Unfitted hybrid high-order methods

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collaboration and support: CEA

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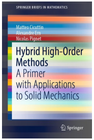
Outline

- **HHO methods in a nutshell**
- **Links to other methods**
- **Unfitted HHO**
- **Stabilization by polynomial extension**

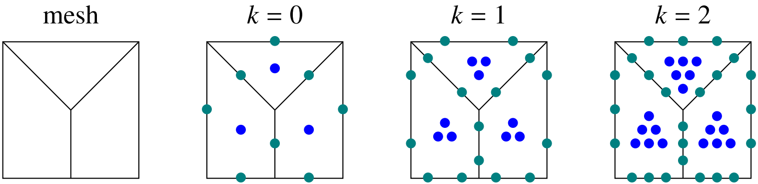
HHO methods in a nutshell

- **Seminal references:** [Di Pietro, AE, Lemaire 14; Di Pietro, AE 15]
- **Two textbooks**
  - HHO on polytopal meshes [Di Pietro, Droniou 20]
  - A primer with application to solid mechanics [Cicuttin, AE, Pignet 21]



Basic ideas

- Degrees of freedom (dofs) located on mesh cells and faces
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces



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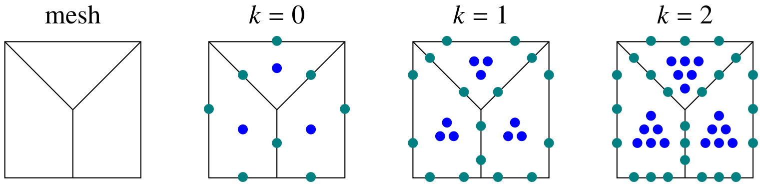

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- One adds a local stabilization to weakly enforce the matching of cell dofs trace with face dofs
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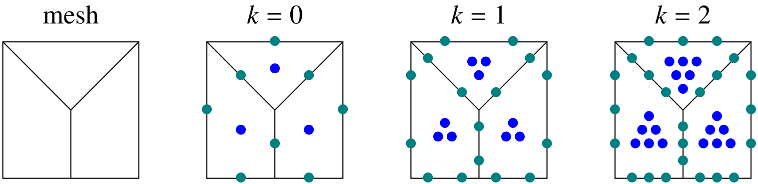

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- The global problem is assembled cellwise as in FEM
- Generalization to higher order of ideas from Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

Gradient reconstruction and stabilization

- Mesh cell $T \in \mathcal{T}$, cell dofs $u_T \in \mathbb{P}^k(T)$, face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

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- Local potential reconstruction $R_T : \hat{U}_T \longrightarrow \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla R_T(\hat{u}_T), \nabla q)_T = -(u_T, \Delta q)_T + (u_{\partial T}, \nabla q \cdot \mathbf{n}_T)_{\partial T}, \quad \forall q \in \mathbb{P}^{k+1}(T)/\mathbb{R}$$

together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

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- Local gradient reconstruction $\mathbf{G}_T(\hat{u}_T) := \nabla R_T(\hat{u}_T) \in \nabla \mathbb{P}^{k+1}(T)$
- Local stabilization operator acting on $\delta_{\hat{u}_T} := u_T|_{\partial T} - u_{\partial T}$
  - penalizing $S_{\partial T}(\delta_{\hat{u}_T}) := \delta_{\hat{u}_T}$ is suboptimal (too much stab.) ...
  - one optimal choice for equal-order polynomials is

$$S_{\partial T}(\delta_{\hat{u}_T}) := \Pi^k_{\partial T}\Big(\delta_{\hat{u}_T} - \underbrace{((I - \Pi^k_T) R_T(0, \delta_{\hat{u}_T}))|_{\partial T}}_{\text{HHO high-order correction}}\Big)$$

Local bilinear form

- Local bilinear form for Poisson model problem (recall $\delta_{\hat{u}_T} := u_T|_{\partial T} - u_{\partial T}$)

$$a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_T(\hat{u}_T), \mathbf{G}_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\delta_{\hat{u}_T}), S_{\partial T}(\delta_{\hat{w}_T}))_{\partial T}$$

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- Local H^1-like seminorm for hybrid variables

$$|\hat{u}_T|^2_{\hat{U}_T} := \|\nabla u_T\|^2_T + h_T^{-1}\|\delta_{\hat{u}_T}\|^2_{\partial T}$$

Notice that $|\hat{u}_T|_{\hat{U}_T} = 0 \implies u_T = u_{\partial T} = c$

- Stability and boundedness

$$\alpha|\hat{u}_T|^2_{\hat{U}_T} \le a_T(\hat{u}_T, \hat{u}_T) \le \omega|\hat{u}_T|^2_{\hat{U}_T}, \quad \forall \hat{u}_T \in \hat{U}_T$$

Assembly of discrete problem

- Global dofs $\hat{u}_h = (u_{\mathcal{T}}, u_{\mathcal{F}})$ ($\mathcal{T}$:= {mesh cells}, $\mathcal{F}$:= {mesh faces})

$$\hat{U}_h := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F}), \quad \mathbb{P}^k(\mathcal{T}) := \bigtimes_{T\in\mathcal{T}} \mathbb{P}^k(T), \quad \mathbb{P}^k(\mathcal{F}) := \bigtimes_{F\in\mathcal{F}} \mathbb{P}^k(F)$$

- Dirichlet conditions enforced on face boundary dofs

$$\hat{U}_{h0} := \{\hat{v}_h \in \hat{U}_h \mid v_F = 0 \ \forall F \subset \partial\Omega\}$$

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- Dirichlet conditions enforced on face boundary dofs

$$\hat{U}_{h0} := \{\hat{v}_h \in \hat{U}_h \mid v_F = 0 \; \forall F \subset \partial\Omega\}$$

- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T\in\mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \quad \forall \hat{w}_h \in \hat{U}_{h0}$$

(only cell component of test function used on rhs)

Algebraic realization and static condensation

- Algebraic realization

$$\begin{bmatrix} \mathsf{A}_{\mathcal{T}\mathcal{T}} & \mathsf{A}_{\mathcal{T}\mathcal{F}} \\ \mathsf{A}_{\mathcal{F}\mathcal{T}} & \mathsf{A}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{U}_{\mathcal{T}} \\ \mathsf{U}_{\mathcal{F}} \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}} \\ 0 \end{bmatrix}$$

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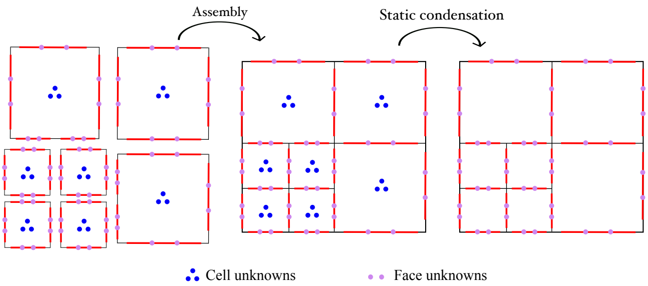
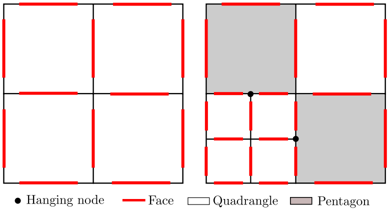
- Cell dofs can be eliminated locally by static condensation
  - global problem couples only face dofs
  - cell dofs recovered by local post-processing

- Summary



Main assets of HHO methods

- General meshes: polytopal cells, hanging nodes (as dG and VEM)



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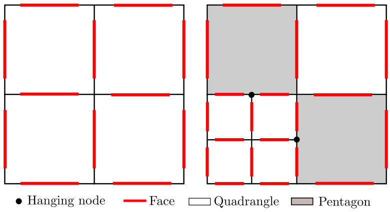

- Local conservation
  - (as any reasonable method) admits algebraically balanced fluxes on faces
  - (as any face-based method) local (cell based) post-processing

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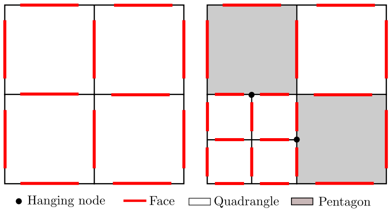

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- Attractive computational costs
  - only face dofs are globally coupled
- More complex problems
  - stab. weight only positive and element-based (no face-based diffusion)
  - no call to nonlinear behavior law at interfaces
  - symmetric (variational) formulation

Error estimates

- Full-regularity solutions (in $H^{k+2}(\Omega)$)
  - $O(h^{k+1})$ H^1-error estimate (face dofs of order $k \geq 0$)
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- Less regularity?
  - $O(h^t)$ H^1-error estimate if $u \in H^{1+t}(\Omega)$, $\forall t \in (\frac{1}{2}, k+1]$
  - for $t \in (0, \frac{1}{2})$, see [AE, Guermond 21 (FoCM)]
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- Main consistency property: Introduce reduction operator

$$\hat{I}_T : H^1(T) \to \hat{U}_T, \qquad \hat{I}_T(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v|_{\partial T}))$$

Then,

- $R_T \circ \hat{I}_T = \mathcal{E}_T^{k+1}$ is the elliptic projection onto $\mathbb{P}^{k+1}(T)$
- $h_T^{-\frac{1}{2}} \|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim \|\nabla(v - \mathcal{E}_T^{k+1}(v))\|_T$

$$\Longrightarrow \quad \|\nabla(v - R_T(\hat{I}_T(v)))\|_T + h_T^{-\frac{1}{2}} \|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim h_T^{k+1} |v|_{H^{k+2}(T)}$$

Variants

- Variant on gradient reconstruction $\mathbf{G}_T : \hat{U}_T \to \mathbb{P}^k(T;\mathbb{R}^d)$ s.t.

$$(\mathbf{G}_T(\hat{u}_T), \mathbf{q})_T = -(u_T, \mathrm{div}\, \mathbf{q})_T + (u_{\partial T}, \mathbf{q}\cdot\mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

  - same scalar mass matrix for each component of $\mathbf{G}_T(\hat{u}_T)$
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- Variants on cell dofs and stabilization
  - mixed-order setting: $k \geq 0$ for face dofs and $l := (k+1)$ for cell dofs
  - this variant allows for the simpler Lehrenfeld–Schöberl HDG stabilization

$$S_{\partial T}(\delta_{\hat{u}_T}) := \Pi^k_{\partial T}(\delta_{\hat{u}_T})$$

  - another variant is $k \geq 1$ for face dofs and $(k-1)$ for cell dofs

Links to other methods

$$\text{HHO} \equiv \text{WG} \equiv \text{HDG} \equiv \text{ncVEM}$$

- [Cockburn, Di Pietro, AE 16 (M2AN)]
  [Di Pietro, Droniou, Manzini 18 (JCP)], [Cicuttin, AE, Pignet 21 (SpringerBriefs)]
- !! Different devising viewpoints should be mutually enriching !!

Weak Galerkin (WG)

- WG methods devised in [Wang, Ye 13] (vast litterature...)
- Similar devising of HHO and WG
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- WG often uses **plain least-squares stabilization**

$$S_{\partial T}^{\mathrm{WG}}(\delta_{\hat{u}_T}) := \delta_{\hat{u}_T} \quad \text{vs.} \quad S_{\partial T}^{\mathrm{HHO}}(\delta_{\hat{u}_T}) := \begin{cases} \Pi_{\partial T}^{k}\big(\delta_{\hat{u}_T} - \big((I - \Pi_T^k)R_T(0, \delta_{\hat{u}_T})\big)|_{\partial T}\big) & (l = k) \\ \Pi_{\partial T}^{k}(\delta_{\hat{u}_T}) & (l = k+1) \end{cases}$$

- Plain least-squares stabilization leads to $O(h^k)$ H^1-error bounds
  - $O(h^{k+1})$ bounds require face polynomials of order $(k+1)$
  - fails for pcw. constant approximation

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  - the local equation for the dual variable is the **grad. rec. formula** in HHO!
  - one passes from HDG to HHO formulation by static condensation of dual variable

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- HHO is an HDG method!
  - this bridge uncovers HHO numerical flux trace

$$\widehat{\mathbf{q}}_{\partial T}(\hat{u}_T) = -\mathbf{G}_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(S^{\star}_{\partial T} \circ S_{\partial T})(\delta_{\hat{u}_T})$$

  - HHO novelty: use of reconstruction in stabilization (equal-order case)

- One HHO benefit: simpler analysis based on L^2-projections (standard HDG projection works on simplicial meshes)

Nonconforming virtual elements

- ncVEM devised in [Ayuso, Manzini, Lipnikov 16]

- Virtual space

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- HHO dof space $\hat{U}_T$ with $l := k - 1$ isomorphic to virtual space $\mathcal{V}_T$
  - virtual reconstruction operator $\mathcal{R}_T : \hat{U}_T \to \mathcal{V}_T$
  - $\hat{\mathcal{J}}_T : \mathcal{V}_T \to \hat{U}_T$: restriction of reduction operator to virtual space
  - then, $\hat{\mathcal{J}}_T \circ \mathcal{R}_T = I_{\hat{U}_T}$ and $\mathcal{R}_T \circ \hat{\mathcal{J}}_T = I_{\mathcal{V}_T}$

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- Stabilization controls energy-norm of noncomputable remainder
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- Further link to Multiscale Hybrid Mixed (MHM methods) [Chaumont, AE, Lemaire, Valentin 22]; see also [Lemaire 21]

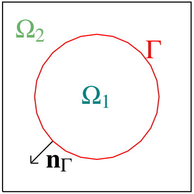
HHO on unfitted meshes

- [Burman, AE 18 (SINUM)], [Burman, Cicuttin, Delay, AE 21 (SISC)]

Model problem

- Model problem with curved interface (class C^2 for simplicity)
- Find $u \in H^1(\Omega_1 \cup \Omega_2)$ s.t.

$$
\begin{aligned}
-\nabla\cdot(\kappa\nabla u) &= f && \text{in } \Omega_1 \cup \Omega_2 \\
[\![u]\!]_\Gamma &= g_D && \text{on } \Gamma \\
[\![\kappa\nabla u]\!]_\Gamma \cdot \mathbf{n}_\Gamma &= g_N && \text{on } \Gamma \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$



- data $f \in L^2(\Omega_1 \cup \Omega_2)$, $g_D \in H^{\frac{1}{2}}(\Gamma)$, $g_N \in L^2(\Gamma)$
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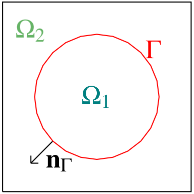

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- Weak formulation

$$\text{Find } u \in V_{g_D} \quad : \quad a(u,w) = \ell(w) \qquad \forall w \in V_0$$

with $V_{g_D} := \{v \in H^1(\Omega_1 \cup \Omega_2) \mid [\![v]\!]_\Gamma = g_D \text{ on } \Gamma,\ v = 0 \text{ on } \partial\Omega\}$, $V_0 = H^1_0(\Omega)$, and

$$a(u,w) := \sum_{i\in\{1,2\}} \kappa_i(\nabla u_i, \nabla w_i)_{\Omega_i}, \qquad \ell(w) := (f,w)_\Omega + (g_N, w)_\Gamma$$

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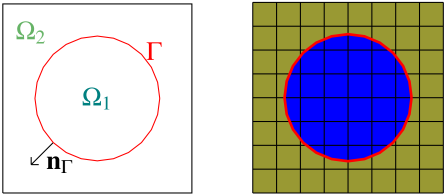
  (also explored in ncVEM context [Beirao da Veiga, Liu, Mascotto, Russo 24])
- Alternative idea: use unfitted meshes
  - background mesh very simple to devise
  - curved interface can cut arbitrarily through mesh cells
  - numerical method must deal with ill cut cells



Unfitted HHO

- Well developed paradigm for unfitted FEM
  - double nodal dofs in cut cells and use a consistent Nitsche's penalty technique to enforce jump conditions [Hansbo, Hansbo 02]
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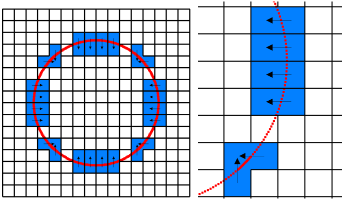
- Everything readily extends to domains with curved boundary

Agglomeration procedure

- Circular interface



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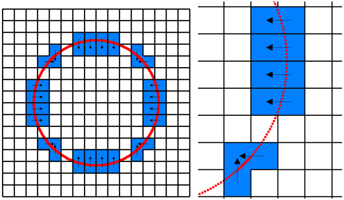
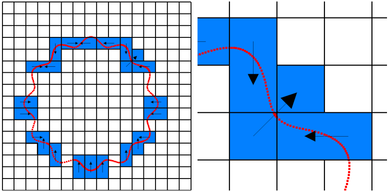


- Flower-like interface



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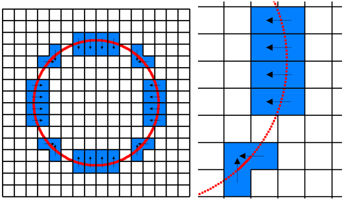

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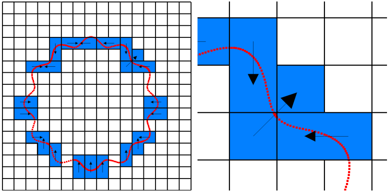

- Usual numerical analysis tools available on agglomerated mesh
  - discrete inverse and trace inequalities, optimal polynomial approximation
  - precise statements in [Burman, AE 18]

Local dofs

- Partition of $\mathcal{T}$ into cut and uncut cells

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- Doubling of HHO unknowns in cut cells

$$\hat{u}_T := (\hat{u}_{T^1}, \hat{u}_{T^2}) := (u_{T^1}, u_{(\partial T)^1}, u_{T^2}, u_{(\partial T)^2}) \in \widehat{\mathcal{U}}_T := \widehat{\mathcal{U}}_{T^1} \times \widehat{\mathcal{U}}_{T^2}$$

with $\widehat{\mathcal{U}}_{T^i} := \mathbb{P}^{k+1}(T^i) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^i})$ and $\mathbb{P}^k(\mathcal{F}_{(\partial T)^i}) := \bigtimes_{F^i \in \mathcal{F}_{(\partial T)^i}} \mathbb{P}^k(F^i)$

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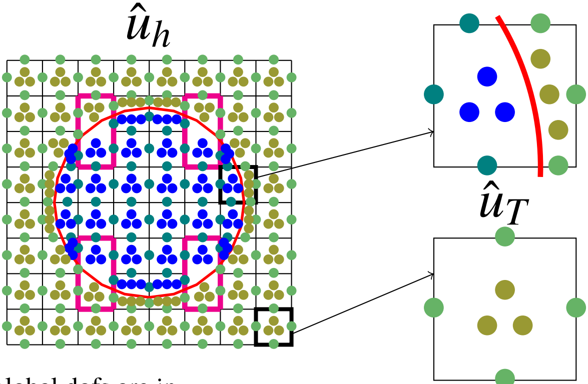
- Similar notation in uncut cells:

$$\hat{u}_T := (u_T, u_{\partial T}, 0, 0) \ \forall T \in \mathcal{T}^1, \qquad \hat{u}_T := (0, 0, u_T, u_{\partial T}) \ \forall T \in \mathcal{T}^2$$

Global dofs



- The global dofs are in

$$\hat{u}_h \in \hat{U}_h := \bigtimes_{T^1 \in \mathcal{T}^1} \mathbb{P}^{k+1}(T^1) \times \bigtimes_{F^1 \in \mathcal{F}^1} \mathbb{P}^{k}(F^1) \times \bigtimes_{T^2 \in \mathcal{T}^2} \mathbb{P}^{k+1}(T^2) \times \bigtimes_{F^2 \in \mathcal{F}^2} \mathbb{P}^{k}(F^2)$$

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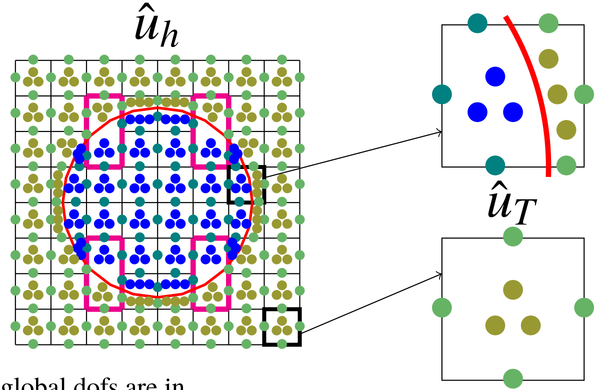
Global dofs



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- Set to zero all the face components attached to $\partial\Omega$ (Dirichlet BCs)
- All the cell dofs locally eliminated by static condensation
- Only face dofs globally coupled

Local gradient reconstruction

- General ideas
  - a gradient is reconstructed in each sub-cell
  - the two gradient reconstructions are independent
  - jump across interface is accounted for in gradient reconstruction

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- $\boldsymbol{G}_{T^i}^k : \widehat{\mathcal{U}}_T \to \mathbb{P}^k(T^i;\mathbb{R}^d)$ s.t., for all $\boldsymbol{q} \in \mathbb{P}^k(T^i;\mathbb{R}^d)$ and $i \in \{1,2\}$,

$$(\boldsymbol{G}_{T^i}^k(\hat{u}_T), \boldsymbol{q})_{T^i} := (\nabla u_{T^i}, \boldsymbol{q})_{T^i} + (u_{(\partial T)^i} - u_{T^i}, \boldsymbol{q}\cdot\mathbf{n}_T)_{(\partial T)^i} - \delta_{i1}(\llbracket u_T \rrbracket_\Gamma, \boldsymbol{q}\cdot\mathbf{n}_\Gamma)_{T^\Gamma}$$

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- Numbering of sub-domains so that $\kappa_1 \leq \kappa_2$
  - non-symmetric inclusion of $[\![u_T]\!]_\Gamma$ allows for robustness when $\kappa_1 \ll \kappa_2$
  - inclusion of $[\![u_T]\!]_\Gamma$ in both gradient reconstructions possible when $\kappa_1 \approx \kappa_2$

Stabilization

- Usual HHO stabilization on sub-faces in each sub-domain (LS in mixed-order setting)

$$s_h^{\circ}(\hat{v}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \frac{\kappa_i}{h_T} (\Pi^k_{(\partial T)^i}(v_{T^i}) - v_{(\partial T)^i}, \Pi^k_{(\partial T)^i}(w_{T^i}) - w_{(\partial T)^i})_{(\partial T)^i}$$

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- Total stabilization

$$s_h(\hat{v}_h, \hat{w}_h) := s_h^\circ(\hat{v}_h, \hat{w}_h) + s_h^\Gamma(\hat{v}_h, \hat{w}_h)$$

Global assembly

- Discrete bilinear form

$$a_h(\hat{v}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \kappa_i(\boldsymbol{G}^k_{T^i}(\hat{v}_T), \boldsymbol{G}^k_{T^i}(\hat{w}_T))_{T^i} + s_h(\hat{v}_h, \hat{w}_h)$$

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- Right-hand side devised to ensure consistency

$$\ell_h(\hat{w}_h) := \sum_{T\in\mathcal{T}} \sum_{i\in\{1,2\}} (f, w_{T^i})_{T^i} + \sum_{T\in\mathcal{T}^{\mathrm{cut}}} (g_N, w_{T^2})_{T^\Gamma}$$
$$+ \sum_{T\in\mathcal{T}^{\mathrm{cut}}} \kappa_1 (g_D, h_T^{-1} [\![w_T]\!]_\Gamma - \boldsymbol{G}^k_{T^1}(\hat{w}_T)\cdot \mathbf{n}_\Gamma)_{T^\Gamma}$$

Notice that both jump conditions are enforced weakly

Discrete problem and error estimate

- Discrete problem

$$\hat{u}_h \in \widehat{\mathcal{U}}_{h0} \quad : \quad a_h(\hat{u}_h, \hat{w}_h) = \ell_h(\hat{w}_h) \quad \forall \hat{w}_h \in \widehat{\mathcal{U}}_{h0}$$

- Stability and consistency properties can be established
  - see [Burman, AE 18 (SINUM)] for details

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- Main error estimate: Assume $u \in H^s(\Omega_1 \cup \Omega_2)$ with $s \in (\frac{3}{2}, k+2]$. Then,

$$\left\{ \sum_{T \in \mathcal{T}} \sum_{i \in \{1,2\}} \kappa_i \|\nabla(u_i - u_{T^i})\|_{T^i}^2 \right\}^{\frac{1}{2}} \lesssim h^{s-1} \sum_{i \in \{1,2\}} \kappa_i^{\frac{1}{2}} |u_i|_{H^s(\Omega_i)}$$

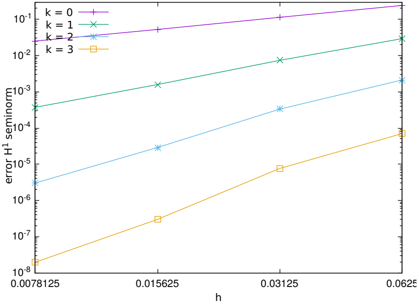
  reaching $O(h^{k+1})$-convergence rates in H^1

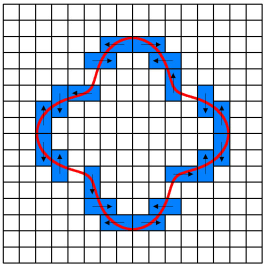
Test case with jump

- Flower-like interface, exact solution with jump

$$u(x_1, x_2) := \begin{cases} \sin(\pi x_1)\sin(\pi x_2) & \text{in } \Omega_1 \\ \sin(\pi x_1)\sin(\pi x_2) + 2 + x^3 y^3 & \text{in } \Omega_2 \end{cases}$$

- Optimal $O(h^{k+1})$ convergence rates in H^1





Stabilization by polynomial extension

- [Burman, AE, Mottier 25 (arXiv)]

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- Stabilization by polynomial extension used in other contexts, e.g.,
  - Lagrange multipliers with FEM [Haslinger, Renard 09]
  - shifted boundary [Main, Scovazzi 18] and boundary correction [Burman, Hansbo, Larson 18] methods with FEM
  - isogeometric methods on trimmed geometries [Buffa, Puppi, Vázquez 20]
  - unfitted VEM [Bertoluzza, Pennacchio, Prada 22; Hou, Liu, Wang 24]

Well-cut and ill-cut cells

- Partition of cut cells of the original unfitted mesh

$$\mathcal{T}^{\text{cut}} = \mathcal{T}^{\text{OK}} \cup \mathcal{T}^{\text{KO}}$$

- Fix parameter $\vartheta \in (0, 1)$, then $T \in \mathcal{T}^{\text{OK}}$ if T^i contains a ball of radius ϑh_T for all $i \in \{1, 2\}$

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- [Burman, AE 18; Lemma 6.2] shows that, if
  - h small enough w.r.t. interface curvature,
  - ϑ small enough w.r.t. mesh regularity parameter,

  the above ball condition can only fail on at most one sub-cell of T

- Partition of cut cells as

$$\mathcal{T}^{\mathrm{cut}} = \mathcal{T}^{\mathrm{OK}} \cup \mathcal{T}^{\mathrm{KO},1} \cup \mathcal{T}^{\mathrm{KO},2}$$

Pairing operator

- For every ill-cut cell $S \in \mathcal{T}^{\text{KO}}$, find a well-cut cell T in $\Delta(S)$

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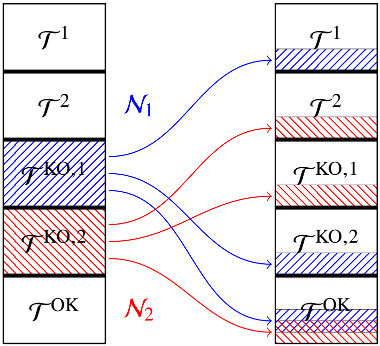
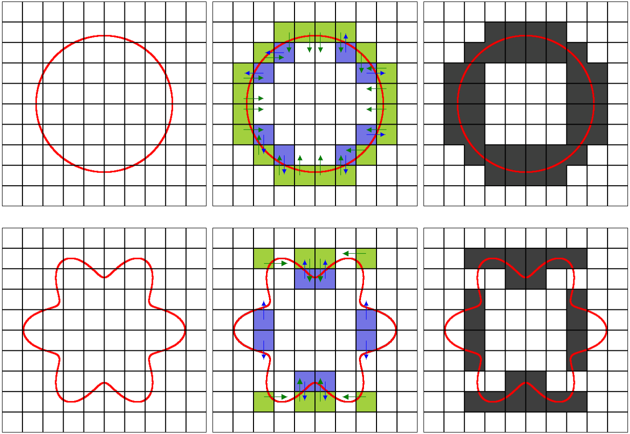


Polynomial extension vs. Cell agglomeration



Local gradient reconstruction

- Enlarge stencil for local gradient reconstruction to

$$\hat{u}_T^{\mathcal{N}} := (\hat{u}_T, (\hat{u}_S)_{S \in \mathcal{N}^{-1}(T)}) \in \widehat{\mathcal{U}}_T^{\mathcal{N}} := \widehat{\mathcal{U}}_T \times \bigtimes_{S \in \mathcal{N}^{-1}(T)} \widehat{\mathcal{U}}_S$$

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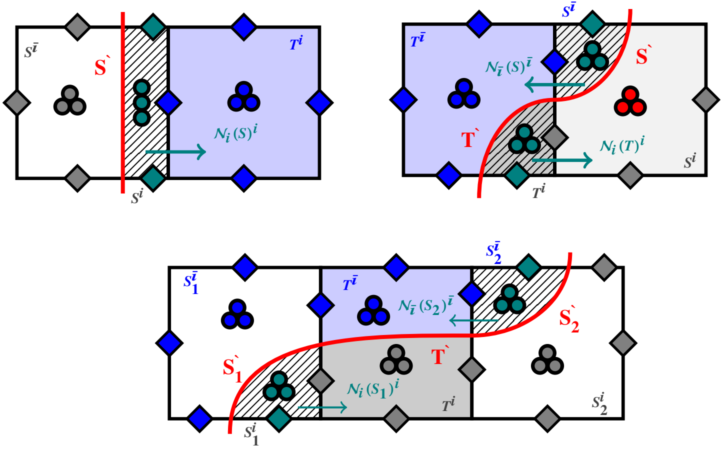
- If the ball condition fails, then simply set

$$\boldsymbol{G}_{T^i}^k(\hat{u}_T^{\mathcal{N}}) := \nabla u_{T^i}$$

Some examples



Stabilization

- Keep usual HHO stabilization inside sub-domains and Nitsche-like penalty at interface

$$s_h^\circ(\hat{v}_h, \hat{w}_h), \qquad s_h^\Gamma(\hat{v}_h, \hat{w}_h)$$

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- Add stabilization to connect cell dofs of well- and ill-cut cells in the spirit of direct ghost penalty method [Preuss 18; Lehrenfeld, Olshanski 19]

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Discrete problem and error estimate

- Global assembly

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with ℓ_h defined so as to ensure consistency

- Main error estimate: Assume $u \in H^s(\Omega_1 \cup \Omega_2)$ with $s \in (\frac{3}{2}, k+2]$. Then,

$$\left\{ \sum_{T\in\mathcal{T}} \sum_{i\in\{1,2\}} \kappa_i \|\nabla(u_i - u_{T^i})\|^2_{T^i} \right\}^{\frac{1}{2}} \lesssim h^{s-1} \sum_{i\in\{1,2\}} \kappa_i^{\frac{1}{2}} |u_i|_{H^s(\Omega_i)}$$

reaching $O(h^{k+1})$-convergence rates in H^1

Main analysis tools

- Inverse inequalities: For all $(T, i) \in \mathcal{P}_h^{\mathrm{OK}}$ and all $\phi \in \mathbb{P}^\ell(T^i; \mathbb{R})$,

$$\sum_{S \in \{T\} \cup \mathcal{N}_i^{-1}(T)} \left\{ \|\phi^+\|_S + h_S^{\frac{1}{2}} \|\phi^+\|_{(\partial S)^i \cup S^\Gamma} \right\} \lesssim \|\phi\|_{T^i}$$

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$$I_{T^i}^{k+1}(v_i) := \Pi_T^{k+1}(E_i^s(v_i))|_{T^i} \in \mathbb{P}^{k+1}(T^i)$$

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- Under the mild assumption $\mathrm{conv}(T) \subset \Delta(T)$, $I_{T^i}^{k+1}$ has optimal approximation properties

$$\sum_{S \in \{T\} \cup \mathcal{N}_i^{-1}(T)} \left\{ \|v_i - I_{T^i}^{k+1}(v_i)^+\|_{S^i} + h_S^{\frac{1}{2}} \|v_i - I_{T^i}^{k+1}(v_i)^+\|_{(\partial S)^i} \ldots \right\} \lesssim h_T^s |E_i^s(v_i)|_{H^s(\Delta(T))}$$

Implementation aspects

- Nontrivial modifications of global assembly module
- Modal (centered and scaled) bases attached to sub-cells
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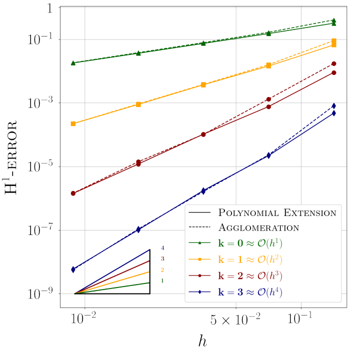
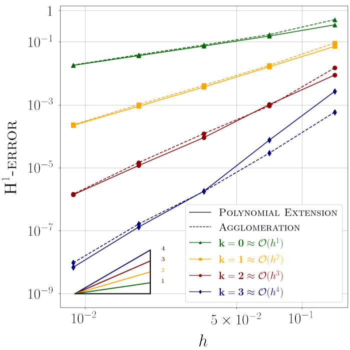
- Pairing operator guarantees locality
- Quadratures in cut cells based on sub-triangulation, using a pcw. linear approximation of interface into 2^r segments (to be improved!)

Convergence rates on smooth solutions

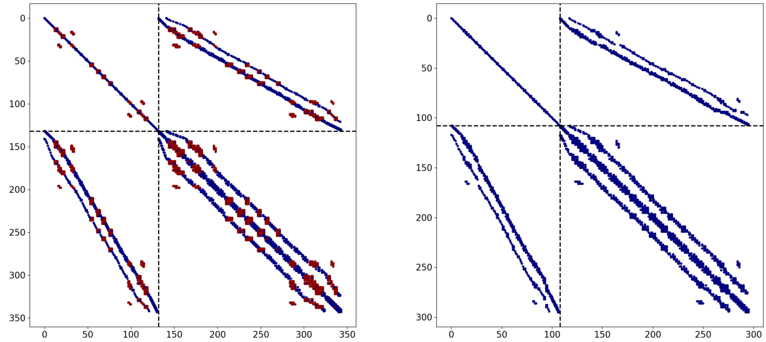
- $u(x, y) = \sin(\pi x)\sin(\pi y)$, $g_D = g_N = 0$, $\kappa_1 = \kappa_2 = 1$
- Comparison between polynomial extension (solid) and cell agglomeration (dashed)
- Circular (left) and flower-like (right) interface





Comparison of matrix sparsity profiles

- Polynomial extension (left) vs. cell agglomeration (right)

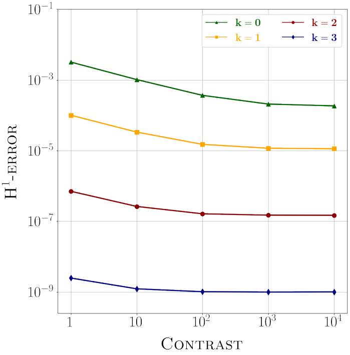


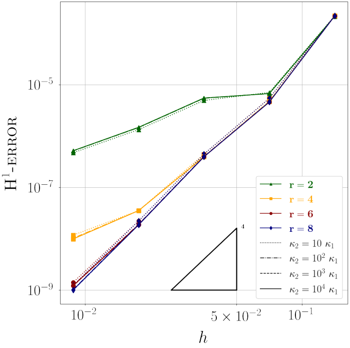
Solutions with contrasted diffusivity

- Circular interface, $g_D = g_N = 0$, in polar coordinates (ρ, θ)

$$u_1(\rho) = \frac{\rho^6}{\kappa_1}, \qquad u_2(\rho) = \frac{\rho^6}{\kappa_2} + R^6\left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right)$$

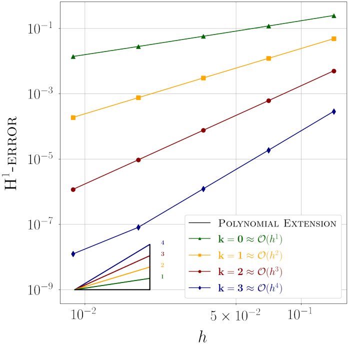
- Left: Error vs. diffusivity contrast, $\kappa_2 = 10^m \kappa_1$, $m \in \{0{:}4\}$, finest mesh
- Right: Error vs. h for sub-triangulation parameter $r \in \{2, 4, 6, 8\}$

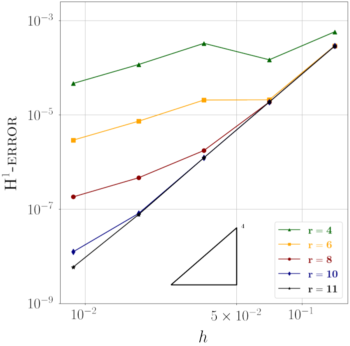




Solution with non-polynomial jumps

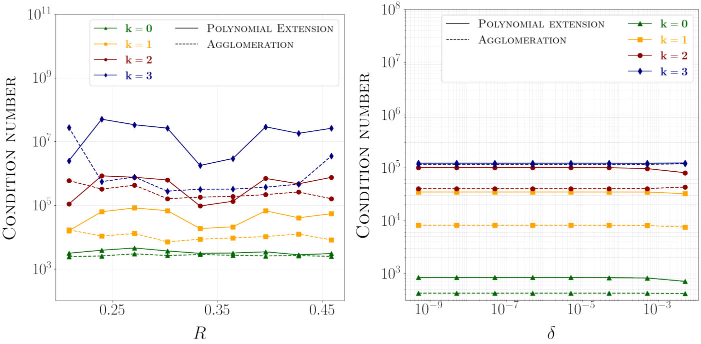
- $u_1(x,y) = \cos(y)e^x$, $u_2(x,y) = \sin(\pi x)\sin(\pi y)$, $\kappa_1 = \kappa_2 = 1$
- Left: Error vs. h, $k \in \{0,1,2,3\}$, $r = 10$
- Right: Error vs. h, $k = 3$, $r \in \{4,6,8,10,11\}$





Conditioning of stiffness matrix

- Left: Circular interface with radius $R = \frac{1}{3} + \frac{i}{32}$, $i \in \{-4, \ldots, 4\}$
- Right: Square interface, distance to mesh 0.5×10^{-p}, $p \in \{1, \ldots, 5\}$



- **Robust conditioning** for severe ill-cuts

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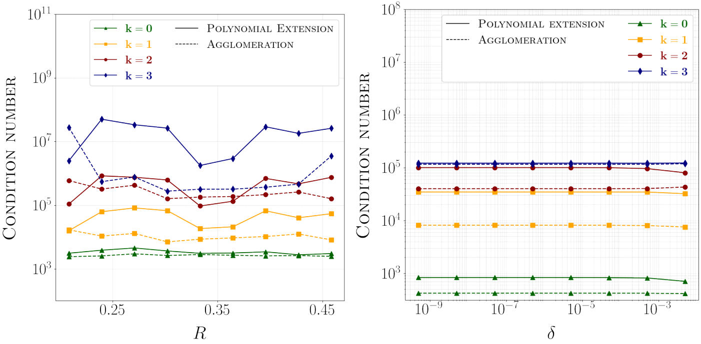

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!! Thank you for your attention !!