Convergence of ERK-dG approximations of the first-order form of Maxwell's equations with low regularity

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Outline

- Setting and examples
- Main result
- Discontinuous Galerkin (dG) space discretization
- Explicit Runge–Kutta (ERK) time discretization
- Convergence proof (sketch)
- References
  - [AE & JLG, hal-05142005, 2025]
  - [AE & JLG, SINUM, **63**, 661-684, 2025]

Setting and examples

- Abstract setting
- Example 1: Maxwell's equations
- Example 2: Acoustic wave equation

Abstract setting

- Hilbert space L
- Unbounded linear operator $A : D(A) \subset L \to L$, maximal, monotone
- Evolution problem

$$\partial_t u + A(u) = 0 \qquad u(0) = u^0 \in D(A)$$

Abstract setting

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- Unbounded linear operator $A : D(A) \subset L \longrightarrow L$, maximal, monotone
- Evolution problem

$$\partial_t u + A(u) = 0 \qquad u(0) = u^0 \in D(A)$$

- **Hille–Yosida Theorem:**

$$\exists! u \in C^0([0, +\infty); D(A)) \cap C^1([0, +\infty); L)$$

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- **Additional structural property:** Dense subspace with compact embedding, $H^s \Subset L$, s.t.

$$X := \operatorname{im}(A) \cap D(A) \subset H^s$$

$D(A)$ **does not embed compactly in** L**!**

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- **Main conclusion:** If $u^0 \in X$,

$$u(t) \in X \subset H^s \quad \forall t \geq 0$$

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- Recall $u \in C^0([0, +\infty); D(A)) \cap C^1([0, +\infty); L)$ (Hille–Yosida)
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- Proof by **energy techniques (a priori estimates)**
  - take one time derivative of (*) and test with $\partial_t u$
  - take one more time derivative of (*) and test with $\partial_{tt} u$

Example 1: Maxwell's equations

- Space-time PDEs posed on $D \times J$ with $D \subset \mathbb{R}^3$, $J := (0, T)$
- Given ICs $(\boldsymbol{H}^0, \boldsymbol{E}^0)$, find $(\boldsymbol{H}, \boldsymbol{E}) : D \times J \to \mathbb{R}^3 \times \mathbb{R}^3$ s.t.

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\begin{aligned}
&\partial_t(\mu \boldsymbol{H}) + \nabla\times\boldsymbol{E} = \boldsymbol{0} && \text{(Faraday)}\\
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- Observe that $\partial_t(\nabla \cdot (\mu \boldsymbol{H})) = 0$ and $\partial_t(\nabla \cdot (\epsilon \boldsymbol{E})) = -\nabla \cdot \boldsymbol{j} = \partial_t \rho$
  - if $\nabla \cdot (\mu \boldsymbol{H}^0) = 0$, then $\nabla \cdot (\mu \boldsymbol{H}) = 0$ at all times
  - if $\nabla \cdot (\epsilon \boldsymbol{E}^0) = \rho_0$, then $\nabla \cdot (\epsilon \boldsymbol{E}) = \rho$ at all times

  One says that Gauss's laws are involutions

Functional setting

- Graph spaces for gradient, curl, or divergence

$$H^1(D) = H(\mathbf{grad}; D), \quad \boldsymbol{H}(\mathbf{curl}; D) := \{\boldsymbol{h} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{h} \in \boldsymbol{L}^2(D)\}, \quad \boldsymbol{H}(\mathrm{div}; D)$$

- Hilbert spaces equipped with natural graph norm, e.g.,

$$\|\boldsymbol{h}\|^2_{\boldsymbol{H}(\mathbf{curl};D)} := \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2} + \ell_D^2 \|\nabla\times\boldsymbol{h}\|^2_{\boldsymbol{L}^2}$$

(ℓ_D: global length scale to be dimensionally consistent)

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- De Rham sequences (with and without BC)

$$H_0^1(D) \xrightarrow{\nabla_0} \boldsymbol{H}_0(\mathbf{curl}; D) \xrightarrow{\nabla_0\times} \boldsymbol{H}_0(\mathrm{div}; D) \xrightarrow{\nabla_0\cdot} L_0^2(D)$$

$$L^2(D) \xleftarrow{\nabla\cdot} \boldsymbol{H}(\mathrm{div}; D) \xleftarrow{\nabla\times} \boldsymbol{H}(\mathbf{curl}; D) \xleftarrow{\nabla} H^1(D)$$

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- All operators have closed range

- Pairs of adjoint operators: $(\nabla_0, -\nabla\cdot)$, $(\nabla_0\times, \nabla\times)$, $(\nabla_0\cdot, -\nabla)$

Rewriting in Hille–Yosida framework

- To fix ideas, enforce zero Dirichlet BC on magnetic field

$$\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl}; D), \quad \boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}; D)$$

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$$\partial_t(\mu \boldsymbol{H}) + \nabla \times \boldsymbol{E} = \mathbf{0}$$
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$$u := (\boldsymbol{H}, \boldsymbol{E}) \qquad A(u) := \left(\tfrac{1}{\mu}\nabla{\times}\boldsymbol{E}, -\tfrac{1}{\epsilon}\nabla_0{\times}\boldsymbol{H}\right)$$
$$L := \boldsymbol{L}^2(D){\times}\boldsymbol{L}^2(D) \qquad D(A) := \boldsymbol{H}_0(\mathbf{curl}; D){\times}\boldsymbol{H}(\mathbf{curl}; D)$$

with $((\boldsymbol{H}, \boldsymbol{E}), (\boldsymbol{h}, \boldsymbol{e}))_L := (\mu \boldsymbol{H}, \boldsymbol{h})_{\boldsymbol{L}^2(D)} + (\epsilon \boldsymbol{E}, \boldsymbol{e})_{\boldsymbol{L}^2(D)}$

Functional setting for involutions

- Rewriting of involutions using Closed Range Theorem (orthogonalities meant in $\boldsymbol{L}^2$) [Hiptmair 02]

$$\mu \boldsymbol{H} \in \operatorname{im}(\nabla\times) = \ker(\nabla_0\times)^{\perp}, \quad \epsilon \boldsymbol{E} \in \operatorname{im}(\nabla_0\times) = \ker(\nabla\times)^{\perp}$$

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- **Topology-blind statements!**
  - $\ker(\nabla_0\times)^\perp \subset \ker(\nabla\cdot)$ with equality iff $\Gamma := \partial D$ is connected
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See [Dautray, Lions 90; Amrouche, Bernardi, Dauge, Girault, 98]

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$$\boldsymbol{X}_{\mu 0} := \{\boldsymbol{h} \in \boldsymbol{H}_0(\mathbf{curl}; D) \mid \mu \boldsymbol{h} \in \ker(\nabla_0\times)^\perp\}$$
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- Assuming $(\mu \boldsymbol{H}^0, \epsilon \boldsymbol{E}^0) \in \operatorname{im}(\nabla\times) \times \operatorname{im}(\nabla_0\times)$, we have

$$(\boldsymbol{H}(t), \boldsymbol{E}(t)) \in X := \boldsymbol{X}_{\mu 0} \times \boldsymbol{X}_{\epsilon} \quad \forall t \geq 0$$

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See [Weber, 80; Birman & Solomyak, 87; Costabel, 90; Amrouche, Bernardi, Dauge, Girault, 98; Jochmann, 99; Bonito, Guermond & Luddens, 13]

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$$(\partial_t \boldsymbol{H}, \partial_t \boldsymbol{E}) \in L^\infty(J; \boldsymbol{H}^s(D) \times \boldsymbol{H}^s(D))$$
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- Given ICs $(p^0, \boldsymbol{v}^0)$, find $(p, \boldsymbol{v}) : D \times J \to \mathbb{R} \times \mathbb{R}^3$ s.t.

$$\partial_t(\kappa p) + \nabla \cdot \boldsymbol{v} = 0, \quad \partial_t(\rho \boldsymbol{v}) + \nabla p = \mathbf{0}$$

- Material properties: κ (compressibility), ρ (density)

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$$u := (p, \boldsymbol{v}) \qquad A(u) := (\tfrac{1}{\kappa}\nabla_0{\cdot}\boldsymbol{v}, \tfrac{1}{\rho}\nabla p)$$

$$L := L^2(D){\times}\boldsymbol{L}^2(D) \qquad D(A) = H^1(D){\times}\boldsymbol{H}_0(\mathrm{div}; D)$$

with $((p, \boldsymbol{v}), (q, \boldsymbol{w}))_L := (\kappa p, q)_{L^2(D)} + (\rho \boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{L}^2(D)}$

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- **Minimal regularity setting** (well prepared IC)

$$(p, \boldsymbol{v}) \in L^\infty(J; H^1(D)\times\boldsymbol{H}^s(D))$$
$$(\partial_t p, \partial_t \boldsymbol{v}) \in L^\infty(J; H^1(D)\times\boldsymbol{H}^s(D))$$
$$(\partial_{tt} p, \partial_{tt} \boldsymbol{v}) \in L^\infty(J; L^2(D)\times\boldsymbol{L}^2(D))$$

Main result

- Recall $\partial_t u + A(u) = 0$ and $X := \mathrm{im}(A) \cap D(A) \subset H^s$
- Recall minimal regularity setting: For all $T > 0$ with $J := (0, T)$,

$$u \in L^\infty(J; H^s), \quad \partial_t u \in L^\infty(J; H^s), \quad \partial_{tt} u \in L^\infty(J; L)$$

Space semi-discretization

- Finite dimensional subspace $L_h \subset L$ (h: mesh size)
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- For Maxwell's and acoustic wave equations, these properties can be realized using dG with stabilization (upwind) on simplicial meshes with $s \in (0, \frac{1}{2}]$ [AE & JLG 25]

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- **Comments:**
  - convergence rate $O(\tau + h^s)$, optimal rate given regularity of u
  - bound grows linearly in time
  - Sobolev regularity index can be in $(0, \frac{1}{2}]$
  - so far, restriction $s > \frac{1}{2}$ in the literature (unrealistic for Maxwell's equations with heterogeneous materials)
    [Ciarlet & Zou 99], [Fezoui, Lanteri, Lohrengel & Piperno 05], [Li 09]

dG space discretization

- Focus on Maxwell's equations
- Discrete curls and stabilization
- Ritz projection

Mesh and broken polynomial spaces

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- **Nonconforming approximation:** $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \not\subset \boldsymbol{H}_0(\mathbf{curl}; D)$
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- dG textbooks: [Hesthaven & Warburton 08; Di Pietro & AE, 12]
- dG methods for Maxwell's equations:
  [Perugia, Schötzau, Monk 02; Houston, Perugia, Schneebeli, Schötzau 05]
  [Chaumont-Frelet, AE 25]

Jumps and stabilization

- **Mesh faces** $F \in \mathcal{F}_h$
  - interface $F = \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$, unit normal $\boldsymbol{n}_F$ pointing from K_l to K_r
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- **Stabilization bilinear forms**

$$s_h^{\mathrm{H}}(\boldsymbol{H}_h, \boldsymbol{h}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} \qquad s_h^{\mathrm{E}}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h^\circ} ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}$$

  Jump seminorms: $|\boldsymbol{h}_h|_h^{\mathrm{H}} := s_h^{\mathrm{H}}(\boldsymbol{h}_h, \boldsymbol{h}_h)^{\frac{1}{2}}$, $|\boldsymbol{e}_h|_h^{\mathrm{E}} := s_h^{\mathrm{E}}(\boldsymbol{e}_h, \boldsymbol{e}_h)^{\frac{1}{2}}$

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- **Literature:**
  - Discrete gradient for elliptic problems: [Bassi et al., 97], [Brezzi et al., 00]
  - Weak consistency, compactness [Burman & AE, 08; Buffa & Ortner, 09; Di Pietro & AE, 09]
  - Maxwell's equations [Perugia, Schötzau & Monk, 02], [Chaumont-Frelet, AE 25]

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- **Space semi-discrete Maxwell's equations:** $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in C^1(\overline{J}; L_h)$ s.t.

$$\partial_t(\epsilon \boldsymbol{E}_h, \mu \boldsymbol{H}_h) + A_h(\boldsymbol{H}_h, \boldsymbol{E}_h) = \boldsymbol{0}, \quad (\boldsymbol{H}_h(0), \boldsymbol{E}_h(0)) = (\boldsymbol{H}_h^0, \boldsymbol{E}_h^0)$$

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- Discrete space $L_h := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$
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$$a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) := -(\boldsymbol{C}_h^k(\boldsymbol{E}_h), \boldsymbol{h}_h)_{\boldsymbol{L}^2} + (\boldsymbol{C}_{h,0}^k(\boldsymbol{H}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2} + \kappa_{\mathrm{H}} s_h^{\mathrm{H}}(\boldsymbol{H}_h, \boldsymbol{h}_h) + \kappa_{\mathrm{E}} s_h^{\mathrm{E}}(\boldsymbol{E}_h, \boldsymbol{e}_h)$$

$\Longrightarrow$ **positive weights mean upwinding!**

- **Discrete operator** $A_h : L_h \to L_h$ s.t.

$$\big(A_h(\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)_{\boldsymbol{L}^2\times\boldsymbol{L}^2} := a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)$$

- **Space semi-discrete Maxwell's equations:** $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in C^1(\overline{J}; L_h)$ s.t.

$$\partial_t(\epsilon \boldsymbol{E}_h, \mu \boldsymbol{H}_h) + A_h(\boldsymbol{H}_h, \boldsymbol{E}_h) = \boldsymbol{0}, \quad (\boldsymbol{H}_h(0), \boldsymbol{E}_h(0)) = (\boldsymbol{H}_h^0, \boldsymbol{E}_h^0)$$

- Assuming $(\boldsymbol{E}_h^0, \boldsymbol{H}_h^0) \in \mathrm{im}(A_h)$, we have

$$(\epsilon \boldsymbol{E}_h(t), \mu \boldsymbol{H}_h(t)) \in \mathrm{im}(A_h) = \ker(A_h^{\mathsf{T}})^{\perp} \quad \forall t \geq 0$$

$\Longrightarrow$ $\mathrm{im}(A_h)$ **is a genuine subset of** L_h**!**

Discrete involutions

- **Nédélec spaces** of order $k \geq 0$: $\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h)$, $\boldsymbol{P}^{\mathrm{c}}_{k}(\mathcal{T}_h)$ (with BCs or not)
- Curl-free subspaces **(they are non-trivial!)**

$$\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h) := \{\boldsymbol{h}_h \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h) \mid \nabla_0{\times}\boldsymbol{h}_h = \mathbf{0}\} = \boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h) \cap \ker(\nabla_0{\times})$$

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- **Lemma:** $\ker(A_h^{\mathsf{T}}) = \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h) \times \boldsymbol{P}^{\mathrm{c}}_{k}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)$
  $\Longrightarrow$ **upwinding used to prove this result!**
- **Discrete involution-aware subspaces:**

$$\boldsymbol{X}_{\mu0,h} := \{\boldsymbol{h}_h \in \boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h) \mid \mu\boldsymbol{h}_h \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^{\perp}\}$$

$$\boldsymbol{X}_{\epsilon,h} := \{\boldsymbol{e}_h \in \boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h) \mid \epsilon\boldsymbol{e}_h \in \boldsymbol{P}^{\mathrm{c}}_{k}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^{\perp}\}$$

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- **Discrete involutions:** Assuming $(\boldsymbol{H}_h(0), \boldsymbol{E}_h(0)) \in \boldsymbol{X}_{\mu 0,h} \times \boldsymbol{X}_{\epsilon,h}$,

$$(\boldsymbol{H}_h(t), \boldsymbol{E}_h(t)) \in \boldsymbol{X}_{\mu 0,h} \times \boldsymbol{X}_{\epsilon,h} \quad \forall t \geq 0$$

Ritz projection

- Recall inner product and norm in $L := \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$

$$\|(\boldsymbol{f}, \boldsymbol{g})\|_L^2 := \|\mu^{\frac{1}{2}}\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}}\boldsymbol{g}\|_{\boldsymbol{L}^2(D)}^2$$

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or, in more abstract way, $A_h \circ \Pi_h = \Pi_h^{\mathrm{b}} \circ A$

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- **Theorem:** [AE & JLG 25]

$$\|(\boldsymbol{H}, \boldsymbol{E}) - \Pi_h(\boldsymbol{H}, \boldsymbol{E})\|_L \leq c h^s \left\{ \mu_0^{\frac{1}{2}} |\boldsymbol{H}|_{\boldsymbol{H}^s(D)} + \epsilon_0^{\frac{1}{2}} |\boldsymbol{E}|_{\boldsymbol{H}^s(D)} \right\}$$

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- **Theorem:** [AE & JLG 25]

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- This result crucially relies on the discrete involutions (discrete Gauss's laws), which require upwinding
  - importance of upwinding already recognized in engineering literature [Hesthaven, Warburton 02], [Alvarez, Angulo, Rubio, Garcia 12]

ERK time discretization

- ERK3 and ERK4
- Stability result (abstract)
- Application to ERK3 and ERK4

ERK3 and ERK4

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$$u_h^{n+1} = P_3(-\tau A_h)(u_h^n), \qquad u_h^{n+1} = P_4(-\tau A_h)(u_h^n)$$

with $P_l(z) := \sum_{m\in\{0:l\}} \frac{1}{m!} z^m$ (Taylor approximation of e^z at order l)

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  - see also [Imperiale, Joly 25+] for some further advances

A quadratic identity

- **Lemma.** [AE & JLG 25+] Let $s \geq 1$ and (arbitrary) polynomial $\mathcal{P}(z) := \sum_{k\in\{0:s\}} a_k z^k$. Repeated integration by parts gives

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- $(a_k^L)_{k\in\{1:s\}}$ and $(a_{kl}^S)_{k,l\in\{0:s-1\}}$ s.t. $a_{kl}^S = a_{lk}^S$ and

$$a_k^L = a_k^2 + 2 \sum_{m=1}^{\min(k,s-k)} (-1)^m a_{k+m} a_{k-m}, \qquad \forall k \in \{1:s\}$$

$$a_{kl}^S = 2 \sum_{m=1}^{\min(l+1,s-k)} (-1)^{k+l+m} a_{k+m} a_{l-m+1}, \quad \forall k \in \{0:s-1\},\ \forall l \in \{0:k\}$$

Examples

- Taylor polynomials associated with ERK2, ERK3 and ERK4

$$\text{ERK2} \qquad a^L = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \qquad a^S = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{ERK3} \qquad a^L = \begin{pmatrix} 0 \\ -\frac{1}{12} \\ \frac{1}{36} \end{pmatrix} \qquad a^S = \begin{pmatrix} -2 & 1 & -\frac{1}{3} \\ 1 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

$$\text{ERK4} \qquad a^L = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{72} \\ \frac{1}{576} \end{pmatrix} \qquad a^S = \begin{pmatrix} -2 & 1 & -\frac{1}{3} & \frac{1}{12} \\ 1 & -\frac{2}{3} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{1}{4} & -\frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & -\frac{1}{12} & \frac{1}{24} & -\frac{1}{72} \end{pmatrix}$$

Abstract stability result

- **Lemma.** [AE & JLG 25+] Assume
  - A_h bounded & dissipative: $\|A_h(v_h)\|_L \le c_A h^{-1} \|v_h\|_L$, $(A_h(v_h), v_h)_L \ge 0$

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  - there is $r \ge 1$ s.t.

$$\text{(i)} \quad a_r^L < 0 \quad \& \quad a_l^L = 0 \ \ \forall l \in \{1{:}r-1\}$$

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Then we have

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- Conditions (i) and (ii) hold for $P_3(z)$ (ERK3) with $r = 2$
- Condition (i) holds for $P_4(z)$ (ERK4) with $r = 3$, but condition (ii) fails
- Conditions (i) and (ii) hold for $P_4(z)^2$ with $r = 3$

$$a^L = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{36} \\ \vdots \end{pmatrix} \qquad (a^S_{kl})_{k,l\in\{0:3\}} \propto \begin{pmatrix} -144 & 144 & -96 & 48 \\ 144 & -192 & 144 & -78 \\ -96 & 144 & -114 & 65 \\ 48 & -78 & 65 & -37 \end{pmatrix} \leq 0$$

Convergence proof (sketch)

- Focus on ERK3
- Discrete errors
- Residuals and error equations

Discrete errors

- Generic form for ERK3: Setting $u_h^{n,0} := u_h^n$, compute

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\begin{aligned}
u_h^{n,1} &= u_h^{n,0} - \tau A_h u_h^{n,0} \\
u_h^{n,2} &= u_h^{n,1} - \tfrac{1}{2}\tau A_h (u_h^{n,1} - u_h^{n,0}) \\
u_h^{n+1} &= u_h^{n,2} - \tfrac{1}{3}\tau A_h (u_h^{n,2} - u_h^{n,1})
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- Define discrete errors using **Ritz projection** Π_h

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\begin{aligned}
e_h^{n} &:= u_h^{n,0} - \Pi_h\big(u(t^n)\big) \\
e_h^{n,1} &:= u_h^{n,1} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big) \\
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- **Comments:**
  - no need to exploit Taylor expansions beyond first-order (low regularity)
  - only stability of ERK3 plays a role

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- **Comments:**
  - no need to exploit Taylor expansions beyond first-order (low regularity)
  - only stability of ERK3 plays a role
  - discrete errors defined using Π_h^{b}: optimal approximation properties clearly hold, but space consistency errors pollute error bound

Residuals and error equations

- The key property of the Ritz projection is

$$\Pi_h^{\mathrm{b}}\big(\partial_t u(t)\big) = -A_h\big(\Pi_h\big(u(t)\big)\big) \quad \forall t \geq 0$$

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- Simple algebra leads to the discrete error equations

$$\begin{aligned} e_h^{n,1} &= e_h^{n,0} - \tau A_h e_h^{n,0} + \tau R_h^{n,0} \\ e_h^{n,2} &= e_h^{n,1} - \tfrac{1}{2}\tau A_h(e_h^{n,1} - e_h^{n,0}) + \tau R_h^{n,1} \\ e_h^{n,3} &= e_h^{n,2} - \tfrac{1}{3}\tau A_h(e_h^{n,2} - e_h^{n,1}) + \tau R_h^{n,2} \end{aligned}$$

with residuals

$$\begin{aligned} R_h^{n,0} &:= (\Pi_h^{\mathrm{b}} - \Pi_h)\big(\partial_t u(t^n)\big) \\ R_h^{n,1} &:= \tau \Pi_h^{\mathrm{b}}\big(\partial_{tt} u(t^n)\big) \\ R_h^{n,2} &:= \tau^{-1}\Pi_h\big(u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n)\big) \end{aligned}$$

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\end{aligned}$$

- The residuals do not involve space derivatives!!

Error bound

- Error equations rewrite

$$e_h^{n+1} = P_3(-\tau A_h)(e_h^n) + \tau \sum_{m\in\{0:2\}} Q_{3,m}(-\tau A_h)(R_h^{n,m})$$

with $Q_{3,0}(z) := 1 - \frac{1}{2}z + \frac{1}{6}z^2$, $Q_{3,1}(z) := 1 - \frac{1}{3}z$, $Q_{3,2}(z) := 1$

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- Triangle inequality + ERK3 stability + CFL condition

$$\|e_h^{n+1}\|_L \le \|P_3(-\tau A_h)(e_h^n)\|_L + \tau \sum_{m\in\{0:2\}} \|Q_{3,m}(-\tau A_h)(R_h^{n,m})\|_L$$
$$\le \|e_h^n\|_L + \tau \sum_{m\in\{0:2\}} \|R_h^{n,m}\|_L$$

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- Bound residuals and sum over n

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- Bound residuals and sum over n

!! Thank you for your attention !!