Computing risk averse equilibrium in incomplete market

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International Symposium on Mathematical Programming
Bordeaux, France, July, 2018

CERMICS - EPOC

Uncertainty on electricity market

- Today, wholesale electricity markets takes the form of an auction that matches supply and demand
- But, the demand cannot be predicted with absolute certainty. Day-ahead markets must be augmented with balancing ones
- To reduce CO_2 emissions and increase the penetration of renewables, there are increasing amounts of electricity from intermittent sources such as wind and solar
- Equilibrium on the market are then set in a stochastic setting

Objective

- We want to study risk averse equilibrium in incomplete market
- We need a quick recall on what is the difference between
  - optimization and equilibrium problems ?
  - risk neutral and risk averse ?
  - complete and incomplete markets ?

Social Planner or Equilibrium[1]

[1] Illustration idea from Pierre Fraigniaud

Social Planner or Equilibrium[1]

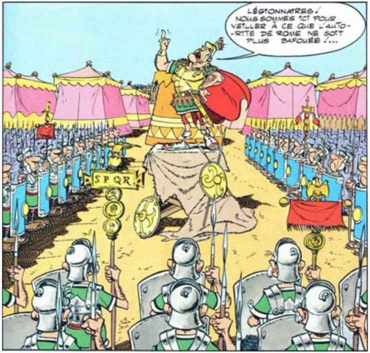


Figure 1: Social planner

[1] Illustration idea from Pierre Fraigniaud

Social Planner or Equilibrium[1]

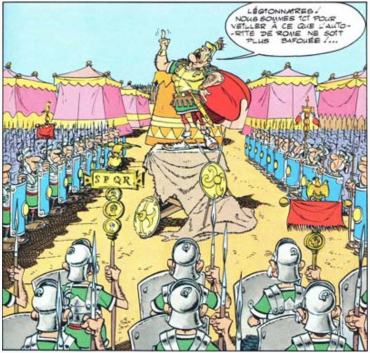


Figure 1: Social planner

Figure 2: Equilibrium

[1] Illustration idea from Pierre Fraigniaud

Optimization and uncertainty

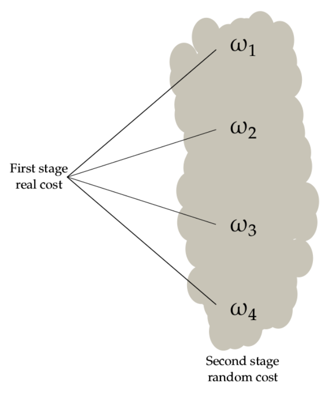


Figure 3: Aggregating uncertainty with a risk measure to obtain real value

To do optimization, we aggregate uncertainty using a risk measure which turns a random variable into a real number

- the expectation $\mathbb{E}_{\mathbb{P}}$: risk neutral
- a risk measure $\mathbb{F}$: risk averse
  - Worst Case
  - Best Case
  - Quantile
  - Median
  - Any convex combination

Complete market and incomplete market

Definition

A complete market is a market in which the number of different Arrow–Debreu securities equals the number of states of nature

- We will define an Arrow-Debreu security later
- We will retain for the moment that

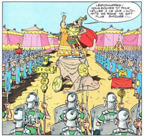
Complete market

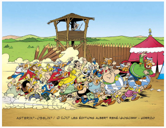
Stage 1	Stage 2
buy and sell contracts	buy and sell products

Incomplete market

Stage 1	Stage 2
do nothing	buy and sell products

Relations between Optimization and Equilibrium problems





	Optimization with Social Planner		Equilibirum
Risk Neutral $\mathbb{E}_{\mathbb{P}}$	RnSp	$\Leftrightarrow$	RnEq
Risk Averse $\mathbb{F}$	RaSp	$\Rightarrow$	RaEq-AD

- Two questions
  - What about the reverse statement ?
  - What about equilibrium in risk averse incomplete markets ?

Result on multistage stochastic equilibrium

- In Philpott, Ferris, and Wets (2013), the authors present a framework for multistage stochastic equilibria
- They show an equivalence between global risk neutral optimization problem and equilibrium in risk-neutral market. This allows us to decompose per agent
- We also mention the results of Ralph and Smeers (2015) concerning the risk averse case with complete markets

Multiple equilibrium in a incomplete market

- We show a reverse statement in the risk averse case with complete markets
- We present a toy problem with agreable properties (strong concavity of utility) that displays multiple equilibrium
- Classical computing methods fail to find all equilibria

Outline

Ingredients of the toy problem

Optimization (social planner) and equilibrium problems

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium

Outline

Ingredients of the problem

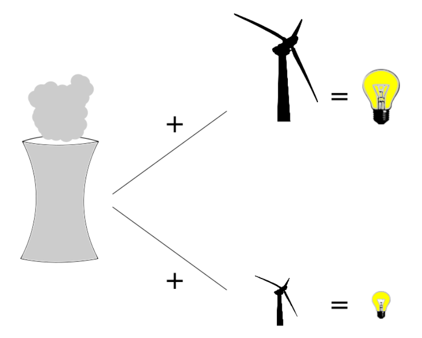


Figure 4: Illustration of the toy problem

- Two time-step market
- One good traded
- Two agents:
  producer and consumer
- Finite number of scenario
  $\omega \in \Omega$
- Consumption
  on second stage only

Producer's welfare and Consumer's welfare

- Step 1: production of x at a marginal cost cx
- Step 2: random production $\mathbf{x}_r$ at uncertain marginal cost $\mathbf{c}_r\mathbf{x}_r$

$$\underbrace{W_p(\omega)}_{\text{producer's welfare}} = -\underbrace{\frac{1}{2}cx^2}_{\text{cost step 1}} \underbrace{-\frac{1}{2}\mathbf{c}_r(\omega)\mathbf{x}_r(\omega)^2}_{\text{cost step 2}}$$

- Step 1: no consumption $\varnothing$
- Step 2: random consumption $\mathbf{y}$ at marginal utility $\mathbf{V} - \mathbf{r}\mathbf{y}$

$$\underbrace{W_c(\omega)}_{\text{consumer's welfare}} = \underbrace{\mathbf{V}(\omega)\mathbf{y}(\omega) - \frac{1}{2}\mathbf{r}(\omega)\mathbf{y}(\omega)^2}_{\text{consumer's utility at step 2}}$$

Outline

Optimization (social planner) and equilibrium problems

Optimization problem

Equilibrium problems

Social planner's welfare

The welfare of the social planner can be defined by

$$\underbrace{\boldsymbol{W}_{sp}(\omega)}_{\text{Social planner's welfare}} = \underbrace{\boldsymbol{W}_{p}(\omega)}_{\text{Producer's welfare}} + \underbrace{\boldsymbol{W}_{c}(\omega)}_{\text{Consumer's welfare}}$$

Risk neutral social planner problem

Given a probability distribution $\mathbb{P}$ on Ω, we can define a risk neutral social planner problem

$$
\begin{aligned}
\mathrm{RnSp}(\mathbb{P}):\ \max_{x,\mathbf{x}_r,\mathbf{y}} \quad & \underbrace{\mathbb{E}_{\mathbb{P}}[\boldsymbol{W}_{sp}]}_{\text{expected welfare}} \\
\text{s.t.} \quad & \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}}, \quad \forall \omega \in \Omega
\end{aligned}
$$

Risk averse social planner problem

Given a risk measure $\mathbb{F}$, we can define a risk averse social planner problem

$$\mathrm{RaSp}(\mathbb{F}): \quad \max_{x,\mathbf{x}_r,\mathbf{y}} \quad \underbrace{\mathbb{F}[\boldsymbol{W}_{sp}]}_{\text{risk adjusted welfare}}$$

$$\text{s.t.} \quad \underbrace{x + \mathbf{x}_r(\omega)}_{\text{supply}} = \underbrace{\mathbf{y}(\omega)}_{\text{demand}} \;, \quad \forall \omega \in \Omega$$

Coherent risk measures

We study coherent risk measures defined by
(see Artzner, Delbaen, Eber, and Heath (1999))

$$\mathbb{F}[\boldsymbol{Z}] = \min_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\boldsymbol{Z}]$$

where $\mathcal{Q}$ is a convex set of probability distributions over Ω

Risk averse social planner problem with polyhedral risk measure

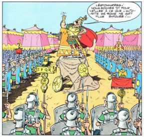
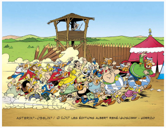
- If $\mathbb{Q}$ is a polyhedron defined by K extreme points $(\mathbb{Q}_k)_{k \in [\![1;K]\!]}$, then the risk measure $\mathbb{F}$ is said to be polyhedral and is defined by

$$\mathbb{F}[\boldsymbol{Z}] = \min_{\mathbb{Q}_1, \dots, \mathbb{Q}_K} \mathbb{E}_{\mathbb{Q}_k}[\boldsymbol{Z}]$$

- The problem RaSp($\mathbb{F}$) where $\mathbb{F}$ is polyhedral can be written in a more convenient form for optimization

$$\begin{aligned} \max_{\theta, x, \mathbf{x}_r, \mathbf{y}} \ & \theta \\ \text{s.t. } & \theta \le \mathbb{E}_{\mathbb{Q}_k}[\boldsymbol{W}_{sp}] \ , \ \ k \in [\![1;K]\!] \\ & x + \mathbf{x}_r(\omega) = \mathbf{y}(\omega) \ , \ \ \forall \omega \in \Omega \end{aligned}$$

We have presented Optimization problems

	 Optimization with Social Planner	 Equilibirum
Risk Neutral	RnSp	RnEq
Risk Averse	RaSp	RaEq(-AD)

Optimization (social planner) and equilibrium problems

Optimization problem

Equilibrium problems

Agent are price takers

Definition

An agent is *price taker* if she acts as if she has no influence on the price.

In the remain of the presentation, we consider that agents are price takers

Definition risk neutral equilibrium

Definition ((See Arrow and Debreu (1954) or Uzawa (1960)))
Given a probability $\mathbb{P}$ on Ω, a risk neutral equilibrium $\mathrm{RnEq}(\mathbb{P})$ is a set of prices $\{\boldsymbol{\pi}(\omega)\,,\ \omega \in \Omega\}$ such that there exists a solution to the system

$$
\begin{aligned}
\mathrm{RnEq}(\mathbb{P}):\quad & \max_{x,\mathbf{x}_r} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\,(x + \mathbf{x}_r)\Big]}_{\text{expected profit}} \\
& \max_{\mathbf{y}} \quad \underbrace{\mathbb{E}_{\mathbb{P}}\big[\boldsymbol{W}_c - \boldsymbol{\pi}\,\mathbf{y}\big]}_{\text{expected utility}} \\
& \underbrace{0 \le x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \ge 0}_{\text{market clears}}\,,\quad \forall \omega \in \Omega
\end{aligned}
$$

Remark on complementarity constraints

- Complementarity constraints are defined by

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0 \ , \ \ \forall \omega \in \Omega$$

- If $\boldsymbol{\pi} > 0$ then supply $=$ demand
- If $\boldsymbol{\pi} = 0$ then supply $\geq$ demand

Definition of risk averse equilibrium

Definition

Given two risk measures $\mathbb{F}_p$ and $\mathbb{F}_c$, a risk averse equilibrium $\mathrm{RaEq}(\mathbb{F}_p, \mathbb{F}_c)$ is a set of prices $\{\boldsymbol{\pi}(\omega) : \omega \in \Omega\}$ such that there exists a solution to the system

$$
\begin{aligned}
\mathrm{RaEq}(\mathbb{F}_p, \mathbb{F}_c)\text{:} \quad & \max_{x, \mathbf{x}_r} \quad \underbrace{\mathbb{F}_p\Big[\boldsymbol{W}_p + \boldsymbol{\pi}\,(x + \mathbf{x}_r)\Big]}_{\text{risk adjusted profit}} \\
& \max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\big[\boldsymbol{W}_c - \boldsymbol{\pi}\,\mathbf{y}\big]}_{\text{risk adjusted consumption}} \\
& \underbrace{0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0}_{\text{market clears}}\,, \quad \forall \omega \in \Omega
\end{aligned}
$$

- If $\mathbb{F}_p = \mathbb{F}_c$ then we write $\mathrm{RaEq}(\mathbb{F})$

Consumer is insensitive to the choice of risk measure

Assuming that the risk measure $\mathbb{F}_c$ of the consumer is monotonic, she can optimize scenario per scenario as she has no first stage decision

$$\max_{\mathbf{y}} \quad \underbrace{\mathbb{F}_c\big[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}\big]}_{\text{risk adjusted consumption}}$$

$$\Updownarrow$$

$$\forall \omega \in \Omega\ , \quad \max_{\mathbf{y}(\omega)} \quad \underbrace{\boldsymbol{W}_c(\omega) - \boldsymbol{\pi}(\omega)\mathbf{y}(\omega)}_{\text{scenario independant}}$$

Risk averse equilibrium with polyhedral risk measure

If the risk measure $\mathbb{F}$ is polyhedral, then RaEq($\mathbb{F}$) reads

$$\mathrm{RaEq:}\ \max_{\theta, x, \mathbf{x}_r} \quad \theta$$

$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\right], \quad \forall k \in [\![1; K]\!]$$

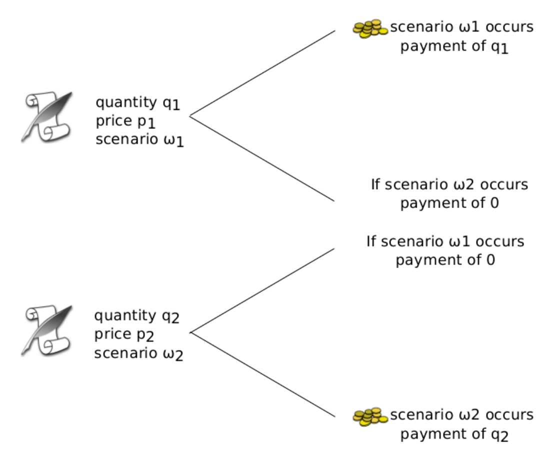
$$\max_{\mathbf{y}(\omega)} \quad \boldsymbol{W}_c(\omega) - \boldsymbol{\pi}\mathbf{y}(\omega), \quad \forall \omega \in \Omega$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0, \quad \forall \omega \in \Omega$$

Definition of an Arrow-Debreu security

Definition

An *Arrow-Debreu security* for node $\omega \in \Omega$ is a contract that charges a price $\mu(\omega)$ in the first stage, to receive a payment of 1 in scenario ω.



Risk averse equilibrium with trading

A *risk trading equilibrium* is sets of prices $\{\boldsymbol{\pi}(\omega)\ ,\ \ \omega \in \Omega\}$ and $\{\boldsymbol{\mu}(\omega)\ ,\ \ \omega \in \Omega\}$ such that there exists a solution to the system:

$$\text{RaEq-AD:} \max_{x,\mathbf{x}_r} \quad - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{a}(\omega)}_{\text{value of contracts purchased}} \quad + \mathbb{F}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r) + \mathbf{a}\right]$$

$$\max_{\phi,\mathbf{y}} \quad - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{b}(\omega)}_{\text{value of contracts purchased}} \quad + \mathbb{F}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y} + \mathbf{b}\right]$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0\ , \quad \forall \omega \in \Omega$$

$$\underbrace{0 \leq -\mathbf{a}(\omega) - \mathbf{b}(\omega)}_{\text{"supply} \geq \text{demand"}} \perp \boldsymbol{\mu}(\omega) \geq 0\ , \quad \forall \omega \in \Omega$$

RaEq with trading and polyhedral risk measure

A *risk trading equilibrium* is sets of prices $\{\boldsymbol{\pi}(\omega)\ ,\ \omega \in \Omega\}$ and $\{\boldsymbol{\mu}(\omega)\ ,\ \omega \in \Omega\}$ such that there exists a solution to the system:

$$\text{RaEq-AD:}\ \max_{\theta, x, \mathbf{x}_r} \quad \theta - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{a}(\omega)}_{\text{value of contracts purchased}}$$

$$\text{s.t.} \quad \theta \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r) + \mathbf{a}\right]\ ,\ \ \forall k \in [\![1; K]\!]$$

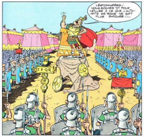
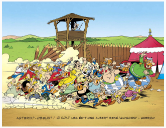
$$\max_{\phi, \mathbf{y}} \quad \phi - \underbrace{\sum_{\omega \in \Omega} \boldsymbol{\mu}(\omega)\mathbf{b}(\omega)}_{\text{value of contracts purchased}}$$

$$\text{s.t.} \quad \phi \leq \mathbb{E}_{\mathbb{Q}_k}\left[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y} + \mathbf{b}\right]\ ,\ \ \forall k \in [\![1; K]\!]$$

$$0 \leq x + \mathbf{x}_r(\omega) - \mathbf{y}(\omega) \perp \boldsymbol{\pi}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

$$\underbrace{0 \leq -\mathbf{a}(\omega) - \mathbf{b}(\omega)}_{\text{"supply} \geq \text{demand"}} \perp \boldsymbol{\mu}(\omega) \geq 0\ ,\ \ \forall \omega \in \Omega$$

We have presented Equilibrium problems

	 Optimization with Social Planner	 Equilibirum
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Outline

Ingredients of the toy problem

Optimization (social planner) and equilibrium problems

Links between optimization problems and equilibrium problems

Multiple risk averse equilibrium

RnSp($\mathbb{P}$) is equivalent to RnEq($\mathbb{P}$)

Proposition

Let $\mathbb{P}$ be a probability measure over Ω.
The elements $(x^\star \mathbf{x}_r^\star, \mathbf{y}_r^\star)$ are optimal solutions to RnSp($\mathbb{P}$) if and only if there exist non trivial equilibrium prices $\boldsymbol{\pi}$ for RnEq($\mathbb{P}$) with associated optimal controls $(x^\star, \mathbf{x}_r^\star, \mathbf{y}^\star)$

Corollary

If producer's criterion and consumer's criterion are strictly concave, then RnSp($\mathbb{P}$) admit a unique solution and RnEq($\mathbb{P}$) admit a unique equilibrium.

RaEq-AD is equivalent to RaSp

Theorem

Let $(x^\sharp, \mathbf{x}_r^\sharp, \mathbf{y}_r^\sharp)$ *be optimal solutions to RaSp, with associated worst case probability measure* $\boldsymbol{\mu}$*. Then there exists prices* $\boldsymbol{\pi}$ *such that* $(\boldsymbol{\pi}, \boldsymbol{\mu})$ *forms a risk trading equilibrium for RaEq-AD with optimal solutions* $(x^\sharp, \mathbf{x}_r^\sharp, \mathbf{y}_r^\sharp)$

- We adapt a result of Ralph and Smeers (2015)

Theorem

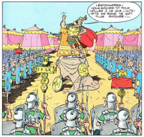
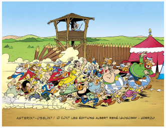
Let $(\boldsymbol{\pi}, \boldsymbol{\mu})$ *be equilibrium prices such that* $(x^\sharp, \mathbf{x}_r^\sharp, \mathbf{y}_r^\sharp, \mathbf{a}, \mathbf{b}, \theta, \phi)$ *solves RaEq-AD. Then* $(x^\sharp, \mathbf{x}_r^\sharp, \mathbf{y}_r^\sharp)$ *solves RaSP, with worst case measure* $\boldsymbol{\mu}$*.*

Uniqueness of equilibrium

Corollary

If both the producer's and consumer's criterion are strictly concave and some technical assumptions, then RaSp admits a unique solution and RaEq-AD admits a unique equilibrium

Summing up equivalences

	 Optimization with Social Planner		 Equilibirum
Risk Neutral $\mathbb{E}_{\mathbb{P}}$	RnSp	$\Leftrightarrow$	RnEq
Risk Averse $\mathbb{F}$	RaSp	$\Leftrightarrow$	$\underbrace{\text{RaEq-AD}}_{\text{complete market}}$

- This leads to result about uniqueness of equilibrium and methods of decomposition
- What can we say about $\underbrace{\text{RaEq}}_{\text{incomplete market}}$?

Outline

Multiple risk averse equilibrium

Numerical results

Analytical results

Recall on the problem

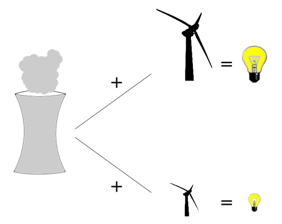


Figure 5: Illustration of the toy problem

Recall:

- Two time-step market
- One good traded
- Two agents
- Consumption on second stage only

We focus on:

- Two scenarios ω_1 and ω_2
- Two prices: π_1 and π_2
- Five controls: x, x_1, x_2, y_1 and y_2
- Two probabilities $(\underline{p}, 1 - \underline{p})$ and $(\bar{p}, 1 - \bar{p})$
- $\underline{p} = \frac{1}{4}$, $\bar{p} = \frac{3}{4}$
- prices $0 < \pi_1 < \pi_2$

Computing an equilibrium with GAMS

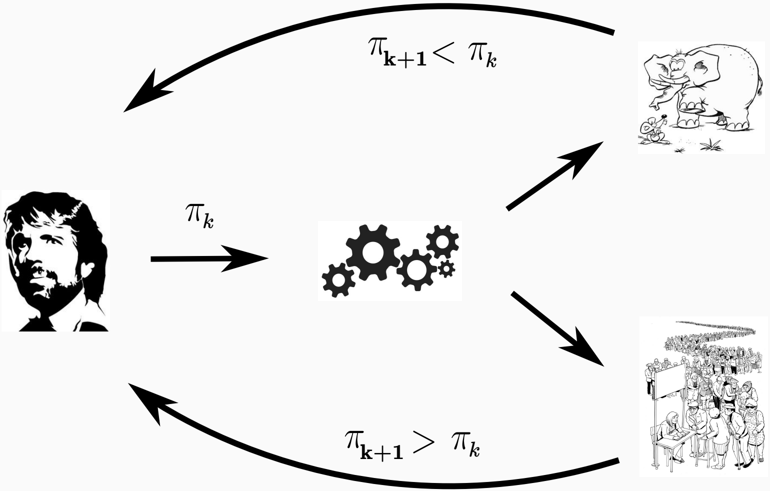
- GAMS with the solver PATH in the EMP framework
  (See Britz et al. (2013), Brook et al. (1988), Ferris and Munson (2000) and Ferris et al. (2009))

- different starting points defined by a grid 100×100 over the square $[1.220; 1.255] \times [2.05; 2.18]$

- We find one equilibrium defined by

$$\boldsymbol{\pi} = (\pi_1, \pi_2) = (1.23578; 2.10953)$$

A second algorithm : the idea of tâtonnement method



Walras's tâtonnement algorithm (See Uzawa (1960))

Then we compute the equilibrium using a tâtonnement algorithm

Data: MAX-ITER, $(\pi_1^0, \pi_2^0), \tau$

Result: A couple $(\pi_1^\star, \pi_2^\star)$ approximating equilibrium price $\pi_\sharp$

for *k from* 0 *to MAX-ITER* **do**

  Compute an optimal decision for each player given a price :

    $x\ ,\ \ x_1, x_2 = \arg\max \mathbb{F}\big[\boldsymbol{W}_p + \boldsymbol{\pi}(x + \mathbf{x}_r)\big];$

    $y(\omega) = \arg\max \mathbb{F}[\boldsymbol{W}_c - \boldsymbol{\pi}\mathbf{y}];$

  Update the price :

    $\pi_1 = \pi_1 - \tau \max\big\{0; y_1 - (x + x_1)\big\};$

    $\pi_2 = \pi_2 - \tau \max\big\{0; y_2 - (x + x_2)\big\};$

end

return (π_1, π_2)

Algorithm 1: Walras' tâtonnement

Computing equilibria with Walras's tâtonnement

- Running Walras's tâtonnement algorithm starting from $(1.25; 2.06)$, respectively from $(1.22; 2.18)$, with 100 iterations and a step size of 0.1, we find two new equilibria

$$\boldsymbol{\pi} = (1.2256; 2.0698) \text{ and } \boldsymbol{\pi} = (1.2478; 2.1564)$$

- An alternative tatônnement method called FastMarket (see Facchinei and Kanzow (2007)) find the same equilibria

Summing up about computing equilibrium

	Equilibrium prices	Risk adjusted welfares
red (Tâtonnement)	$(1.2478; 2.1564)$	$(2.113; 0.845)$
blue (GAMS)	$(1.2358; 2.1095)$	$(2.134; 0.821)$
green (Tâtonnement)	$(1.2256; 2.0698)$	$(2.152; 0.798)$

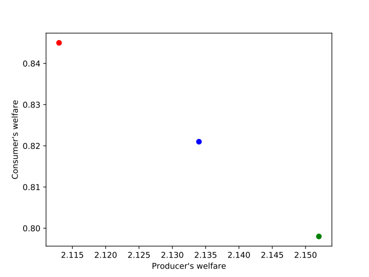


Figure 6: Representation of equilibrium in terms of welfare

- No equilibrium dominates an other

Multiple risk averse equilibrium

Numerical results

Analytical results

Optimal control of agents with respect to a price π

There are three regimes

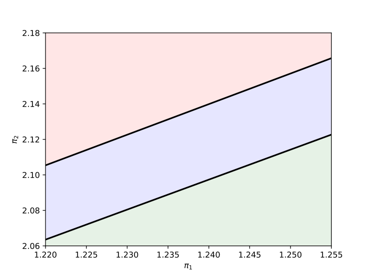


Figure 7: Illustration of the three regimes

condition	$x^\sharp$	$x_i^\sharp$	$y_i^\sharp$
$x_c \leq \frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i-\pi_i}{r_i}$
$\frac{\mathbb{E}_{\bar{p}}\left[\boldsymbol{\pi}\right]}{c} \leq x_c \leq \frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c}$	x_c	$\frac{\pi_i}{c_i}$	$\frac{V_i-\pi_i}{r_i}$
$\frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c} \leq x_c$	$\frac{\mathbb{E}_{\underline{p}}\left[\boldsymbol{\pi}\right]}{c}$	$\frac{\pi_i}{c_i}$	$\frac{V_i-\pi_i}{r_i}$

Table 1: Optimal control for producer and consumer problems

$$\text{where } x_c(\boldsymbol{\pi}) = \frac{1}{2(\pi_1 - \pi_2)}\left(\frac{\pi_2^2}{2c_2} - \frac{\pi_1^2}{2c_1}\right)$$

Excess production function

- We have optimal control as a function of price in three regions
- We loof for prices (π_1, π_2) such that supply $=$ demands
- The complementarity constraints are satisfied if

$$0 = z_i(\boldsymbol{\pi}) = \underbrace{x^\sharp(\boldsymbol{\pi}) + x_i^\sharp(\boldsymbol{\pi}) - y_i^\sharp(\boldsymbol{\pi})}_{\text{market clears for equilibrium prices}} \quad , \quad \underbrace{i \in \{1, 2\}}_{\text{scenarios}}$$

- This excess functions have three regime
- In the green and red part the equation is linear, in the blue part the equation is quadratic.

Regimes of excess production function in scenario 1 ($z_1(\pi) = 0$)

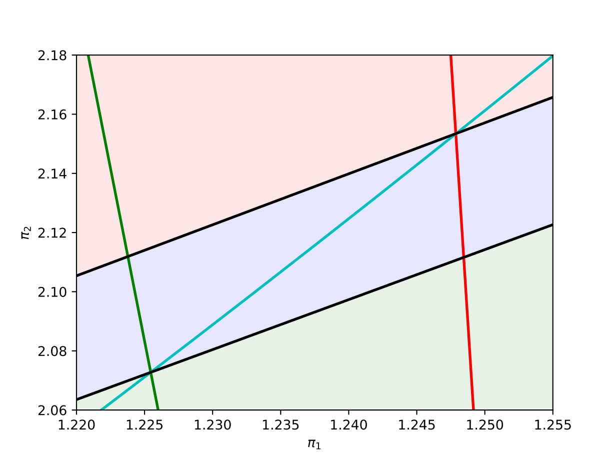


Figure 8: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Regimes of excess production function in scenario 1 ($z_2(\pi) = 0$)

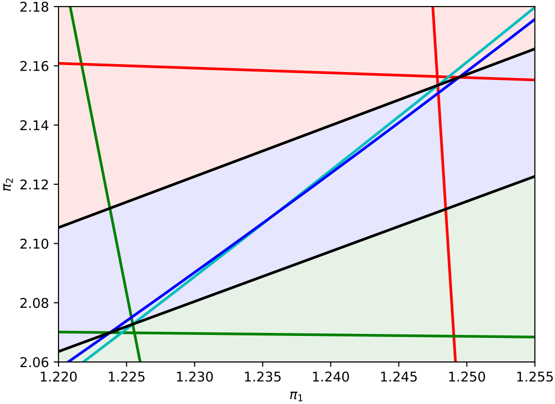


Figure 9: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (red equilibrium)

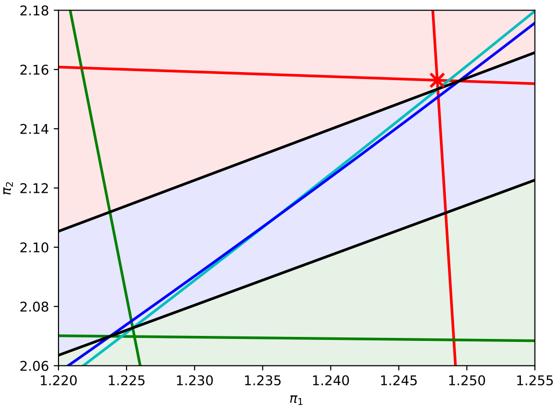


Figure 10: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (blue equilibrium)

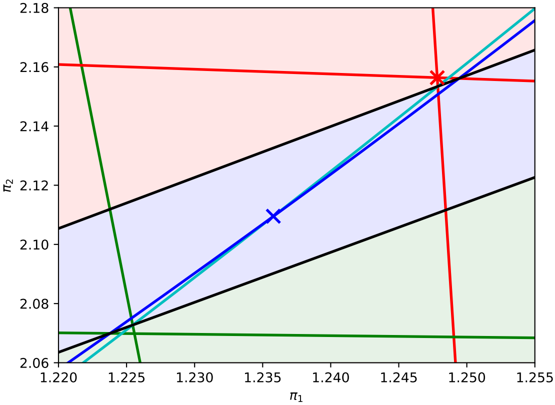


Figure 11: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Representation of analytical solutions (green equilibrium)

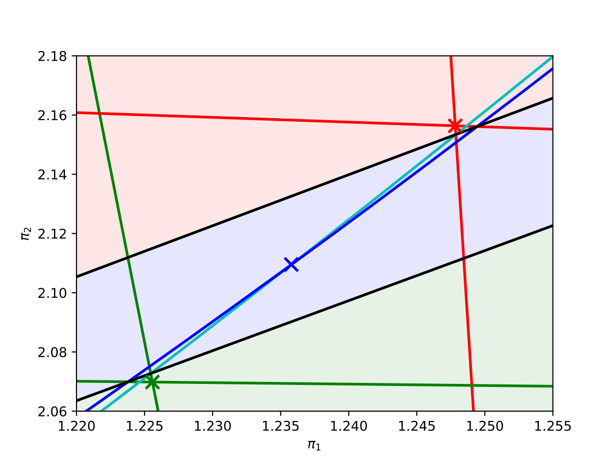


Figure 12: Null excess function per scenario manifold for $V_1 = 4$, $V_2 = \frac{48}{5}$, $c = \frac{23}{2}$, $c_1 = 1$, $c_2 = \frac{7}{2}$, $r_1 = 2$, $r_2 = 10$.

Some interesting remarks

Remark

The PATH solver find the blue equilibrium, while the tatônnements methods find equilibrium green and red. Interestingly it can be shown that the blue equilibrium is unstable in the sense that the dynamical system driven by $\pi' = z(\pi)$ is unstable around the blue equilibrium.

Remark

There exists a set of non-zero measure of parameters $V_1, V_2, c, c_1, c_2, r_1$, and r_2 (albeit small), that have three distinct equilibrium with the same properties.

Remark

We can show that the blue equilibrium is a convex combination of red and green equilibrium.

Stability of equilibriums (red equilibrium)

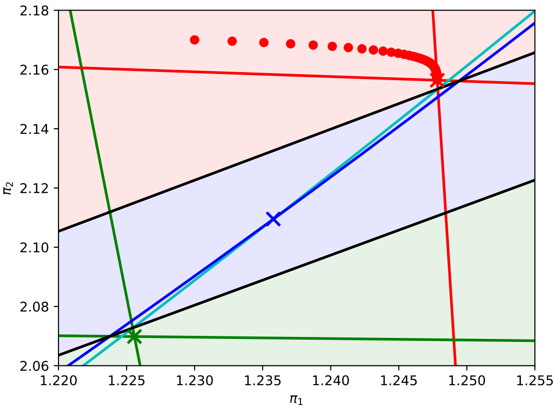


Figure 13: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (blue equilibrium)

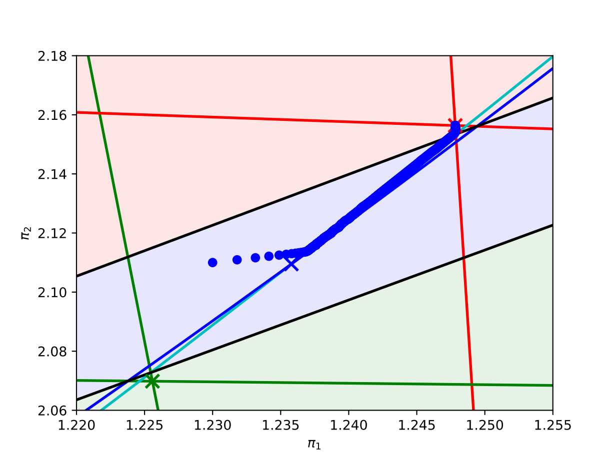


Figure 14: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (green equilibrium)

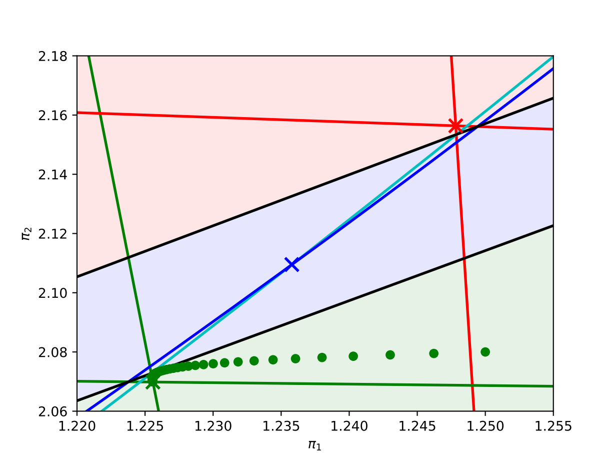


Figure 15: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Stability of equilibriums (vector field)

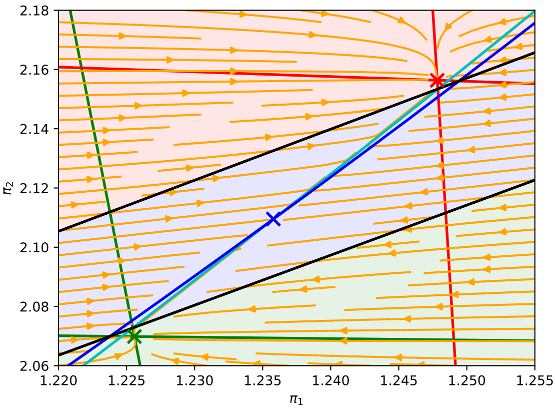


Figure 16: Representation of vector field $\boldsymbol{\pi}' = z(\pi)$ around green equilibrium

Conclusion

In this talk we have

- shown an equivalence between risk averse social planner problem and risk trading equilibrium
  (respectively risk neutral equivalence)
- given theorems of uniqueness of equilibrium
- shown non uniqueness of equilibrium in incomplete market

On going work

- Extend the counter example with multiple agents and scenarios
- Do we have uniqueness with bounds on the number of Arrow-Debreu securities exchanged ?

References I

K. J. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econometrica: Journal of the Econometric Society*, pages 265–290, 1954.

P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

W. Britz, M. Ferris, and A. Kuhn. Modeling water allocating institutions based on multiple optimization problems with equilibrium constraints. *Environmental modelling & software*, 46:196–207, 2013.

A. Brook, D. Kendrick, and A. Meeraus. Gams, a user's guide. *ACM Signum Newsletter*, 23(3-4):10–11, 1988.

References II

F. Facchinei and C. Kanzow. Generalized nash equilibrium problems. *4OR: A Quarterly Journal of Operations Research*, 5 (3):173–210, 2007.

M. C. Ferris and T. S. Munson. Complementarity problems in gams and the path solver. *Journal of Economic Dynamics and Control*, 24(2):165–188, 2000.

M. C. Ferris, S. P. Dirkse, J.-H. Jagla, and A. Meeraus. An extended mathematical programming framework. *Computers & Chemical Engineering*, 33(12):1973–1982, 2009.

H. Gérard, V. Leclère, and A. Philpott. On risk averse competitive equilibrium. *arXiv preprint arXiv:1706.08398*, 2017.

References III

A. Philpott, M. Ferris, and R. Wets. Equilibrium, uncertainty and risk in hydro-thermal electricity systems. *Mathematical Programming*, pages 1–31, 2013.

D. Ralph and Y. Smeers. Risk trading and endogenous probabilities in investment equilibria. *SIAM Journal on Optimization*, 25(4):2589–2611, 2015.

H. Uzawa. Walras' tatonnement in the theory of exchange. *The Review of Economic Studies*, 27(3):182–194, 1960.