Agent Consistency for decomposition of stochastic optimization problems

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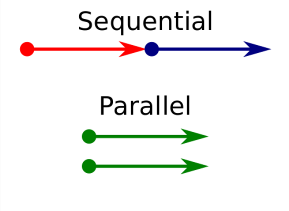
Problem and motivation

- Multistage and multi agent stochastic optimization problems are naturally large scale
- Decomposition-coordination methods make it possible to tackle such problems numerically



- Time Consistency is known to be a key ingredient for dynamic programming
- We introduce Agent Consistency that is a key ingredient for parallel computing

Idea of Time Consistency in a deterministic setting

- You are offered the choice of two desserts with a fixed meal
- You have preferences
  - over desserts
  - over main course+dessert

- You are Time Consistent if after the main course,
  you stick to your previous choice of dessert
  as if one individual is two consecutive agent

- Time consistency is a form of stability over time

- Time consistency is closely related to dynamic programming

Idea of Agent Consistency in a deterministic case

- You are now offered the choice of two main courses and two desserts
- You have preferences
  - over main courses
  - over desserts
  - over main course+dessert

- You are Agent Consistent if the menu that you prefer is composed of the main course that you prefer and the dessert that you prefer as if one individual is two parallel agent

- Agent consistency is a form of stability over product set

- Agent consistency is closely related to parallel computing

Our goals

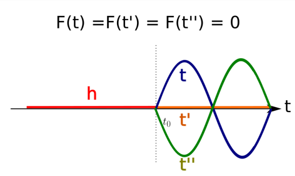
- We first define an abstract framework for Time Consistency that connects a disparate literature and prove an equivalence between Time Consistency and Nested Formula
- We recover dynamic programming
- We generalize Time Consistency to Consistency for binary relations and define Agent Consistency
- We recover parallel computing
- We apply our results to equilibrium on energy markets

Outline

Definition of Weak Time Consistency

- Let $\mathbb{H}$ (headset), $\mathbb{T}$ (tailset), $\mathbb{A}$ and $\mathbb{F}$ be four sets
- Let A (aggregator) and F (factor) be the two mappings:

$$\underbrace{A : \mathbb{H} \times \mathbb{T} \rightarrow \mathbb{A}}_{\text{head+tail assessment}} \;,\quad \underbrace{F : \mathbb{T} \rightarrow \mathbb{F}}_{\text{tail assessment}}$$

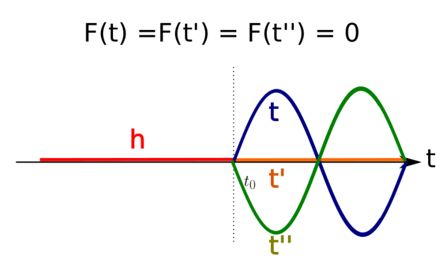


Definition (Weak Time Consistency)

The couple aggregator-factor (A, F) is said to satisfy *Weak Time Consistency (WTC)* if we have

$$F(t) = F(t') \Rightarrow A(h, t) = A(h, t')$$

Examples



- The mapping F averages the end of the process
- Consider two aggregators:
  - A_1 averages the entire process
  - A_2 returns the maximum of the process
- (A_1, F) are WTC but not (A_2, F)

Characterization of Weak Time Consistency

Theorem (Nested decomposition of WTC mappings)

The couple aggregator-factor (A, F) *is WTC if and only if there exist a mapping* $S^{F,A}$ *such that the following Nested Formula between mappings holds true:*

$$A(h, t) = S^{F,A}(h, F(t))$$

Remark

The mapping $S^{F,A}$ is unique on $\mathbb{H} \times \mathrm{Im}(F)$ and called subaggregator. It is defined by

$$\begin{aligned} S^{F,A} : \ \mathbb{H} \times \mathrm{Im}(F) &\to \ \mathbb{A} \\ (h, f) &\mapsto \ S^{F,A}(h, f) = \{A(h, t) \mid F(t) = f\} \end{aligned}$$

Application to dynamic programming

- Under technical assumptions (monotony and infimum achieved), the following dynamic programming equation holds true

$$\underbrace{\bigwedge_{h\in\mathbb{H},t\in\mathbb{T}} A(h,t)}_{\text{global optimization}} = \underbrace{\bigwedge_{h\in\mathbb{H}} S^{F,A}\big(h, \bigwedge_{t\in\mathbb{T}} F(t)\big)}_{\text{sequential optimization}}$$

1. We first solve $\bigwedge_{t\in\mathbb{T}} F(t)$ and denote the solution $f^\sharp$
2. Then we optimize $\bigwedge_{h\in\mathbb{H}} S^{F,A}(h, f^\sharp)$

A disparate literature under this framework

	Article	Objects	Head	Tail	Assessment
Time Consistency	Kreps and Porteus	Lottery	Lottery from 1 to s	Lottery from $s+1$ to T	Expected utility
	Epstein and Schneider	Lottery	Lottery from 1 to s	Lottery from $s+1$ to T	Not necessarily expected utility
	Ruszczyński	Process	Process from 1 to s	Process from $s+1$ to T	Dynamic risk measure
	Artzner et al.	Process	Process from 1 to τ	Process from τ to T, τ stopping time	Coherent risk measure
Nested Formula	Shapiro	Process	Process from 1 to s	Process from $s+1$ to T	Coherent risk measure
	Ruszczynski and Shapiro	Process	Process from 1 to s	Process from $s+1$ to T	Coherent risk measure
	De Lara and Leclère	Process	Process from 1 to s	Process from $s+1$ to T	Dynamic risk measure

Table 1: Sketch of papers selected on Time Consistency and Nested Formulas

Additional assumption to Time Consistency among authors

	Article	Monotony	Translation invariance	Convexity
Time Consistency	(Kreps and Porteus, 1978)	Yes	No	Yes
	(Kreps and Porteus, 1979)	Yes	No	Yes
	(Epstein and Schneider, 2003)	Yes	No	Yes
	(Ruszczyński, 2010)	Yes	Yes	No
	(Artzner, Delbaen, Eber, Heath, and Ku, 2007)	Yes	Yes	Yes
Nested Formula	(Shapiro, 2016)	Yes	Yes	Yes
	(Ruszczynski and Shapiro, 2006)	Yes	Yes	Yes
	(De Lara and Leclère, 2016)	Yes	No	No

Table 2: Most common assumptions in the selection of papers on Time Consistency and Nested Formula

Under technical assumptions on the mappings A and F,
(detailed in Gérard, De Lara, and Chancelier (2017))
we can show that the subaggregator is monotone, continuous, convex, positively homogeneous and/or translation invariant

From equality to inequality: Usual and Strong Time Consistency

	Weak $\Leftarrow$	Usual $\Leftarrow$	Strong
Definition	$F(t) = F(t')$ $\Downarrow$ $A(h,t) = A(h,t')$	$F(t) \leq F(t')$ $\Downarrow$ $A(h,t) \leq A(h,t')$	$h \leq h'$, $F(t) \leq F(t')$ $\Downarrow$ $A(h,t) \leq A(h',t')$
Characterization in terms of subaggregator	$S^{F,A}$ is a mapping	$S^{F,A}$ is a mapping increasing in its second argument	$S^{F,A}$ is a mapping increasing in both arguments

Table 3: Characterization of Time Consistency in terms of subaggregator

Conclusion on Time Consistency

- Time Consistency is a notion widely discussed in various fields, ranging from economics to mathematics
- We have presented a framework of Weak Time Consistency which allows us to prove under minimal assumptions an equivalence with a Nested Formula,
- We have derived analytical properties of the subaggregator
- We believe that this makes the notion easy to handle and that it opens the way for extensions

Outline

Agent consistency as extension of Time Consistency

- Time Consistency relies upon two dimensions that we are going to generalize
- First the set $\mathbb{H}$ plays a particular role and represents elements "that occurs before" elements of the set $\mathbb{T}$. We now consider sets in a symmetric way
- Second, we have used mappings to compare elements. We now use binary relations

We want to obtain a formula of the kind

$$\underbrace{\bigwedge_{(s_a \in \mathbb{S}_a)_{a \in \mathcal{A}}} A\big((s_a)_{a \in \mathcal{A}}\big)}_{\text{global optimization}} = \underbrace{S\Big(\big(\bigwedge_{s_a \in \mathbb{S}_a} F_a(s_a)\big)_{a \in \mathcal{A}}\Big)}_{\text{parallel optimization}}$$

Consistency for binary relations

Definition

- $\mathcal{A}$ is a set of agents
- For each $a \in \mathcal{A}$, $(\mathbb{S}_a)_{a\in\mathcal{A}}$ is a set
- For each $a \in \mathcal{A}$, $\mathcal{S}_a$ is a relation on the set $\mathbb{S}_{\mathcal{A}}$
- $\mathcal{R}$ is a relation on the product set $\Pi_{a\in A}\mathbb{S}_a$

The tuple of relations $\big((\mathcal{S}_a)_{a\in A}, \mathcal{R}\big)$ is said to be *consistent* if

$$\underbrace{\Pi_{a\in\mathcal{A}}\mathcal{S}_a}_{\text{product of relations}} \subset \underbrace{\mathcal{R}}_{\text{relation over product set}}$$

When the tuple is consistent, we say that relations $(\mathcal{S}_a)_{a\in\mathcal{A}}$ are *factors* of the relation $\mathcal{R}$

Time Consistency included in Consistency

Proposition

Let $A : \mathbb{H} \times \mathbb{T} \to \mathbb{A}$ *and* $F : \mathbb{T} \to \mathbb{F}$ *be two mappings.*
(A, F) *is WTC if and only if the triplet* $(\Delta_{\mathbb{H}}, \mathbb{T}/F, \mathbb{H} \times \mathbb{T}/A)$ *is consistent where*

- $\Delta_{\mathbb{H}}$ *is the equality relation on* $\mathbb{H}$,
- $\mathbb{T}/F$ *is the equivalence relation on* $\mathbb{T}$ *induced* F, *that is,*

$$t \ \mathcal{T} \ t' \Leftrightarrow F(t) = F(t')$$

- $\mathbb{H} \times \mathbb{T}/A$ *is the equivalence relation on* $\mathbb{H} \times \mathbb{T}$ *induced by* A, *that is,*

$$(h, t) \ \mathcal{R} \ (h', t') \Leftrightarrow A(h, t) = A(h', t')$$

Strong agent consistency (SAC)

- Let $(\mathbb{S}_a)_{a\in\mathcal{A}}$ be a collection of sets
- Let $\mathbb{A}$ and $(\mathbb{F}_a)_{a\in\mathcal{A}}$ be sets equipped with orders denoted by $\leq$
- Let $A : \Pi_{a\in\mathcal{A}}\mathbb{S}_a \to \mathbb{A}$ and $F_a : \mathbb{S}_a \to \mathbb{F}_a$ be mappings

Definition (Definition of strong agent consistency)

The tuple $(A, (F_a)_{a\in\mathcal{A}})$ is said to satisfy Strong Agent Consistency (SAC) if we have

$$F_a(s_a) \leq F_a(s'_a) \Rightarrow A\big((s_a)_{a\in\mathcal{A}}\big) \leq A\big((s'_a)_{a\in\mathcal{A}}\big)$$

Parallel computing with Strong Agent Consistency

Proposition (Nested decomposition for SAC mappings)

The tuple $(A, (F_a)_{a\in\mathcal{A}})$ is Strong Agent Consistent if and only if there exists a mapping S increasing in all arguments such that we have the Nested Formula

$$A\big((s_a)_{a\in\mathcal{A}}\big) = S\Big(\big(F_a(s_a)\big)_{a\in\mathcal{A}}\Big)$$

Proposition

Under technical assumptions (monotony and infimum achieved), the following parallel computing equation holds true

$$\underbrace{\bigwedge_{(s_a\in\mathbb{S}_a)_{a\in\mathcal{A}}} A\big((s_a)_{a\in\mathcal{A}}\big)}_{\textit{global optimization}} = \underbrace{S\Big(\big(\bigwedge_{s_a\in\mathbb{S}_a} F_a(s_a)\big)_{a\in\mathcal{A}}\Big)}_{\textit{parallel optimization}}$$

Outline

Remark on economy

- In economy, prices are instrument used to coordinate agents to share a ressource
- Does the prices play also a role to make the preferences of a group of agent consistent ?

Applications of Agent Consistency to Equilibrium

Ingredients of the problem

Agent Consistency and Equilibrium with risk neutral agents

Agent Consistency and Equilibrium with risk averse agents

Ingredients of the problem

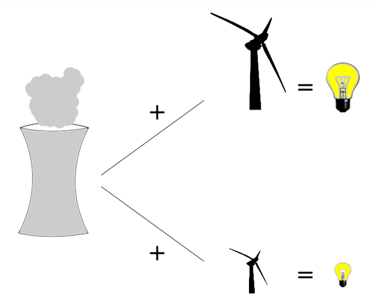


Figure 1: Illustration of the toy problem

- Two time-step market
- One good traded
- Two agents:
  producer and consumer
- Finite number of scenarios
  $\omega \in \Omega$
- Consumption
  on second stage only

In the remain of this talk, we consider that agents are price takers i.e. they act as if they have no influence on the price.

Producer's welfare and Consumer's welfare

- Step 1: production of x at a marginal cost cx
- Step 2: random production $\mathbf{x}_r$ at uncertain marginal cost $\mathbf{c}_r\mathbf{x}_r$

$$\underbrace{\boldsymbol{W}_p(\omega)}_{\text{producer's welfare}} = -\underbrace{\frac{1}{2}cx^2}_{\text{cost step 1}} - \underbrace{\frac{1}{2}\mathbf{c}_r(\omega)\mathbf{x}_r(\omega)^2}_{\text{cost step 2}}$$

- Step 1: no consumption $\varnothing$
- Step 2: random consumption $\mathbf{y}$ at marginal utility $\mathbf{V} - \mathbf{ry}$

$$\underbrace{\boldsymbol{W}_c(\omega)}_{\text{consumer's welfare}} = \underbrace{\mathbf{V}(\omega)\mathbf{y}(\omega) - \frac{1}{2}\mathbf{r}(\omega)\mathbf{y}(\omega)^2}_{\text{consumer's utility at step 2}}$$

Social planner's welfare

The welfare of the social planner is defined by

$$\underbrace{\boldsymbol{W}_p(\omega)}_{\text{Producer's welfare}} \quad + \quad \underbrace{\boldsymbol{W}_c(\omega)}_{\text{Consumer's welfare}}$$

Equilibrium and social planner problems (See Arrow and Debreu or Uzawa)

Given a probability $\mathbb{P}$ on Ω, a risk neutral social planner problem and an risk neutral equilibrium are defined by

$$\max_{x,\mathbf{x}_r,\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_p + \boldsymbol{W}_c]}_{\text{expected welfare}}$$

$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}$$

$$\max_{x,\mathbf{x}_r} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{expected profit}}$$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\right]}_{\text{expected utility}}$$

$$\underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}$$

Decomposing social's planner criterion

When we dualize the constraint, the social planner's problem for a system of price $\boldsymbol{\pi}$ reads

$$\max_{x,\mathbf{x}_r,\mathbf{y}} \mathbb{E}_{\mathbb{P}}\left[\boldsymbol{W}_p + \boldsymbol{W}_c + \boldsymbol{\pi}(x + \mathbf{x}_r - \mathbf{y})\right]$$

and we naturally have

$$\underbrace{\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{W}_p + \boldsymbol{W}_c + \boldsymbol{\pi}(x + \mathbf{x}_r - \mathbf{y})\right]}_{\text{translated central planner problem}}$$

$$=$$

$$\underbrace{\mathbb{E}_{\mathbb{P}}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{producer problem}} + \underbrace{\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\right]}_{\text{consumer problem}}$$

Behind coordination, agent consistency ?

- Modifying the criterion of the social planner make it possible to align preferences of social planner with the ones of producer and consumer
- We obtain strong agent consistency
- For equilibrium prices $\boldsymbol{\pi}^\sharp$, the term $\boldsymbol{\pi}^\sharp(x + \mathbf{x}_r - \mathbf{y})$ vanishes at the optimum

Applications of Agent Consistency to Equilibrium

Ingredients of the problem

Agent Consistency and Equilibrium with risk neutral agents

Agent Consistency and Equilibrium with risk averse agents

Equilibrium and social planner problems

Given three risk measures $\mathbb{F}$, $\mathbb{F}_p$ and $\mathbb{F}_c$ [1], a risk averse social planner problem and an risk averse equilibrium are defined by

$$\max_{x,\mathbf{x}_r,\mathbf{y}} \quad \underbrace{\mathbb{F}[\boldsymbol{W}_p + \boldsymbol{W}_c]}_{\text{risk adjusted welfare}}$$

$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}$$

$$\max_{x,\mathbf{x}_r} \quad \underbrace{\mathbb{F}_p\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\left(x + \mathbf{x}_r\right)\Big]}_{\text{risk adjusted profit}}$$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\right]}_{\text{risk adjusted consumption}}$$

$$\underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}$$

- When we dualize does price $\boldsymbol{\pi}$ go inside $\mathbb{F}_{sp}$?
- If so, is $\mathbb{F}_{sp}\big(\boldsymbol{W}_c + \boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r - \mathbf{y})\big)$ decomposable ?

[1] A risk measure is numerical mapping $\mathbb{F} : \Omega \to \mathbb{R}$

Consumer is insensitive to the choice of risk measure

- If $\mathbb{F}_c$ is monotonic, consumer can optimize scenario per scenario and we have latitude to chose the risk measure $\mathbb{F}_c$

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{risk adjusted consumption}}$$

$$\Updownarrow$$

$$\forall \omega \in \Omega\,, \quad \max_{\mathbf{y}(\omega)} \quad \underbrace{\boldsymbol{W}_c(\omega) - \boldsymbol{\pi}(\omega)\mathbf{y}(\omega)}_{\text{scenario independant}}$$

- An idea: assume that the risk measure $\mathbb{F}_p$ has the form

$$\mathbb{F}_p(\boldsymbol{X}) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\boldsymbol{X}]$$

does there exists $\mathbb{Q}^\sharp$ to apply risk neutral case ?

Outline

Conclusion

- In this talk we have
  - presented a general framework of Time Consistency adapted to dynamic programming
    and we have shown an equivalence with Nested Formula
  - presented a more general framework of
    Consistency for binary relations adapted parallel computing
- ongoing work
  - discuss connections between coordination by price and consistency

More results can be found
in https://arxiv.org/abs/1711.08633
and in my future PhD thesis.

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