_T(\hat{u}_T), \mathbf{G}_T(\hat{w}_T))_{\frac{1}{\rho};T} + \tau_{\partial T}(S_{\partial T}(\delta_{\hat{u}_T}), S_{\partial T}(\delta_{\hat{w}_T}))_{\partial T}$$

- Global bilinear form a_h on HHO space $\hat{U}_{h0}$ (with Dirichlet BCs)

HHO space semi-discretization

- Space semi-discrete form: Find $\hat{p}_h \in C^2(\overline{J}; \hat{U}_{h0})$ s.t.

$$(\partial_{tt} p_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + a_h(\hat{p}_h(t), \hat{w}_h) = (f(t), w_{\mathcal{T}})_{\Omega}, \quad \forall \hat{w}_h \in \hat{U}_{h0}\, \forall t \in J$$

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$$\mathfrak{E}_h(t) := \frac{1}{2}\|\partial_t p_{\mathcal{T}}(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t))\|^2_{\frac{1}{\rho};\Omega} + \frac{1}{2} s_h(\hat{p}_h(t), \hat{p}_h(t))$$

Stabilization is taken into account in the energy definition

- HDG methods for wave equation in second-order form [Cockburn, Fu, Hungria, Ji, Sanchez, Sayas 18]

Algebraic realization

- Bases for $\mathbb{P}^l(\mathcal{T})$ and $\mathbb{P}^k(\mathcal{F}) \Longrightarrow$ vectors $(\mathsf{P}_{\mathcal{T}}(t), \mathsf{P}_{\mathcal{F}}(t)) \in \mathbb{R}^{N_{\mathcal{T}}} \times \mathbb{R}^{N_{\mathcal{F}}}$

$$\begin{bmatrix} \mathsf{M}_{\mathcal{T}\mathcal{T}} \partial_{tt} \mathsf{P}_{\mathcal{T}}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \mathsf{A}_{\mathcal{T}\mathcal{T}} & \mathsf{A}_{\mathcal{T}\mathcal{F}} \\ \mathsf{A}_{\mathcal{F}\mathcal{T}} & \mathsf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{P}_{\mathcal{T}}(t) \\ \mathsf{P}_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}}(t) \\ 0 \end{bmatrix}$$

- Mass matrix $\mathsf{M}_{\mathcal{T}\mathcal{T}}$ and stiffness submatrix $\mathsf{A}_{\mathcal{T}\mathcal{T}}$ are block-diagonal
- Stiffness submatrix $\mathsf{A}_{\mathcal{F}\mathcal{F}}$ is only sparse: face dofs from the same cell are coupled together owing to reconstruction

Error analysis

- [Burman, Duran, AE, Steins 21 (JSC)] proves (for smooth solutions)
  - $\|\partial_t p - \partial_t p_{\mathcal{T}}\|_{L^\infty(J;L^2(\frac{1}{\kappa};\Omega))} + \|\nabla p - \mathbf{G}_{\mathcal{T}}(\hat{p}_h)\|_{L^2(J;L^2(\frac{1}{\rho};\Omega))} \lesssim h^{k+1}$
  - $\|\Pi^l_{\mathcal{T}}(p) - p_{\mathcal{T}}\|_{L^\infty(J;L^2(\frac{1}{\rho};\Omega))} \lesssim h^{k+2}$ (under full elliptic regularity)
- Some comments on proofs
  - adapt ideas from FEM analysis [Dupont 73; Wheeler 73; Baker 76]
  - simpler than HDG (which needs special initialization)
  - applies to DG using discr. gradients (revisit [Grote, Schneebeli, Schötzau 06])

Newmark schemes

- Newmark scheme with parameters $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$
  - implicit, second-order, unconditionally stable
  - p, $\partial_t p$, $\partial_{tt} p$ are approximated by hybrid pairs $\hat{p}_h^n, \hat{v}_h^n, \hat{a}_h^n \in \hat{U}_{h0}$, $\forall n \geq 0$

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- Discrete energy is exactly conserved
- **Improvements on leapfrog scheme** [Steins, AE, Jamond, Drui 23 (M2AN)]
  - plain leapfrog not efficient: needs inverting stiffness submatrix $\mathsf{A}_{\mathcal{F}\mathcal{F}}$
  - one can use an iterative method exploiting bock-diagonal structure of face-face penalty submatrix
  - convergence guaranteed if stabilization scaled with large enough weight
    - sharp estimate depending on trace inequality constant (h-independent)
    - mild impact on CFL condition despite increased stiffness (up to factor of 2)
  - computational performances
    - close-to-optimal value of weight easy to set
    - generally outperforms plain leapfrog, especially for nonlinear problems
    - mixed-order HHO setting more efficient than equal-order

First-order formulation in time

- Introduce velocity $v := \partial_t p$ and dual variable $\boldsymbol{\sigma} := \frac{1}{\rho}\nabla p$
- Weak form: $\forall(\boldsymbol{\tau}, w) \in L^2(\Omega;\mathbb{R}^d) \times H^1_0(\Omega)$, $\forall t \in J$,

$$\begin{cases} (\partial_t \boldsymbol{\sigma}(t), \boldsymbol{\tau})_{\rho;\Omega} - (\nabla v(t), \boldsymbol{\tau})_\Omega = 0 & \leftrightarrow \quad \rho\partial_t\boldsymbol{\sigma} - \nabla v = 0 \\ (\partial_t v(t), w)_{\frac{1}{\kappa};\Omega} + (\boldsymbol{\sigma}(t), \nabla w)_\Omega = (f(t), w)_\Omega & \leftrightarrow \quad \frac{1}{\kappa}\partial_t v - \operatorname{div}\boldsymbol{\sigma} = f \end{cases}$$

- Energy balance: $\mathfrak{E}(t) = \mathfrak{E}(0) + \int_0^t (f(s), v(s))_\Omega ds$ with

$$\mathfrak{E}(t) := \frac{1}{2}\|v(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\boldsymbol{\sigma}(t)\|^2_{\rho;\Omega}$$

HHO space semi-discretization

- $\hat{v}_h \in C^1(\bar{J}; \hat{U}_{h0})$ and $\boldsymbol{\sigma}_{\mathcal{T}} \in C^1(\bar{J}; \mathbf{S}_{\mathcal{T}})$ with $\mathbf{S}_{\mathcal{T}} := \mathbb{P}^k(\mathcal{T}; \mathbb{R}^d)$
- Space semi-discrete form:

$$\begin{cases} (\partial_t \boldsymbol{\sigma}_{\mathcal{T}}(t), \boldsymbol{\tau}_{\mathcal{T}})_{\rho;\Omega} - (\mathbf{G}_{\mathcal{T}}(\hat{v}_h(t)), \boldsymbol{\tau}_{\mathcal{T}})_{\Omega} = 0 \\ (\partial_t v_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + (\boldsymbol{\sigma}_{\mathcal{T}}(t), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\Omega} + \tilde{s}_h(\hat{v}_h(t), \hat{w}_h) = (f(t), w_{\mathcal{T}})_{\Omega} \end{cases}$$

- Stabilization $\tilde{s}_h(\cdot,\cdot)$ with weight $\tilde{\tau}_{\partial T} = O(h_T^{-\alpha})$, one takes $\alpha \in \{0, 1\}$

HHO space semi-discretization

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- Stabilization $\tilde{s}_h(\cdot,\cdot)$ with weight $\tilde{\tau}_{\partial T} = O(h_T^{-\alpha})$, one takes $\alpha \in \{0, 1\}$
- Energy balance: $\mathfrak{E}_h(t) := \frac{1}{2}\|v_{\mathcal{T}}(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\boldsymbol{\sigma}_{\mathcal{T}}(t)\|^2_{\rho;\Omega}$

$$\mathfrak{E}_h(t) + \int_0^t \tilde{s}_h(\hat{v}_h(s), \hat{v}_h(s))ds = \mathfrak{E}_h(0) + \int_0^t (f(s), v_{\mathcal{T}}(s))_\Omega ds$$

Stabilization acts as a dissipative mechanism

- HDG methods for wave equation in first-order form [Nguyen, Peraire, Cockburn 11; Stranglmeier, Nguyen, Peraire, Cockburn 16]

Algebraic realization

- Component vectors $\mathsf{Z}_{\mathcal{T}}(t) \in \mathbb{R}^{M_{\mathcal{T}}}$ and $(\mathsf{V}_{\mathcal{T}}(t), \mathsf{V}_{\mathcal{F}}(t)) \in \mathbb{R}^{N_{\mathcal{T}} \times N_{\mathcal{F}}}$

$$\begin{bmatrix} \mathsf{M}^{\sigma}_{\mathcal{T}\mathcal{T}} \partial_t \mathsf{Z}_{\mathcal{T}}(t) \\ \mathsf{M}_{\mathcal{T}\mathcal{T}} \partial_t \mathsf{V}_{\mathcal{T}}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathsf{G}_{\mathcal{T}} & -\mathsf{G}_{\mathcal{F}} \\ \mathsf{G}^{\dagger}_{\mathcal{T}} & \mathsf{S}_{\mathcal{T}\mathcal{T}} & \mathsf{S}_{\mathcal{T}\mathcal{F}} \\ \mathsf{G}^{\dagger}_{\mathcal{F}} & \mathsf{S}_{\mathcal{F}\mathcal{T}} & \mathsf{S}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{Z}_{\mathcal{T}}(t) \\ \mathsf{V}_{\mathcal{T}}(t) \\ \mathsf{V}_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathsf{F}_{\mathcal{T}}(t) \\ 0 \end{bmatrix}$$

- Mass matrices $\mathsf{M}^{\sigma}_{\mathcal{T}\mathcal{T}}$ and $\mathsf{M}_{\mathcal{T}\mathcal{T}}$ are block-diagonal
- Key point: stab. submatrix $\mathsf{S}_{\mathcal{F}\mathcal{F}}$ block-diagonal only if $l = k + 1$
  - for $l = k$, high-order HHO correction in stabilization destroys this property (couples all faces of the same cell)
  - mixed-order HHO setting recommended for explicit schemes!

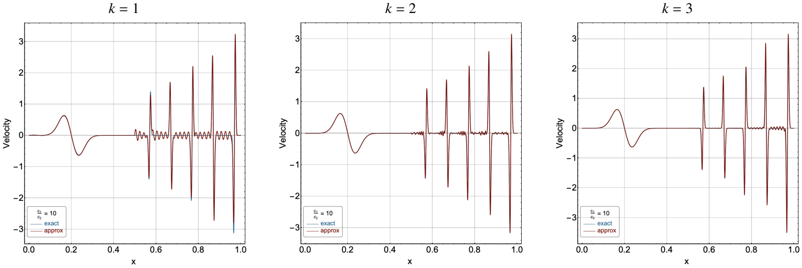
Runge–Kutta (RK) schemes

- Natural choice for first-order formulation in time
  - single diagonally implicit RK: SDIRK($s, s+1$) (s stages, order $(s+1)$)
  - explicit RK: ERK(s) (s stages, order s)

- ERK schemes subject to CFL stability condition $\frac{c\Delta t}{h} \leq \beta(s)\mu(k)$
  - $\beta(s)$ slightly increases with $s \in \{2, 3, 4\}$
  - $\mu(k)$ essentially behaves as $(k+1)^{-1}$ w.r.t. polynomial degree

1D heterogeneous media

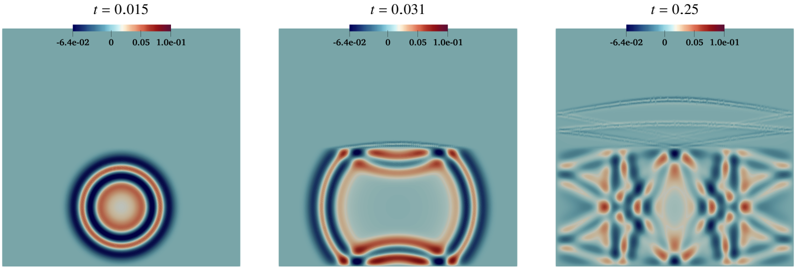
- 1D test case, $\Omega_1 = (0, 0.5)$, $\Omega_2 = (0.5, 1)$, $c_1/c_2 = 10$
  - initial Gaussian profile in Ω_1
  - analytical solution available (series)
- Benefits of increasing polynomial degree
  - Newmark scheme, equal-order, $k \in \{1, 2, 3\}$, $h = 0.1 \times 2^{-8}$, $\Delta t = 0.1 \times 2^{-9}$
  - HHO-Newmark solution at $t = \frac{1}{2}$ (after reflection/transmission at $x = \frac{1}{2}$)



2D heterogeneous media

- 2D test case, Ricker (Mexican hat) wavelet
  - $\Omega_1 = (0,1) \times (0, \frac{1}{2})$, $\Omega_2 = (0,1) \times (\frac{1}{2}, 1)$, $c_1/c_2 = 5$
  - $p_0 = 0$, $v_0 = -\frac{4}{10}\sqrt{\frac{10}{3}}\left(1600\, r^2 - 1\right) \pi^{-\frac{1}{4}} \exp\left(-800 r^2\right)$,
    $r^2 = (x - x_c)^2 + (y - y_c)^2$, $(x_c, y_c) = (\frac{1}{2}, \frac{1}{4}) \in \Omega_1$
  - semi-analytical solution (infinite media): `gar6more2d` software (INRIA)

- HHO-SDIRK(3,4) velocity profiles
  - mixed-order, $k = 5$, polygonal meshes
  - $\Delta t = 0.025 \times 2^{-6}$ (four times larger than Newmark for similar accuracy)



Wave propagation across interface

- Subdomains $\Omega_1, \Omega_2 \subset \Omega$, interface Γ, jump $[\![a]\!]_\Gamma = a_{|\Omega_1} - a_{|\Omega_2}$
- Acoustic wave propagation across interface

$$\begin{cases} \frac{1}{\kappa}\partial_{tt}p - \mathrm{div}\left(\frac{1}{\rho}\nabla p\right) = f & \text{in } J \times (\Omega_1 \cup \Omega_2) \\ [\![p]\!]_\Gamma = 0, \ [\![\frac{1}{\rho}\nabla p]\!]_\Gamma \cdot \mathbf{n}_\Gamma = 0 & \text{on } J \times \Gamma \end{cases}$$

- Use main ideas from elliptic interface problems
  - mixed-order setting $l = k + 1$
  - distinct gradient reconstructions $\mathbf{G}_{T_i}$ in $\mathbb{P}^k(T_i;\mathbb{R}^d)$, $i \in \{1,2\}$
  - $O(1)$ penalty parameter
  - LS stabilization on $(\partial T)^i$, $i \in \{1,2\} \Longrightarrow s_{T_i}(\cdot,\cdot)$
- Unfitted HHO-Newmark, ERK and SDIRK available

Fitted-unfitted comparison

- 2D heterogeneous test case with **flat interface**
  - $\Omega_1 := (-\frac{3}{2}, \frac{3}{2}) \times (-\frac{3}{2}, 0)$, $\Omega_2 := (-\frac{3}{2}, \frac{3}{2}) \times (0, \frac{3}{2})$
  - Ricker wavelet centered at $(0, \frac{2}{3}) \in \Omega_2$, sensor $S_1 = (\frac{3}{4}, -\frac{1}{3}) \in \Omega_1$
  - fitted and unfitted HHO behave similarly, both benefit from increasing k

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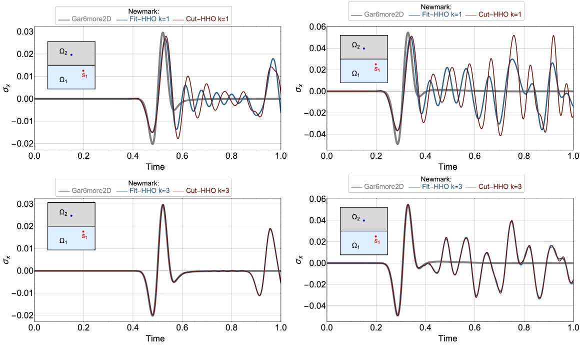
  - fitted and unfitted HHO behave similarly, both benefit from increasing k
- HHO-Newmark, σ_x signals
  - comparison of semi-analytical and HHO (fitted or unfitted) solutions
  - $k = 1$ (top) and $k = 3$ (bottom)
  - $c_2/c_1 = \sqrt{3}$ (low contrast, left) or $c_2/c_1 = 8\sqrt{3}$ (high contrast, right)

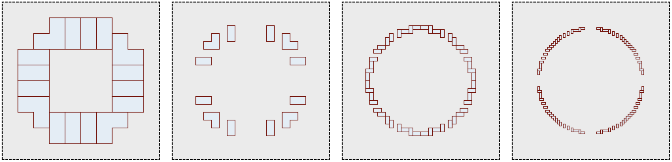


CFL condition for ERK (1/2)

- Homogeneous test case, flat interface
- CFL condition for ERK(s): $\frac{c\Delta t}{h} \leq \beta(s)\mu(k)$
  - $\beta(s)$ mildly depends on the number of stages
  - $\mu(k)$ behaves as $(k+1)^{-1}$ and is quantified by solving a generalized eigenvalue problem with the mass and stiffness matrices
- Additional jump penalties in unfitted HHO only mildly impact $\mu(k)$

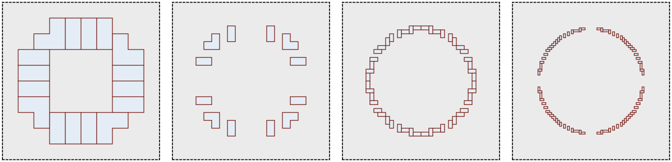
k	0	1	2	3
Fitted-HHO	0.118	0.0522	0.0338	0.0229
Unfitted-HHO	0.0765	0.0373	0.0232	0.0159
Ratio	1.5	1.4	1.5	1.4

- Homogeneous test case, circular interface
  - study of impact of agglomeration parameter θ_{agg} on $\mu(k)$
  - "ill cut" cells flagged if relative area of any subcell falls below θ_{agg}
- Agglomerated cells for $\theta_{agg} = 0.3$ on a sequence of refined quad meshes

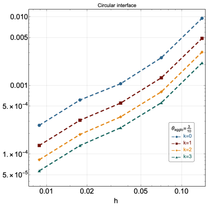


CFL condition for ERK (2/2)

- Homogeneous test case, circular interface
  - study of impact of agglomeration parameter θ_{agg} on $\mu(k)$
  - "ill cut" cells flagged if relative area of any subcell falls below θ_{agg}
- Agglomerated cells for $\theta_{agg} = 0.3$ on a sequence of refined quad meshes



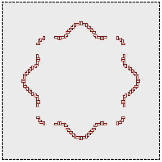
- Behavior of $h\mu(k)$ and impact of θ_{agg} on $\mu(k)$
  - tolerating ill cut cells deteriorates the CFL condition

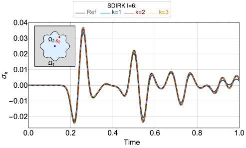


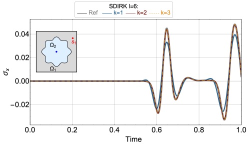
k	0	1	2	3
$\theta_{agg} = 0.5$	0.042	0.022	0.014	0.0099
$\theta_{agg} = 0.3$	0.030	0.015	0.0094	0.0065
Ratio	1.4	1.5	1.5	1.5
$\theta_{agg} = 0.1$	0.017	0.0087	0.0055	0.0039
Ratio	2.5	2.6	2.6	2.5

Flower-like interface

- Agglomerated cells for a flower-like interface (quad mesh, $h = 2^{-5}$), HHO-SDIRK(3,4) signal for σ_x at two sensors, $k \in \{1, 2, 3\}$, $c_2/c_1 = \sqrt{3}$

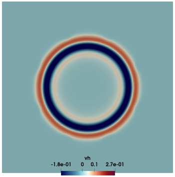




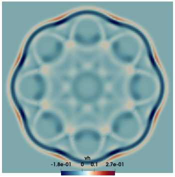


- Pressure isovalues, SDIRK(3,4), $k = 3$, $h = 0.1 \times 2^{-8}$, $\Delta t = 2^{-6}$

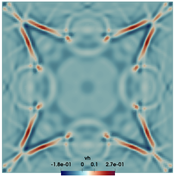
$t = 0.25$



$t = 0.5$



$t = 1$



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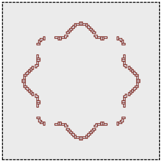

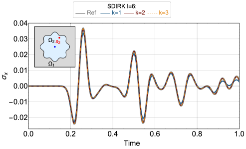

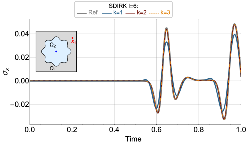

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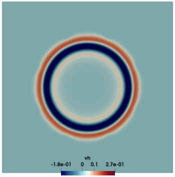
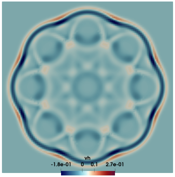


$t = 0.5$



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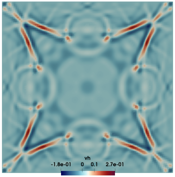


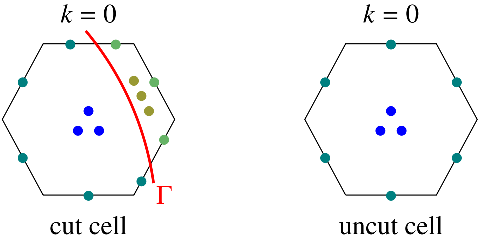
!! Thank you for your attention !!

Competition: Newmark vs. RK

- All schemes deliver same max. rel. error on a sensor at $(\frac{1}{2}, \frac{2}{3})$
- Disclaimer: preliminary results! (off-the-shelf solvers)
- If no direct solvers allowed, ERK(4) wins despite CFL restriction
- With direct solvers, SDIRK(3,4) wins
- RK schemes more efficient than Newmark scheme
- for SDIRK(3,4), $\tilde{\tau}_{\partial T} = O(h_T^{-\alpha})$, $\alpha = 1$ more accurate/expensive than $\alpha = 0$

scheme	(l, k)	α	solver	t/step	steps	time	err
ERK(4)	$(6, 5)$	0	n/a	0.410	5,120	2,099	2.23
Newmark	$(7, 6)$	1	iter	56.74	2,560	58,265	2.15
SDIRK$(3, 4)$	$(6, 5)$	1	iter	31.24	640	5,639	2.21
SDIRK$(3, 4)$	$(6, 5)$	0	iter	22.52	640	2,200	4.45
Newmark	$(7, 6)$	1	direct	0.515	2,560	1,318	2.15
SDIRK$(3, 4)$	$(6, 5)$	1	direct	1.579	640	1,010	2.21

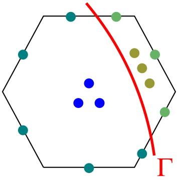
Local dofs



- Mesh still composed of polygonal cells (with planar faces)
- Decomposition of cut cells: $\overline{T} = \overline{T_1} \cup \overline{T_2}$, $T^\Gamma = T \cap \Gamma$
- Decomposition of cut faces: $\partial(T_i) = (\partial T)^i \cup T^\Gamma$, $i \in \{1, 2\}$
- Local dofs (no dofs on T^Γ!)

$$\hat{u}_T = (u_{T_1}, u_{T_2}, u_{(\partial T)^1}, u_{(\partial T)^2}) \in \mathbb{P}^{k+1}(T_1) \times \mathbb{P}^{k+1}(T_2) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^2})$$

Gradient reconstruction in cut cells



- Gradient reconstruction $\mathbf{G}_{T_i}(\hat{u}_T) \in \mathbb{P}^k(T_i;\mathbb{R}^d)$ in each subcell
  - **(Option 1)** Independent reconstruction in each subcell

$$(\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\,\mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{T_i}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

  - **(Option 2)** Reconstruction mixing data from both subcells

$$(\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\,\mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{T_{3-i}}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

- Both options avoid Nitsche's consistency terms
  - $O(1)$ penalty parameter

Local bilinear form in cut cells

- Local bilinear form

$$a_T(\hat{u}_T, \hat{w}_T) := \sum_{i\in\{1,2\}} \left\{ \kappa_i (\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{G}_{T_i}(\hat{w}_T))_{T_i} + s_{T_i}(\hat{u}_T, \hat{w}_T) \right\} + s_T^\Gamma(u_T, w_T)$$

- LS stabilization inside each subdomain

$$s_{T_i}(\hat{u}_T, \hat{w}_T) := \kappa_i h_{T_i}^{-1} (\Pi^k_{(\partial T)^i}(\delta_{\hat{u}_{T_i}}), \delta_{\hat{w}_{T_i}})_{(\partial T)^i}$$

- Interface bilinear form

$$s_T^\Gamma(u_T, w_T) := \eta \kappa_1 h_T^{-1} ([\![u_T]\!]_\Gamma, [\![w_T]\!]_\Gamma)_{T^\Gamma} \text{ with } \eta = O(1)$$

- The use of two gradient reconstructions allows for robustness w.r.t. contrast ($\kappa_1 \ll \kappa_2$)
  - use option 1 in Ω_1 and option 2 in Ω_2
  - a_T is symmetric, but Ω_1/Ω_2 do not play symmetric roles

Error analysis

- Multiplicative and discrete trace inequalities [Burman, AE 18]
  - for any cut cell T, there is a ball $T^\dagger$ of size $O(h_T)$ containing T and a finite number of its neighbors, and s.t. all $T \cap \Gamma$ is visible from a point in $T^\dagger$
  - small ball with diameter $O(h_T)$ present on both sides of interface
  - achievable using local cell agglomeration if mesh fine enough

Error estimate

Assuming that $u|_{\Omega_i} \in H^{1+t}(\Omega_i)$ with $t \in (\frac{1}{2}, k+1]$,

$$\sum_T \sum_{i\in\{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|^2_{T_i} \leq Ch^{2t} \sum_{i\in\{1,2\}} \kappa_i |u|^2_{H^{t+1}(\Omega_i)}$$

Convergence order $O(h^{k+1})$ if $u|_{\Omega_i} \in H^{k+2}(\Omega_i)$