Decomposition-coordination methods for optimization under risk

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Why are we studying decomposition methods

- Multistage stochastic optimization problems (SOP)
  are large scale due to
  - the number of scenarios
  - the number of time steps
  - the number of agents, units
- Decomposition methods can make such problems more tractable
- Having in mind possible applications to the management of electricity networks under risk (black-out, failures, reliability...),
  what happens when the traditional mathematical expectation
  is replaced by a risk measure?

The path we are going to follow

In the talk, we develop two resolution methods

- by time decomposition (nested formulation)
- by scenario decomposition (duality)

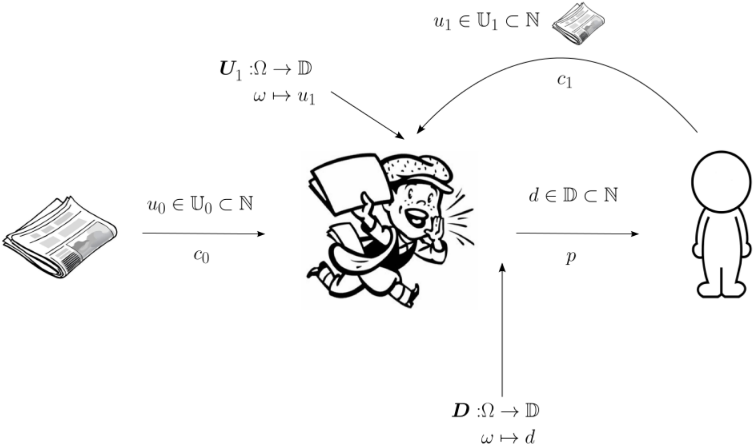
Outline

1. Working out the example of a risk averse newsvendor problem
2. Resolution by time decomposition
3. Resolution by dualization of the non-anticipativity constraints

Outline of the section

1. Working out the example of a risk averse newsvendor problem
2. Resolution by time decomposition
3. Resolution by dualization of the non-anticipativity constraints

Presentation of the two stage newsvendor

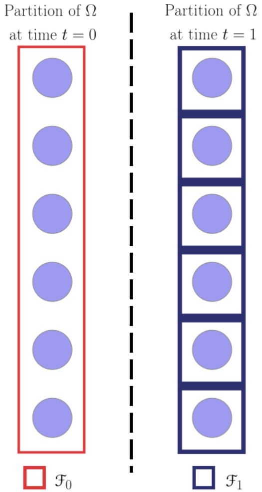


Cost function

We consider the following cost function:

$$
\begin{aligned}
j(u_0, u_1, d) = & \, c_0 \underbrace{u_0}_{\text{order } t=0} - p \min\{u_0, d\} \\
& + c_1 \underbrace{u_1}_{\text{order } t=1} - p \min\Big\{u_1, \underbrace{d - \min\{u_0, d\}}_{\text{unsatisfied demand}}\Big\}
\end{aligned}
$$

Probability space and demand



- We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Ω finite
- The demand $\boldsymbol{D}$ is a random variable on Ω
- The demand $\boldsymbol{D}$ is revealed after the initial order

How we express the non-anticipativity constraints

- We consider a first control $u_0 \in \mathbb{U}$
  to express that it does not depend on $\boldsymbol{D}$;
  we denote that by
  $$\boldsymbol{U}_0 \preceq \sigma\big(\{\emptyset, \Omega\}\big)$$
- We consider a recourse control $\boldsymbol{U}_1 \in \mathbb{U}^{\Omega}$
  because it is allowed to depend on $\boldsymbol{D}$;
  we denote that by
  $$\boldsymbol{U}_1 \preceq \sigma(\boldsymbol{D})$$

Introduction of a risk measure

We denote by $\mathbb{F}$ a risk measure

- $\mathbb{F}[\boldsymbol{X}] = \mathbb{E}_{\mathbb{P}}[\boldsymbol{X}]$: risk neutral case
- $\mathbb{F}[\boldsymbol{X}] = \max_{\omega \in \Omega} \boldsymbol{X}(\omega)$: worst case
- Conditional Value-at-risk definition

$$\mathrm{CVAR}_{\beta}[\boldsymbol{X}] = \inf_{s \in \mathbb{R}} \left\{ \frac{\mathbb{E}_{\mathbb{P}}\left[(\boldsymbol{X} - s)^{+}\right]}{1 - \beta} + s \right\} , \quad \beta \in [0, 1[$$

Risk averse formulation with non-anticipativity constraints

$$\underbrace{\min_{u_0 \in \mathbb{U}, \boldsymbol{U}_1 \in \mathbb{U}^{\Omega}}}_{\text{non-anticipativity constraints}} \overbrace{\mathbb{F}}^{\text{risk measure}} \Big[\underbrace{j}_{\text{criterion}} \big(\overbrace{u_0, \boldsymbol{U}_1}^{\text{controls}}, \overbrace{\boldsymbol{D}}^{\text{demand}} \big) \Big]$$

What will follow

- On a toy problem, we have presented
  - how we formulate a risk averse problem
  - the CVAR risk measure for numerical applications
- Now we develop methods to solve more general stochastic optimization problems under risk
  - by time decomposition
  - by scenario decomposition

Assumptions for the two decompositions methods

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

- The set Ω is finite
- The probability $\mathbb{P}$ charges every point of Ω

Outline of the section

Temporal structure and statement of the problem

Temporal structure of information

We consider two $\sigma-$fields

$$\mathcal{F}_0 \subset \mathcal{F}_1 \ , \ \ \mathcal{F}_0 \neq \mathcal{F}_1$$

We write a two stage optimization problem

$$\min_{\boldsymbol{U}_0 \in \mathbb{U}_0^\Omega, \boldsymbol{U}_1 \in \mathbb{U}_1^\Omega} \overbrace{\mathbb{F}}^{\text{risk measure}} \left[j(\underbrace{\boldsymbol{U}_0, \boldsymbol{U}_1}_{\text{controls}}, \underbrace{\boldsymbol{W}}_{\text{uncertainty}}) \right]$$

subject to non-anticipativity constraints

$$\text{s.t.} \begin{cases} \boldsymbol{U}_0 \preceq \mathcal{F}_0 \\ \boldsymbol{U}_1 \preceq \mathcal{F}_1 \end{cases}$$

Time consistency is an extension of tower property

Remark

With $\mathcal{F}_0 \subset \mathcal{F}_1$, we have the following tower property

$$\mathbb{E}_{\mathbb{P}}[\boldsymbol{U}] = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\big[\mathbb{E}_{\mathbb{P}}[\boldsymbol{U} \mid \mathcal{F}_1] \bigm| \mathcal{F}_0\big]\Big]$$

Definition

A dynamic risk measure $(\mathbb{F}, \mathbb{F}_0, \mathbb{F}_1)$ has the time consistency property if

$$\mathbb{F}[\boldsymbol{U}] = \mathbb{F}\big[\underbrace{\mathbb{F}_0[\boldsymbol{U}]}_{\mathcal{F}_0-\text{measurable}}\big] = \mathbb{F}\Big[\mathbb{F}_0\big[\underbrace{\mathbb{F}_1[\boldsymbol{U}]}_{\mathcal{F}_1-\text{measurable}}\big]\Big] , \quad \forall \boldsymbol{U} \in \mathbb{U}^{\Omega}$$

Time decomposition in the risk neutral case

Proposition (Time decomposition)

When $\mathcal{F}_0 \subset \mathcal{F}_1$, the problem

$$\min_{\boldsymbol{U}_0 \in \mathbb{U}_0^{\Omega}, \boldsymbol{U}_1 \in \mathbb{U}_1^{\Omega}} \mathbb{E}_{\mathbb{P}}\big[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\big]$$

$$s.t. \begin{cases} \boldsymbol{U}_0 \preceq \mathcal{F}_0 \\ \boldsymbol{U}_1 \preceq \mathcal{F}_1 \end{cases}$$

is equivalent to

$$\mathbb{E}_{\mathbb{P}}\Big[\min_{u_0 \in \mathbb{U}_0} \mathbb{E}_{\mathbb{P}}\Big[\min_{u_1 \in \mathbb{U}_1} \mathbb{E}_{\mathbb{P}}\big[j(u_0, u_1, \boldsymbol{W}) \bigm| \mathcal{F}_1\big] \Bigm| \mathcal{F}_0\Big]\Big]$$

Algorithm: Backward resolution and dynamic programming

Extension of time decomposition to the risk averse case

Proposition

- *if* $\mathcal{F}_0 \subset \mathcal{F}_1$
- *if there exists a dynamic risk measure* $(\mathbb{F}, \mathbb{F}_0, \mathbb{F}_1)$ *which is time consistent*
- *if* $(\mathbb{F}, \mathbb{F}_0)$ *are monotonous*

$$\min_{\boldsymbol{U}_0 \in \mathbb{U}_0^\Omega, \boldsymbol{U}_1 \in \mathbb{U}_1^\Omega} \mathbb{F}\left[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\right]$$

$$s.t. \begin{cases} \boldsymbol{U}_0 \preceq \mathcal{F}_0 \\ \boldsymbol{U}_1 \preceq \mathcal{F}_1 \end{cases}$$

is equivalent to

$$\mathbb{F}\left[\min_{u_0 \in \mathbb{U}_0} \mathbb{F}_0\left[\min_{u_1 \in \mathbb{U}_1} \mathbb{F}_1\left[j(u_0, u_1, \boldsymbol{W})\right]\right]\right]$$

Comment on time decomposition

- Conclusion
  - We have extended the time decomposition method
    from the risk neutral to the risk averse case
- Perspectives
  - It can be difficult to exhibit time consistent dynamic risk measures
  - We look for weaker assumptions
    that however make possible a dynamic programming equation

Outline of the section

From non-anticipativity to measurability constraints

- Time decomposition requires $\mathcal{F}_0 \subset \mathcal{F}_1$ to arrive to a nested formulation
- We no longer require this assumption for scenario decomposition
- The constraints

$$\begin{aligned} \boldsymbol{U}_0 &\preceq \mathcal{F}_0 \\ \boldsymbol{U}_1 &\preceq \mathcal{F}_1 \end{aligned}$$

are now called measurability constraints

Statement of the SOP

- In the risk-neutral case, we rewrite the measurability constraints as

$$\min_{\boldsymbol{U}_0 \in \mathbb{U}_0^\Omega, \boldsymbol{U}_1 \in \mathbb{U}_1^\Omega} \mathbb{E}_{\mathbb{P}}\big[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\big]$$

$$\text{s.t.} \begin{cases} \mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_0 \mid \mathcal{F}_0] = \boldsymbol{U}_0 \\ \mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_1 \mid \mathcal{F}_1] = \boldsymbol{U}_1 \end{cases}$$

- Primal problem

$$\min_{\boldsymbol{U}_0, \boldsymbol{U}_1} \max_{\pi_0, \pi_1} \mathbb{E}_{\mathbb{P}}\big[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\big] + \sum_{i \in \{0,1\}} \boldsymbol{\pi}_i \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_i \mid \mathcal{F}_i] - \boldsymbol{U}_i\big)$$

- Dual problem

$$\max_{\pi_0, \pi_1} \min_{\boldsymbol{U}_0, \boldsymbol{U}_1} \mathbb{E}_{\mathbb{P}}\big[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\big] + \sum_{i \in \{0,1\}} \boldsymbol{\pi}_i \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_i \mid \mathcal{F}_i] - \boldsymbol{U}_i\big)$$

Resolution by dualization in the risk-neutral case

Proposition

Under technical assumptions, primal and dual problems are equivalent. They are also equivalent to

$$\max_{\boldsymbol{\pi}_0 \in (\mathbb{R}^{n_0})^\Omega, \boldsymbol{\pi}_1 \in (\mathbb{R}^{n_1})^\Omega} \mathbb{E}_{\mathbb{P}} \Big[\min_{u_0 \in \mathbb{U}_0, u_1 \in \mathbb{U}_1} j(u_0, u_1, \boldsymbol{W}) + \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{\pi}_0 \mid \mathcal{F}_0] - \boldsymbol{\pi}_0\big) u_0 + \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{\pi}_1 \mid \mathcal{F}_1] - \boldsymbol{\pi}_1\big) u_1 \Big]$$

Scheme of the algorithm (Progressive hedging)

- For fixed multiplier π, the problem is solved scenario by scenario
- Update of the multiplier π

Statement of the problem in the risk averse case

$$\min_{\boldsymbol{U}_0 \in \mathbb{U}_0^\Omega, \boldsymbol{U}_1 \in \mathbb{U}_1^\Omega} \mathbb{F}\Big[j(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{W})\Big]$$

$$\text{s.t.} \begin{cases} \mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_0 \mid \mathcal{F}_0] = \boldsymbol{U}_0 \\ \mathbb{E}_{\mathbb{P}}[\boldsymbol{U}_1 \mid \mathcal{F}_1] = \boldsymbol{U}_1 \end{cases}$$

Dualization in the risk averse case

Proposition

If the risk measure $\mathbb{F}$ has the following form (coherent risk measure)

$$\mathbb{F}[\boldsymbol{U}] = \max_{\mathbb{Q}\in\mathfrak{Q}_{\mathbb{F}}} \mathbb{E}_{\mathbb{Q}}[\boldsymbol{U}]$$

with $\mathfrak{Q}_{\mathbb{F}}$ a closed convex set of probability laws and under technical assumptions, primal and dual problems are equivalent to

$$\begin{aligned}\max_{\boldsymbol{\pi}_0\in(\mathbb{R}^{n_0})^{\Omega},\boldsymbol{\pi}_1\in(\mathbb{R}^{n_1})^{\Omega}} \max_{\mathbb{Q}\in\mathfrak{Q}_{\mathbb{F}}} \mathbb{E}_{\mathbb{P}}\Big[&\min_{u_0\in\mathbb{U}_0,u_1\in\mathbb{U}_1} j(u_0,u_1,\boldsymbol{W})\frac{d\mathbb{Q}}{d\mathbb{P}} \\ &+ \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{\pi}_0 \mid \mathcal{F}_0] - \boldsymbol{\pi}_0\big)u_0 \\ &+ \big(\mathbb{E}_{\mathbb{P}}[\boldsymbol{\pi}_1 \mid \mathcal{F}_1] - \boldsymbol{\pi}_1\big)u_1\Big]\end{aligned}$$

Comment on scenario decomposition

- Conclusion
  - We have extended the scenario decomposition method from the risk neutral to the risk averse case
- Perspectives
  - We want to use scenario decomposition to solve multi-agents problems

Conclusion and perspectives

- Conclusion
  - We studied stochastic optimization problems with risk and time
  - We presented two results of decomposition in the risk averse case
- Perspectives
  - We are currently working on time consistent risk measures and their extensions to arrive at nested formulations
  - We are trying to extend time and scenario decompositions to solve multi-agent problems