L^2-stability of explicit Runge–Kutta methods with SUPG stabilization for transient transport problems

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SFB Workshop, Vienna, 16-19/02/2026

Outline

- Setting
- Main result and illustrations
- Abstract proof: ERK3 and ERK4
- Pedestrian proof: ERK3
- Reference
  - [AE & JLG, hal-05363804, 2025]

Setting

- Model problem
- Finite element setting and SUPG
- Time discretization using ERK

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- **Hille–Yosida Theorem:**

$$\exists! u \in C^0(J; V) \cap C^1(J; L)$$

Reaction-advection equation

- $\mathcal{D} \subset \mathbb{R}^d$, open, bounded, connected, Lipschitz subset
- $\boldsymbol{\beta} \in \boldsymbol{L}^\infty(\mathcal{D})$, $\nabla\cdot\boldsymbol{\beta} \in L^\infty(\mathcal{D})$, $\mu \in L^\infty(\mathcal{D})$ with $\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} \geq \mu_0 > 0$ in $\mathcal{D}$

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with $\partial\mathcal{D}^\pm := \{\boldsymbol{x} \in \partial\mathcal{D} \mid \pm(\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{x}) > 0\}$

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- $\mathcal{L} : V \to L$ is maximal monotone [AE & JLG, 06]
- Monotonicity (dissipativity) takes the form

$$\langle \mathcal{L}(v), v\rangle_{L^2} = \int_{\mathcal{D}} \left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) v^2 dx + \int_{\partial\mathcal{D}^+} (\boldsymbol{\beta}\cdot\boldsymbol{n}) v^2 ds \geq 0 \quad \forall v \in V$$

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- Literature
  - [Brooks & Hughes 82 (SUPG)], [Hughes, Franca & Hulbert 89 (GaLS)]
  - SUPG = GaLS in the absence of diffusion
  - analysis in [Johnson, Nävert & Pitkäranta 84]: $O(h^{k+\frac{1}{2}})$ L^2-error bound
  - textbooks: [Roos, Stynes & Tobiska 08; AE & JLG 04 & 21 (Vol III)]

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    - SUPG-norm $|||v_h|||^2 := \|v_h\|_{L^2}^2 + \delta^2 \|\mathcal{L}(v_h)\|_{L^2}^2$
    - main stability result (assuming $\delta^2 \lesssim \tau$)

$$|||u_h^n||| \leq C \left\{ |||u_h^0||| + t^n \|f\|_{C^0([t^0,t^n];L^2)} \right\}$$

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  - Further work
    - CN [Burman 10], dG and cPG in time [Ahmed & Matthies 15-16]
    - space-time LS [Besson & de Montmollin 04; Perrochet & Azérad 95]

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- Forward Euler will not work
- ERK3 and ERK4 have better stability properties . . .

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$$\begin{array}{c|ccccc} 0 & 0 & & & & \\ c_2 & a_{2,1} & 0 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ c_s & a_{s,1} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & \cdots & b_{s-1} & b_s \end{array} \longrightarrow \widetilde{A} := (\widetilde{a}_{ij})_{i,j\in\{1:s\}} = \begin{pmatrix} a_{2,1} & & & \\ \vdots & \ddots & & \\ a_{s,1} & \cdots & a_{s,s-1} & \\ a_{s+1,1} & \cdots & a_{s+1,s-1} & a_{s+1,s} \end{pmatrix}$$

  - $a_{s+1,j} := b_j$ for all $j \in \{1{:}s\}$
  - order condition implies $\det(\widetilde{A}) = \frac{1}{s!}$, i.e., $\widetilde{A}$ invertible
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- ERK + plain Galerkin: For all $i \in \{1:s\}$ and all $v_h \in V_h$, (with $u_h^{n,0} := u_h^n$)

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Main result and illustrations

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- More precise result for ERK4:

$$\|\!|u_h^n|\!\| \le \max(\|\!|u_h^0|\!\|, \|\!|u_h^1|\!\|) + Ct^n \|f\|_{C^0([t^0,t^n];L^2)}$$

CFL condition gives $\|\!|u_h^1|\!\| \le C\|\!|u_h^0|\!\|$

2D transport with smooth solution

- $\mathcal{D} := (0,1)^2$, $\boldsymbol{\beta} := (1,1)^\top$, Gaussian-like IC
- $\gamma = 0.04$, $c_{\mathcal{L}} = ((k+1)(k+2))^{\frac{1}{2}}$, $k \in \{1,2,3\}$
- Courant number for each ERK3 stage set to 0.25

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- L^1-error at $T = 0.5$

$\mathbb{P}_1$ approximation			$\mathbb{P}_2$ approximation			$\mathbb{P}_3$ approximation		
I	L^1 err	rate	I	L^1 err	rate	I	L^1 err	rate
961	2.97E-01	2.96	961	2.76E-01	–	961	6.26E-01	–
3721	4.95E-02	2.65	1681	1.48E-01	2.22	3721	9.38E-02	2.80
8281	1.44E-02	3.08	6561	3.92E-02	1.95	8281	2.65E-02	3.16
14641	5.50E-03	3.39	14641	1.66E-02	2.14	14641	8.58E-03	3.96
32761	1.26E-03	3.66	32761	6.20E-03	2.45	32761	1.61E-03	4.15

- CV rates better than, or close to, expected rate $O(h^{k+\frac{1}{2}})$

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- Classical IC profile combining cylinder, cone and smooth bumps
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- Courant number for each ERK3 stage set to 0.25
- L^1-error at $T = 1$

$\mathbb{P}_1$ approximation			$\mathbb{P}_2$ approximation			$\mathbb{P}_3$ approximation		
I	L^1 err	rate	I	L^1 err	rate	I	L^1 err	rate
453	3.92E-01	–	458	3.43E-01	–	460	4.50E-01	–
1716	2.39E-01	0.75	1746	1.95E-01	0.84	1711	2.44E-01	0.93
6497	1.35E-01	0.85	6735	1.12E-01	0.82	6751	1.29E-01	0.93
25049	8.04E-02	0.77	25734	6.75E-02	0.75	27091	7.54E-02	0.78
98752	4.72E-02	0.78	99690	3.98E-02	0.78	99859	4.59E-02	0.76

- Close to expected rate $O(h)$ in L^1-norm
- Errors and rates for $\mathbb{P}_2$ and $\mathbb{P}_3$ better than [Guermond, Nazarov & Popov 24]
  $\Longrightarrow$ benefit of using linear stabilization for nonsmooth problems

Fourier analysis in 1D

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$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3, \qquad \Lambda(\theta) := \lambda \frac{2\gamma(\cos(\theta) - 1) - \mathrm{i}\sin(\theta)}{\frac{1}{3}(\cos(\theta) + 2) - \mathrm{i}\gamma \sin(\theta)}$$

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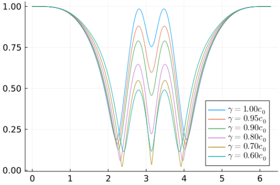
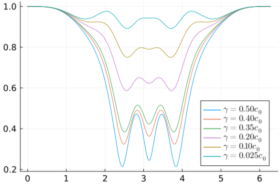
- $|g_3(\theta)|$ as a function of θ for decreasing values of $\lambda\gamma$





- $|g_3(\theta)| \leq 1$ for all $\lambda \in (0, 1]$ and $\lambda\gamma \in (0, c_0 := 0.196]$

Abstract proof: ERK3 and ERK4

Stability of ERK3 and ERK4 schemes (1/2)

- Equip V_h with inner product $(\!(\bullet, \bullet)\!)$, associated norm $|\!|\!| \bullet |\!|\!| := (\!(\bullet, \bullet)\!)^{\frac{1}{2}}$

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- Consider linear operator $C_h : V_h \longrightarrow V_h$ satisfying
  - (dissipativity) $(\!(C_h(v_h), v_h)\!) \geq 0$
  - (inverse inequality) $|\!|\!|C_h(v_h)|\!|\!| \leq c_* h^{-1} |\!|\!|v_h|\!|\!|$

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- Canonical ERK scheme: For all $i \in \{1{:}s\}$ and all $v_h \in V_h$,

$$(\!(u_h^{n,i} - u_h^{n,0}, w_h)\!) + \tau_n \sum_{j \in \{1:i\}} \widetilde{a}_{i,j} (\!(C_h(u_h^{n,j-1}) - g_h^{n,j-1}, w_h)\!) = 0$$

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- **Theorem.** [AE & JLG 25] Let $P_s(z) := \sum_{l \in \{0:s\}} \frac{1}{l!} z^l$. There is $\sigma_0 > 0$ so that, for all $\lambda \leq \sigma_0$,

$$|\!|\!|P_3(-\tau_n C_h)|\!|\!|_{\mathcal{L}(V_h)} \leq 1, \qquad |\!|\!|P_4(-\tau_n C_h)^2|\!|\!|_{\mathcal{L}(V_h)} \leq 1$$

Seminal ideas in [Sun & Shu 19; Xu, Zhang, Shu & Wang 19]

Stability of ERK3 and ERK4 schemes (2/2)

- s-stage ERK scheme rewrites

$$u_h^{n+1} = P_s(-\tau_n C_h)(u_h^n) + \tau_n \sum_{r\in\{1:s\}} (-\tau_n C_h)^{r-1}(\Phi_r(g_h))$$

with $\Phi_r(g_h) := \sum_{j\in\{1:s-r+1\}}(bA^{r-1})_j\, g_h^{n,j-1}$

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- **Corollary.** (above theorem, triangle ineq., inverse ineq., CFL condition)

$$\text{(ERK3)} \qquad |||u_h^{n+1}||| \leq |||u_h^n||| + C\tau_n \max_{t\in[t^n,t^{n+1}]} \|g_h(t)\|_{L^2}$$

$$\text{(ERK4)} \qquad |||u_h^{n+1}||| \leq |||u_h^{n-1}||| + C\tau_n \max_{t\in[t^{n-1},t^{n+1}]} \|g_h(t)\|_{L^2}$$

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- Summing over n gives

$$\text{(ERK3)} \qquad \|u_h^n\| \le \|u_h^0\| + Ct^n \|g_h\|_{C^0([t^0,t^n];L^2)}$$

$$\text{(ERK4)} \qquad \|u_h^n\| \le C\left\{\|u_h^0\| + t^n \|f\|_{C^0([t^0,t^n];L^2)}\right\}$$

Fitting ERK-SUPG into canonical form

- Recall ERK-SUPG scheme

$$\langle u_h^{n,i} - u_h^{n,0}, v_h + \delta\mathcal{L}(v_h)\rangle_{L^2} + \tau_n \sum_{j\in\{1:i\}} \widetilde{a}_{i,j} \langle \mathcal{L}(u_h^{n,j-1}) - f^{n,j-1}, v_h + \delta\mathcal{L}(v_h)\rangle_{L^2} = 0$$

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- We reach the canonical form by defining $C_h : V_h \to V_h$ s.t.

$$((C_h(v_h), w_h)) := ((\mathcal{A}_h(v_h), B_h^{-1}(w_h)))$$

(We shall see that B_h is indeed invertible.)

Properties of C_h and g_h

- C_h is dissipative

$$((C_h(v_h), v_h)) \geq 0 \quad \forall v_h \in V_h$$

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- These bounds **prove our main result** for ERK3 and ERK4 for $\gamma \leq \frac{1}{2}$

Proof insights: Properties of B_h

- Recall $(\!(B_h(v_h), w_h)\!) := \langle v_h + \delta\mathcal{L}(v_h), w_h \rangle_{L^2}, \forall v_h, w_h \in V_h$

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  2. $\implies (\!(B_h(v_h) - v_h, w_h)\!) = \delta \langle \mathcal{L}(v_h), w_h - \delta \mathcal{L}(w_h) \rangle_{L^2}$

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  2. $\Longrightarrow (\!(B_h(v_h) - v_h, w_h)\!) = \delta\langle\mathcal{L}(v_h), w_h - \delta\mathcal{L}(w_h)\rangle_{L^2}$
  3. $\mathcal{L}$ dissip. $\Longrightarrow \|w_h - \delta\mathcal{L}(w_h)\|_{L^2}^2 = |\!|\!|w_h|\!|\!|^2 - 2\delta\langle\mathcal{L}(w_h), w_h\rangle_{L^2} \leq |\!|\!|w_h|\!|\!|^2$

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  2. $\implies ((B_h(v_h) - v_h, w_h)) = \delta\langle\mathcal{L}(v_h), w_h - \delta\mathcal{L}(w_h)\rangle_{L^2}$
  3. $\mathcal{L}$ dissip. $\implies \|w_h - \delta\mathcal{L}(w_h)\|_{L^2}^2 = |||w_h|||^2 - 2\delta\langle\mathcal{L}(w_h), w_h\rangle_{L^2} \leq |||w_h|||^2$
  4. (Step 2 + CS) $\implies ((B_h(v_h) - v_h, w_h)) \leq \delta\|\mathcal{L}(v_h)\|_{L^2}|||w_h|||$

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- **Property 2:** $|\!|\!|B_h(v_h)|\!|\!| \geq (1-\gamma)|\!|\!|v_h|\!|\!|$
  1. $|\!|\!|B_h(v_h) - v_h|\!|\!| \leq \delta\|\mathcal{L}(v_h)\|_{L^2} \leq \gamma c_{\mathcal{L}}^{-1} h c_{\mathcal{L}} h^{-1}\|v_h\|_{L^2} \leq \gamma\|v_h\|_{L^2}$

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- **Property 2:** $|\!|\!| B_h(v_h) |\!|\!| \geq (1 - \gamma) |\!|\!| v_h |\!|\!|$
  1. $|\!|\!| B_h(v_h) - v_h |\!|\!| \leq \delta \|\mathcal{L}(v_h)\|_{L^2} \leq \gamma c_{\mathcal{L}}^{-1} h c_{\mathcal{L}} h^{-1} \|v_h\|_{L^2} \leq \gamma \|v_h\|_{L^2}$
  2. $|\!|\!| B_h(v_h) |\!|\!| \geq |\!|\!| v_h |\!|\!| - |\!|\!| B_h(v_h) - v_h |\!|\!| \geq |\!|\!| v_h |\!|\!| - \gamma \|v_h\|_{L^2} \geq (1 - \gamma) |\!|\!| v_h |\!|\!|$

Proof insights: Properties of B_h

- Recall $((B_h(v_h), w_h)) := \langle v_h + \delta\mathcal{L}(v_h), w_h\rangle_{L^2}, \forall v_h, w_h \in V_h$

- **Property 1:** $\|\!|B_h(v_h) - v_h|\!\| \le \delta\|\mathcal{L}(v_h)\|_{L^2}$
  1. $((B_h(v_h), w_h)) = ((v_h, w_h)) + \delta\langle\mathcal{L}(v_h), w_h - \delta\mathcal{L}(w_h)\rangle_{L^2}$
  2. $\implies ((B_h(v_h) - v_h, w_h)) = \delta\langle\mathcal{L}(v_h), w_h - \delta\mathcal{L}(w_h)\rangle_{L^2}$
  3. $\mathcal{L}$ dissip. $\implies \|w_h - \delta\mathcal{L}(w_h)\|^2_{L^2} = \|\!|w_h|\!\|^2 - 2\delta\langle\mathcal{L}(w_h), w_h\rangle_{L^2} \le \|\!|w_h|\!\|^2$
  4. (Step 2 + CS) $\implies ((B_h(v_h) - v_h, w_h)) \le \delta\|\mathcal{L}(v_h)\|_{L^2}\|\!|w_h|\!\|$

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  2. $\|\!|B_h(v_h)|\!\| \ge \|\!|v_h|\!\| - \|\!|B_h(v_h) - v_h|\!\| \ge \|\!|v_h|\!\| - \gamma\|v_h\|_{L^2} \ge (1-\gamma)\|\!|v_h|\!\|$

- **Corollary.** B_h invertible for $\gamma < 1$

Proof insights: C_h is dissipative

- Recall $((C_h(v_h), v_h)) = ((\mathcal{A}_h(v_h), B_h^{-1}(v_h)))$ and set $w_h := B_h^{-1}(v_h)$ so that

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- $-\frac{1}{2\delta} |\!|\!| B_h(w_h) - w_h |\!|\!|^2 + \frac{\delta}{2} \| \mathcal{L}(w_h) \|_{L^2}^2 \geq 0$ and $\mathcal{L}$ dissipative $\implies ((C_h(v_h), v_h)) \geq 0$

Pedestrian proof: ERK3

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- Test ERK stages by
  - the solution at different stages (coercivity argument)
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- Can we reproduce this for explicit schemes with multiple stages?

ERK rewriting in incremental form

- Recall ERK scheme

$$\langle u_h^{n,i} - u_h^{n,0}, v_h + \delta\mathcal{L}(v_h)\rangle_{L^2} + \tau_n \sum_{j\in\{1:i\}} \widetilde{a}_{i,j} \langle \mathcal{L}(u_h^{n,j-1}) - f^{n,j-1}, v_h + \delta\mathcal{L}(v_h)\rangle_{L^2} = 0$$

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- Setting $\Delta u_h^{n,i} := u_h^{n,i} - u_h^{n,i-1}$ and $\widehat{a}_{i,j} := \widetilde{a}_{i,j} - \widetilde{a}_{i-1,j}$ gives

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- **Compact notation in $\mathbb{R}^s$:**

$$\langle [\Delta u_h^n], [v_h] + \delta\mathcal{L}[v_h]\rangle_{\ell^2(L^2)} + \tau_n\langle \widehat{A}(\mathcal{L}[\check{u}_h^n] - [\check{f}^n]), [v_h] + \delta\mathcal{L}[v_h]\rangle_{\ell^2(L^2)} = 0$$

with

$$[\Delta u_h^n] := \begin{pmatrix} \Delta u_h^{n,1} \\ \vdots \\ \Delta u_h^{n,s} \end{pmatrix} \qquad [\check{u}_h^n] := \begin{pmatrix} u_h^{n,0} \\ \vdots \\ u_h^{n,s-1} \end{pmatrix} \qquad [\check{f}^n] := \begin{pmatrix} f^{n,0} \\ \vdots \\ f^{n,s-1} \end{pmatrix}$$

Incrementally r-quasi-stable matrices

- Recall incremental ERK scheme

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- We want to test with $[v_h] := B[\check{u}_h^n]$ and with $[v_h] := \frac{\delta}{\tau_n} B' [\Delta u_h^n]$ for suitable matrices $B, B' \in \mathbb{R}^{s,s}$

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- B is said to be **incrementally r-quasi-stable** if there is $r \in \{2{:}s\}$ s.t.

$$\langle B[\breve{u}_h^n], [\Delta u_h^n]\rangle_{\ell^2} = \frac{1}{2}\left\{|u_h^{n,s}|^2 - |u_h^{n,0}|^2\right\} + \sum_{l \in \{r:s\}} \beta_l |\Delta u_h^{n,l}|^2, \qquad \beta_r > 0$$

Incrementally r-quasi-stable matrices

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- We want to test with $[v_h] := B[\check{u}_h^n]$ and with $[v_h] := \frac{\delta}{\tau_n} B' [\Delta u_h^n]$ for suitable matrices $B, B' \in \mathbb{R}^{s,s}$

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$$\langle B[\check{u}_h^n], [\Delta u_h^n] \rangle_{\ell^2} = \frac{1}{2} \left\{ |u_h^{n,s}|^2 - |u_h^{n,0}|^2 \right\} + \sum_{l \in \{r:s\}} \beta_l |\Delta u_h^{n,l}|^2, \qquad \beta_r > 0$$

- Example for $s = 3$ [Burman, AE & Fernandez]

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}, \qquad B \begin{pmatrix} u_h^{n,0} \\ u_h^{n,1} \\ u_h^{n,2} \end{pmatrix} \cdot \begin{pmatrix} u_h^{n,1} - u_h^{n,0} \\ u_h^{n,2} - u_h^{n,1} \\ u_h^{n,3} - u_h^{n,2} \end{pmatrix} = \frac{1}{2} \left\{ |u_h^{n,3}|^2 - |u_h^{n,0}|^2 \right\} + \frac{1}{6} |\Delta u_h^{n,2}|^2 - \frac{1}{2} |\Delta u_h^{n,3}|^2$$

The matrices C and B'

- We require that the matrix $C := B^\mathsf{T}\widehat{A} \in \mathbb{R}^{s,s}$ is symmetric
- We require that $\langle C[\check{u}_h^n], [\check{u}_h^n]\rangle_{\ell^2} \geq -\kappa \sum_{l\in\{r:s\}} |\Delta u_h^{n,l}|^2$ for some $\kappa \geq 0$

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  - One can check that

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!! Thank you for your attention !!