# Edge finite element approximation of Maxwell's equations with low regularity solutions

Alexandre Ern and Jean-Luc Guermond

**Université Paris-Est, CERMICS, ENPC and INRIA, Paris**

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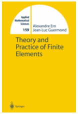
# Outline

- Maxwell's equations
- Edge FEM discretization
- Analysis tools
- Back to Maxwell's equations
- Nonconforming approximation of elliptic PDEs

# Disclaimer/announcement

- Some of the contents are possibly textbook material ...
- **Shrinking-based mollification operators**
  - you can live without them ...
- **Averaging quasi-interpolation**
  - decay rates for best-approximation with minimal Sobolev regularity
- **Nonconforming error analysis** (elliptic PDEs)
  - novel extension of the flux at faces
- New Finite Element book(s) (Fall 2018)
  - 10 chapters of 50 pages $\rightarrow$ 65 chapters of 14 pages with exercices




$\Longrightarrow$

# Maxwell's equations

- Lipschitz polyhedron $D \subset \mathbb{R}^3$ with simple topology
- Model problem: Find $\boldsymbol{A} : D \to \mathbb{C}^3$ s.t.

$$\mu \boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}, \qquad \boldsymbol{A}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0} \text{ (for simplicity)}$$

- Assumptions on $\mu$ and $\kappa$
  - **boundedness:** $\mu, \kappa \in L^\infty(D;\mathbb{C})$, set $\mu_\sharp = \|\mu\|_{L^\infty}$, $\kappa_\sharp = \|\kappa\|_{L^\infty}$
  - **positivity:** there are real numbers $\theta$, $\mu_\flat > 0$, $\kappa_\flat > 0$ s.t.

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\mu(\boldsymbol{x})) \geq \mu_\flat, \qquad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\kappa(\boldsymbol{x})) \geq \kappa_\flat$$

  - **heterogeneous medium**: $\mu$ and $\kappa$ can have jumps, but are pcw. smooth ($W^{1,\infty}$) on a Lipschitz partition of $D$
  - no tracking of contrast factors $\mu_{\sharp/\flat} = \mu_\sharp/\mu_\flat$, $\kappa_{\sharp/\flat} = \kappa_\sharp/\kappa_\flat$
- Assumptions on source term: $\boldsymbol{f} \in \boldsymbol{L}^2(D)$ and $\nabla\cdot\boldsymbol{f} = 0 \Longrightarrow$

$$\nabla\cdot(\mu\boldsymbol{A}) = 0$$

# Two examples

- Time-harmonic regime (frequency $\omega$)
- Source current $\boldsymbol{j}_s$
- **Helmholtz problem: $\boldsymbol{A} = \boldsymbol{E}$**

$$\mu = -\omega^2\epsilon + i\omega\sigma, \quad \kappa = \hat{\mu}^{-1}, \quad \boldsymbol{f} = -i\omega\boldsymbol{j}_s$$

$\epsilon$: electric permittivity, $\hat{\mu}$: magnetic permeability, $\sigma$: electric conductivity

- **Eddy-current problem: $\boldsymbol{A} = \boldsymbol{H}$**

$$\mu = i\omega\hat{\mu}, \quad \kappa = \sigma^{-1}, \quad \boldsymbol{f} = \nabla\times(\sigma^{-1}\boldsymbol{j}_s)$$

# Basic functional setting

- $\boldsymbol{V}_0 = \boldsymbol{H}_0(\text{curl}; D) = \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(D),\ \boldsymbol{v}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}\}$
  - norm $\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\text{curl};D)} = \|\boldsymbol{v}\|^2_{L^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{v}\|^2_{L^2(D)}$
  - $\ell_D$ is a characteristic length of $D$ (for dimensional coherence)
- Weak formulation: Find $\boldsymbol{A} \in \boldsymbol{V}_0$ s.t. $a(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b})$, $\forall \boldsymbol{b} \in \boldsymbol{V}_0$

$$a(\boldsymbol{A}, \boldsymbol{b}) = \int_D (\mu\boldsymbol{A}\cdot\overline{\boldsymbol{b}} + \kappa\nabla\times\boldsymbol{A}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x, \quad \ell(\boldsymbol{b}) = \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x$$

- $a(\cdot,\cdot)$ is bounded and coercive on $\boldsymbol{V}_0$ (Lax–Milgram Lemma)

$$Re(\mathrm{e}^{i\theta}a(\boldsymbol{b}, \boldsymbol{b})) \geq \min(\mu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\text{curl};D)}$$

- Coercivity parameter **not robust w.r.t.** $\mu_\flat$; this is relevant
  - in the low-frequency limit for the eddy-current problem
  - in the limit $\sigma \ll \omega\epsilon$ with $\kappa \in \mathbb{R}$ for the Helmholtz problem

# Control on the divergence

- Since $\nabla\cdot\boldsymbol{f} = 0$, we have $\nabla\cdot(\mu\boldsymbol{A}) = 0$, so that

$$\boldsymbol{A} \in \boldsymbol{X}_{0\mu} = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid (\mu\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_0\}, \quad M_0 = H_0^1(D)$$

- **Poincaré(–Steklov) inequality**

$$\exists \check{C}_{\mathrm{P},D} > 0 \ \text{ s.t. } \ \check{C}_{\mathrm{P},D}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\mu}$$

  - $\check{C}_{\mathrm{P},D}$ depends on $D$ and contrast factor $\mu_{\sharp/\flat}$
  - in $H_0^1(D)$, see [Poincaré 1894; Steklov 1897]

- On $\boldsymbol{X}_{0\mu}$, the coercivity of $a(\cdot,\cdot)$ is **robust w.r.t.** $\mu_\flat$

$$\begin{aligned}\Re(e^{i\theta}a(\boldsymbol{b},\boldsymbol{b})) &\ge \mu_\flat\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} \ge \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} \\ &\ge \frac{1}{2}\kappa_\flat(\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \check{C}^2_{\mathrm{P},D}\ell_D^{-2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)}) \\ &\ge \frac{1}{2}\kappa_\flat\ell_D^{-2}\min(1, \check{C}^2_{\mathrm{P},D})\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}\end{aligned}$$

# Regularity pickup on $\boldsymbol{A}$

- $\exists s > 0$ and $\check{C}_D$ (depending on $D$ and contrast factor $\mu_{\sharp/\flat}$) s.t.

$$\check{C}_D \ell_D^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} \leq \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\mu}$$

  with $\|\cdot\|_{\boldsymbol{H}^s} = (\|\cdot\|_{L^2}^2 + \ell_D^{2s}|\cdot|_{H^s}^2)^{1/2}$ and Sobolev–Slobodeckij seminorm
- $\implies \boldsymbol{A} \in \boldsymbol{H}^s(D)$, $s > 0$, and typically $s < \frac{1}{2}$
- Proofs in [Jochmann 99] and [Bonito, Guermond, Luddens 13]
  - earlier results by [Birman, Solomyak 87; Costabel 90] for constant $\mu$

$$\boldsymbol{X}_0 = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid \nabla\cdot\boldsymbol{b} = 0\} \hookrightarrow \boldsymbol{H}^s(D)$$

    with $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1]$ for a Lipschitz polyhedron [Amrouche, Bernardi, Dauge, Girault 98]

# Regularity pickup on $\nabla\times\boldsymbol{A}$

- Let $\boldsymbol{V} = \boldsymbol{H}(\mathrm{curl}; D)$, $M_* := \{q \in H^1(D) \mid (q,1)_{L^2(D)} = 0\}$, and

$$\boldsymbol{X}_{*\kappa^{-1}} = \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; \mathbf{D}) \mid (\kappa^{-1}\boldsymbol{b}, \nabla \mathbf{m})_{\boldsymbol{L}^2(\mathbf{D})} = \mathbf{0},\ \forall \mathbf{m} \in \mathbf{M}_*\}$$

- $\exists s' > 0$ and $\check{C}'_D$ (depending on $D$ and contrast factor $\kappa_{\sharp/\flat}$) s.t.

$$\check{\mathbf{C}}'_{\mathbf{D}} \ell_{\mathbf{D}}^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^{s'}(\mathbf{D})} \le \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(\mathbf{D})}, \quad \forall \boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}$$

- The field $\boldsymbol{R} = \kappa\nabla\times\boldsymbol{A}$ is in $\boldsymbol{X}_{*\kappa^{-1}}$, so that $\boldsymbol{R} \in \boldsymbol{H}^{s'}(D)$
- Multiplier property: $|\kappa^{-1}\boldsymbol{\xi}|_{\boldsymbol{H}^\tau(D)} \le C_{\kappa^{-1}} |\boldsymbol{\xi}|_{\boldsymbol{H}^\tau(D)}$, $\forall \boldsymbol{\xi} \in \boldsymbol{H}^\tau(D)$
- Letting $\sigma := \min(s, s', \tau) \in (0, \frac{1}{2})$, we conclude that

$$\boldsymbol{A} \in \boldsymbol{H}^\sigma(D), \qquad \nabla\times\boldsymbol{A} \in \boldsymbol{H}^\sigma(D)$$

# Finite element setting

- Shape-regular sequence of affine simplicial meshes $(\mathcal{T}_h)_{h>0}$
- De Rham sequence for canonical FE spaces

$$P^{\mathrm{g}}(\mathcal{T}_h) \xrightarrow{\nabla} \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \xrightarrow{\nabla\times} \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) \xrightarrow{\nabla\cdot} P^{\mathrm{b}}(\mathcal{T}_h)$$

  - Lagrange/Nédélec/Raviart–Thomas/dG FEM spaces
  - conforming in $H^1(D)/\boldsymbol{H}(\mathrm{curl};D)/\boldsymbol{H}(\mathrm{div};D)/L^2(D)$
  - degrees $(k+1)/k/k/k$

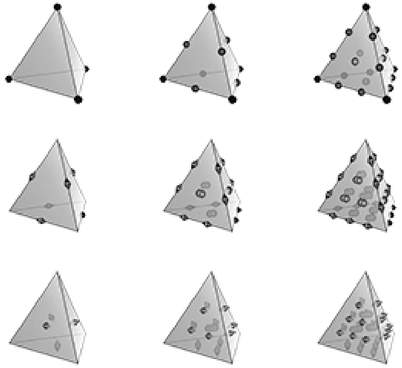
- Similar sequence with BCs

$$P_0^{\mathrm{g}}(\mathcal{T}_h) \xrightarrow{\nabla} \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) \xrightarrow{\nabla\times} \boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h) \xrightarrow{\nabla\cdot} P_0^{\mathrm{b}}(\mathcal{T}_h)$$

with $P_0^{\mathrm{g}}(\mathcal{T}_h) = P^{\mathrm{g}}(\mathcal{T}_h) \cap H_0^1(D)$, $\boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) = \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl};D)$, etc.

- Unified notation: $P(\mathcal{T}_h), P_0(\mathcal{T}_h)$ with $\mathbb{R}^q$-valued functions, $q \in \{1,3\}$

# Periodic table of finite elements [Arnold & Logg 14]



$H^1$

| | |
|---|---|
| point evaluation | $(k \geq 1)$ |
| edge integral | $(k \geq 2)$ |
| face integral | $(k \geq 3)$ |
| cell integral | $(k \geq 4)$ |

$\boldsymbol{H}(\mathrm{curl})$

| | |
|---|---|
| edge integral | $(k \geq 0)$ |
| face integral | $(k \geq 1)$ |
| cell integral | $(k \geq 2)$ |

$\boldsymbol{H}(\mathrm{div})$

| | |
|---|---|
| face integral | $(k \geq 0)$ |
| cell integral | $(k \geq 1)$ |

# Maxwell's equations

- Conforming edge FEM approximation in $\boldsymbol{V}_{h0} = \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) \subset \boldsymbol{V}_0$
- Discrete problem: Find $\boldsymbol{A}_h \in \boldsymbol{V}_{h0}$ s.t. $a(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$, $\forall \boldsymbol{b}_h \in \boldsymbol{V}_{h0}$
- The discrete problem is well-posed (Lax–Milgram Lemma)
- **Main questions** to be addressed
  - $\mu_\flat$-robust coercivity in the discrete setting
  - error estimates for $\boldsymbol{A} \in \boldsymbol{H}^{\sigma}(D)$, $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{\sigma}(D)$, $\sigma \in (0, \frac{1}{2})$

# Robust coercivity

- Since $\nabla P_0^{\mathrm{g}}(\mathcal{T}_h) \subset \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)$, we do have a discrete control on the divergence of $\boldsymbol{A}_h$

$$\boldsymbol{A}_h \in \boldsymbol{X}_{h0\mu} = \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\mu \boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in P_0^{\mathrm{g}}(\mathcal{T}_h)\}$$

  but $\boldsymbol{X}_{h0\mu}$ **is not a subspace** of $\boldsymbol{X}_{0\mu}$ ...

- One needs a discrete PS inequality in $\boldsymbol{X}_{h0\mu}$
  - one can invoke a **discrete compactness argument** [Kikuchi 89; Caorsi, Fernandes, Raffetto 00; Monk & Demkowicz 01]
  - alternatively, one invokes **commuting quasi-interpolation operators** [Arnold, Falk & Winther 10]

# Error estimates

- The canonical interpolation operators commute with differential operators ... but have **poor stability properties**
- For edge elements, stability only holds in $\boldsymbol{H}^s(D)$, $s > 1$ $(d = 3)$
  - using [Amrouche et al. 98] shows stability in $\{\boldsymbol{v} \in \boldsymbol{H}^s(D), s > \frac{1}{2}, \nabla\times\boldsymbol{v} \in L^p(D), p > 2\}$ [Boffi, Gastaldi 06]
  - regularity barrier $s > \frac{1}{2}$ still remains ...
- To approximate fields in $\boldsymbol{H}^s(D)$, $s > 0$, we shall invoke averaging quasi-interpolation operators from [AE, Guemond, 15-17]

# FE analysis tools

- **Commuting quasi-interpolation** $\mathcal{J}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$

$$\|v - \mathcal{J}_h(v)\|_{L^p(D;\mathbb{R}^q)} \leq c \inf_{v_h \in P(\mathcal{T}_h)} \|v - v_h\|_{L^p(D;\mathbb{R}^q)}$$

  - [Schöberl 01; Christiansen & Winther 08]

- **Averaging quasi-interpolation** $\mathcal{I}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$

$$\inf_{v_h \in P(\mathcal{T}_h)} \|v - v_h\|_{L^p(D;\mathbb{R}^q)} \leq \|v - \mathcal{I}_h(v)\|_{L^p(D;\mathbb{R}^q)} \leq c\, h^s |v|_{W^{s,p}(D)}$$

  - for $H^1$-conforming FEM [Clément 75; Scott, Zhang 90]

# Commuting quasi-interpolation

There exist operators $\mathcal{J}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$ s.t.

- $\mathcal{J}_h$ leaves $P(\mathcal{T}_h)$ pointwise invariant ($\mathcal{J}_h \circ \mathcal{J}_h = \mathcal{J}_h$)
- $\|\mathcal{J}_h\|_{\mathcal{L}(L^p;L^p)} \le c$, $\forall p \in [1,\infty]$
- $\mathcal{J}_h$ commutes with the standard differential operators

$$\begin{tikzcd}
H^1(D) \arrow[r, "\nabla"] \arrow[d, "\mathcal{J}_h^{\mathrm{g}}"] & \boldsymbol{H}(\mathrm{curl};D) \arrow[r, "\nabla\times"] \arrow[d, "\mathcal{J}_h^{\mathrm{c}}"] & \boldsymbol{H}(\mathrm{div};D) \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{J}_h^{\mathrm{d}}"] & L^2(D) \arrow[d, "\mathcal{J}_h^{\mathrm{b}}"] \\
P^{\mathrm{g}}(\mathcal{T}_h) \arrow[r, "\nabla"] & \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \arrow[r, "\nabla\times"] & \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) \arrow[r, "\nabla\cdot"] & P^{\mathrm{b}}(\mathcal{T}_h)
\end{tikzcd}$$

- **Stability and polynomial invariance imply approximation**

$$\|v - \mathcal{J}_h(v)\|_{L^p(D;\mathbb{R}^q)} \le c \inf_{v_h \in P(\mathcal{T}_h)} \|v - v_h\|_{L^p(D;\mathbb{R}^q)}$$

A similar construction is possible with **boundary prescription**

$$\mathcal{J}_{h0} : L^1(D;\mathbb{R}^q) \to P_0(\mathcal{T}_h)$$

# Main ideas of the construction

- See [Schöberl 01, 05; Christiansen 07, Christiansen & Winther 08]
- Compose canonical interpolation $\widehat{\mathcal{I}}_h$ operator with some **mollification** operator $\mathcal{K}_\delta$, $\delta > 0$

$$L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{K}_\delta} C^\infty(\overline{D};\mathbb{R}^q) \xrightarrow{\widehat{\mathcal{I}}_h} P(\mathcal{T}_h)$$

- $\widehat{\mathcal{J}}_h := \widehat{\mathcal{I}}_h \circ \mathcal{K}_\delta$ achieves stability and commutation
  - $\widehat{\mathcal{J}}_h$ is invertible on $P(\mathcal{T}_h)$ if $\delta \leq ch$, $c$ small enough
  - on shape-regular meshes, $\delta$ is a (smooth) space-dependent function []
  - $\mathcal{J}_h := (\widehat{\mathcal{J}}_h|_{P(\mathcal{T}_h)})^{-1} \circ \widehat{\mathcal{J}}_h$ **satisfies all the required properties**
- Boundary conditions can be prescribed

# Shrinking-based mollification [AE, Guermond 16]

- Globally transversal field $\boldsymbol{j} \in \boldsymbol{C}^\infty(\mathbb{R}^d)$ to $D$ [Hofmann, Mitrea, Taylor 07]
- Shrinking map $\varphi_\delta : \mathbb{R}^d \ni \boldsymbol{x} \mapsto \boldsymbol{x} - \delta \boldsymbol{j}(\boldsymbol{x}) \in \mathbb{R}^d$: There is $r > 0$ s.t.

$$\varphi_\delta(D) + B(\mathbf{0}, \delta r) \subset D, \qquad \forall \delta \in [0,1]$$

- **The shrinking technique avoids invoking extensions outside $D$**
- Shrinking-based mollification operators inspired from [Schöberl 01]

$$(\mathcal{K}_\delta^{\mathrm{g}} f)(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \rho(\boldsymbol{y}) f(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

$$(\mathcal{K}_\delta^{\mathrm{c}} \boldsymbol{g})(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \rho(\boldsymbol{y}) \mathbb{J}_\delta^{\mathsf{T}}(\boldsymbol{x}) \boldsymbol{g}(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y, \quad \text{etc.}$$

with $\mathbb{J}_\delta(\boldsymbol{x})$ the Jacobian matrix of $\varphi$ at $\boldsymbol{x} \in D$ and $\rho$ is a smooth kernel supported in $B(\mathbf{0}, 1)$

# Averaging quasi-interpolation

- Finite element generation
- Main result: no boundary prescription
- Main result with boundary prescription

# Finite element generation

- Reference finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ of degree $k \geq 0$
  - $\mathbb{P}_{k,d}(\widehat{K}; \mathbb{R}^q) \subset \widehat{P} \subset W^{1,\infty}(\widehat{K}; \mathbb{R}^q)$
  - reference shape functions $\{\widehat{\theta}_i\}_{i\in\mathcal{N}}$ and dof's $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$
- For any mesh cell $K \in \mathcal{T}_h$, we consider
  - an affine geometric map $\boldsymbol{T}_K : \widehat{K} \to K$
  - a functional map $\psi_K : L^1(K; \mathbb{R}^q) \to L^1(\widehat{K}; \mathbb{R}^q)$ s.t.

$$\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$$

  for some matrix $\mathbb{A}_K \in \mathbb{R}^{d\times d}$ (Piola transformations)

- FE generation in each mesh cell $K \in \mathcal{T}_h$

$$(K, P_K, \Sigma_K), \quad P_K = \psi_K^{-1} \circ \widehat{P}, \quad \Sigma_K = \widehat{\Sigma} \circ \psi_K$$

$\Longrightarrow$ local shape functions $\{\theta_{K,i}\}_{i\in\mathcal{N}}$ and dof's $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$

# Finite element spaces

- Broken (or dG) FE space

$$P^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^\infty(D;\mathbb{R}^q) \mid v_{h|K} \in P_K,\ \forall K \in \mathcal{T}_h\}$$

leading to $P^{\mathrm{g,b}}(\mathcal{T}_h)$ for $H^1$-conf. FE, $\boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h)$ for $\boldsymbol{H}(\mathrm{curl})$-conf. FE, etc.

- $H^1$-, $\boldsymbol{H}(\mathrm{curl})$-, and $\boldsymbol{H}(\mathrm{div})$-conforming subspaces

$$\begin{aligned}
P^{\mathrm{g}}(\mathcal{T}_h) &= \{v_h \in P^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F = 0,\ \forall F \in \mathcal{F}_h^\circ\} \\
\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) &= \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F \times \boldsymbol{n}_F = \boldsymbol{0},\ \forall F \in \mathcal{F}_h^\circ\} \\
\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) &= \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F \cdot \boldsymbol{n}_F = 0,\ \forall F \in \mathcal{F}_h^\circ\}
\end{aligned}$$

where $\mathcal{F}_h^\circ$ collects the mesh interfaces

# Fundamental property of face dof's

- Let $K \in \mathcal{T}_h$ be a mesh cell, let $F \in \mathcal{F}_K$ be a face of $K$
- Face unisolvence: $\exists$ nonempty subset $\mathcal{N}_{K,F} \subset \mathcal{N}$ s.t., for all $p \in P_K$,

$$[\,\sigma_{K,i}(p) = 0,\ \forall i \in \mathcal{N}_{K,F}\,] \iff [\,\gamma_{K,F}(p) = 0\,]$$

where $\gamma_{K,F}$ is one of the above trace operators from $K$ to $F$

- This implies that for all $i \in \mathcal{N}_{K,F}$, there is a unique linear map $\sigma_{K,F,i} : P_{K,F} := \gamma_{K,F}(P_K) \to \mathbb{R}$ s.t. $\sigma_{K,i} = \sigma_{K,F,i} \circ \gamma_{K,F}$
- The fundamental property is that there is $c$, uniform, s.t.

$$|\sigma_{K,F,i}(q)| \le c\,\|\mathbb{A}_K\|_{\ell^2}\|q\|_{L^\infty(F;\mathbb{R}^t)} \qquad \forall q \in P_{K,F},\ \forall i \in \mathcal{N}_{K,F}$$

**This assumption is satisfied by all FE elements from de Rham complex (all degree, all type, all kind)**

# Two-step construction procedure

$$\mathcal{I}_h : L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{I}_h^\sharp} P^{\mathrm{b}}(\mathcal{T}_h) \xrightarrow{\mathcal{I}_h^{\mathrm{av}}} P(\mathcal{T}_h)$$

- First apply the projection operator $\mathcal{I}_h^\sharp$ onto the **broken FE space**
  - $L^2$-orthogonal or oblique projection
  - $\mathcal{I}_h^\sharp$ enjoys local stability and approximation properties
- Then stitch the result by **averaging dof's** using $\mathcal{I}_h^{\mathrm{av}}$
  - the averaging step only handles discrete functions
- Some literature
  - nodal-averaging for scalar FEM has a long history [Oswald 93; Brenner 93; Hoppe, Wohlmuth 96; Karakashian, Pascal 03; Burman, AE 07 ...]
  - see also [Peterseim 14], [Kornhuber & Yserentant 16] for recent two-step construction in scalar-valued case

# Averaging operator (1)

- Recall the local shape functions $\theta_{K,i}$, $\forall (K,i) \in \mathcal{T}_h \times \mathcal{N}$
- Global shape functions $\varphi_a$, $\forall a \in \mathcal{A}_h$
  - connectivity array $\mathrm{a} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h$ s.t. $\varphi_{\mathrm{a}(K,i)|K} = \theta_{K,i}$
  - connectivity set $\mathcal{C}_a := \{(K,i) \in \mathcal{T}_h \times \mathcal{N} \mid \mathrm{a}(K,i) = a\}$
- $\mathcal{I}_h^{\mathrm{av}} : P^{\mathrm{b}}(\mathcal{T}_h) \to P(\mathcal{T}_h)$ is defined by averaging dof's

$$\mathcal{I}_h^{\mathrm{av}}(v_h)(\boldsymbol{y}) = \sum_{a \in \mathcal{A}_h} \left( \frac{1}{\#(\mathcal{C}_a)} \sum_{(K,i) \in \mathcal{C}_a} \sigma_{K,i}(v_{h|K}) \right) \varphi_a(\boldsymbol{y})$$

# Averaging operator (2)

- Bound on averaging error

$$|v_h - \mathcal{I}_h^{\mathrm{av}}(v_h)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{\frac{1}{p}-m} \sum_{F\in\mathcal{F}_K^\circ} \|[\![v_h]\!]_F\|_{L^p(F;\mathbb{R}^t)}$$

for all $m \in \{0{:}k+1\}$, all $p \in [1,\infty]$, all $v_h \in P^{\mathrm{b}}(\mathcal{T}_h)$

  - $\mathcal{F}_K^\circ$ is the collection of mesh interfaces sharing a dof with $K$

- A discrete trace inequality shows that $\mathcal{I}_h^{\mathrm{av}}$ is $L^p$-stable on $P^{\mathrm{b}}(\mathcal{T}_h)$

$$\|\mathcal{I}_h^{\mathrm{av}}(v_h)\|_{L^p(K;\mathbb{R}^q)} \le c\, \|v_h\|_{L^p(D_K;\mathbb{R}^q)}$$

  - $D_K$ collects all the mesh cells sharing a dof with $K$

# Theorem

$\mathcal{I}_h : L^1(D;\mathbb{R}^q) \to P(\mathcal{T}_h)$

- leaves $P(\mathcal{T}_h)$ pointwise invariant ($\mathcal{I}_h \circ \mathcal{I}_h = \mathcal{I}_h$)
- $\|\mathcal{I}_h\|_{\mathcal{L}(L^p;L^p)} \le c$, $\forall p \in [1,\infty]$
- has optimal local approximation properties

$$|v - \mathcal{I}_h(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{s-m} |v|_{W^{s,p}(D_K;\mathbb{R}^q)}$$

for all $s \in [0, k+1]$ and $m \in \{0{:}\lfloor s \rfloor\}$, all $p \in [1,\infty)$ ($p \in [1,\infty]$ if $s \in \mathbb{N}$), all $K \in \mathcal{T}_h$, all $v \in W^{s,p}(D_K;\mathbb{R}^q)$

In particular, we infer that

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p(D;\mathbb{R}^q)} \le c\, h^s |v|_{W^{s,p}(D;\mathbb{R}^q)}$$

# Polynomial approximation in $D_K$

$$\inf_{p\in\mathbb{P}_{k,d}} |v-p|_{W^{m,p}(D_K)} \leq c\, h_K^{s-m} |v|_{W^{s,p}(D_K)}$$

- Poincaré(–Steklov) in $W^{s,p}(D_K)$, $s \in (0,1)$ (direct proof)

$$\|v - \underline{v}_{D_K}\|_{L^p(D_K)} \leq c\, h_U^s |v|_{W^{s,p}(D_K)}$$

with $\underline{v}_{D_K} = \frac{1}{|D_K|}\int_{D_K} v\,\mathrm{d}x$

- Poincaré(–Steklov) in $W^{s,p}(D_K)$, $s = 1$

$$\|v - \underline{v}_{D_K}\|_{L^p(D_K)} \leq c\, h_K |v|_{W^{1,p}(D_K)}$$

  - $D_K$ possibly nonconvex, cannot use the result from [Bebendorf 03]
  - break $D_K$ into sub-simplices and combine PS in simplices with multiplicative trace inequality (see also [Veeser & Verfürth 12])

# Main result with boundary prescription

- Two-step construction

$$\mathcal{I}_{h0} : L^1(D;\mathbb{R}^q) \xrightarrow{\mathcal{I}_h^\sharp} P^{\mathrm{b}}(\mathcal{T}_h) \xrightarrow{\mathcal{I}_{h0}^{\mathrm{av}}} P_0(\mathcal{T}_h)$$

  - BCs enforced at the second stage (on polynomials) by zeroing out the components of $\mathcal{I}_{h0}^{\mathrm{av}}(v_h)$ attached to boundary dof's

- **Theorem**
  - $\mathcal{I}_{h0}$ leaves $P_0(\mathcal{T}_h)$ pointwise invariant
  - $\|\mathcal{I}_{h0}\|_{\mathcal{L}(L^p;L^p)} \le c$, $\forall p \in [1,\infty]$
  - best approximation: for all $s \in [0, k+1]$

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p} \le \begin{cases} c\, h^s |v|_{W^{s,p}}, & \forall v \in W^{s,p}_{0,\gamma}(D;\mathbb{R}^q) \text{ if } sp > 1 \\ c\, h^s \ell_D^{-s} \|v\|_{W^{s,p}}, & \forall v \in W^{s,p}(D;\mathbb{R}^q) \text{ if } sp < 1 \end{cases}$$

  where $W^{s,p}_{0,\gamma}(D;\mathbb{R}^q) = \{v \in W^{s,p}(D;\mathbb{R}^q) \mid \gamma(v) = 0\}$
  - localized versions and bounds on higher-order norms available

# Comments on the case $sp < 1$

$$\inf_{w_h \in P_0(\mathcal{T}_h)} \|v - w_h\|_{L^p} \leq c\, h^s \ell_D^{-s} \|v\|_{W^{s,p}(D;\mathbb{R}^q)}$$

- $v$ is not smooth enough to have a trace on $\partial D$, it can even blow up
- Yet, we can achieve an $h$-optimal decay estimate of best approximation w.r.t. discrete functions **with boundary prescription**
- The reason is that $v$ cannot blow up too fast (as $\rho^{-s}$, $\rho = d(\cdot, \partial D)$)
  - see [Grisvard 85]
  - estimate cannot be localized close to $\partial D$
- This result seems to be new even in the $H^1$-conforming setting
  - see [Ciarlet Jr. 13] for Scott–Zhang operator and $sp > 1$

# Robust coercivity for Maxwell's equations

- Recall $\boldsymbol{X}_{0\mu} = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid (\mu\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in H^1_0(D)\}$ and that

$$\check{C}_{\mathrm{P},D}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \leq \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \quad \forall\boldsymbol{b} \in \boldsymbol{X}_{0\mu}$$

- Recall $\boldsymbol{X}_{h0\mu} = \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\mu\boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in P^{\mathrm{g}}_0(\mathcal{T}_h)\}$ and that $\boldsymbol{X}_{h0\mu}$ is not a subspace of $\boldsymbol{X}_{0\mu}$
- Letting $\check{C}_{\mathrm{P},\mathcal{T}_h} := \mu_{\sharp/\flat}^{-1}\|\mathcal{J}^{\mathrm{c}}_{h0}\|^{-1}_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\check{C}_{\mathrm{P},D}$, we have

$$\check{C}_{\mathrm{P},\mathcal{T}_h}\ell_D^{-1}\|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} \leq \|\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}, \quad \forall\boldsymbol{b}_h \in \boldsymbol{X}_{h0\mu}$$

# $\boldsymbol{H}$(curl)-error estimate

- $\boldsymbol{A}_h \in \boldsymbol{X}_{h0\mu}$ is s.t. $a(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h)$, $\forall \boldsymbol{b}_h \in \boldsymbol{X}_{h0\mu}$
- Discrete PS inequality yields $\mu_\flat$-robust coercivity on $\boldsymbol{X}_{h0\mu}$
- Standard techniques lead to

$$\begin{aligned} \|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} &\lesssim \inf_{\boldsymbol{b}_h \in \boldsymbol{X}_{h0\mu}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \\ &\lesssim \inf_{\boldsymbol{b}_h \in \boldsymbol{V}_{h0}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \end{aligned}$$

where hidden constants depend on the contrast factors $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, and the magnetic Reynolds number $\gamma_{\mathrm{m}} = \mu_\sharp \ell_D^{-2} \kappa_\sharp^{-1}$

# Convergence rates

- Convergence rates follow from

$$
\begin{aligned}
\inf_{\boldsymbol{b}_h \in \boldsymbol{V}_{h0}} \|\boldsymbol{A} - \boldsymbol{b}_h\|^2_{\boldsymbol{H}(\mathrm{curl};D)} &\le \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{H}(\mathrm{curl};D)} \\
&= \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2 \|\nabla\times\boldsymbol{A} - \nabla\times\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} \\
&= \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2 \|\nabla\times\boldsymbol{A} - \mathcal{J}^{\mathrm{d}}_{h0}(\nabla\times\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} \\
&\le c \inf_{\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2 \inf_{\boldsymbol{d}_h \in \boldsymbol{P}^{\mathrm{d}}_0(\mathcal{T}_h)} \|\nabla\times\boldsymbol{A} - \boldsymbol{d}_h\|^2_{\boldsymbol{L}^2(D)} \\
&\le c\|\boldsymbol{A} - \mathcal{I}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2 \|\nabla\times\boldsymbol{A} - \mathcal{I}^{\mathrm{d}}_{h0}(\nabla\times\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)}
\end{aligned}
$$

and we can now use the decay estimates for averaging quasi-int.

# $\boldsymbol{L}^2$-error estimate

- Recall that $\boldsymbol{A} \in \boldsymbol{H}^\sigma(D)$, $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^\sigma(D)$, $\sigma \in (0, \frac{1}{2})$
- Main steps of the proof for $\boldsymbol{L}^2$-error estimate
  - duality argument + bound on curl-preserving lifting
  - main obstruction: dual solution is only in $\boldsymbol{H}^\sigma(D)$
  - see [Zhong, Shu, Wittum, Xu 09] with assumption $\sigma > \frac{1}{2}$
- **New result** [AE, Guermond 17]

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}} \left(\|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2} + h^\sigma \ell_D^{-\sigma} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{curl})}\right)$$

where hidden constant depends on $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, $\kappa_\sharp C_{\kappa^{-1}}$, $\gamma_{\mathrm{m}}$

# Elliptic PDEs with contrasted coefficients

- Lipschitz polyhedron $D$ in $\mathbb{R}^d$, source term $f \in L^q(D)$, $q > \frac{2d}{2+d}$
  - $q > 1$ if $d = 2$, $q = \frac{6}{5}$ if $d = 3$, one can always take $q = 2$
  - $L^q(D) \hookrightarrow (H^1(D))'$ (minimal requirement is $q > \frac{2d}{2+d}$)
- $\lambda \in L^\infty(D)$, **uniformly positive** and **pcw. constant on a Lipschitz polyhedral partition** of $D$
  - possible extensions: $\lambda$ tensor-valued and pcw. Lipschitz
- Weak formulation: Find $u \in H_0^1(D)$ s.t., for all $w \in H_0^1(D)$,

$$a(u, w) := \int_D \boldsymbol{\sigma}(u) \cdot \nabla w \, \mathrm{d}x = \int_D f w \, \mathrm{d}x =: \ell(w), \qquad \boldsymbol{\sigma}(u) := \lambda \nabla u$$

- Modest elliptic regularity pickup: $u \in H^{1+r}(D)$, $r > 0$

# Nonconforming approximation

- Shape-regular sequence of (simplicial affine) meshes $(\mathcal{T}_h)_{h>0}$
- Mesh faces $\mathcal{F}_h = \mathcal{F}_h^\circ \cup \mathcal{F}_h^\partial$: interfaces $\mathcal{F}_h^\circ$ and boundary faces $\mathcal{F}_h^\partial$
  - $F \in \mathcal{F}_h$ is oriented by the unit normal vector $\boldsymbol{n}_F$
  - $[\![\cdot]\!]_F$ is the **jump** across $F \in \mathcal{F}_h^\circ$ or the value at $F \in \mathcal{F}_h^\partial$
- Broken polynomial space ($k \geq 0$)

$$P_k^{\mathrm{b}}(\mathcal{T}_h) = \{v_h \in L^\infty(D) \mid v_{h|K} \in \mathbb{P}_k, \forall K \in \mathcal{T}_h\}$$

- Broken gradient $\nabla_h : (H^1(D) + P_k^{\mathrm{b}}(\mathcal{T}_h)) \to L^2(D;\mathbb{R}^d)$
  - $\nabla_h v = \nabla v$ on $H^1(D)$
  - $(\nabla_h v_h)_{|K} = \nabla(v_{h|K})$ on $P_k^{\mathrm{b}}(\mathcal{T}_h)$, for all $K \in \mathcal{T}_h$
- **Broken bilinear form** on $P_k^{\mathrm{b}}(\mathcal{T}_h) \times P_k^{\mathrm{b}}(\mathcal{T}_h)$

$$a_h(v_h, w_h) := \int_D \boldsymbol{\sigma}_h(v_h)\cdot\nabla_h w_h \,\mathrm{d}x, \qquad \boldsymbol{\sigma}_h(v_h) := \lambda \nabla_h v_h$$

# Examples of nonconforming methods (I)

- **Crouzeix–Raviart finite elements**
  - $V_h := P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h) = \{v_h \in P^{\mathrm{b}}_1(\mathcal{T}_h) \mid \int_F [\![v_h]\!]_F \,\mathrm{d}s = 0, \forall F \in \mathcal{F}_h\}$
  - discrete problem: Find $u_h \in V_h$ s.t., for all $w_h \in V_h$,

$$b_h(u_h, w_h) := a_h(u_h, w_h) = \ell(w_h)$$

- **Nitsche's boundary penalty with conforming FEM**
  - $V_h := P^{\mathrm{g}}_k(\mathcal{T}_h) = \{v_h \in P^{\mathrm{b}}_k(\mathcal{T}_h) \mid [\![v_h]\!]_F = 0, \forall F \in \mathcal{F}^{\circ}_h\}$
  - functions in $V_h$ can be nonzero at the boundary $\partial D$
  - discrete problem: Find $u_h \in V_h$ s.t., for all $w_h \in V_h$,

$$b_h(u_h, w_h) := a(u_h, w_h) - n_h(u_h, w_h) + s_h(u_h, w_h) = \ell(w_h)$$

  - consistency term (one can symmetrize)

$$n_h(v_h, w_h) = \sum_{F \in \mathcal{F}^{\partial}_h} \int_F (\boldsymbol{n}\cdot\nabla v_h) w_h \,\mathrm{d}s$$

  - stabilization $s_h(v_h, w_h) = \sum_{F \in \mathcal{F}^{\partial}_h} \eta_0 \frac{\lambda_{K_F}}{h_F} \int_F v_h w_h \,\mathrm{d}s$ ($\eta_0$ large enough)

# Examples of nonconforming methods (II)

- **Discontinuous Galerkin**
  - discrete problem: find $u_h \in V_h := P_k^{\mathrm{b}}(\mathcal{T}_h)$ s.t., for all $w_h \in V_h$,

$$b_h(u_h, w_h) := a_h(u_h, w_h) - n_h(u_h, w_h) + s_h(u_h, w_h) = \ell(w_h)$$

  - consistency term (one can symmetrize)

$$n_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \int_F \boldsymbol{n}_F \cdot \{\nabla_h v_h\}_\theta [\![ w_h ]\!] \, \mathrm{d}s$$

  - stabilization $s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \eta_0 \frac{\lambda_F}{h_F} \int_F [\![ v_h ]\!] [\![ w_h ]\!] \, \mathrm{d}s$, $\lambda_F := \frac{2\lambda_{K_l}\lambda_{K_r}}{\lambda_{K_l}+\lambda_{K_r}}$ for all $F = \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$, $\eta_0$ large enough (independent of $\lambda$)

- **Robustness w.r.t. contrast: weighted averages**

$$\{\phi\}_\theta = \theta_{F,K_l}\phi_{|K_l} + \theta_{F,K_r}\phi_{|K_r}, \quad \theta_{F,K_l}, \theta_{F,K_r} \in [0,1], \quad \theta_{F,K_l} + \theta_{F,K_r} = 1$$

  - $\theta_{F,K_l} = \theta_{F,K_r} = \frac{1}{2}$ recovers usual averages
  - **diffusion-dependent averages**: $\theta_{F,K_l} = \frac{\lambda_{K_r}}{\lambda_{K_l}+\lambda_{K_r}}$, see [Dryja 03; Burman & Zunino 06; Di Pietro, AE, JLG 08]

# Quasi-optimal error estimate

- **Quasi-minimal regularity space** $V_{\mathrm{S}} \subset V$, assume $u \in V_{\mathrm{S}}$
  - $V_\sharp := V_{\mathrm{S}} + V_h \ni (u - u_h)$, with norm $\|\cdot\|_{V_\sharp}$ (unbounded in $V$)
  - discrete norm equivalence: $\|v_h\|_{V_\sharp} \le c_\sharp \|v_h\|_{V_h}$, $\forall v_h \in V_h$
- **Bounded extension** of $n_h$ to $n_\sharp$ on $V_\sharp \times V_h$ s.t. $\forall w_h \in V_h$,

$$n_\sharp(v_h, w_h) = n_h(v_h, w_h), \qquad \forall v_h \in V_h$$

$$n_\sharp(v, w_h) = \int_D \{(\nabla\cdot\boldsymbol{\sigma}(v))w_h + \boldsymbol{\sigma}(v)\cdot\nabla_h w_h\}\,\mathrm{d}x, \quad \forall v \in V_{\mathrm{S}}$$

$$|n_\sharp(v, w_h)| \le \omega \|v\|_{V_\sharp} \|w_h\|_{V_h}$$

- **Quasi-optimal error estimate**

$$\|u - u_h\|_{V_\sharp} \le c \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}$$

- See also [Zanotti PhD Thesis 17; Veeser & Zanotti, 17-]
  - **energy-norm** estimates, $f \in H^{-1}(D)$
  - requires to **modify RHS** $\ell(w_h)$ (using, e.g., bubble functions)
  - assumes (so far) constant diffusion coefficient $\lambda$

# Choice of space $V_{\mathrm{S}}$

- Let $\boldsymbol{\sigma} \in L^2(D;\mathbb{R}^d)$ with $\nabla\cdot\boldsymbol{\sigma} \in L^2(D)$
  - $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_K)$ can be given a meaning in $H^{-\frac{1}{2}}(\partial K)$ by Green's formula
  - this object **cannot be localized** to the faces composing $\partial K$
- The classical route is to set $V_{\mathrm{S}} := H^{1+r}(D)$, $r > \frac{1}{2}$
  - $[u \in V_{\mathrm{S}}] \Longrightarrow [\boldsymbol{\sigma}(u)_{|F} \in L^1(F;\mathbb{R}^d),\ \forall F \in \mathcal{F}_h]$
  - one can set $n_\sharp(v, w_h) := \sum_{F\in\mathcal{F}_h} \int_F \boldsymbol{n}_F\cdot\{\nabla v\}_\theta [\![w_h]\!]\,\mathrm{d}s$, $\forall v \in V_{\mathrm{S}}$
  - the ansatz $r > \frac{1}{2}$ is **unrealistic** for heterogeneous diffusion
- Letting $p > 2$, $q > \frac{2d}{2+d}$, we are going to work in

$$V_{\mathrm{S}} := \{v \in H^1_0(D) \mid \boldsymbol{\sigma}(v) \in L^p(D;\mathbb{R}^d),\ \nabla\cdot\boldsymbol{\sigma}(v) \in L^q(D)\}$$

  - **realistic** choice since $[u \in H^{1+r}(D),\ r > 0] \Longrightarrow [\boldsymbol{\sigma}(v) \in L^p(D;\mathbb{R}^d)]$ and $\nabla\cdot\boldsymbol{\sigma}(u) = f \in L^q(D)$

$$\|v\|^2_{V_\sharp} = \sum_{K\in\mathcal{T}_h} \lambda_K \|\nabla v\|^2_{L^2} + \lambda_K^{-1}\big(h_K^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{\sigma}\|_{L^p} + h_K^{1+d(\frac{1}{2}-\frac{1}{q})}\|\nabla\cdot\boldsymbol{\sigma}\|_{L^q}\big)^2$$

# Face-to-cell lifting operators

- Let $K \in \mathcal{T}_h$ be a mesh cell (outward normal $\boldsymbol{n}_K$), face $F \subset \partial K$
- $\exists$ **stable face-to-cell lifting** operator based on zero-extension

$$L_F^K : W^{\frac{1}{t},t'}(F) \to W^{1,t'}(K) \hookrightarrow W^{1,p'}(K) \cap L^{q'}(K)$$

with $t \in (2,p]$ be s.t. $q \geq \frac{td}{t+d}$

- Let $\boldsymbol{\sigma} \in L^p(K;\mathbb{R}^d)$, $p > 2$, with $\nabla{\cdot}\boldsymbol{\sigma} \in L^q(K)$, $q > \frac{2d}{2+d}$
- Local normal component $(\boldsymbol{\sigma}{\cdot}\boldsymbol{n}_K)_{|F} \in (W^{\frac{1}{t},t'}(F))'$: $\forall \phi \in W^{\frac{1}{t},t'}(F)$

$$\langle (\boldsymbol{\sigma}{\cdot}\boldsymbol{n}_K)_{|F}, \phi \rangle := \int_K \Big( \boldsymbol{\sigma}{\cdot}\nabla L_F^K(\phi) + (\nabla{\cdot}\boldsymbol{\sigma}) L_F^K(\phi) \Big)\,\mathrm{d}x$$

# Devising $n_\sharp$

- Recall $V_{\mathrm{s}} := \{v \in H^1_0(D) \mid \boldsymbol{\sigma}(v) \in L^p(D;\mathbb{R}^d),\ \nabla\cdot\boldsymbol{\sigma}(v) \in L^q(D)\}$
- Recall $V_\sharp = V_{\mathrm{s}} + V_h$ with $V_h = P^{\mathrm{b}}_k(\mathcal{T}_h)$
- For all $(v, w_h) \in V_\sharp \times V_h$, we set

$$n_\sharp(v, w_h) := \sum_{F\in\mathcal{F}_h} \sum_{K\in\mathcal{T}_F} \epsilon_{K,F}\theta_{K,F}\langle(\boldsymbol{\sigma}(v)_{|K}\cdot\boldsymbol{n}_K)_{|F}, [\![w_h]\!]\rangle$$

with $\epsilon_{K,F} = \boldsymbol{n}_K\cdot\boldsymbol{n}_F = \pm 1$ and diffusion-dependent weights $\theta_{K,F}$

- **Boundedness** (robust w.r.t. $\lambda$)

$$|n_\sharp(v, w_h)| \le \omega\|v\|_{V_\sharp} s_h(w_h, w_h)^{\frac{1}{2}}$$

($s_h$ uses harmonic average $\lambda_F$ and is controlled by stability norm)

# The two key properties

- The following holds true:

$$n_\sharp(v_h, w_h) = n_h(v_h, w_h), \qquad \forall v_h \in V_h \quad (\mathrm{A})$$

$$n_\sharp(v, w_h) = \int_D \left\{(\nabla\cdot\boldsymbol{\sigma}(v))w_h + \boldsymbol{\sigma}(v)\cdot\nabla_h w_h\right\} \mathrm{d}x, \quad \forall v \in V_\mathrm{s} \quad (\mathrm{B})$$

- Property $(\mathrm{A})$ results from elementary manipulations (we work with pcw. polynomials)
- Property $(\mathrm{B})$ is a bit more subtle
  - being based on a **"density argument"**, it is "part of the folklore"
  - we believe it deserves a rigorous proof
  - this proof completes previous literature "claims", e.g., [Cai, Ye, Zhang, SINUM, 2011, p. 1767]

$$\langle\nabla\phi\cdot\boldsymbol{n}, g\rangle = \langle\nabla\phi\cdot\boldsymbol{n}, v_g\rangle_{\partial K} = (\Delta\phi, v_g)_K + (\nabla\phi, \nabla v_g)_K$$

(first equality could be a definition and second one could deserve a proof)

# The density argument

- Based on commuting mollification operators

$$(\mathcal{K}^{\mathrm{g}}_{\delta} v)(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \zeta(\boldsymbol{y}) v(\varphi_{\delta}(\boldsymbol{x}) + (\delta\rho)\boldsymbol{y}) \,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{c}}_{\delta} \boldsymbol{\theta})(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \zeta(\boldsymbol{y}) \mathbb{J}^{\mathsf{T}}_{\delta}(\boldsymbol{x}) \boldsymbol{\theta}(\ldots) \,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{d}}_{\delta} \boldsymbol{\sigma})(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \zeta(\boldsymbol{y}) \det(\mathbb{J}_{\delta}(\boldsymbol{x})) \mathbb{J}^{-1}_{\delta}(\boldsymbol{x}) \boldsymbol{\sigma}(\ldots) \,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{b}}_{\delta} f)(\boldsymbol{x}) := \int_{B(\mathbf{0},1)} \zeta(\boldsymbol{y}) \det(\mathbb{J}_{\delta}(\boldsymbol{x})) f(\ldots) \,\mathrm{d}y$$

$\mathbb{J}_{\delta}(\boldsymbol{x})$: Jacobian of $\varphi$ at $\boldsymbol{x} \in D$; $\zeta$: smooth kernel in $B(\mathbf{0},1)$

- Proof of key property $(\mathrm{B})$: evaluate in two ways (Green's formula)

$$\sum_{F \in \mathcal{F}_h} \sum_{K \in \mathcal{T}_F} \epsilon_{K,F} \theta_{K,F} \langle (\mathcal{K}^{\mathrm{d}}_{\delta}(\boldsymbol{\sigma}(v))_{|K} \cdot \boldsymbol{n}_K)_{|F}, [\![ w_h ]\!] \rangle$$

and pass to limit $\delta \to 0$ using commuting pty. $\nabla \cdot (\mathcal{K}^{\mathrm{d}}_{\delta}(\boldsymbol{\sigma})) = \mathcal{K}^{\mathrm{b}}_{\delta}(\nabla \cdot \boldsymbol{\sigma})$

# Summary

- New quasi-interpolation operators for FEM best-approximation
- Optimal $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{L}^2$-estimates for Maxwell's equations with Sobolev regularity $H^s$, $s \in (0, \frac{1}{2})$
- Nonconforming error estimates for elliptic PDEs with Sobolev regularity $H^{1+s}$, $s \in (0, \frac{1}{2})$

**Thank you for your attention**