Hybrid high-order methods for the wave equation

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collaboration and support: CEA

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Outline

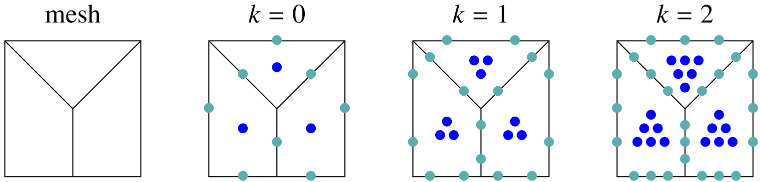
Hybrid high-order (HHO) methods ...

- in a nutshell
- for wave propagation
- on unfitted meshes (curved interfaces/boundary)

HHO in a nutshell

Basic ideas

- Introduced in [Di Pietro, AE, Lemaire 14; Di Pietro, AE 15]
- Degrees of freedom (dofs) located on mesh cells and faces
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces



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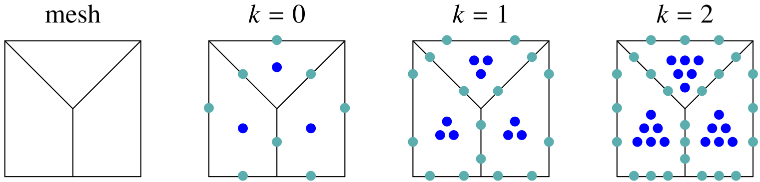

- In each cell, one devises a local gradient reconstruction operator
- One adds a local stabilization to weakly enforce the matching of cell dofs trace with face dofs
- The global problem is assembled cellwise as in FEM

Gradient reconstruction and stabilization

- Mesh cell $T \in \mathcal{T}$, cell dofs $u_T \in \mathbb{P}^k(T)$, face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Local potential reconstruction $R_T : \hat{U}_T \to \mathbb{P}^{k+1}(T)$ s.t.

$$(\nabla R_T(\hat{u}_T), \nabla q)_T = -(u_T, \Delta q)_T + (u_{\partial T}, \nabla q \cdot \mathbf{n}_T)_{\partial T}, \quad \forall q \in \mathbb{P}^{k+1}(T)/\mathbb{R}$$

together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

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together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$

- Local gradient reconstruction $\mathbf{G}_T(\hat{u}_T) := \nabla R_T(\hat{u}_T) \in \nabla \mathbb{P}^{k+1}(T)$
- Local stabilization operator acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \Pi^k_{\partial T}\Big(\delta - \underbrace{\big((I - \Pi^k_T) R_T(0, \delta)\big)|_{\partial T}}_{\text{high-order correction}}\Big)$$

Local bilinear form

- Local bilinear form for Poisson model problem

$$a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_T(\hat{u}_T), \mathbf{G}_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$$

- Stability and boundedness

$$\alpha \|\hat{u}_T\|_{\hat{U}_T}^2 \le a_T(\hat{u}_T, \hat{u}_T) \le \omega \|\hat{u}_T\|_{\hat{U}_T}^2, \quad \forall \hat{u}_T \in \hat{U}_T$$

with $\|\hat{u}_T\|_{\hat{U}_T}^2 := \|\nabla u_T\|_T^2 + h_T^{-1}\|u_T|_{\partial T} - u_{\partial T}\|_{\partial T}^2$

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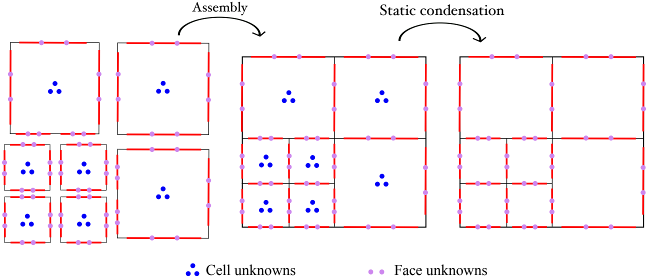
with $\|\hat{u}_T\|^2_{\hat{U}_T} := \|\nabla u_T\|^2_T + h_T^{-1} \|u_T|_{\partial T} - u_{\partial T}\|^2_{\partial T}$

- Reduction operator $\hat{I}_T(v) := (\Pi^k_T(v), \Pi^k_{\partial T}(v|_{\partial T})) \in \hat{U}_T$, $\forall v \in H^1(T)$
- Main consistency properties
  - $h_T^{-1}\|v - R_T(\hat{I}_T(v))\|_T + \|\nabla(v - R_T(\hat{I}_T(v)))\|_T \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$
  - $h_T^{-\frac{1}{2}}\|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$

Assembly and static condensation



- Global dofs $\hat{u}_h = (u_{\mathcal{T}}, u_{\mathcal{F}})$ ($\mathcal{T}$:= {mesh cells}, $\mathcal{F}$:= {mesh faces})

$$\hat{U}_h := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F}), \quad \mathbb{P}^k(\mathcal{T}) := \bigtimes_{T\in\mathcal{T}} \mathbb{P}^k(T), \quad \mathbb{P}^k(\mathcal{F}) := \bigtimes_{F\in\mathcal{F}} \mathbb{P}^k(F)$$

- Global assembly: $\sum_{T\in\mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = \sum_{T\in\mathcal{T}} (f, w_T)_T$
- Dirichlet conditions can be directly enforced on the face boundary dofs
- Cell dofs are eliminated locally by static condensation
  - global problem couples only face dofs
  - cell dofs recovered by local post-processing

Main characteristics

- General meshes: polytopal cells, hanging nodes
- Optimal error estimates (smooth solutions)
  - $O(h^{k+1})$ H^1-error estimate (face dofs of order $k \geq 0$)
  - $O(h^{k+2})$ L^2-error estimate (with full elliptic regularity)

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  - more generally, $O(h^t)$ H^1-error estimate if $u \in H^{1+t}(\Omega)$, $t \in (\frac{1}{2}, k+1]$
  - for $t \in (0, \frac{1}{2})$, see [AE, Guermond 21 (FoCM)]

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- Local conservation
  - optimally convergent and algebraically balanced fluxes on faces
  - as any face-based method, balance at cell level
- Attractive computational costs
  - only face dofs are globally coupled
  - compact stencil

Variants

- Variant on gradient reconstruction $\mathbf{G}_T : \hat{U}_T \to \mathbb{P}^k(T;\mathbb{R}^d)$ s.t.

$$(\mathbf{G}_T(\hat{u}_T), \mathbf{q})_T = -(u_T, \mathrm{div}\ \mathbf{q})_T + (u_{\partial T}, \mathbf{q}\cdot\mathbf{n}_T)_{\partial T}, \quad \forall \mathbf{q} \in \mathbb{P}^k(T;\mathbb{R}^d)$$

  - same scalar mass matrix for each component of $\mathbf{G}_T(\hat{u}_T)$
  - useful for nonlinear problems
    [Di Pietro, Droniou 17; Botti, Di Pietro, Sochala 17; Abbas, AE, Pignet 18]

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- Variants on cell dofs and stabilization
  - mixed-order setting: $k \geq 0$ for face dofs and $(k+1)$ for cell dofs
  - this variant allows for the simpler Lehrenfeld–Schöberl HDG stabilization

$$S_{\partial T}(\hat{u}_T) := \Pi^k_{\partial T}(\delta)$$

  - another variant is $k \geq 1$ for face dofs and $(k-1)$ for cell dofs

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

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  - weak gradient $\leftrightarrow$ HHO grad. rec.
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- HHO equivalent (up to stab.) to ncVEM [Ayuso, Manzini, Lipnikov 16]
  - HHO dof space $\hat{U}_T$ isomorphic to virtual space $\mathcal{V}_T$

$$\mathbb{P}^{k+1}(T) \subsetneq \mathcal{V}_T := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^k(T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

  - HHO grad. rec. $\leftrightarrow$ computable gradient projection
  - stabilization controls energy-norm of noncomputable remainder
  - see [Cockburn, Di Pietro, AE 16; Di Pietro, Droniou, Manzini 18; Lemaire 21]]

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- Different devising viewpoints should be mutually enriching

Applications, libraries, textbooks

- Broad area of applications (non-exhaustive list...)
  - **solid mechanics**: nonlinear elasticity, hyperlasticity and plasticity, contact, Tresca friction, obstacle pb
  - **fluid mechanics/porous media**: Stokes, NS, poroelasticity, fractures
  - Leray-Lions, spectral pb, H^{-1}-loads, magnetostatics, de Rham complexes
- Libraries
  - industry (`code_aster`, `code_saturne`, EDF R&D), ongoing developments at CEA
  - academia: `diskpp (C++)` (ENPC/INRIA `github.com/wareHHOuse`), HArD::Core (Monash/Montpellier `github.com/jdroniou/HArDCore`)
- Textbooks
  - Di Pietro, Droniou, The HHO method for polytopal meshes. Design, analysis and applications (Springer, 2020)
  - Cicuttin, AE, Pignet, HHO methods. A primer with application to solid mechanics (Springer Briefs, 2021)

HHO for wave propagation

- Second-order formulation in time: Newmark schemes
- First-order formulation in time: RK schemes
- [Burman, Duran, AE 21 (CAMC)], [Burman, Duran, AE, Steins 21 (JSC)]

Second-order formulation in time

- Domain $\Omega \subset \mathbb{R}^d$, time interval $J := (0, T_{\mathrm{f}})$, $T_{\mathrm{f}} > 0$
- Acoustic wave equation with wave speed $c := \sqrt{\kappa/\rho}$

$$\frac{1}{\kappa}\partial_{tt}p - \operatorname{div}\left(\frac{1}{\rho}\nabla p\right) = f \quad \text{in } J \times \Omega$$

  Everything can be extended to elastodynamics

- Weak form: Under mild regularity assumptions on the data,

$$(\partial_{tt}p(t), w)_{\frac{1}{\kappa};\Omega} + (\nabla p(t), \nabla w)_{\frac{1}{\rho};\Omega} = (f(t), w)_{\Omega}, \quad \forall w \in H_0^1(\Omega)\, \forall t \in J$$

- Energy balance: $\mathfrak{E}(t) = \mathfrak{E}(0) + \int_0^t (f(s), \partial_t p(s))_{\Omega} ds$ with

$$\mathfrak{E}(t) := \frac{1}{2}\|\partial_t p(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\nabla p(t)\|^2_{\frac{1}{\rho};\Omega}$$

HHO space semi-discretization (1/3)

- Local cell dofs in $\mathbb{P}^{k'}(T)$, $k' \in \{k, k+1\}$, and local face dofs in $\mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T = (u_T, u_{\partial T}) \in \hat{U}_T := \mathbb{P}^{k'}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

- Local gradient reconstruction $\mathbf{G}_T(\hat{u}_T) \in \mathbb{P}^k(T; \mathbb{R}^d)$ (or in $\nabla \mathbb{P}^{k+1}(T)$)
- Local stabilization acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \begin{cases} \Pi^k_{\partial T}\big(\delta - \big((I - \Pi^k_T) R_T(0, \delta)\big)|_{\partial T}\big) & \text{if } k' = k \\ \Pi^k_{\partial T}(\delta) & \text{if } k' = k+1 \end{cases}$$

- Local bilinear form

$$a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_T(\hat{u}_T), \mathbf{G}_T(\hat{w}_T))_{\frac{1}{\rho};T} + \tau_{\partial T}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$$

with $\tau_{\partial T} := (\rho_{|T} h_T)^{-1}$

HHO space semi-discretization (2/3)

- Global dofs $\hat{u}_h = (u_{\mathcal{T}}, u_{\mathcal{F}}) \in \hat{U}_h := \mathbb{P}^{k'}(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F})$
- Global assembly leading to

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) := (\mathbf{G}_{\mathcal{T}}(\hat{u}_h), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\frac{1}{\rho};\Omega} + s_h(\hat{u}_h, \hat{w}_h)$$

- Dirichlet conditions can be directly enforced on the face boundary dofs

$$\hat{U}_{h0} := \mathbb{P}^{k'}(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F}^\circ)$$

with $\mathcal{F}^\circ := \{\text{mesh interfaces}\}$

HHO space semi-discretization (3/3)

- Wave equation in space semi-discrete form: $\hat{p}_h \in C^2(\bar{J}; \hat{U}_{h0})$ s.t.

$$(\partial_{tt} p_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + a_h(\hat{p}_h(t), \hat{w}_h) = (f(t), w_{\mathcal{T}})_{\Omega}, \quad \forall \hat{w}_h \in \hat{U}_{h0}\ \forall t \in J$$

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- Energy balance: $\mathfrak{E}_h(t) = \mathfrak{E}_h(0) + \int_0^t (f(s), \partial_t p_{\mathcal{T}}(s))_{\Omega} ds$ with

$$\mathfrak{E}_h(t) := \frac{1}{2}\|\partial_t p_{\mathcal{T}}(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t))\|^2_{\frac{1}{\rho};\Omega} + \frac{1}{2} s_h(\hat{p}_h(t), \hat{p}_h(t))$$

  Stabilization is taken into account in the energy definition

- HDG methods for wave equation in second-order form [Cockburn, Fu, Hungria, Ji, Sanchez, Sayas 18]

Algebraic realization

- Bases for $\mathbb{P}^{k'}(\mathcal{T})$ and $\mathbb{P}^{k}(\mathcal{F})$, component vector $(\mathsf{P}_{\mathcal{T}}(t), \mathsf{P}_{\mathcal{F}}(t)) \in \mathbb{R}^{N_{\mathcal{T}} \times N_{\mathcal{F}}}$

$$\begin{bmatrix} \mathsf{M}_{\mathcal{T}\mathcal{T}} \partial_{tt} \mathsf{P}_{\mathcal{T}}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} \mathsf{K}_{\mathcal{T}\mathcal{T}} & \mathsf{K}_{\mathcal{T}\mathcal{F}} \\ \mathsf{K}_{\mathcal{F}\mathcal{T}} & \mathsf{K}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{P}_{\mathcal{T}}(t) \\ \mathsf{P}_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} \mathsf{F}_{\mathcal{T}}(t) \\ 0 \end{bmatrix}$$

- Mass matrix $\mathsf{M}_{\mathcal{T}\mathcal{T}}$ and stiffness submatrix $\mathsf{K}_{\mathcal{T}\mathcal{T}}$ are block-diagonal
- Stiffness submatrix $\mathsf{K}_{\mathcal{F}\mathcal{F}}$ is only sparse: face dofs from the same cell are coupled together owing to reconstruction

Error analysis

- Assuming a smooth solution,
  - $\|\partial_t p - \partial_t p_{\mathcal{T}}\|_{L^\infty(J;L^2(\frac{1}{\kappa};\Omega))} + \|\nabla p - \mathbf{G}_{\mathcal{T}}(\hat{p}_h)\|_{L^2(J;L^2(\frac{1}{\rho};\Omega))}$ decays as $O(h^{k+1})$
  - $\|\Pi^{k'}_{\mathcal{T}}(p) - p_{\mathcal{T}}\|_{L^\infty(J;L^2(\frac{1}{\rho};\Omega))}$ decays as $O(h^{k+2})$ under (full) elliptic reg.

- Some comments on proofs
  - adapt ideas for FEM analysis from [Dupont 73; Wheeler 73; Baker 76]
  - simpler than for HDG (avoids HDG projection which needs a special initialization in HDG scheme)
  - could be re-used in DG setting using discrete gradients (revisiting [Grote, Schneebeli, Schötzau 06])

Newmark scheme

- Newmark scheme with parameters $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$
  - implicit, second-order, unconditionally stable
  - p, $\partial_t p$, $\partial_{tt} p$ are approximated by hybrid pairs $\hat{p}_h^n, \hat{v}_h^n, \hat{a}_h^n \in \hat{U}_{h0}$, $\forall n \geq 0$
- Each time-step implemented as usual

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- Each time-step implemented as usual
- Discrete energy is exactly conserved
- Central FD scheme is not efficient: inversion of stiffness submatrix $\mathsf{K}_{\mathcal{F}\mathcal{F}}$

First-order formulation in time

- Introduce velocity $v := \partial_t p$ and dual variable $\boldsymbol{\sigma} := \frac{1}{\rho}\nabla p$

$$\begin{cases} \rho\partial_t\boldsymbol{\sigma} - \nabla v = 0 \\ \frac{1}{\kappa}\partial_t v - \operatorname{div}\boldsymbol{\sigma} = f \end{cases} \quad \text{in } J \times \Omega$$

- Weak form: $\forall(\boldsymbol{\tau}, w) \in L^2(\Omega;\mathbb{R}^d) \times H^1_0(\Omega)$, $\forall t \in J$,

$$\begin{cases} (\partial_t\boldsymbol{\sigma}(t), \boldsymbol{\tau})_{\rho;\Omega} - (\nabla v(t), \boldsymbol{\tau})_\Omega = 0 \\ (\partial_t v(t), w)_{\frac{1}{\kappa};\Omega} + (\boldsymbol{\sigma}(t), \nabla w)_\Omega = (f(t), w)_\Omega \end{cases}$$

- Energy balance: $\mathfrak{E}(t) = \mathfrak{E}(0) + \int_0^t (f(s), v(s))_\Omega ds$ with

$$\mathfrak{E}(t) := \frac{1}{2}\|v(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\boldsymbol{\sigma}(t)\|^2_{\rho;\Omega}$$

HHO space semi-discretization

- $\hat{v}_h \in C^1(\overline{J}; \hat{U}_{h0})$ and $\boldsymbol{\sigma}_{\mathcal{T}} \in C^1(\overline{J}; \mathbf{S}_{\mathcal{T}})$ with $\mathbf{S}_{\mathcal{T}} := \mathbb{P}^k(\mathcal{T}; \mathbb{R}^d)$
- Space semi-discrete form:

$$\begin{cases}(\partial_t \boldsymbol{\sigma}_{\mathcal{T}}(t), \boldsymbol{\tau}_{\mathcal{T}})_{\rho;\Omega} - (\mathbf{G}_{\mathcal{T}}(\hat{v}_h(t)), \boldsymbol{\tau}_{\mathcal{T}})_{\Omega} = 0 \\ (\partial_t v_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + (\boldsymbol{\sigma}_{\mathcal{T}}(t), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\Omega} + \tilde{s}_h(\hat{v}_h(t), \hat{w}_h) = (f(t), w_{\mathcal{T}})_{\Omega}\end{cases}$$

- Stabilization $\tilde{s}_h(\cdot,\cdot)$ with weight $\tilde{\tau}_{\partial T} = (\rho c)^{-1}_{|T}$, i.e., $\tilde{\tau}_{\partial T} = O(1)$

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- Energy balance: $\mathfrak{E}_h(t) + \int_0^t \tilde{s}_h(\hat{v}_h(s), \hat{v}_h(s))ds = \mathfrak{E}_h(0) + \int_0^t (f(s), v_{\mathcal{T}}(s))_{\Omega} ds$

$$\mathfrak{E}_h(t) := \frac{1}{2}\|v_{\mathcal{T}}(t)\|^2_{\frac{1}{\kappa};\Omega} + \frac{1}{2}\|\boldsymbol{\sigma}_{\mathcal{T}}(t)\|^2_{\rho;\Omega}$$

Stabilization acts as a dissipative mechanism

- HDG methods for wave equation in first-order form [Nguyen, Peraire, Cockburn 11; Stranglmeier, Nguyen, Peraire, Cockburn 16]

Algebraic realization

- Component vectors $\mathsf{Z}_{\mathcal{T}}(t) \in \mathbb{R}^{M_{\mathcal{T}}}$ and $(\mathsf{V}_{\mathcal{T}}(t), \mathsf{V}_{\mathcal{F}}(t)) \in \mathbb{R}^{N_{\mathcal{T}} \times N_{\mathcal{F}}}$

$$\begin{bmatrix} \mathsf{M}^{\sigma}_{\mathcal{T}\mathcal{T}} \partial_t \mathsf{Z}_{\mathcal{T}}(t) \\ \mathsf{M}_{\mathcal{T}\mathcal{T}} \partial_t \mathsf{V}_{\mathcal{T}}(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathsf{G}_{\mathcal{T}} & -\mathsf{G}_{\mathcal{F}} \\ \mathsf{G}^{\dagger}_{\mathcal{T}} & \mathsf{S}_{\mathcal{T}\mathcal{T}} & \mathsf{S}_{\mathcal{T}\mathcal{F}} \\ \mathsf{G}^{\dagger}_{\mathcal{F}} & \mathsf{S}_{\mathcal{F}\mathcal{T}} & \mathsf{S}_{\mathcal{F}\mathcal{F}} \end{bmatrix} \begin{bmatrix} \mathsf{Z}_{\mathcal{T}}(t) \\ \mathsf{V}_{\mathcal{T}}(t) \\ \mathsf{V}_{\mathcal{F}}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathsf{F}_{\mathcal{T}}(t) \\ 0 \end{bmatrix}$$

- Mass matrices $\mathsf{M}^{\sigma}_{\mathcal{T}\mathcal{T}}$ and $\mathsf{M}_{\mathcal{T}\mathcal{T}}$ are block-diagonal
- Key point: stab. submatrix $\mathsf{S}_{\mathcal{F}\mathcal{F}}$ block-diagonal only if $k' = k + 1$
  - for $k' = k$, high-order HHO correction in stabilization destroys this property (couples all faces of the same cell!)

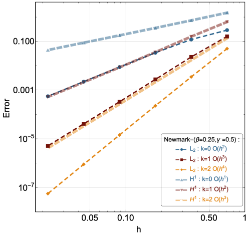
Runge–Kutta (RK) schemes

- Natural choice for first-order formulation in time
  - single diagonally implicit RK: SDIRK$(s, s+1)$ (s stages, order $(s+1)$)
  - explicit RK: ERK(s) (s stages, order s)

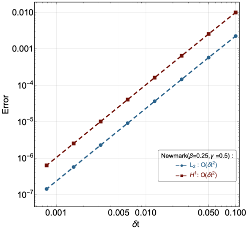
- ERK schemes subject to CFL stability condition $\frac{c\Delta t}{h} \leq \beta(s)\mu(k)$
  - $\beta(s)$ slightly increases with $s \in \{2, 3, 4\}$
  - $\mu(k)$ essentially behaves as $(k+1)^{-1}$ w.r.t. polynomial degree

- Smooth solution
- Newmark scheme (equal-order, quadrilateral mesh)

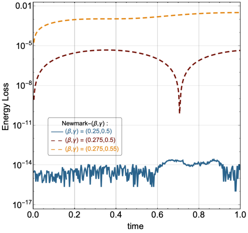
cv. in space



cv. in time

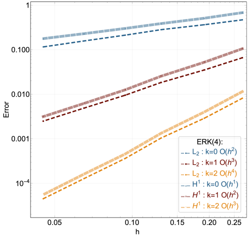


energy cons.

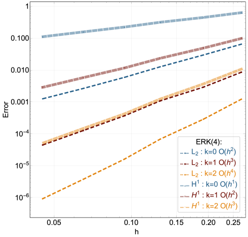


- SDIRK(3,4) and ERK(4) schemes (mixed-order, quad/poly meshes)
  - recall that $\tilde{\tau}_{\partial T} = O(1)$
  - we also consider over-penalty with $\tilde{\tau}_{\partial T} = O(h_T^{-1})$

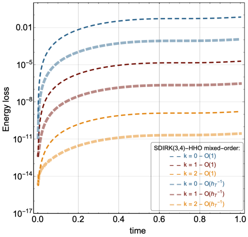
$\tilde{\tau}_{\partial T} = O(1)$, ERK(4), poly mesh



$\tilde{\tau}_{\partial T} = O(h_T^{-1})$, ERK(4), poly mesh



energy, SDIRK(3,4), quad mesh



- Energy dissipation strongly tempered by increasing polynomial degree

Numerical results: homogeneous media (2/2)

- SDIRK(3,4) and ERK(4) schemes (mixed-order, quad/poly meshes)
  - recall that $\tilde{\tau}_{\partial T} = O(1)$
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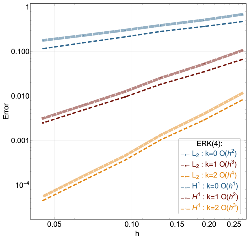
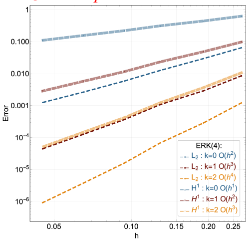
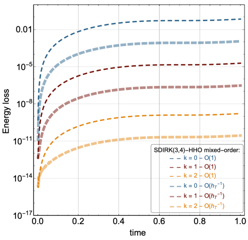


$\tilde{\tau}_{\partial T} = O(h_T^{-1})$, ERK(4), poly mesh



energy, SDIRK(3,4), quad mesh

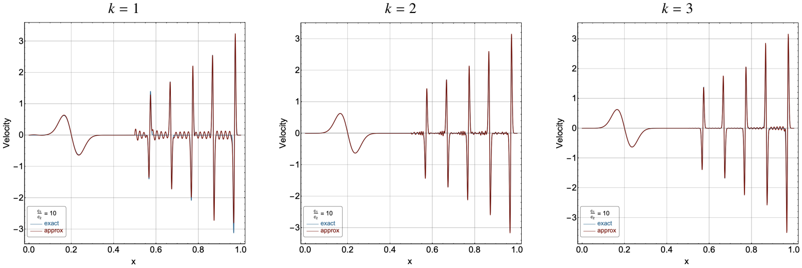


- Energy dissipation strongly tempered by increasing polynomial degree

- Discussion on $\tilde{\tau}_{\partial T}$
  - energy-error decays optimally as $O(h^{k+1})$ for both $\tilde{\tau}_{\partial T}$
    $\Rightarrow$ proof for (HHO, $O(h_T^{-1})$) and HDG, but using different tools
  - L^2-error decays optimally as $O(h^{k+2})$ only for $\tilde{\tau}_{\partial T} = O(h_T^{-1})$
    $\Rightarrow$ HDG, $\tilde{\tau}_{\partial T} = O(1)$, special post-proc. [Cockburn, Quenneville-Bélair 12]
  - $\tilde{\tau}_{\partial T} = O(h_T^{-1})$ worsens CFL condition for ERK schemes

Numerical results: heterogeneous media (1/3)

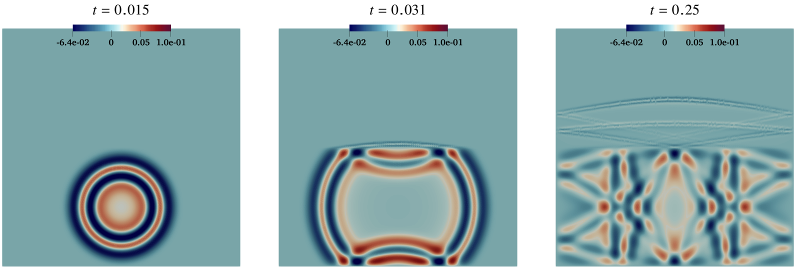
- 1D test case, $\Omega_1 = (0, 0.5)$, $\Omega_2 = (0.5, 1)$, $c_1/c_2 = 10$
  - initial Gaussian profile in Ω_1
  - analytical solution available (series)
- Benefits of increasing polynomial degree
  - Newmark scheme, equal-order, $k \in \{1, 2, 3\}$, $h = 0.1 \times 2^{-8}$, $\Delta t = 0.1 \times 2^{-9}$
  - HHO-Newmark solution at $t = \frac{1}{2}$ (after reflection/transmission at $x = \frac{1}{2}$)



Numerical results: heterogeneous media (2/3)

- 2D test case, Ricker (Mexican hat) wavelet
  - $\Omega_1 = (0,1) \times (0, \frac{1}{2})$, $\Omega_2 = (0,1) \times (\frac{1}{2}, 1)$, $c_1/c_2 = 5$
  - $p_0 = 0$, $v_0 = -\frac{4}{10}\sqrt{\frac{10}{3}}\left(1600\, r^2 - 1\right)\pi^{-\frac{1}{4}} \exp\left(-800 r^2\right)$,
    $r^2 = (x - x_c)^2 + (y - y_c)^2$, $(x_c, y_c) = (\frac{1}{2}, \frac{1}{4}) \in \Omega_1$
  - semi-analytical solution (infinite media): `gar6more2d` software (INRIA)

- HHO-SDIRK(3,4) velocity profiles
  - mixed-order, $k = 5$, polygonal meshes
  - $\Delta t = 0.025 \times 2^{-6}$ (four times larger than Newmark for similar accuracy)



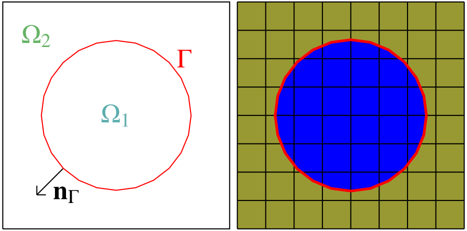
Numerical results: heterogeneous media (3/3)

- Comparison of computational efficiency
  - all schemes tuned to comparable max. rel. error on a sensor at $(\frac{1}{2}, \frac{2}{3})$
  - very preliminary results! (on-the-shelf solvers)
  - if no direct solvers allowed, ERK(4) wins despite CFL restriction
  - with direct solvers, SDIRK(3,4) wins
  - RK schemes more efficient than Newmark scheme
  - for SDIRK(3,4), $\tilde{\tau}_{\partial T} = O(h^{-1})$ more accurate/expensive than $\tilde{\tau}_{\partial T} = O(1)$

scheme	(k', k)	stab	solver	t/step	steps	time	err
ERK(4)	$(6, 5)$	$O(1)$	n/a	0.410	5,120	2,099	2.23
Newmark	$(7, 6)$	$O(h^{-1})$	iter	56.74	2,560	58,265	2.15
SDIRK(3, 4)	$(6, 5)$	$O(h^{-1})$	iter	31.24	640	5,639	2.21
SDIRK(3, 4)	$(6, 5)$	$O(1)$	iter	22.52	640	2,200	4.45
Newmark	$(7, 6)$	$O(h^{-1})$	direct	0.515	2,560	1,318	2.15
SDIRK(3, 4)	$(6, 5)$	$O(h^{-1})$	direct	1.579	640	1,010	2.21

Unfitted meshes

Elliptic interface problem



- Polytopal domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$
- Subdomains $\Omega_1, \Omega_2 \subset \Omega$ with different (contrasted) material properties
- Curved interface Γ, jump $[\![a]\!]_\Gamma = a_{|\Omega_1} - a_{|\Omega_2}$
- Model problem

$$\begin{aligned} -\operatorname{div}(\kappa \nabla u) = f & \quad \text{in } \Omega_1 \cup \Omega_2 \\ [\![u]\!]_\Gamma = g_D, \ [\![\kappa \nabla u]\!]_\Gamma \cdot \mathbf{n}_\Gamma = g_N & \quad \text{on } \Gamma \\ u = 0 & \quad \text{on } \partial\Omega \end{aligned}$$

- Everything can be adapted to a single domain with curved boundary

Motivation for unfitted meshes

- Use of unfitted meshes for interface problems
  - curved interface can cut arbitrarily through mesh cells
  - numerical method must deal with badly cut cells
- Classical FEM on unfitted meshes
  - double unknowns in cut cells and use a consistent Nitsche's penalty technique to enforce jump conditions [Hansbo, Hansbo 02]
  - ghost penalty [Burman 10] to counter bad cuts (gradient jump penalty across faces near curved boundary/interface)

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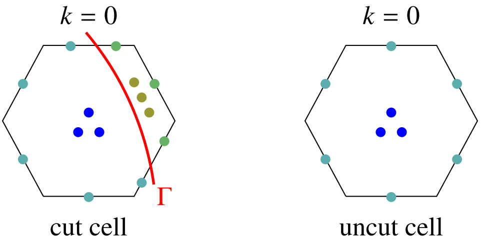
- An alternative to ghost penalty: local cell agglomeration
  - natural for polytopal methods as dG [Sollie, Bokhove, van der Vegt 11; Johansson, Larson 13]
  - cG agglomeration procedure in [Badia, Verdugo, Martín 18]

Unfitted HHO

- Main ideas [Burman, AE 18 (SINUM)]
  - double cell and face dofs in cut cells, no dofs on curved boundary/interface
  - local cell agglomeration to counter bad cuts
  - mixed-order setting: $k \geq 0$ for face dofs and $(k+1)$ for cell dofs

Unfitted HHO

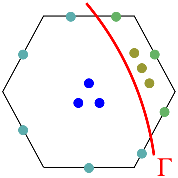
- Main ideas [Burman, AE 18 (SINUM)]
  - double cell and face dofs in cut cells, no dofs on curved boundary/interface
  - local cell agglomeration to counter bad cuts
  - mixed-order setting: $k \geq 0$ for face dofs and $(k+1)$ for cell dofs
- Improvements in [Burman, Cicuttin, Delay, AE 21 (SISC)]
  - novel gradient reconstruction, avoiding that the penalty parameter in Nitsche's method is large enough
  - robust cell agglomeration procedure (guaranteeing locality)
- Stokes interface problems [Burman, Delay, AE 20 (IMANUM)]
- Wave propagation [Burman, Duran, AE 21] `hal-03086432`



- Mesh still composed of polytopal cells (with planar faces)
- Decomposition of cut cells: $\overline{T} = \overline{T_1} \cup \overline{T_2}$, $T^\Gamma = T \cap \Gamma$
- Decomposition of cut faces: $\partial(T_i) = (\partial T)^i \cup T^\Gamma$, $i \in \{1, 2\}$
- Local dofs (no dofs on T^Γ!)

$$\hat{u}_T = (u_{T_1}, u_{T_2}, u_{(\partial T)^1}, u_{(\partial T)^2}) \in \mathbb{P}^{k+1}(T_1) \times \mathbb{P}^{k+1}(T_2) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^1}) \times \mathbb{P}^k(\mathcal{F}_{(\partial T)^2})$$

Gradient reconstruction in cut cells



- Gradient reconstruction $\mathbf{G}_{T_i}(\hat{u}_T) \in \mathbb{P}^k(T_i;\mathbb{R}^d)$ in each subcell
  - **(Option 1)** Independent reconstruction in each subcell

$$(\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\ \mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{T_i}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

  - **(Option 2)** Reconstruction mixing data from both subcells

$$(\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{q})_{T_i} = -(u_{T_i}, \mathrm{div}\ \mathbf{q})_{T_i} + (u_{(\partial T)^i}, \mathbf{q}\cdot\mathbf{n}_T)_{(\partial T)^i} + (u_{T_{3-i}}, \mathbf{q}\cdot\mathbf{n}_{T_i})_{T^\Gamma}$$

- Both options avoid Nitsche's consistency terms
  - no penalty parameter needs to be taken large enough!

Local bilinear form in cut cells

- Local bilinear form

$$a_T(\hat{u}_T, \hat{w}_T) := \sum_{i\in\{1,2\}} \Big\{\kappa_i(\mathbf{G}_{T_i}(\hat{u}_T), \mathbf{G}_{T_i}(\hat{w}_T))_{T_i} + s_{T_i}(\hat{u}_T, \hat{w}_T)\Big\} + s_T^\Gamma(u_T, w_T)$$

- LS stabilization inside each subdomain

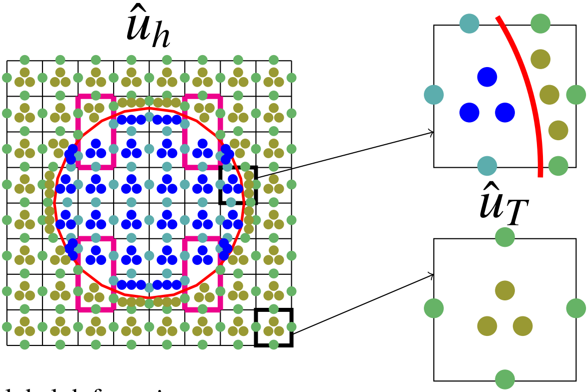
$$s_{T_i}(\hat{u}_T, \hat{w}_T) := \kappa_i h_{T_i}^{-1}(\Pi^k_{(\partial T)^i}(u_{T_i}|_{(\partial T)^i} - u_{(\partial T)^i}), w_{T_i}|_{(\partial T)^i} - w_{(\partial T)^i})_{(\partial T)^i}$$

- Interface bilinear form

$$s_T^\Gamma(u_T, w_T) := \eta\kappa_1 h_T^{-1}([\![u_T]\!]_\Gamma, [\![w_T]\!]_\Gamma)_{T^\Gamma} \text{ with } \eta = O(1)$$

- The use of two gradient reconstructions allows for robustness w.r.t. contrast ($\kappa_1 \ll \kappa_2$)
    - use option 1 in Ω_1 and option 2 in Ω_2
    - a_T is symmetric, but Ω_1/Ω_2 do not play symmetric roles

Global dofs



- The global dofs are in

$$\hat{u}_h \in \hat{U}_h := \bigtimes_{T\in\mathcal{T}^1} \mathbb{P}^{k+1}(T_1) \times \bigtimes_{T\in\mathcal{T}^2} \mathbb{P}^{k+1}(T_2) \times \bigtimes_{F\in\mathcal{F}^1} \mathbb{P}^{k}(F_1) \times \bigtimes_{F\in\mathcal{F}^2} \mathbb{P}^{k}(F_2)$$

- We set to zero all the face components attached to $\partial\Omega$
- We collect in $\hat{u}_T$ all the global unknowns related to a mesh cell T

Global dofs

- Global problem: Find $\hat{u}_h \in \hat{U}_h$ such that

$$a_h(\hat{u}_h, \hat{w}_h) = \ell_h(\hat{w}_h), \quad \forall \hat{w}_h \in \hat{U}_h$$

  with $a_h(\hat{u}_h, \hat{w}_h) = \sum_{T\in\mathcal{T}} a_T(\hat{u}_T, \hat{w}_T)$ and $\ell_h(\hat{w}_h) = \sum_{T\in\mathcal{T}} \ell_T(\hat{w}_T)$ with the consistent rhs

$$\begin{aligned}\ell_T(\hat{w}_T) := (f, w_{T_1})_{T_1} + (f, w_{T_2})_{T_2} + (g_N, w_{T_2})_{T^\Gamma} \\ - \kappa_1(g_D, \mathbf{G}_{T_1}(\hat{w}_T)\cdot\mathbf{n}_\Gamma + \eta h_T^{-1}[\![w_T]\!])_{T^\Gamma}\end{aligned}$$

- All the cell dofs are eliminated locally by static condensation
- Only the face dofs are globally coupled

Error analysis

- Multiplicative and discrete trace inequalities [Burman, AE 18]
  - for any cut cell T, there is a ball $T^\dagger$ of size $O(h_T)$ containing T and a finite number of its neighbors, and s.t. all $T \cap \Gamma$ is visible from a point in $T^\dagger$
  - small ball with diameter $O(h_T)$ present on both sides of interface
  - achievable using local cell agglomeration if mesh fine enough

Error estimate

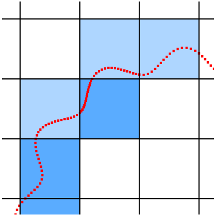
Assuming that $u|_{\Omega_i} \in H^{1+t}(\Omega_i)$ with $t \in (\frac{1}{2}, k+1]$,

$$\sum_T \sum_{i \in \{1,2\}} \kappa_i \|\nabla(u - u_{T_i})\|_{T_i}^2 \leq Ch^{2t} \sum_{i \in \{1,2\}} \kappa_i |u|_{H^{t+1}(\Omega_i)}^2$$

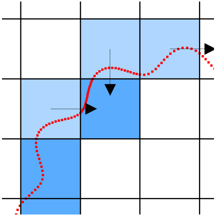
Convergence order $O(h^{k+1})$ if $u|_{\Omega_i} \in H^{k+2}(\Omega_i)$

Agglomeration procedure (1/3)

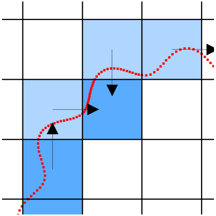
- Three-stage procedure with proven locality in the agglomeration
  1. for any cell KO in Ω_1, find matching partner OK in Ω_2
  2. for any cell KO in Ω_2 not matched, find matching partner OK in Ω_1
  3. rearrange locally partnerships to avoid propagation



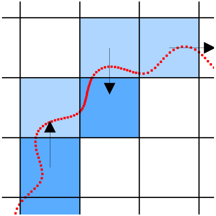
initial mesh



stage 1

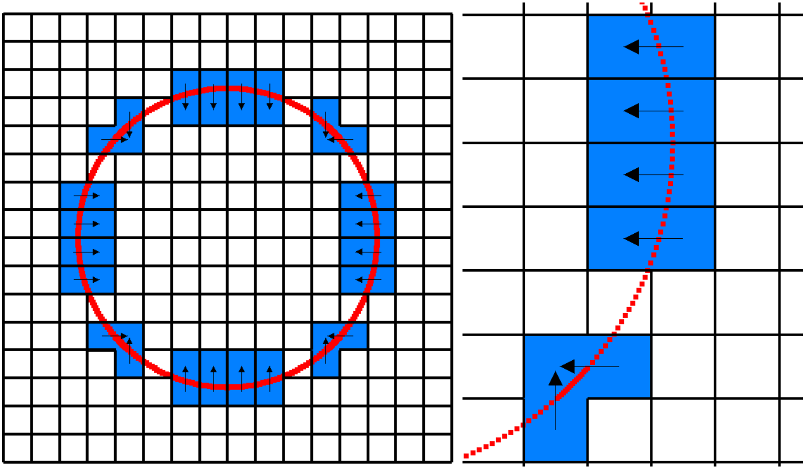


stage 2

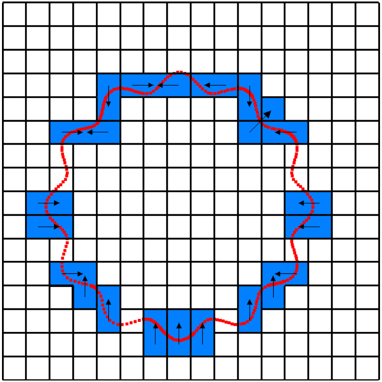


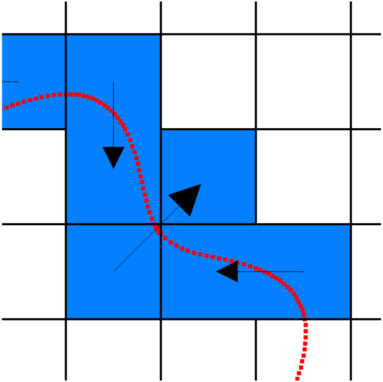
stage 3

- A 16x16 mesh with circular interface



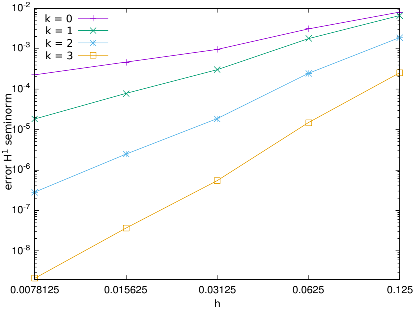
- A 16x16 mesh with flower-like interface

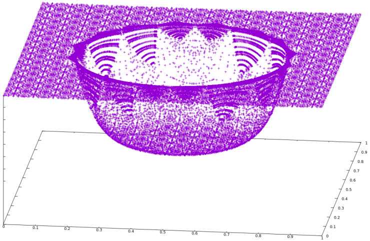




Test case with contrast

- $\kappa_1 = 1$, $\kappa_2 = 10^4$, $g_D = g_N = 0$, $\eta = 1$
- Circular interface ($r^2 = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$)
- Exact solution: $u_1 := \frac{r^6}{\kappa_1}$, $u_2 := \frac{r^6}{\kappa_2} + R^6(\frac{1}{\kappa_1} - \frac{1}{\kappa_2})$

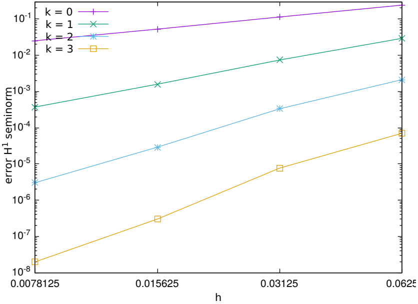


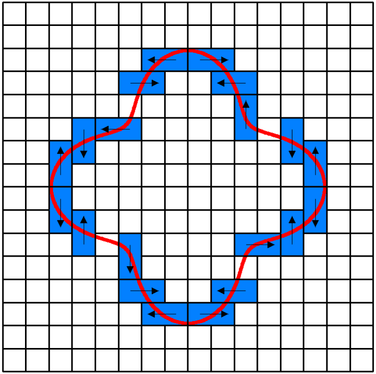


Test case with jump

- Flower-like interface, $\kappa_1 = \kappa_2 = 1$
- Exact solution with jump

$$u(x_1, x_2) := \begin{cases} \sin(\pi x_1)\sin(\pi x_2) & \text{in } \Omega_1 \\ \sin(\pi x_1)\sin(\pi x_2) + 2 + x^3 y^3 & \text{in } \Omega_2 \end{cases}$$





Wave propagation

- Subdomains $\Omega_1, \Omega_2 \subset \Omega$, interface Γ, jump $[\![a]\!]_\Gamma = a_{|\Omega_1} - a_{|\Omega_2}$
- Acoustic wave propagation across interface

$$\begin{cases} \frac{1}{\kappa}\partial_{tt}p - \operatorname{div}\left(\frac{1}{\rho}\nabla p\right) = f & \text{in } J \times (\Omega_1 \cup \Omega_2) \\ [\![p]\!]_\Gamma = 0, \ [\![\frac{1}{\rho}\nabla p]\!]_\Gamma \cdot \mathbf{n}_\Gamma = 0 & \text{on } J \times \Gamma \end{cases}$$

- Main ideas as for elliptic interface problems
  - mixed-order setting $k' = k + 1$
  - distinct gradient reconstructions $\mathbf{G}_{T_i}$ in $\mathbb{P}^k(T_i; \mathbb{R}^d)$, $i \in \{1, 2\}$
  - LS stabilization on $(\partial T)^i$, $i \in \{1, 2\} \Longrightarrow s_{T_i}(\cdot, \cdot)$

Unfitted HHO discretization

- Second-order formulation

$$(\partial_{tt} p_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + (\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t)), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\frac{1}{\rho};\Omega} + s_h^{1,2}(\hat{p}_h(t), \hat{w}_h) + s_h^{\Gamma}(p_{\mathcal{T}}(t), w_{\mathcal{T}}) = (f(t), w_{\mathcal{T}})_{\Omega}$$

  - $s_h^{\Gamma}(p_{\mathcal{T}}(t), w_{\mathcal{T}}) := (\rho_1 h_T)^{-1}([\![p_T]\!]_{\Gamma}, [\![w_T]\!]_{\Gamma})_{T^{\Gamma}}$
  - Algebraic realization and Newmark time-stepping as in fitted case

Unfitted HHO discretization

- Second-order formulation

$$(\partial_{tt} p_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + (\mathbf{G}_{\mathcal{T}}(\hat{p}_h(t)), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\frac{1}{\rho};\Omega} + s_h^{1,2}(\hat{p}_h(t), \hat{w}_h) + s_h^{\Gamma}(p_{\mathcal{T}}(t), w_{\mathcal{T}}) = (f(t), w_{\mathcal{T}})_{\Omega}$$

  - $s_h^{\Gamma}(p_{\mathcal{T}}(t), w_{\mathcal{T}}) := (\rho_1 h_T)^{-1}([\![p_T]\!]_{\Gamma}, [\![w_T]\!]_{\Gamma})_{T^{\Gamma}}$
  - Algebraic realization and Newmark time-stepping as in fitted case

- First-order formulation ($v := \partial_t p$, $\boldsymbol{\sigma} := \frac{1}{\rho}\nabla p$)

$$\begin{cases} (\partial_t \boldsymbol{\sigma}_{\mathcal{T}}(t), \boldsymbol{\tau}_{\mathcal{T}})_{\rho;\Omega} - (\mathbf{G}_{\mathcal{T}}(\hat{v}_h(t)), \boldsymbol{\tau}_{\mathcal{T}})_{\Omega} = 0 \\ (\partial_t v_{\mathcal{T}}(t), w_{\mathcal{T}})_{\frac{1}{\kappa};\Omega} + (\boldsymbol{\sigma}_{\mathcal{T}}(t), \mathbf{G}_{\mathcal{T}}(\hat{w}_h))_{\Omega} + \tilde{s}_h^{1,2}(\hat{v}_h(t), \hat{w}_h) + \tilde{s}_h^{\Gamma}(v_{\mathcal{T}}(t), w_{\mathcal{T}}) = (f(t), w_{\mathcal{T}})_{\Omega} \end{cases}$$

  - $\tilde{s}_h^{\Gamma}(v_{\mathcal{T}}(t), w_{\mathcal{T}}) := \sum_{T \in \mathcal{T}_h} \tilde{\tau}_{\partial T}^{\Gamma}([\![v_T]\!]_{\Gamma}, [\![w_T]\!]_{\Gamma})_{T^{\Gamma}}$
  - $\tilde{\tau}_{\partial T}^{\Gamma} = (\rho_1 c_1)^{-1} = O(1)$ for ERK, and $\tilde{\tau}_{\partial T}^{\Gamma} = O(h_T^{-1})$ for SDIRK
  - Algebraic realization and RK time-stepping as in fitted case

Fitted-unfitted comparison

- 2D heterogeneous test case with **flat interface**
  - $\Omega_1 := (-\frac{3}{2}, \frac{3}{2}) \times (-\frac{3}{2}, 0)$, $\Omega_2 := (-\frac{3}{2}, \frac{3}{2}) \times (0, \frac{3}{2})$
  - Ricker wavelet centered at $(0, \frac{2}{3}) \in \Omega_2$, sensor $S_1 = (\frac{3}{4}, -\frac{1}{3}) \in \Omega_1$
  - fitted and unfitted HHO behave similarly, both benefit from increasing k

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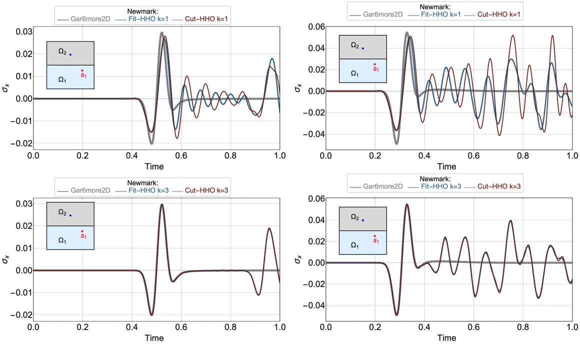
- HHO-Newmark, σ_x signals
  - comparison of semi-analytical and HHO (fitted or unfitted) solutions
  - $k = 1$ (top) and $k = 3$ (bottom)
  - $c_2/c_1 = \sqrt{3}$ (low contrast, left) or $c_2/c_1 = 8\sqrt{3}$ (high contrast, right)



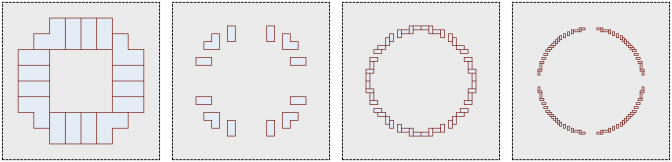
CFL condition for ERK (1/2)

- Homogeneous test case, flat interface
- CFL condition for ERK(s): $\frac{c\Delta t}{h} \leq \beta(s)\mu(k)$
  - $\beta(s)$ mildly depends on the number of stages
  - $\mu(k)$ behaves as $(k+1)^{-1}$ and is quantified by solving a generalized eigenvalue problem with the mass and stiffness matrices
- Additional jump penalties in unfitted HHO only mildly impact $\mu(k)$

k	0	1	2	3
Fitted-HHO	0.118	0.0522	0.0338	0.0229
Unfitted-HHO	0.0765	0.0373	0.0232	0.0159
Ratio	1.5	1.4	1.5	1.4

CFL condition for ERK (2/2)

- Homogeneous test case, circular interface
  - study of impact of agglomeration parameter θ_{agg} on $\mu(k)$
  - "badly cut" cell flagged if relative area of any subcell falls below θ_{agg}
- Agglomerated cells for $\theta_{\text{agg}} = 0.3$ on a sequence of refined quad meshes



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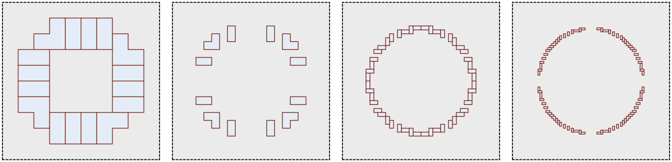
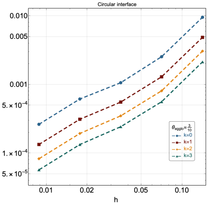


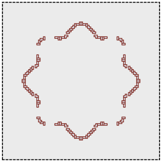
- Behavior of $h\mu(k)$ and impact of θ_{agg} on $\mu(k)$
  - tolerating badly cut cells deteriorates the CFL condition

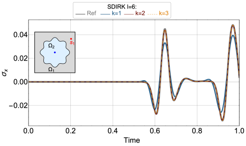


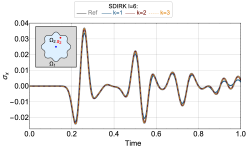
k	0	1	2	3
$\theta_{\text{agg}} = 0.5$	0.042	0.022	0.014	0.0099
$\theta_{\text{agg}} = 0.3$	0.030	0.015	0.0094	0.0065
Ratio	1.4	1.5	1.5	1.5
$\theta_{\text{agg}} = 0.1$	0.017	0.0087	0.0055	0.0039
Ratio	2.5	2.6	2.6	2.5

Flower-like interface

- Agglomerated cells for a flower-like interface (quad mesh, $h = 2^{-5}$), HHO-SDIRK(3,4) signal for σ_x at two sensors, $k \in \{1, 2, 3\}$, $c_2/c_1 = \sqrt{3}$

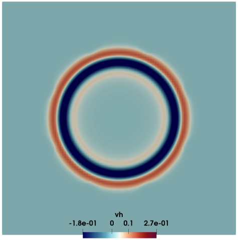




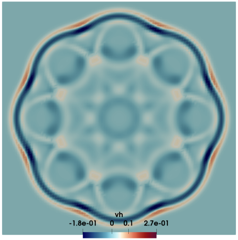


- Pressure isovalues, SDIRK(3,4), $k = 3$, $h = 0.1 \times 2^{-8}$, $\Delta t = 2^{-6}$

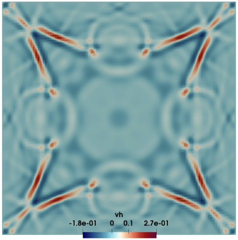
$t = 0.25$



$t = 0.5$



$t = 1$



Some references

- HHO
  - seminal papers [Di Pietro, AE, Lemaire 14; Di Pietro, AE 15]
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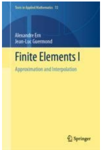

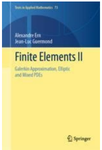

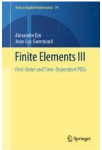

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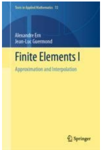

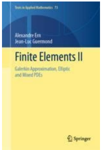

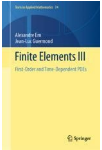

Thank you for your attention!