On the Joint Calibration of SPX and VIX Options

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Motivation

- VIX options started trading in 2006
- How to build a model for the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?
- In 2008, Gatheral was one of the first to investigate this question, and showed that a diffusive model (the double mean-reverting model) could approximately match both markets.
- Later, others have argued that jumps in SPX are needed to fit both markets.
- In this talk, I revisit this problem, trying to answer the following questions:

Does there exist a continuous model on the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?

If so, how to build one such model? If not, why?

Gatheral (2008)

Consistent Modeling of SPX and VIX options

Consistent Modeling of SPX and VIX options

Jim Gatheral

Merrill Lynch

The Fifth World Congress of the Bachelier Finance Society
London, July 18, 2008

Double CEV dynamics

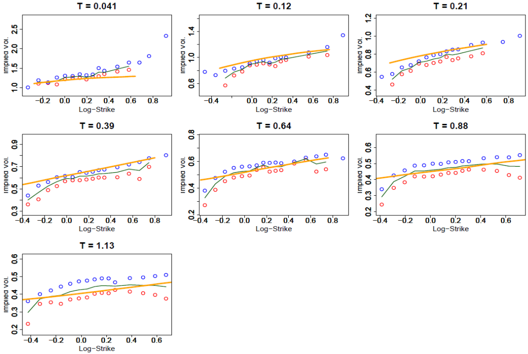
- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned} \frac{dS}{S} &= \sqrt{v}\, dW \\ dv &= -\kappa\,(v - v')\, dt + \eta_1\, v^{\alpha}\, dZ_1 \\ dv' &= -c\,(v' - z_3)\, dt + \eta_2\, {v'}^{\beta}\, dZ_2 \end{aligned} \tag{2}$$

  for any choice of α, $\beta \in [1/2, 1]$.
  - We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
  - the case $\alpha = \beta = 1$ *Double Lognormal*,
  - and the general case *Double CEV*.
- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

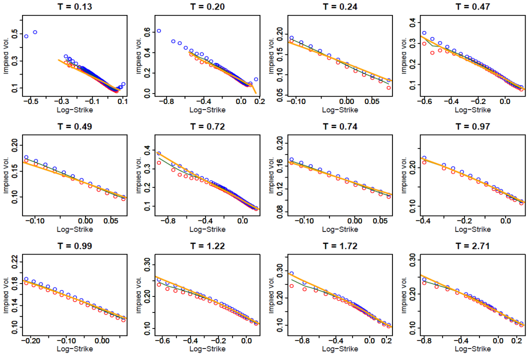
Double CEV fit to VIX options as of 03-Apr-2007

Setting the correlation ρ between volatility factors z_1 and z_2 to its historical average (see later) and iterating on the volatility of volatility parameters ξ_1 and ξ_2 to minimize the differences between model and market VIX option prices, we obtain the following fits (orange lines):

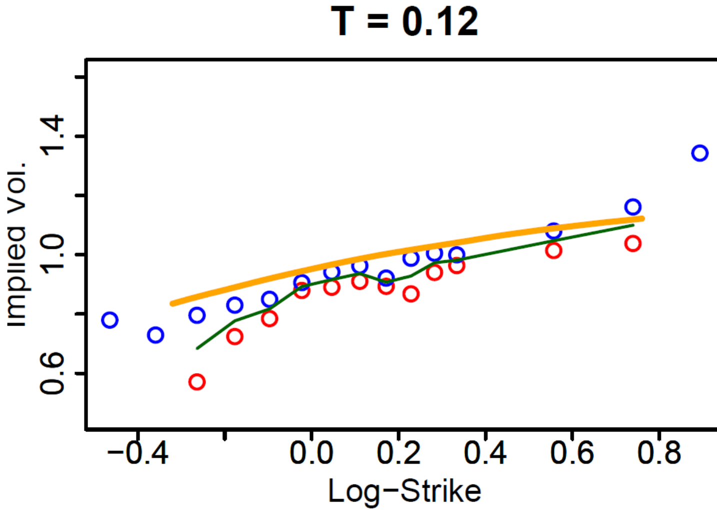


Double CEV fit to SPX options as of 03-Apr-2007

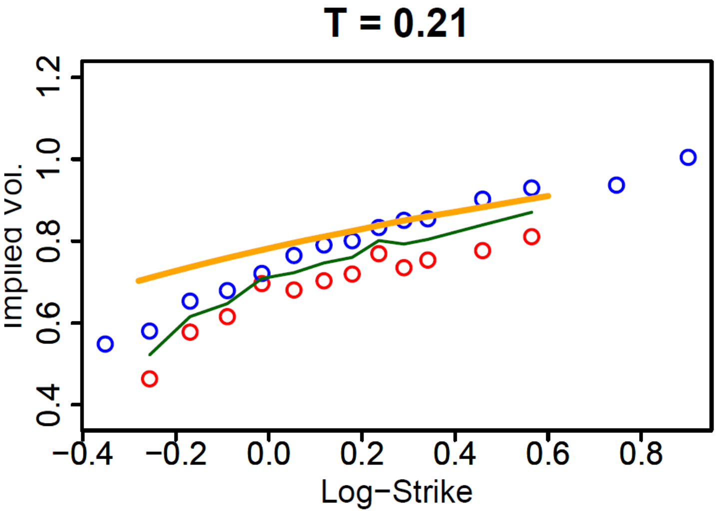
Minimizing the differences between model and market SPX option prices, we find $\rho_1 = -0.9$, $\rho_2 = -0.7$ and obtain the following fits to SPX option prices (orange lines):



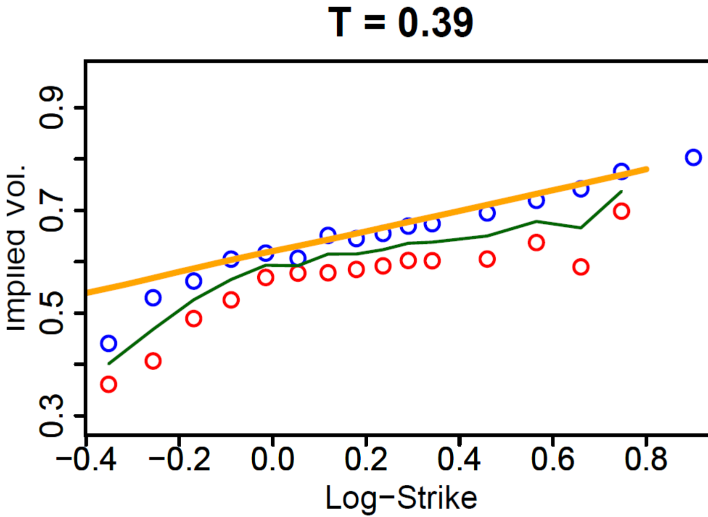
Fit to VIX options



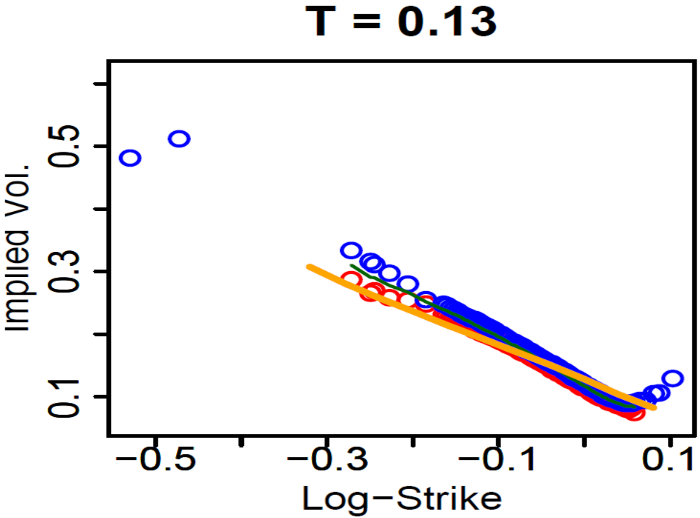
Fit to VIX options



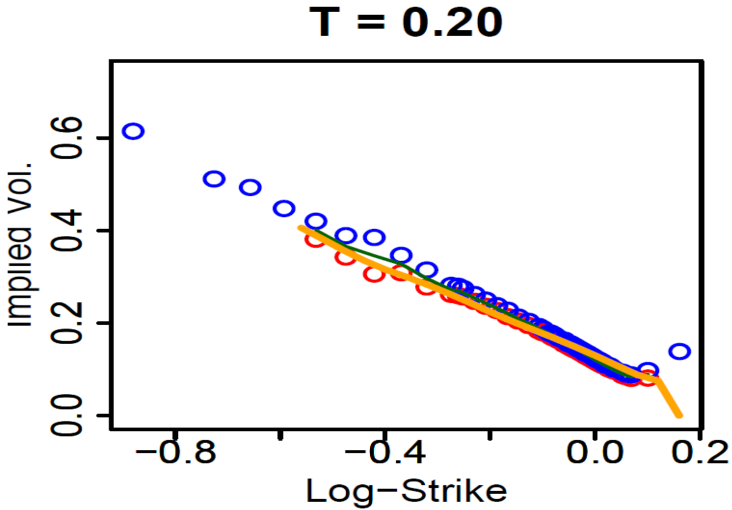
Fit to VIX options



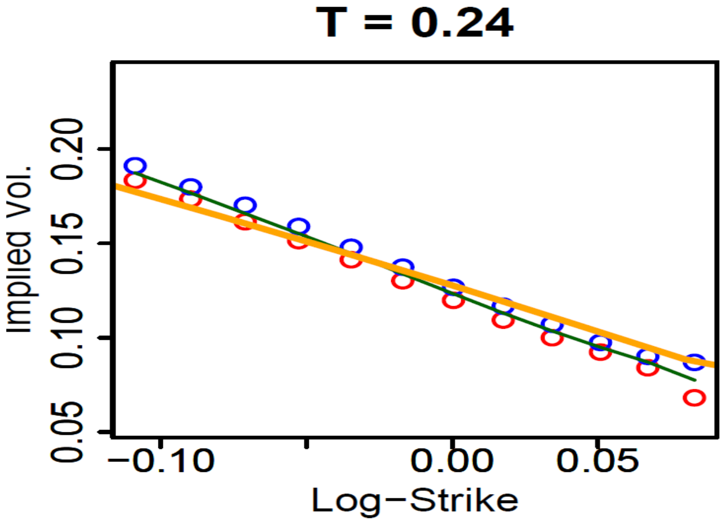
Fit to SPX options



Fit to SPX options



Fit to SPX options



- Joint calibration not so good for short maturities (up to 6 months)
- Unfortunate as these are the most liquid maturities for VIX futures and options
- Vol-of-vol is either too large for VIX market, or too small for SPX market (or both)

Trying with jumps in SPX

Sepp (2012)

Part I. Joint calibration of SPX and VIX skews using jumps

I consider several volatility models to reproduce the volatility skew observed in equity options on the S&P500 index:
Local volatility model (LV)
Jump-diffusion model (JD)
Stochastic volatility model (SV)
Local stochastic volatility model (LSV) with jumps

For each model, I analyze its implied skew for options on the VIX

I show that LV, JD and SV without jumps are not consistent with the implied volatility skew observed in option on the VIX

I show that:
Only the SV model with appropriately chosen jumps can fit the implied VIX skew

Importantly, that only the LSV model with jumps can fit both Equity and VIX option skews

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Baldeaux-Badran (2014)

Consistent Modelling of VIX and Equity Derivatives Using a $3/2$ plus Jumps Model

Jan Baldeaux and Alexander Badran

Abstract

The paper demonstrates that a pure-diffusion 3/2 model is able to capture the observed upward-sloping implied volatility skew in VIX options. This observation contradicts a common perception in the literature that jumps are required for the consistent modelling of equity and VIX derivatives. The pure-diffusion model, however, struggles to reproduce the smile in the implied volatilities of short-term index options. One remedy to this problem is to augment the model by introducing jumps in the index. The resulting 3/2 plus jumps model turns out to be as tractable as its pure-diffusion counterpart when it comes to pricing equity, realized variance and VIX derivatives, but accurately captures the smile in implied volatilities of short-term index options.

Keywords: Stochastic volatility plus jumps model, 3/2 model, VIX derivatives

Baldeaux-Badran (2014)

$$dS_t = S_{t-}\left((r-\lambda\bar{\mu})dt + \rho\sqrt{V_t}dW_t^1 + \sqrt{1-\rho^2}\sqrt{V_t}dW_t^2 + (e^{\xi}-1)dN_t\right), \quad (3)$$

$$dV_t = \kappa V_t(\theta - V_t)dt + \epsilon(V_t^{3/2})dW_t^1, \quad (4)$$

where we denote by N a Poisson process at constant rate λ, by e^{ξ} the relative jump size of the stock and N is adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The distribution of ξ is assumed to be normal with mean μ and variance σ^2. The parameters μ, $\bar{\mu}$, and σ satisfy the following relationship

$$\mu = \log(1+\bar{\mu}) - \frac{1}{2}\sigma^2.$$

Kokholm-Stisen (2015)

$$\frac{dS_t}{S_t} = (r - q - \overline{\mu}\lambda)dt + \sqrt{V_t}dW_t + (e^{J^S} - 1)dN_t \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dZ_t + J^V dN_t \tag{2}$$

where, W_t and Z_t are Wiener processes correlated with coefficient $\rho \in [-1, 1]$ and $\theta, \kappa, \eta \geq 0$. The price and volatility processes have simultaneous jumps with constant arrival intensity $\lambda \geq 0$. The jumps in volatility are independent and identically exponentially distributed with mean $\mu_v \geq 0$. Conditionally, on the jump in volatility, the jump in the price process is normally distributed with:

$$J^V \sim \exp(\mu_v), \quad J^S \mid J^V = y \sim N(\mu_s + \rho_J y, \sigma^2) \tag{3}$$

where $\sigma \geq 0$, $\rho_J \in [-1, 1]$, $\mu_s \in \mathbb{R}$. The martingale condition on the discounted price process imposes that:

$$\overline{\mu} = \frac{e^{\mu_s + \frac{1}{2}\sigma^2}}{1 - \rho_J \mu_v} - 1 \tag{4}$$

Kokholm-Stisen (2015)

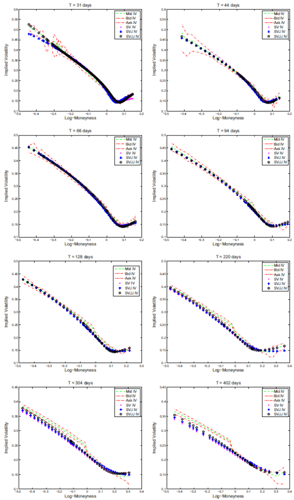

Figure 6.
Fit to the SPX option smiles on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Kokholm-Stisen (2015)

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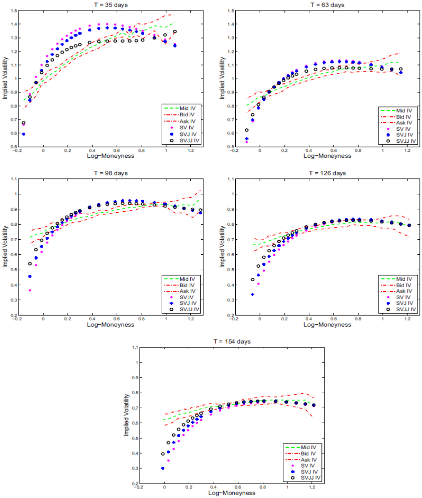


Figure 7.
Fit to the VIX option smiles on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Kokholm-Stisen (2015)

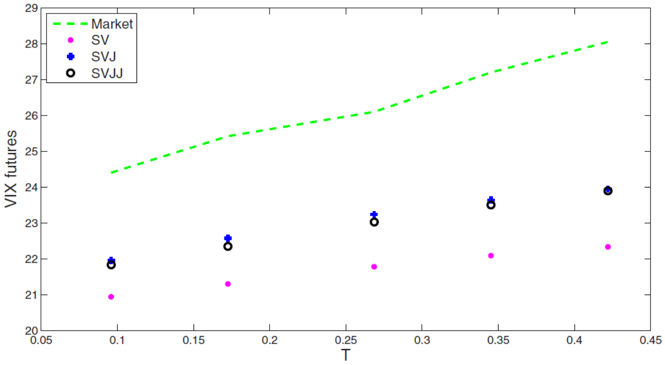


Figure 8.
Fit to the VIX futures on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Bardgett-Gourier-Leippold (2015)

$$dY_t = [-\lambda^{Yv}(v_{t-}, m_{t-})(\theta_Z(1,0,0) - 1) - \frac{1}{2}v_{t-}]dt + \sqrt{v_{t-}}dW_t^Y + dJ_t^Y,$$

$$dv_t = \kappa_v(m_{t-} - v_{t-})dt + \sigma_v\sqrt{v_{t-}}dW_t^v + dJ_t^v,$$

$$dm_t = \kappa_m(\theta_m - m_{t-})dt + \sigma_m\sqrt{m_{t-}}dW_t^m + dJ_t^m,$$

Bardgett-Gourier-Leippold (2015)

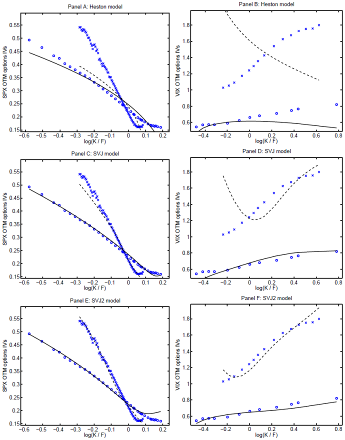


Figure 2: Market and model IVs on May 5, 2010, obtained by a joint calibration on the S&P 500 and VIX option market. Circles represent the market IV for $T = 0.05$ (S&P 500) and $T = 0.04$ (VIX). Crosses represent the market IV for $T = 0.3$ (S&P 500) and $T = 0.36$ (VIX). The dashed line corresponds to the model fit for $T = 0.05$ (S&P 500) and $T = 0.04$ (VIX) while the solid line corresponds to the model fit for $T = 0.3$ (S&P 500) and $T = 0.36$ (VIX). Panels A (S&P 500) and B (VIX) plot the model IVs based on the Heston model. Panels C and D display the corresponding results for the SVJ model, while Panels E and F do so for the SVJ2 model.

Papanicolaou-Sircar (2014)

- Use a regime-switching stochastic volatility model
- Hidden regime θ: continuous time Markov chain

$$
\begin{aligned}
dX_t &= \left(r - \frac{1}{2} f^2(\theta_t) Y_t - \delta\nu(\theta_{t\text{-}})\right) dt + f(\theta_t)\sqrt{Y_t}\, dW_t - \lambda(\theta_t) J_t\, dN_t\ , \\
dY_t &= \kappa(\bar{Y} - Y_t)dt + \gamma\sqrt{Y_t}\, dB_t\ , \\
dN_t &= \mathbb{1}_{[\theta_t \neq \theta_{t-}]},
\end{aligned}
$$

Papanicolaou-Sircar (2014)

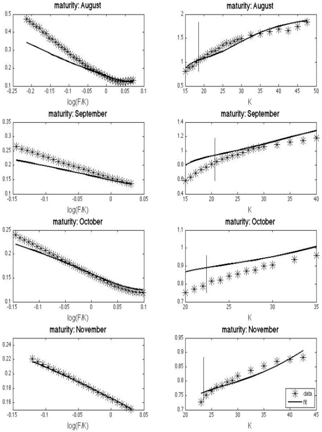


Figure 9: *The implied volatilities of July 27th SPX options (left column) and VIX options (right column), plotted alongside those of a fitted Heston with jumps. The fitted parameter values are given in Table 4. The vertical lines in the plots on the right mark the VIX futures price on the date of maturity.*

Cont-Kokholm (2013)

- Framework *à la* Bergomi:
  1. Model dynamics of forward variances $V_t^{[T_i,T_{i+1}]}$
  2. Given $V_{T_i}^{[T_i,T_{i+1}]}$, model dynamics of SPX
- Simultaneous (Lévy) jumps on forward variances and SPX
- First time a model seems to be able to jointly fit SPX skew and VIX level even for short maturities

Cont-Kokholm (2013)

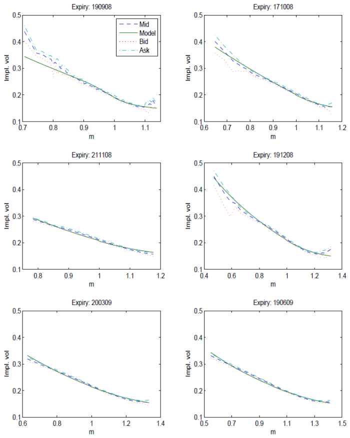


Figure 6: S&P 500 implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/S_t$ on the horizontal axis.

Cont-Kokholm (2013)

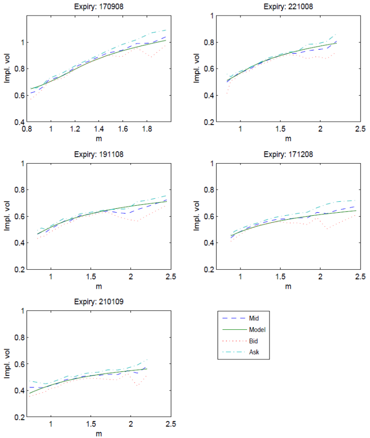


Figure 4: VIX implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the horizontal axis.

Pacati-Pompa-Renò (2015)

$$
\begin{cases}
dx_t = \left[r - q - \lambda\bar{\mu} - \frac{1}{2}\left(\sigma_{1,t}^2 + \phi_t + \sigma_{2,t}^2\right)\right]dt + \sqrt{\sigma_{1,t}^2 + \phi_t}\, dW_{1,t}^S + \sigma_{2,t} dW_{2,t}^S + c_x dN_t \\
d\sigma_{1,t}^2 = \alpha_1(\beta_1 - \sigma_{1,t}^2)dt + \Lambda_1 \sigma_{1,t} dW_{1,t}^\sigma + c_\sigma dN_t + c'_\sigma dN'_t \\
d\sigma_{2,t}^2 = \alpha_2(\beta_2 - \sigma_{2,t}^2)dt + \Lambda_2 \sigma_{2,t} dW_{2,t}^\sigma
\end{cases}
$$

Pacati-Pompa-Renò (2015)

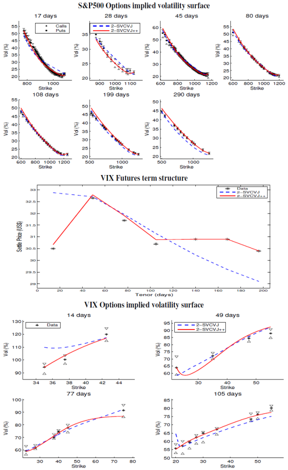


Figure 1: This figure reports market and model implied volatilities for S&P500 (plot at the top) and VIX (plot at the bottom) options, together with the term structure of VIX futures (plot in the middle) on September 2, 2009 obtained calibrating jointly on the three markets the 2-SVCVJ (blue dashed line) and 2-SVCVJ++ (red line). Maturities and tenors are expressed in days and volatilities are in % points and VIX futures settle prices are in US$.

Pacati-Pompa-Renò (2015)

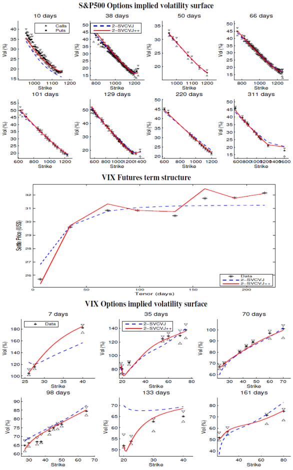


Figure 2: This figure reports market and model implied volatilities for S&P500 (plot at the top) and VIX (plot at the bottom) options, together with the term structure of VIX futures (plot in the middle) on August 11, 2010 obtained calibrating jointly on the three markets the 2-SVCVJ (blue dashed line) and 2-SVCVJ++ (red line). Maturities and tenors are expressed in days and volatilities are in % points and VIX futures settle prices are in US$.

Trying again with no jumps in SPX

Goutte-Ismail-Pham (2017)

- Also use a regime-switching stochastic volatility model
- Hidden regime Z: continuous time Markov chain

$$\begin{cases} dS_t = S_t(r dt + \sqrt{V_t} dW_t^1), \quad S_0 = s \\ dV_t = \kappa(Z_t)(\theta(Z_t) - V_t)dt + \xi(Z_t)\sqrt{V_t} dW_t^2, \quad V_0 = v_0. \end{cases}$$

Goutte-Ismail-Pham (2017)

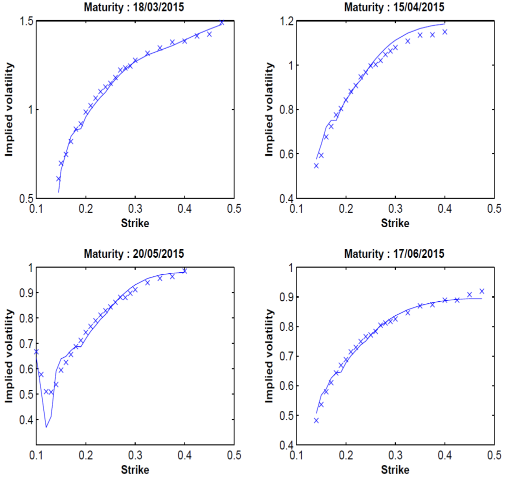


Figure 8: Implied volatilities of February 13, 2015, for VIX call options and the calibrated smile.

Goutte-Ismail-Pham (2017)

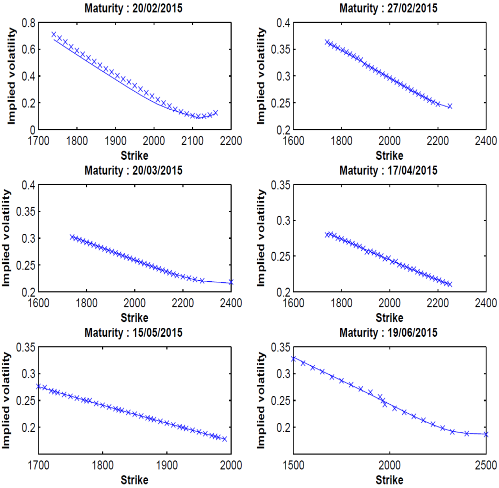


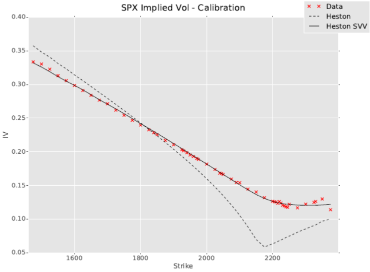
Figure 9: Implied volatilities of February 13, 2015, for S&P 500 call options and the calibrated smile.

Goutte-Ismail-Pham (2017)

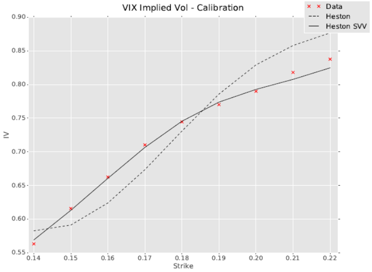
...but problem with SPX market data

Fouque-Saporito (2017)

- Based on Heston model with stochastic vol of vol
- No jumps
- Good fit to both SPX and VIX options... but only for maturities ≥ 4 months



(a) S&P 500



(b) VIX

So does there exist a continuous model on the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?

No answer yet...

Continuous model on SPX calibrated to SPX options

- For simplicity, let us assume zero interest rates, repos, and dividends.
- Let $\mathcal{F}_t$ denote the market information available up to time t.
- We consider continuous models on the SPX index:

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t, \qquad S_0 = x \tag{3.1}$$

- W denotes a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion, (σ_t) is an $(\mathcal{F}_t)$-adapted process such that for all $t \geq 0$, $\int_0^t \sigma_s^2 \, ds < \infty$ a.s., and $x > 0$ is the initial SPX price.
- The local volatility function corresponding to Model (3.1) is the function σ_{loc} defined by

$$\sigma_{\mathrm{loc}}^2(t, S_t) := \mathbb{E}[\sigma_t^2 | S_t]. \tag{3.2}$$

- The corresponding local volatility model is defined by:

$$\frac{dS_t^{\mathrm{loc}}}{S_t^{\mathrm{loc}}} = \sigma_{\mathrm{loc}}(t, S_t^{\mathrm{loc}}) \, dW_t, \qquad S_0^{\mathrm{loc}} = x.$$

Continuous model on SPX calibrated to SPX options

- From Gyöngy (1986), the marginal distributions of the processes $(S_t, t \geq 0)$ and $(S_t^{\mathrm{loc}}, t \geq 0)$ agree:

$$\forall t \geq 0, \quad S_t^{\mathrm{loc}} \stackrel{(d)}{=} S_t. \tag{3.3}$$

- Using Dupire (1994), we conclude that Model (3.1) is calibrated to the full SPX smile if and only if

$$\sigma_{\mathrm{loc}} = \sigma_{\mathrm{lv}} \tag{3.4}$$

where σ_{lv} is the local volatility function derived from market prices of vanilla options on the SPX using Dupire's formula.

- We denote by S^{lv} the market local volatility model is defined by:

$$\frac{dS_t^{\mathrm{lv}}}{S_t^{\mathrm{lv}}} = \sigma_{\mathrm{lv}}(t, S_t^{\mathrm{lv}})\, dW_t, \qquad S_0^{\mathrm{lv}} = x.$$

VIX

- Let $T \geq 0$. By definition, the (idealized) VIX at time T is the implied volatility of a 30 day log-contract on the SPX index starting at T. For continuous models (3.1), this translates into

$$\mathrm{VIX}_T^2 = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\, dt \middle| \mathcal{F}_T\right] = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\sigma_t^2 \middle| \mathcal{F}_T\right] dt \tag{3.5}$$

where $\tau = \frac{30}{365}$ (30 days).

- Since $\mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|\mathcal{F}_T] = \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$, $\mathrm{VIX}_{\mathrm{loc},T}$ satisfies

$$\mathrm{VIX}_{\mathrm{loc},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}]\, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})\, dt \middle| S_T^{\mathrm{loc}}\right].$$

- Similarly,

$$\mathrm{VIX}_{\mathrm{lv},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}})|S_T^{\mathrm{lv}}]\, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}})\, dt \middle| S_T^{\mathrm{lv}}\right].$$

VIX

- The prices at time 0 of the VIX future and the VIX call options with common maturity T in Model (3.1) are respectively given by

$$\mathrm{VIX}_0^{\mathrm{model}}(T) = \mathbb{E}\left[\sqrt{\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt \middle| \mathcal{F}_T\right]}\right], \tag{3.6}$$

$$C_{\mathrm{VIX}}^{\mathrm{model}}(T,K) = \mathbb{E}\left[\left(\sqrt{\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt \middle| \mathcal{F}_T\right]} - K\right)_+\right]. \tag{3.7}$$

- We observe market prices for those instruments, for a list of liquid monthly VIX future maturities T_i, denoted by $\mathrm{VIX}_0^{\mathrm{mkt}}(T_i)$ and $C_{\mathrm{VIX}}^{\mathrm{mkt}}(T_i,K)$, with the most liquid maturities lying below 6 months.
- **Can we find a model satisfying (3.1)-(3.4) and such that for all T_i and K, $\mathrm{VIX}_0^{\mathrm{model}}(T_i) = \mathrm{VIX}_0^{\mathrm{mkt}}(T_i)$ and $C_{\mathrm{VIX}}^{\mathrm{model}}(T_i,K) = C_{\mathrm{VIX}}^{\mathrm{mkt}}(T_i,K)$?**

The case of instantaneous VIX

- $\tau \to 0$: The realized variance over 30 days is then simply replaced by the instantaneous variance, and (3.6)-(3.7) boil down to

$$\mathrm{instVIX}_0^{\mathrm{model}}(T) = \mathbb{E}\left[\sigma_T\right], \tag{4.1}$$

$$C_{\mathrm{instVIX}}^{\mathrm{model}}(T,K) = \mathbb{E}\left[(\sigma_T - K)_+\right]. \tag{4.2}$$

- Reminder: (The distributions of) two random variables X and Y are said to be in convex order if and only if, for any convex function f, $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$. Denoted by $X \leq_c Y$. Both distributions have same mean, but distribution of Y is more "spread" than that of X.
- Assume $\sigma_{\mathrm{loc}} = \sigma_{\mathrm{lv}}$. By conditional Jensen, since $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\mathrm{lv}}^2(t,S_t)$,

$$\forall t \geq 0, \qquad \sigma_{\mathrm{lv}}^2(t,S_t) \leq_c \sigma_t^2.$$

$$X_t := \sigma_{\mathrm{lv}}^2(t, S_t) \qquad \text{and} \qquad Y_t := \sigma_t^2$$

- Conversely, if $X_t \leq_c Y_t$, then there exists a joint distribution π_t of (S_t, σ_t) such that $\mathbb{E}[\sigma_t^2 | S_t] = \sigma_{\mathrm{lv}}^2(t, S_t)$ for all t.
- Indeed, from Strassen's theorem (1965), there exists a joint distribution π'_t of (X_t, Y_t) such that $\mathbb{E}[Y_t | X_t] = X_t$. One then defines π_t as follows: S_t follows the risk-neutral distribution of the SPX for maturity t and, given S_t, $X_t = \sigma_{\mathrm{lv}}^2(t, S_t)$ is known and σ_t^2 is chosen to follow the conditional distribution of Y_t given X_t under π'_t.

- If $\mathrm{instVIX}_0^{\mathrm{mkt}}(t)$ and $C^{\mathrm{mkt}}_{\mathrm{instVIX}}(t,K)$ were accessible, we could imply from the market the distribution of σ_t^2, and compare it to the risk-neutral distribution of $\sigma^2_{\mathrm{lv}}(t,S_t)$.
- A **necessary and sufficient condition** for a jointly calibrating continuous model on the SPX to exist would then simply be that for each t those two market-implied distributions be in the right convex order:

$$\sigma^2_{\mathrm{lv}}(t,S_t) \leq_c \sigma_t^2$$

- Any process defined by $\frac{dS_t}{S_t} = \sigma_t\, dW_t$ where for each t, given S_t, the distribution of σ_t is specified by π_t, is a solution.
- This general construction does not address the issue of the dynamics of (σ_t): σ_t and $\sigma_{t'}$ could be very loosely related for arbitrarily close t and t'.

- In practice, to build a calibrating process, one would discretize time and recursively solve **martingale transport problems**:

$$\mathcal{L}\left(\sigma_{\mathrm{lv}}^2(t_k, S_{t_k})\right) \text{ and } \mathcal{L}\left(\sigma_{t_k}^2\right) \text{ given}, \quad \mathbb{E}[\sigma_{t_k}^2 | \sigma_{\mathrm{lv}}^2(t_k, S_{t_k})] = \sigma_{\mathrm{lv}}^2(t_k, S_{t_k}). \quad (4.3)$$

- Solutions π'_{t_k} to those martingale transport problems include left- and right-curtains (Beiglböck-Juillet, Henry-Labordère), forward-starting solutions to the Skorokhod embedding problems (Dupire), and the local variance gamma model of Carr and Nadtochiy.
- (4.3) is a **new type of application of martingale transport to finance**:
  - Usually, the martingality constraint applies to the underlying at two different dates (Henry-Labordère, Beiglböck, Penkner, Nutz, Touzi, Martini, De Marco, Dolinsky, Soner, Obłój, Stebegg, JG...)
  - Here it applies to **two types of instantaneous variances at a single date**, ensuring that the SPX smile is matched.

- It can already be seen in this limiting case that **it might be impossible to build a continuous model on the SPX that jointly calibrates to SPX and VIX options**. This happens if (and only if) for some t the market-implied distribution of $\sigma_{\mathrm{lv}}^2(t, S_t)$ is "more spread" than that of the instantaneous VIX squared.

The real case

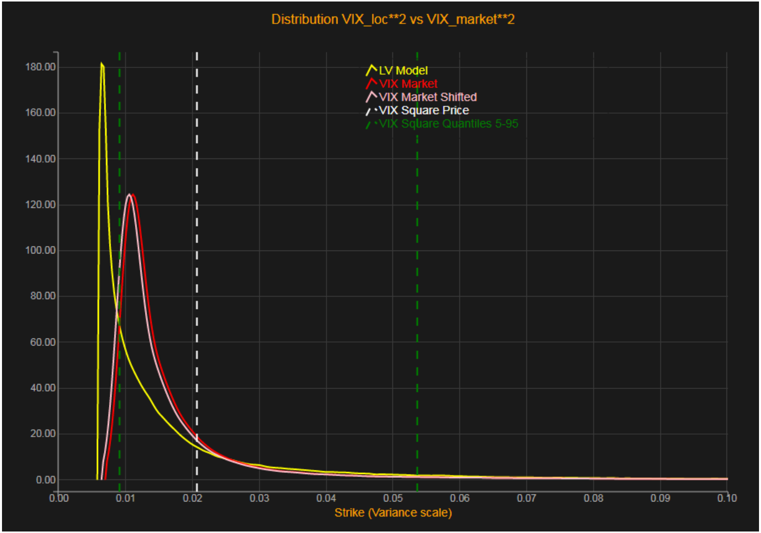
- In reality, squared VIX are not instantaneous variances but the **fair strikes** of **30-day** realized variances.
- Let us look at market data (Sep 21, 2017). We compare the market distributions of

$$\mathrm{VIX}^2_{\mathrm{lv},T} := \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S^{\mathrm{lv}}_t)\,dt \middle| S^{\mathrm{lv}}_T\right]$$

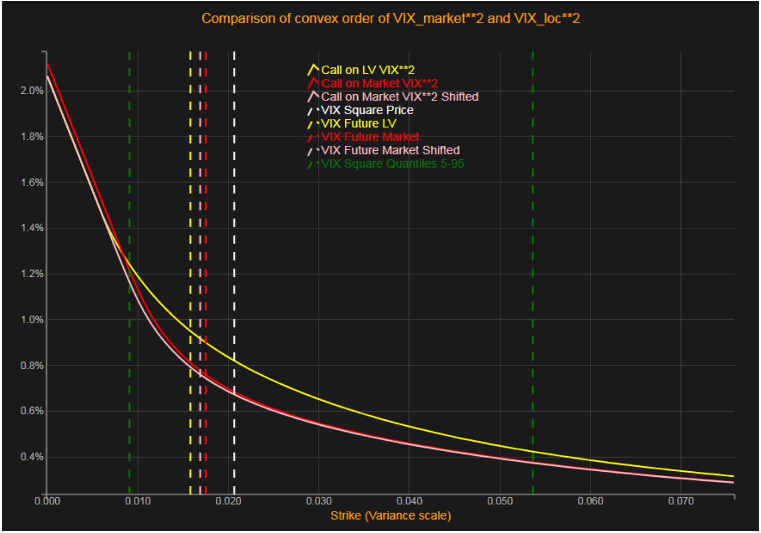
and

$$\mathrm{VIX}^2_{\mathrm{mkt},T} \quad \left(\longleftrightarrow \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\,dt \middle| \mathcal{F}_T\right]\right)$$

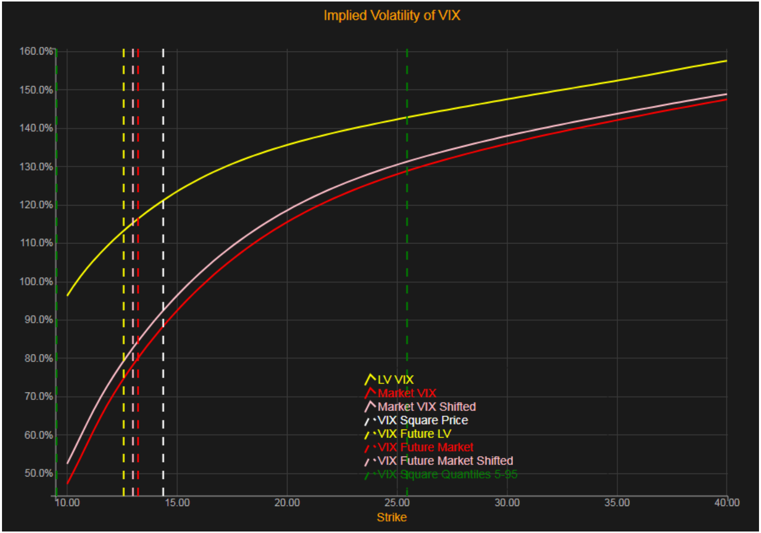
$T = 2$ months


Distribution VIX_loc**2 vs VIX_market**2
LV Model
VIX Market
VIX Market Shifted
VIX Square Price
VIX Square Quantiles 5-95
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Strike (Variance scale)

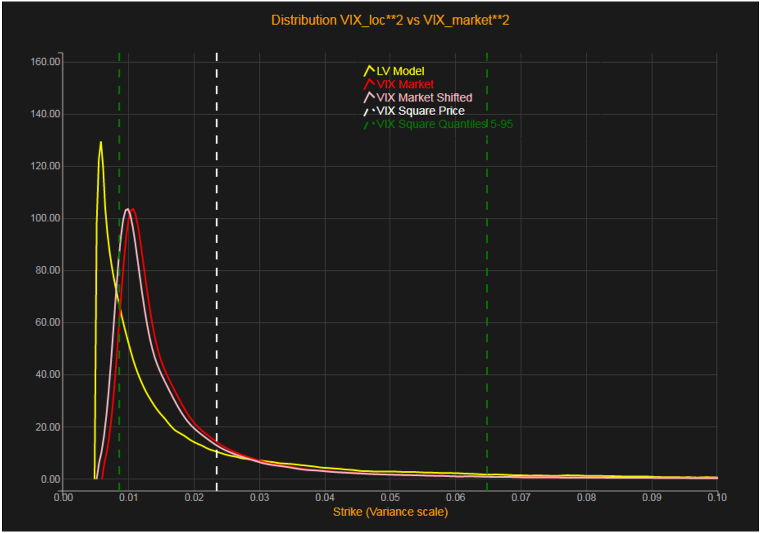
$T = 2$ months


Comparison of convex order of VIX_market**2 and VIX_loc**2
Call on LV VIX**2
Call on Market VIX**2
Call on Market VIX**2 Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95
2.0%
1.8%
1.6%
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
Strike (Variance scale)

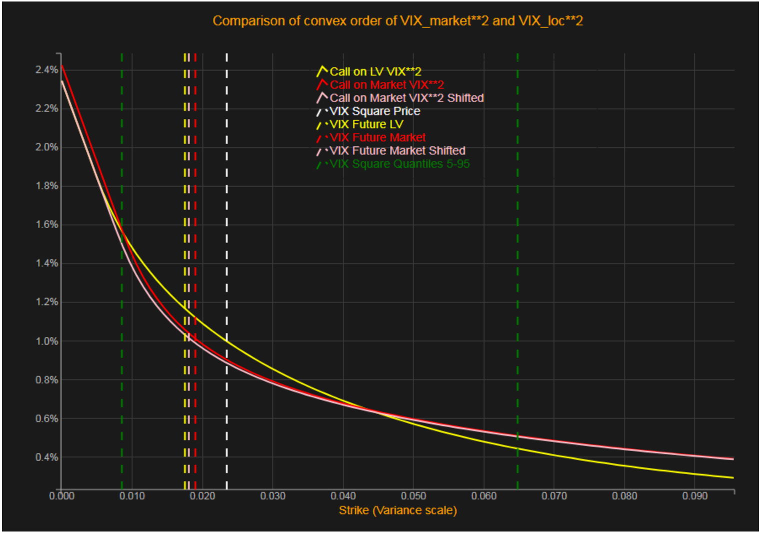
$T = 2$ months


Implied Volatility of VIX
160.0%
150.0%
140.0%
130.0%
120.0%
110.0%
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
10.00
15.00
20.00
25.00
30.00
35.00
40.00
Strike
LV VIX
Market VIX
Market VIX Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95

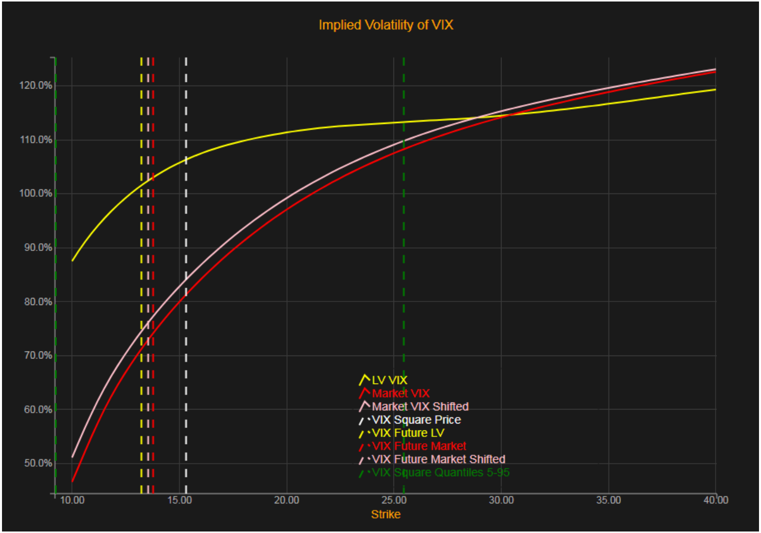
$T = 3$ months


Distribution VIX_loc**2 vs VIX_market**2
LV Model
VIX Market
VIX Market Shifted
VIX Square Price
VIX Square Quantiles 5-95
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Strike (Variance scale)

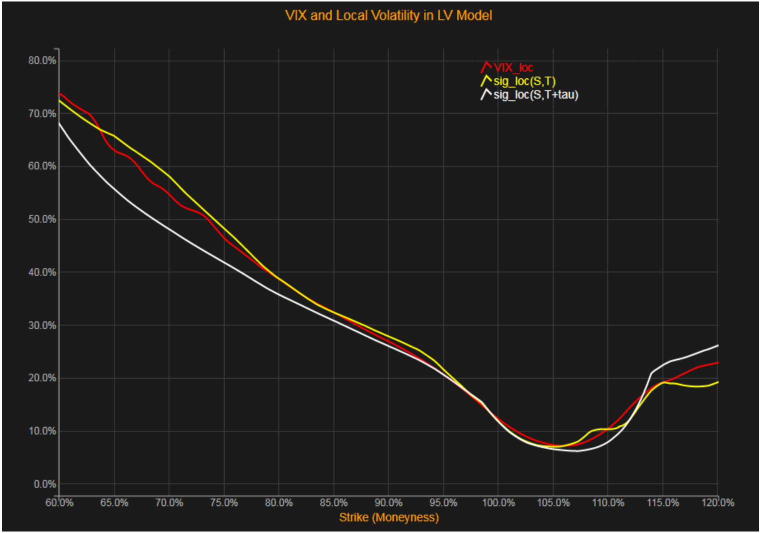
$T = 3$ months


Comparison of convex order of VIX_market**2 and VIX_loc**2
Call on LV VIX**2
Call on Market VIX**2
Call on Market VIX**2 Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95
2.4%
2.2%
2.0%
1.8%
1.6%
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
0.080
0.090
Strike (Variance scale)

$T = 3$ months


Implied Volatility of VIX
120.0%
110.0%
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
10.00
15.00
20.00
25.00
30.00
35.00
40.00
Strike
LV VIX
Market VIX
Market VIX Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95

$T = 3$ months


VIX and Local Volatility in LV Model
VIX_loc
sig_loc(S,T)
sig_loc(S,T+tau)
80.0%
70.0%
60.0%
50.0%
40.0%
30.0%
20.0%
10.0%
0.0%
60.0%
65.0%
70.0%
75.0%
80.0%
85.0%
90.0%
95.0%
100.0%
105.0%
110.0%
115.0%
120.0%
Strike (Moneyness)

The real case

$$\begin{aligned} \mathrm{VIX}^2_{\mathrm{lv},T} &:= \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S^{\mathrm{lv}}_t)\, dt \middle| S^{\mathrm{lv}}_T\right] \\ \mathrm{VIX}^2_T &= \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\, dt \middle| \mathcal{F}_T\right] \end{aligned}$$

- In typical market conditions:

$$\begin{aligned} 1-2 \text{ months}: &\quad \mathrm{VIX}^2_{\mathrm{mkt},T} \le_c \mathrm{VIX}^2_{\mathrm{lv},T} \qquad (5.1) \\ 3-4 \text{ months}: &\quad \mathrm{VIX}^2_{\mathrm{lv},T} \not\le_c \mathrm{VIX}^2_{\mathrm{mkt},T}, \mathrm{VIX}^2_{\mathrm{mkt},T} \not\le_c \mathrm{VIX}^2_{\mathrm{lv},T} \\ 5+ \text{ months}: &\quad \mathrm{VIX}^2_{\mathrm{lv},T} \le_c \mathrm{VIX}^2_{\mathrm{mt},T} \end{aligned}$$

The local volatility model yields a VIX distribution that is "more spread" than the VIX distribution implied from VIX futures and options.

The real case

$$\mathrm{VIX}^2_{\mathrm{lv},T} := \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S^{\mathrm{lv}}_t)\,dt \middle| S^{\mathrm{lv}}_T\right]$$

$$\mathrm{VIX}^2_T = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\,dt \middle| \mathcal{F}_T\right]$$

- One may be tempted to believe that there exists a model of the form (3.1) calibrated to the SPX smile and to all VIX options if and only if for all T_i, $\mathrm{VIX}^2_{\mathrm{lv},T_i} \leq_c \mathrm{VIX}^2_{\mathrm{mkt},T_i}$ — and then conclude that such a model does not exist.
- However:

$$\sigma^2_{\mathrm{lv}}(t, S_t) \leq_c \sigma^2_t \;\;\not\Longrightarrow\;\; \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S_t)\,dt \middle| \mathcal{F}_T\right] \leq_c \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\,dt \middle| \mathcal{F}_T\right]$$

- Sum and $\mathcal{F}_T$ conditioning may undo convex ordering.

Convex order is not preserved under sum

Example

- A trivial almost counterexample:

$$Y_0 = X_0 + Z, \qquad Y_1 = X_1 - Z$$

  with $\mathbb{E}[Z|X_0] = \mathbb{E}[Z|X_1] = 0$ (e.g., Z has zero mean and is independent from (X_0, X_1)).
- Y_0 can be much larger than X_0 in the convex order and Y_1 can be much larger than X_1 in the convex order, if Z has large variance.
- However, $Y_0 + Y_1 = X_0 + X_1$.

Example

- $X_0 = W_{t_1}$, $X_1 = -W_{t_2}$, $Y_0 = W_{t_3}$, and $Y_1 = -W_{t_3}$, with $0 < t_1 < t_2 < t_3$.
- $\mathbb{E}[Y_0|X_0] = X_0$, $\mathbb{E}[Y_1|X_1] = X_1$, hence $X_0 \leq_c Y_0$ and $X_1 \leq_c Y_1$, yet $0 = Y_0 + Y_1 <_c X_0 + X_1$.

Convex order is not preserved under sum

Example

- We generalize the previous example: $G = (X_0, Y_0, X_1, Y_1)$ Gaussian vector.
- We assume that $\mathbb{E}[Y_0|X_0] = X_0$ and $\mathbb{E}[Y_1|X_1] = X_1$, and look for necessary and sufficient conditions under which $X_0 + X_1 \leq_c Y_0 + Y_1$.[1]
- $m_X := \mathbb{E}[X]$, σ_X std dev of X, ρ_{XY} the correlation between X and Y.
- Since G is Gaussian, $\mathbb{E}[Y_i|X_i] = m_{Y_i} + \rho_{X_iY_i}\frac{\sigma_{Y_i}}{\sigma_{X_i}}(X_i - m_{X_i})$ so

$$m_{X_i} = m_{Y_i} \quad \text{and} \quad \sigma_{X_i} = \rho_{X_iY_i}\sigma_{Y_i}. \tag{5.2}$$

In particular, $\rho_{X_iY_i} > 0$. As a consequence, $m_{X_0+X_1} = m_{Y_0+Y_1}$, and since $X_0 + X_1$ and $Y_0 + Y_1$ are Gaussian,

$$X_0 + X_1 \leq_c Y_0 + Y_1 \iff \mathrm{Var}(X_0 + X_1) \leq \mathrm{Var}(Y_0 + Y_1).$$

[1] We ignore trivial cases by assuming that all components of G have positive variance.

Convex order is not preserved under sum

Example

- Now, using the second equation in (5.2), we have

$$\begin{aligned} \mathrm{Var}(X_0 + X_1) &= \sigma_{X_0}^2 + \sigma_{X_1}^2 + 2\rho_{X_0X_1}\sigma_{X_0}\sigma_{X_1} \\ &= \rho_{X_0Y_0}^2\sigma_{Y_0}^2 + \rho_{X_1Y_1}^2\sigma_{Y_i}^2 + 2\rho_{X_0X_1}\rho_{X_0Y_0}\sigma_{Y_0}\rho_{X_1Y_1}\sigma_{Y_1} \end{aligned}$$

so $X_0 + X_1 \leq_c Y_0 + Y_1$ if and only if

$$\sigma_{Y_0}^2(1-\rho_{X_0Y_0}^2) + \sigma_{Y_1}^2(1-\rho_{X_1Y_1}^2) + 2\sigma_{Y_0}\sigma_{Y_1}(\rho_{Y_0Y_1} - \rho_{X_0X_1}\rho_{X_0Y_0}\rho_{X_1Y_1}) \geq 0.$$

In particular, if $\sigma_{Y_0} = \sigma_{Y_1}$, $\rho_{X_iY_i} \neq 1$ for $i \in \{0,1\}$, and

$$\chi := \frac{\rho_{Y_0Y_1} - \rho_{X_0X_1}\rho_{X_0Y_0}\rho_{X_1Y_1}}{\sqrt{1-\rho_{X_0Y_0}^2}\sqrt{1-\rho_{X_1Y_1}^2}} < -1$$

then $X_0 + X_1 \not\leq_c Y_0 + Y_1$.

Convex order is not preserved under conditioning

- The conditioning with respect to $\mathcal{F}_T$ may undo convex ordering too.
- Simple counterexample: if $X \leq_c Y$ with X $\mathcal{F}$-measurable and not constant, and Y independent of $\mathcal{F}$, then $\mathbb{E}[Y] = \mathbb{E}[Y|\mathcal{F}] <_c \mathbb{E}[X|\mathcal{F}] = X$.
- Intuition: Fast mean reversion in (σ_t) may undo convex ordering since $(\sigma_{\mathrm{loc}}(t, S_t^{\mathrm{loc}}))$ does not mean revert. However, the larger the mean reversion, the flatter $S \mapsto \sigma_{\mathrm{loc}}(t, S_t)$.

Inversion of convex ordering

- For Model (3.1) to fit both SPX and VIX option prices, it must satisfy

$$\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma_u^2\,du\middle|\mathcal{F}_T\right]\leq_c \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})\,du\middle|S_T^{\mathrm{loc}}\right] \qquad (5.3)$$

  for T up to a few months, despite the fact that for all $u\geq 0$, $\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})\leq_c \sigma_u^2$.
- Note that $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ and $\mathbb{E}[\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$ have the same mean $\mathbb{E}[\sigma_u^2]$.
- One natural way to achieve (5.3) is to require that

$$\mathbb{E}[\sigma_u^2|\mathcal{F}_T]\leq_c \mathbb{E}[\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}] \qquad (5.4)$$

  for many $u\in(T,T+\tau]$ and hope that this convex ordering of forward instantaneous variances will be preserved when we sum over u.
- When $u=T$, $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]=\sigma_T^2\geq_c\sigma_{\mathrm{loc}}^2(T,S_T^{\mathrm{loc}})=\mathbb{E}[\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}]\Longrightarrow$ We will require that (5.4) holds for all $T\leq\bar{T}$ and $u\in[T+\varepsilon,T+\tau]$.
- When (5.4) holds, the **convex ordering $\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})\leq_c\sigma_u^2$ is actually reversed after conditioning on $\mathcal{F}_T$**:

$$\mathbb{E}[\sigma_u^2|\mathcal{F}_T]\leq_c\mathbb{E}[\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})|\mathcal{F}_T]\leq_c\sigma_{\mathrm{loc}}^2(u,S_u^{\mathrm{loc}})\leq_c\sigma_u^2. \qquad (5.5)$$

One-factor lognormal forward instantaneous variance models

- $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ is a forward instantaneous variance. We denote it by
$$\xi_T^u := \mathbb{E}[\sigma_u^2|\mathcal{F}_T].$$
- It is well known (Dupire, Bergomi, Buehler) that forward instantaneous variances are driftless.
- Second generation stochastic volatility models directly model the dynamics of $(\xi_t^u, t \in [0,u])$ under a risk-neutral measure. The only requirement is that these processes, indexed by u, be nonnegative and driftless.
- For simplicity, let us assume for now that forward instantaneous variances are lognormal and all driven by a single standard one-dimensional $(\mathcal{F}_t)$-Brownian motion Z, correlated with W:
$$\frac{d\xi_t^u}{\xi_t^u} = K(t,u)\,dZ_t. \tag{6.1}$$
- K is called the kernel.
- The SPX dynamics simply reads as (3.1) with $\sigma_t^2 := \xi_t^t$:
$$\frac{dS_t}{S_t} = \sigma_t\,dW_t, \qquad \sigma_t^2 := \xi_t^t, \qquad d\langle W,Z\rangle_t = \rho\,dt.$$

One-factor lognormal forward instantaneous variance models

- The solution to (6.1) is simply

$$\xi_t^u = \xi_0^u \exp\left(\int_0^t K(s,u)\,dZ_s - \frac{1}{2}\int_0^t K(s,u)^2\,ds\right) \tag{6.2}$$

  which yields

$$\sigma_u^2 = \xi_0^u \exp\left(\int_0^u K(s,u)\,dZ_s - \frac{1}{2}\int_0^u K(s,u)^2\,ds\right). \tag{6.3}$$

- For simplicity, let us choose a time-homogeneous kernel

$$K(s,u) = K(u-s).$$

- Financially, we expect the kernel $K : \mathbb{R} \to \mathbb{R}_+$ to be decreasing: The further the instantaneous forward variance maturity u, the less volatile the instantaneous forward variance.

Ingredients needed for inversion of convex order

- Can we choose a kernel K such that

$$\mathbb{E}[\sigma_u^2|\mathcal{F}_T] \leq_c \mathbb{E}[\sigma_{\mathrm{loc}}^2(u, S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$$

  holds for all $T \leq \bar{T}$ and $u \in [T+\varepsilon, T+\tau]$?
- K should make the distribution of $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ "more narrow" than that of $\mathbb{E}[\sigma_{\mathrm{loc}}^2(u, S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$ for those T and u. Two ingredients are needed:
  - **I1**: The knowledge of $\mathcal{F}_T := \sigma(W_s, Z_s, 0 \leq s \leq T)$ should give little information on σ_u^2, so that distribution of $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]$ is narrow.
  - **I2**: The knowledge of S_u should give enough information on σ_u^2, so that $S \mapsto \sigma_{\mathrm{loc}}^2(u, S) = \mathbb{E}[\sigma_u^2|S_u = S]$ varies enough with S and the distribution of $\mathbb{E}[\sigma_{\mathrm{loc}}^2(u, S_u^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$ is not as narrow as that of $\mathbb{E}[\sigma_u^2|\mathcal{F}_T]$.
- Loosely speaking, S_u **should give more information than** $\mathcal{F}_T$ **on** σ_u^2.

Ingredients needed for inversion of convex order

$$\sigma_u^2 = \xi_0^u \exp\left(\int_0^u K(u-s)\,dZ_s - \frac{1}{2}\int_0^u K(u-s)^2\,ds\right).$$

- Ingredients **I1** and **I2** are antagonistic.
- For given T and $u \in (T, T+\tau]$, **I1** requires that $K(u-s)$ be small for all $s \in [0,T]$.
- Conversely, **I2** requires that $K(u-s)$ be large for at least some $s \in [0,u]$. Indeed, the knowledge of S_u gives partial information on $(W_s, 0 \le s \le u)$, which is passed to $(Z_s, 0 \le s \le u)$ through the correlation ρ; this information can impact σ_u^2 only if $K(u-s)$ is large for at least some $s \in [0,u]$.[2]
- For **I1** and **I2** to hold jointly, it is then required that $K(u-s)$ be small for $s \in [0,T]$ and large for $s \in [T,u]$, i.e., that $K(\theta)$ be small for $\theta \in [u-T,u]$ and large for $\theta \in [0,u-T]$.

[2] The knowledge of S_u may also give partial direct information on $(\sigma_s, 0 \le s \le u)$. Indeed, if S_u is extremely large, then many σ_s, $0 \le s \le u$, must have been very large, and σ_u is likely to be large. This explains why the smile has a positive slope at large strikes in stochastic volatility models even if $\rho < 0$. However, for values of S_u close to S_0, the knowledge of S_u is transferred to σ_u^2 mostly through the paths of W and Z up to u.

Ingredients needed for inversion of convex order

- Since this should hold for enough $u \in (T, T+\tau]$, $K(\theta)$ should be very large for $\theta \in [0, \varepsilon]$ and very small for $\theta \geq \varepsilon$, with $\varepsilon \leq \tau$:

$$K(\theta) = \begin{cases} \text{very large if } \ \theta \leq \varepsilon, \\ \text{very small if } \ \theta > \varepsilon \end{cases} \qquad \varepsilon \leq \tau \tag{6.4}$$

- Then **I1** holds for all $u \in [T+\varepsilon, T+\tau]$. **I2** will hold only if $K(\theta)$ is large enough for $\theta \in [0, \varepsilon]$; this is needed for the limited information that S_u gives on $(dZ_s, u-\varepsilon \leq s \leq u)$ to be amplified enough by the kernel K so that it impacts $\mathbb{E}[\sigma_u^2 | S_u]$. Such an extremely fast decreasing kernel K is reminiscent of extremely fast mean-reversion, with characteristic time $\varepsilon \ll \tau = 30$ days.

Two remarks

- The fact that $S \mapsto \sigma_{\rm loc}^2(u,S)$ varies a lot with S does not necessarily mean that the distribution of $\mathbb{E}[\sigma_{\rm loc}^2(u,S_u^{\rm loc})|S_T^{\rm loc}]$ is spread. Precisely, the above procedure describes a model where $\mathbb{E}[\sigma_u^2|\mathcal{F}_T] \ll_c \sigma_u^2$, and one may wonder how much smaller in convex order $\mathbb{E}[\sigma_{\rm loc}^2(u,S_u^{\rm loc})|S_T^{\rm loc}]$ is, compared to $\sigma_{\rm loc}^2(u,S_u^{\rm loc})$. Since $\sigma_{\rm loc}(t,S_t^{\rm loc})$ does not mean revert, contrary to σ_t, we expect $\mathbb{E}[\sigma_{\rm loc}^2(u,S_u^{\rm loc})|S_T^{\rm loc}]$ to be only slightly smaller than $\sigma_{\rm loc}^2(u,S_u^{\rm loc})$ in convex order.
- The smaller T, the less antagonistic **I1** and **I2** are. Indeed, the smaller T, the most information S_u gives on $(dW_s, u-\varepsilon \le s \le u)$, hence on $(dZ_s, u-\varepsilon \le s \le u)$, and finally on σ_u^2, for $u \in [T, T+\tau]$. This is because the smaller T, the smaller u, and the larger the portion of time $\frac{\varepsilon}{u}$ covered by $[u-\varepsilon, u]$. As a consequence, $\mathbb{E}[\sigma_u^2|\mathcal{F}_T] \le_c \mathbb{E}[\sigma_{\rm loc}^2(u,S_u^{\rm loc})|S_T^{\rm loc}]$ is more likely to hold for small T. This is precisely in line with market data: the inversion of convex ordering only holds for short maturities T_i.

Exponential kernel

$$K(\theta) = \omega \exp(-k\theta), \qquad \omega \geq 0,\ k > 0$$

- One-factor Bergomi model (Dupire 1993, Bergomi 2005)
- In this case, ξ_t^u admits a one-dimensional Markov representation: $\xi_t^u = \xi_0^u f^u(t, X_t)$ where

$$f^u(t,x) = \exp\left(\omega e^{-k(u-t)} X_t - \frac{\omega^2}{2} e^{-2k(u-t)} \mathrm{Var}(X_t)\right), \quad \mathrm{Var}(X_t) = \frac{1-e^{-2kt}}{2k}$$

  and the Ornstein-Uhlenbeck process $X_t = \int_0^t e^{-k(t-s)} dZ_s$ follows the Markov dynamics:

$$dX_t = -kX_t\, dt + dZ_t, \quad X_0 = 0. \qquad (6.5)$$

- The Markov property is very convenient: the time t price of SPX (resp. VIX) options are simply functions of (t, S_t, X_t) (resp. of (t, X_t)) that are solutions to second order parabolic linear PDEs; moreover, the numerical simulation of the model is easy as it is enough to simulate the two-dimensional Markov process (S_t, X_t).

Exponential kernel

$$K(\theta) = \omega \exp(-k\theta), \qquad \omega \geq 0,\ k > 0$$

- $k =$ mean-reversion.
- $\omega =$ vol of variance: it is the instantaneous (lognormal) volatility of the instantaneous variance $\sigma_t^2 := \xi_t^t$. Has the dimension of a volatility.
- For (6.4) to hold, we must impose $k \gg \frac{1}{\tau}$ and pick ω large. Fast mean-reversion regime: $\frac{1}{k} \ll \tau$.
- The limiting regime where k and ω tend to $+\infty$ while $\frac{\omega^2}{k}$ is kept constant corresponds to an ergodic limit where (X_t) quickly reaches its stationary distribution $\mathcal{N}(0, \frac{\omega^2}{2k})$. Cf Fouque, Papanicolaou and Sircar (2000).
- In this limit $K(\theta) \sim \sqrt{k} \exp(-k\theta) \longrightarrow 0$ for all $\theta > 0$, while $K(0) \longrightarrow +\infty$. The same holds in any limiting regime where $\omega \sim k^\alpha$. However $\omega \sim \sqrt{k}$ corresponds to the only regime where the variance σ_t^2 has a finite limit, which is the natural regime in finance.

Power-law kernel

$$K(\theta) = \nu\theta^{H-\frac{1}{2}}, \qquad \nu \geq 0,\ H \in \left(0, \frac{1}{2}\right) \tag{6.6}$$

- Rough Bergomi model (Bayer, Friz, Gatheral, 2014). Studied by many at Imperial (Jacquier, Pakkanen, Horvath, Muguruza,... and coauthors...)
- H = Hurst exponent. In this case, $\lim_{\theta\to 0^+} K(\theta) = +\infty$, and

  $$\xi_t^u = \xi_0^u \exp\left(\nu X_t^u - \frac{\nu^2}{2}\mathrm{Var}(X_t^u)\right)$$

  where

  $$X_t^u = \int_0^t (u-s)^{H-\frac{1}{2}}\, dZ_s, \qquad \mathrm{Var}(X_t^u) = \frac{u^{2H} - (u-t)^{2H}}{2H}.$$
- $H > 0$ ensures that $\mathrm{Var}(X_u^u)$ is finite.

Power-law kernel

- No Markov representation for ξ_t^u. The instantaneous variance $\sigma_t^2 := \xi_t^t$ is not a function of Markov processes, nor is it a semimartingale. One cannot write Itô dynamics $d\xi_t^t = \cdots dt + \cdots dZ_t$ for the instantaneous variance, and there is no notion of a dynamic volatility of instantaneous spot variance.
- However we can compare the values of $\mathrm{Var}\left(\ln \frac{\xi_t^u}{\xi_0^u}\right)$ in the power-law kernel model, and in the exponential kernel model:

$$\nu^2 \frac{u^{2H} - (u-t)^{2H}}{2H} \quad \longleftrightarrow \quad \omega^2 e^{-2k(u-t)} \frac{1-e^{-2kt}}{2k} \tag{6.7}$$

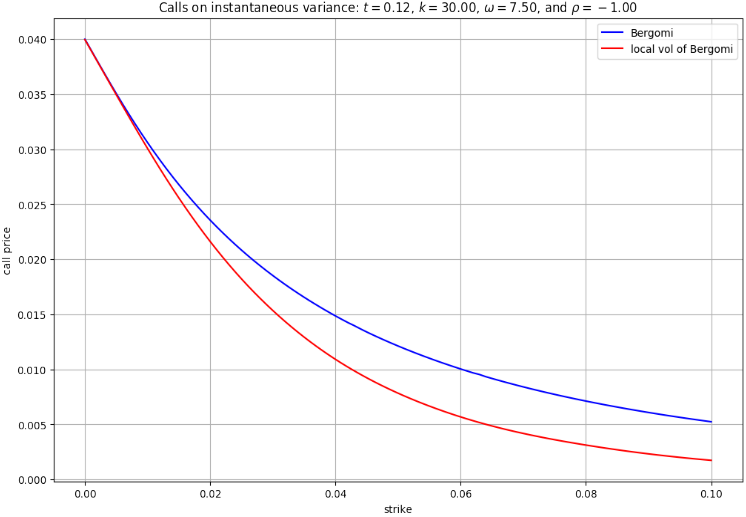
$$u = t \to 0: \quad \nu^2 \frac{t^{2H}}{2H} \quad \longleftrightarrow \quad \omega^2 \frac{1-e^{-2kt}}{2k} \approx \omega^2 t \tag{6.8}$$

  so $\nu \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}}$ can be interpreted as a short term volatility of instantaneous spot variance.
- ν does not have the dimension of a volatility, i.e., time$^{-\frac{1}{2}}$; it is $\nu\theta^{H-\frac{1}{2}}$ that has the dimension of a volatility, so ν has dimension time^{-H}, and $\nu \frac{t^{H-\frac{1}{2}}}{\sqrt{2H}}$ has indeed the dimension of a volatility.

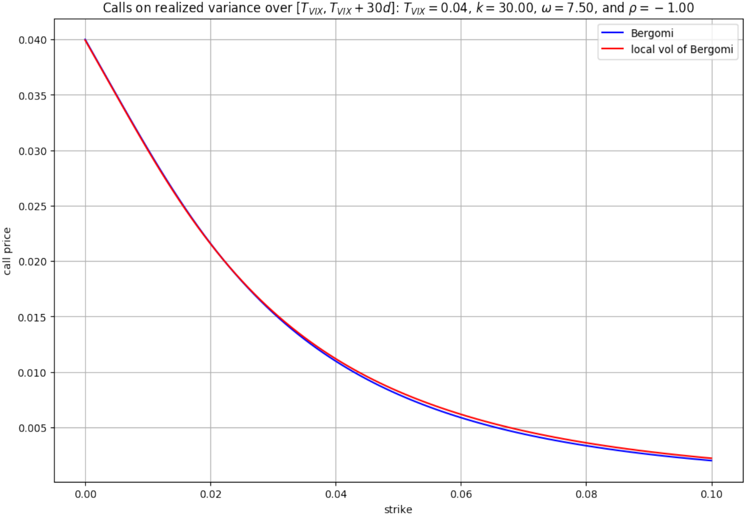
Power-law kernel

- ATM skew in SV models $\sim \rho$ times short term volatility of instantaneous spot variance. Explains why the ATM skew in such rough volatility models behaves like $T^{H-\frac{1}{2}}$ for short maturities T (Alós, Fukasawa...), which is one of the reasons why this model has been introduced (Gatheral, Jaisson, Rosenbaum, Friz, Bayer).
- For (6.4) to hold, we must impose that H be very small. In the limit where H tends to zero, for fixed ν, $\nu^2 \frac{t^{2H}}{2H}$ tends to $+\infty$ for any $t > 0$.
- In order for $\mathrm{Var}(\sigma_t^2)$ to tend to a finite limit, we must impose that $\frac{\nu^2}{2H}$ tend to a finite limit. As a consequence, a natural limiting regime, analogous to the ergodic regime described above in the case of the exponential kernel, consists of letting H and ν tend to zero, with $\frac{\nu^2}{2H}$ kept constant. In this limit $K(\theta) \sim \sqrt{H}\theta^{H-\frac{1}{2}} \longrightarrow 0$ for all $\theta > 0$, while $\lim_{\theta \to 0^+} K(\theta) = +\infty$ for any $H > 0$.

One-factor Bergomi model, $T_i = 0.04$



One-factor Bergomi model



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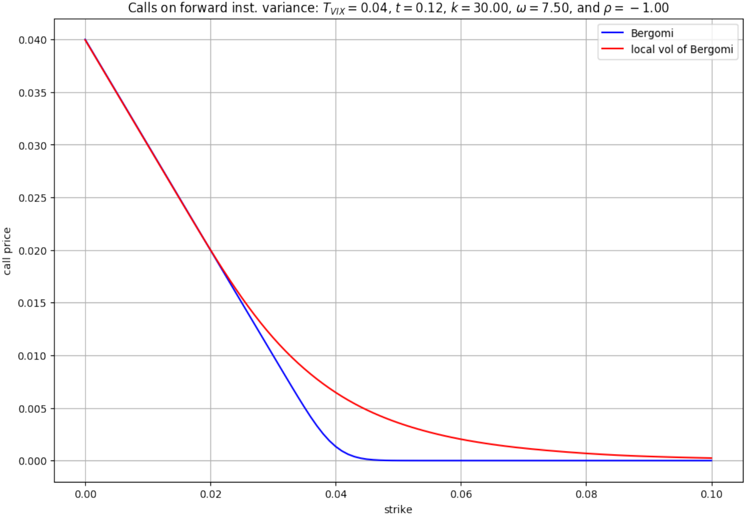

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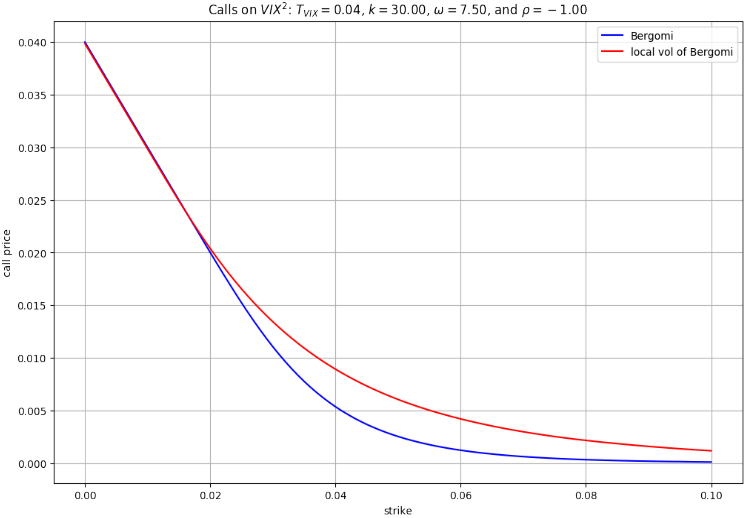

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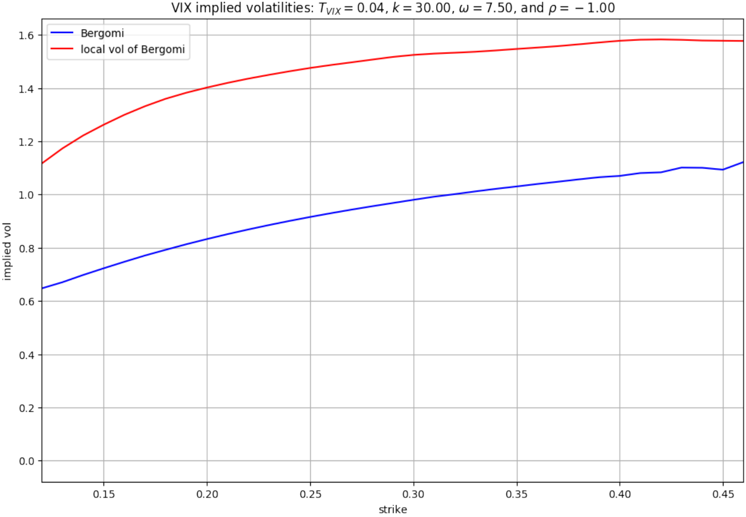
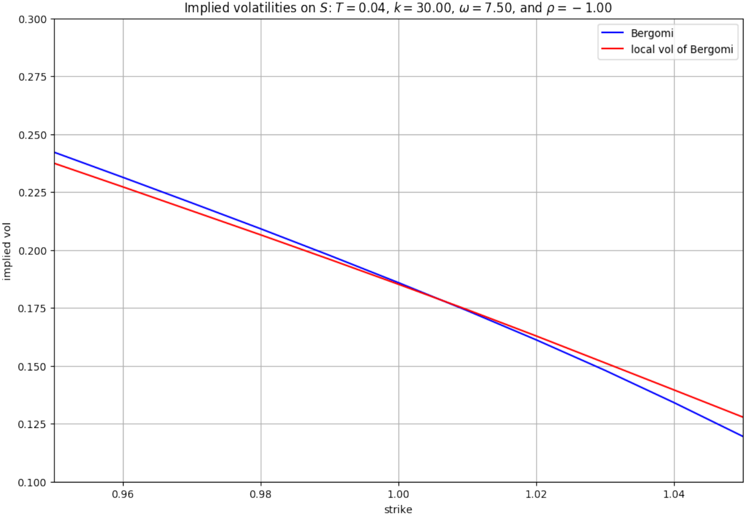
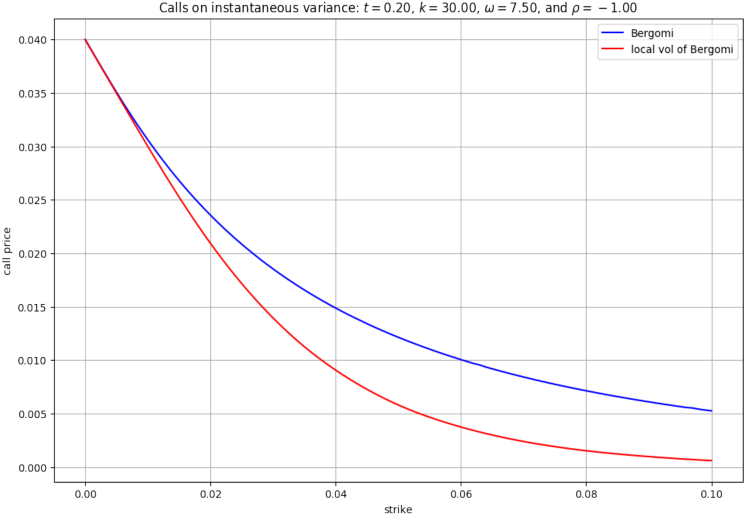


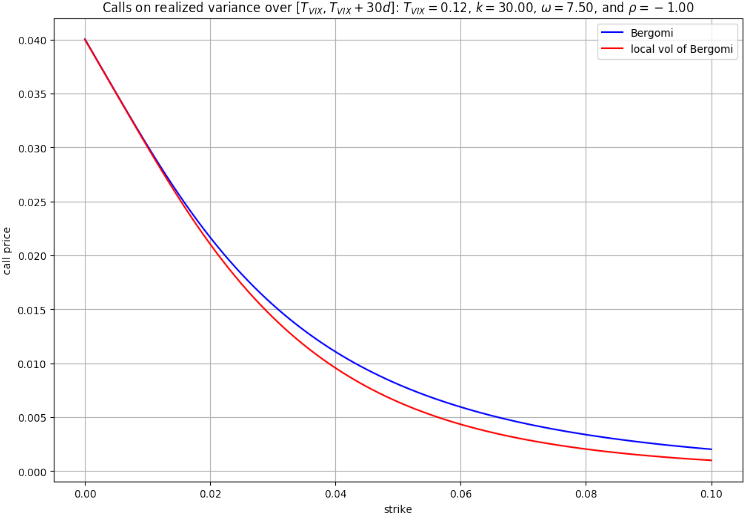
One-factor Bergomi model



One-factor Bergomi model, $T_i = 0.12$



One-factor Bergomi model



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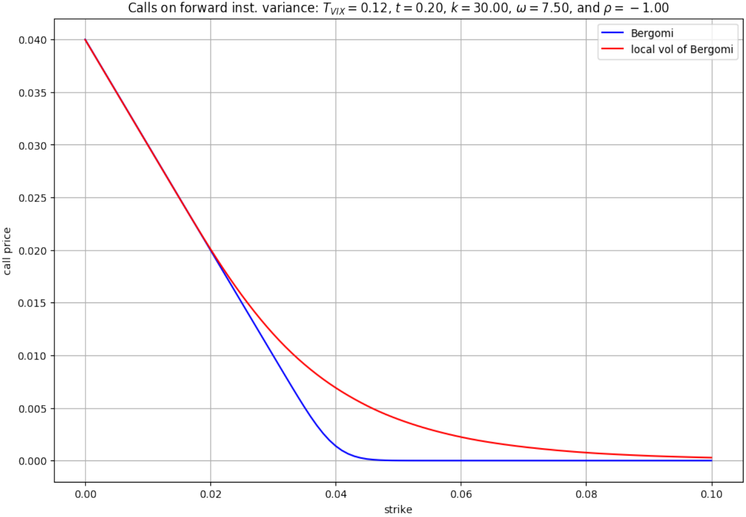

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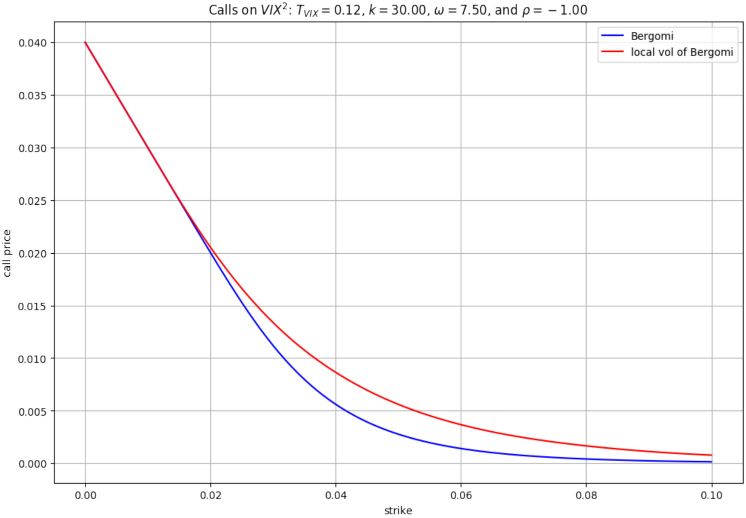

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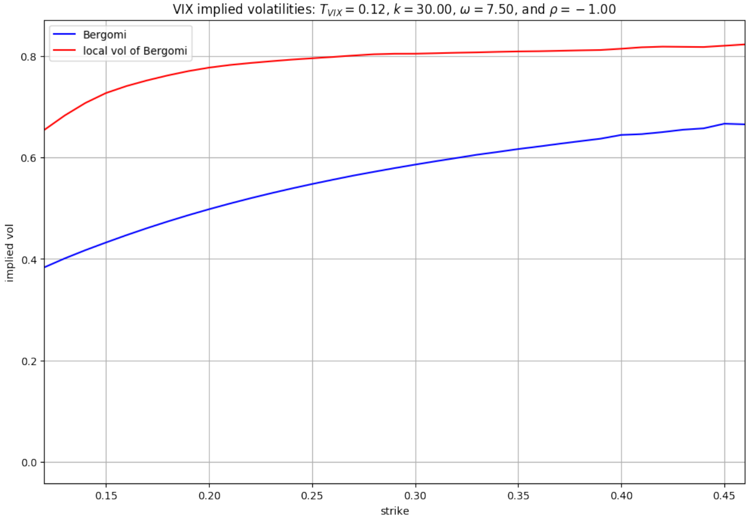

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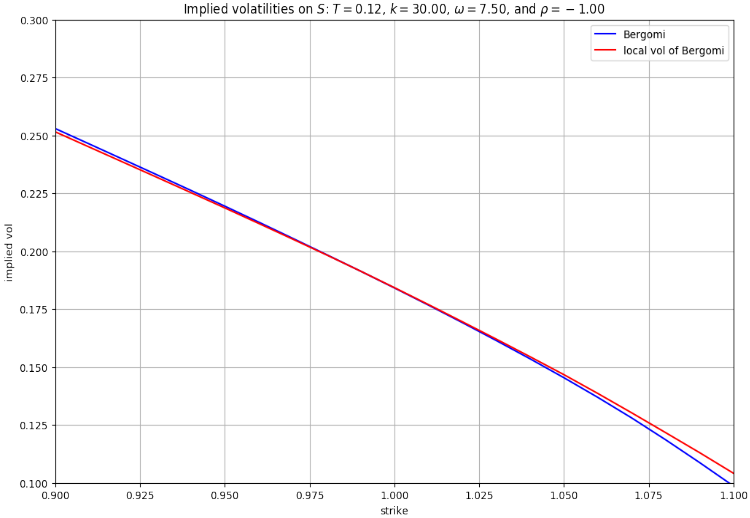
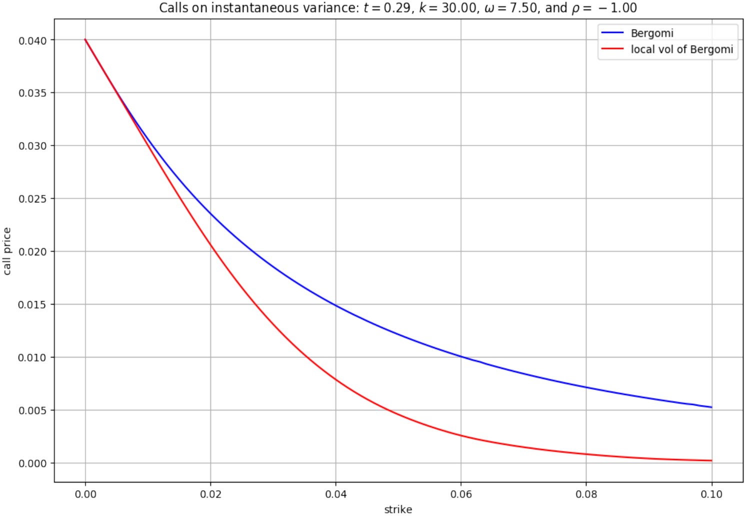


One-factor Bergomi model, $T_i = 0.20$



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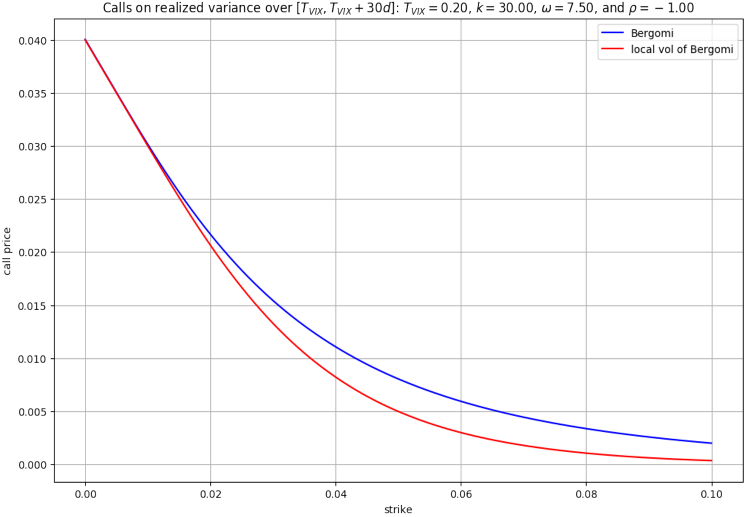

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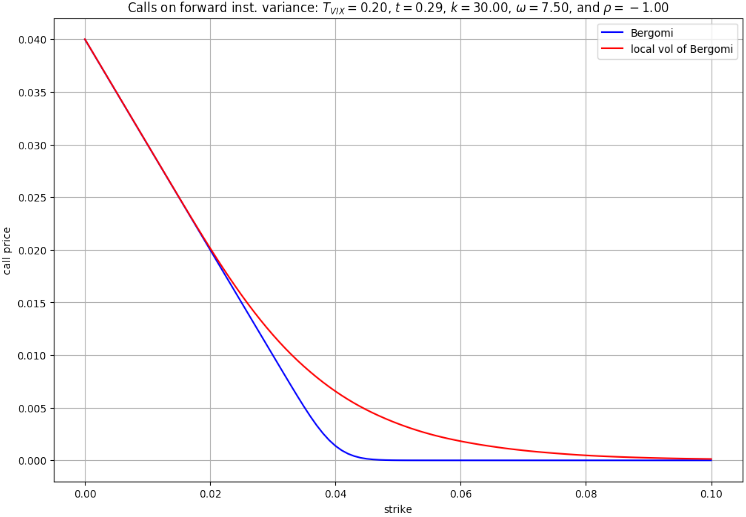

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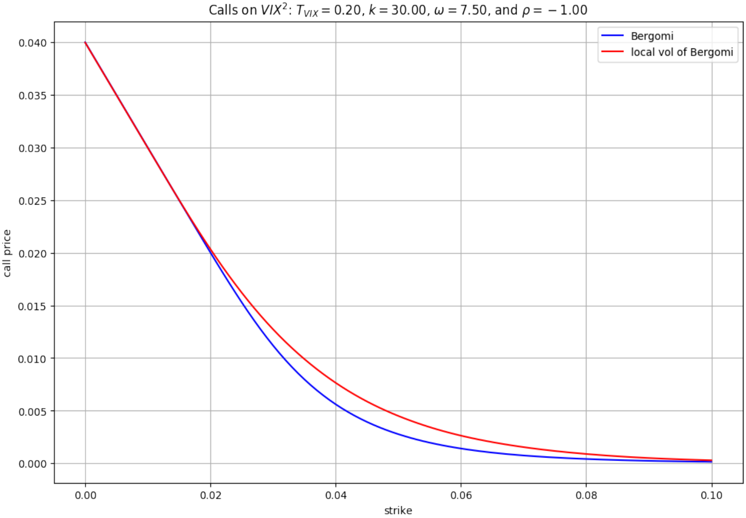

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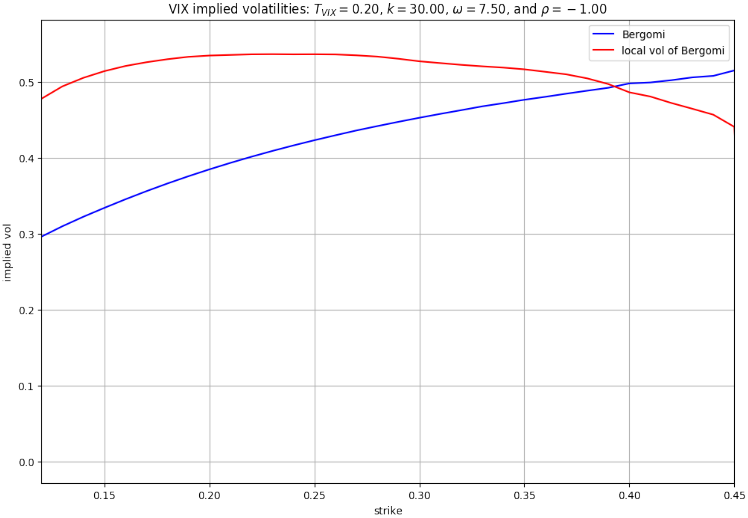

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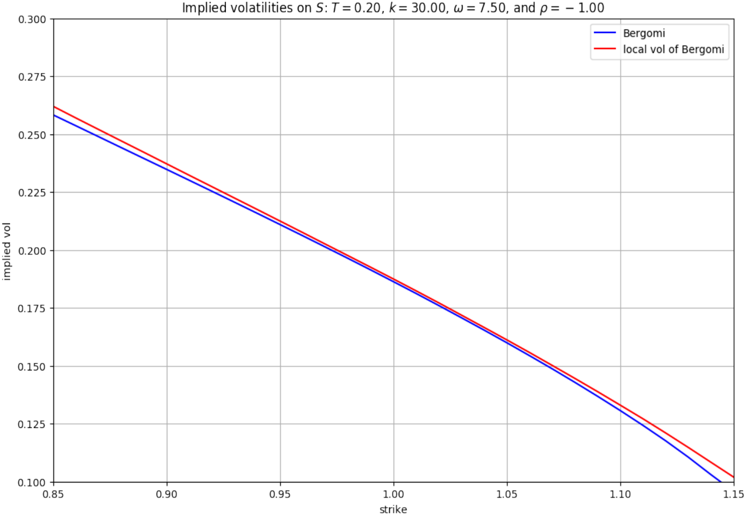
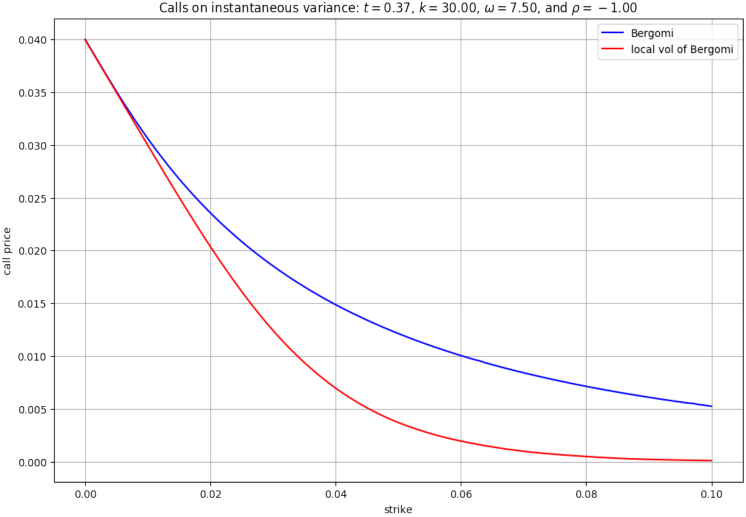


One-factor Bergomi model, $T_i = 0.29$



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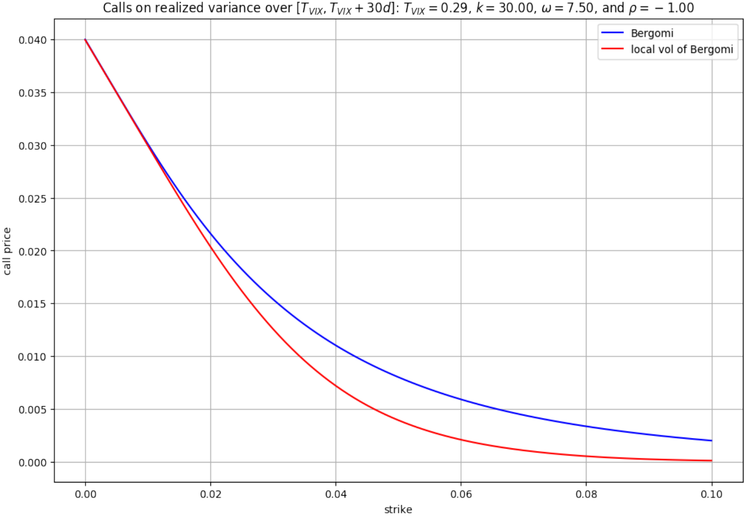

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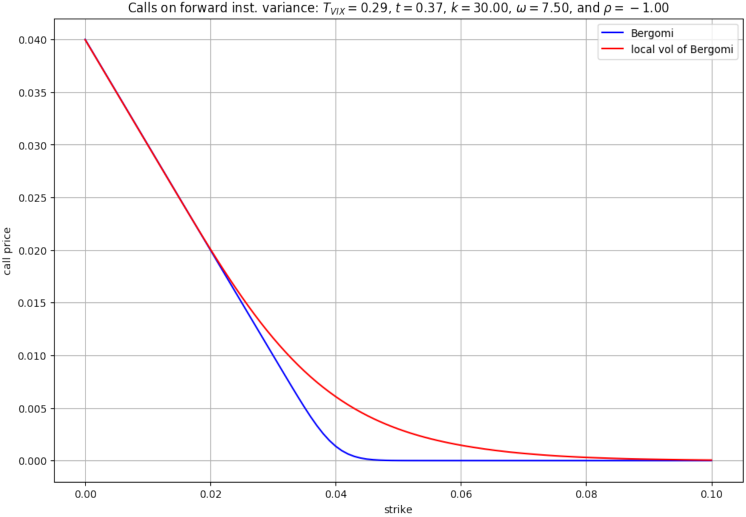

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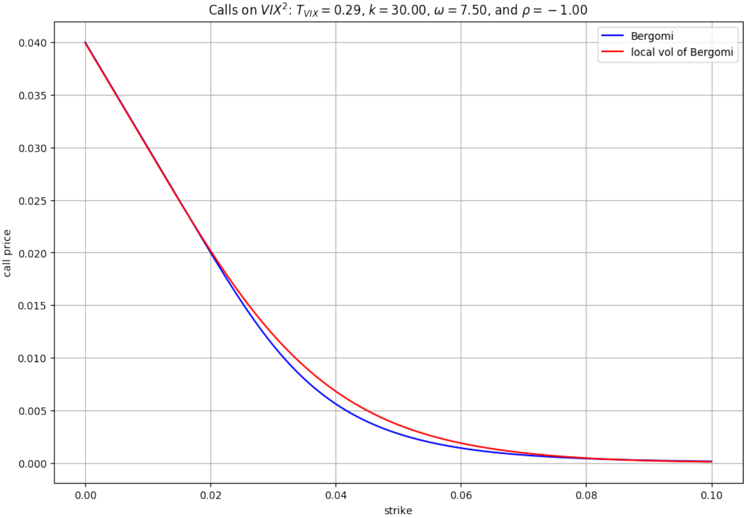

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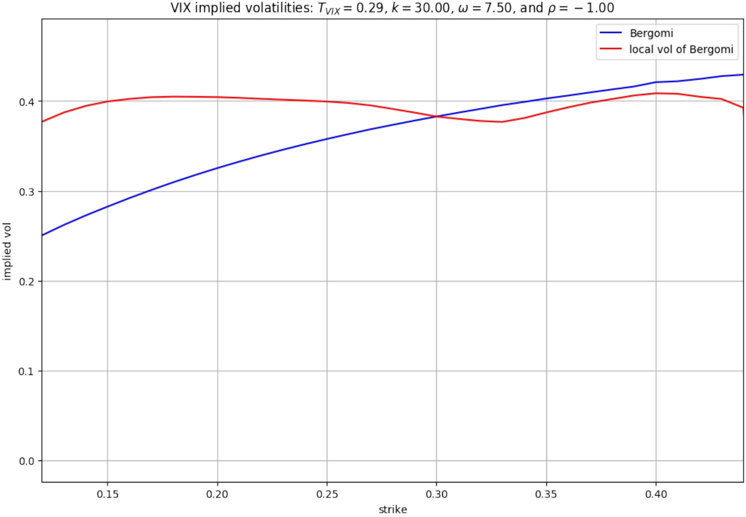

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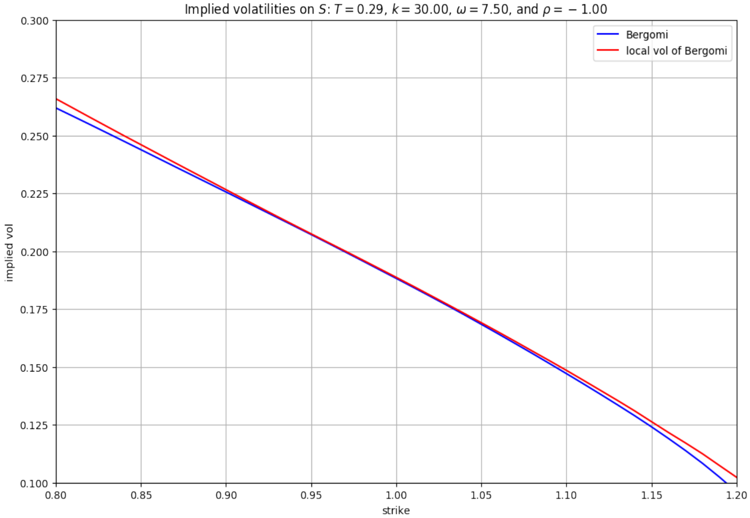
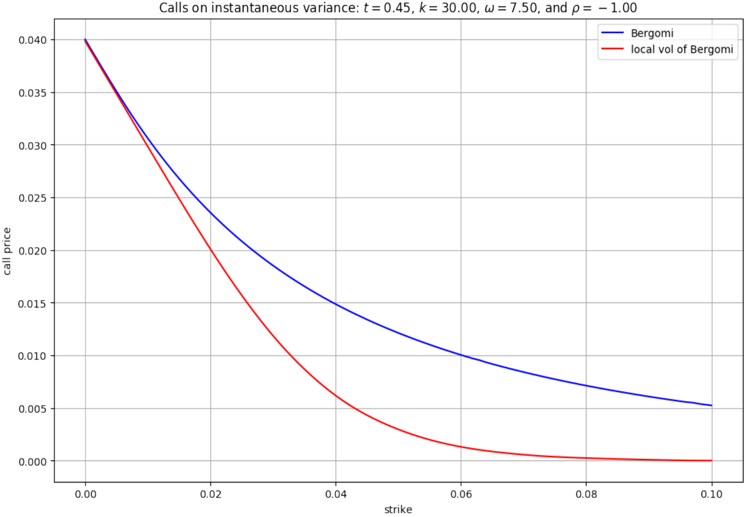


One-factor Bergomi model, $T_i = 0.37$



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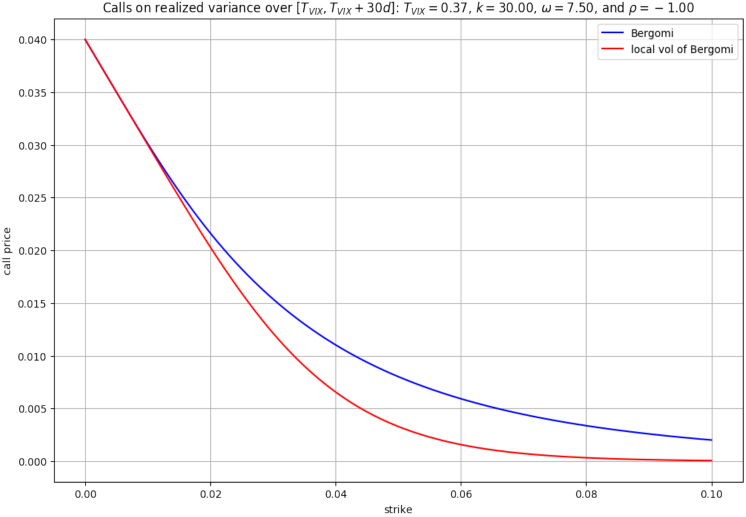

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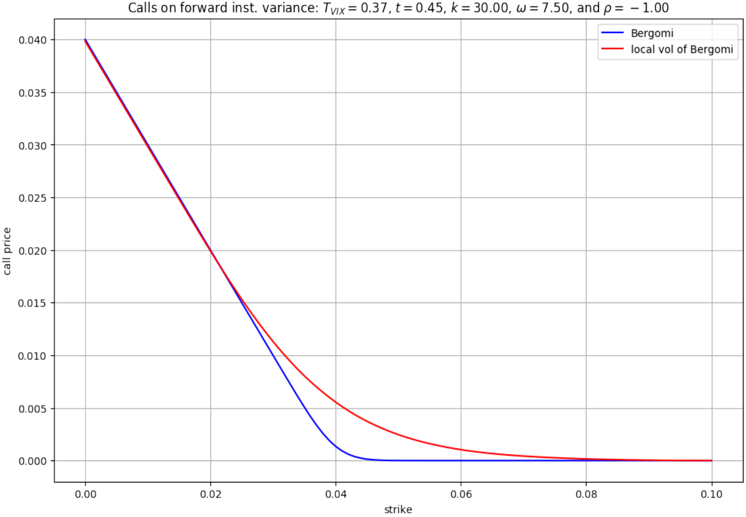

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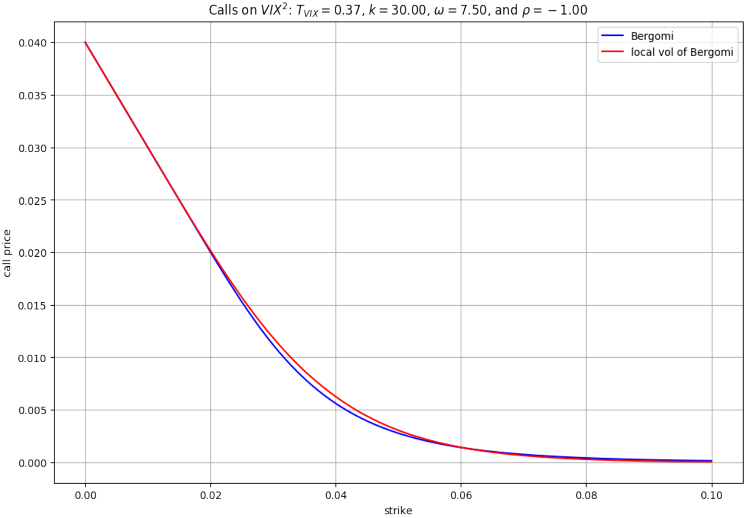
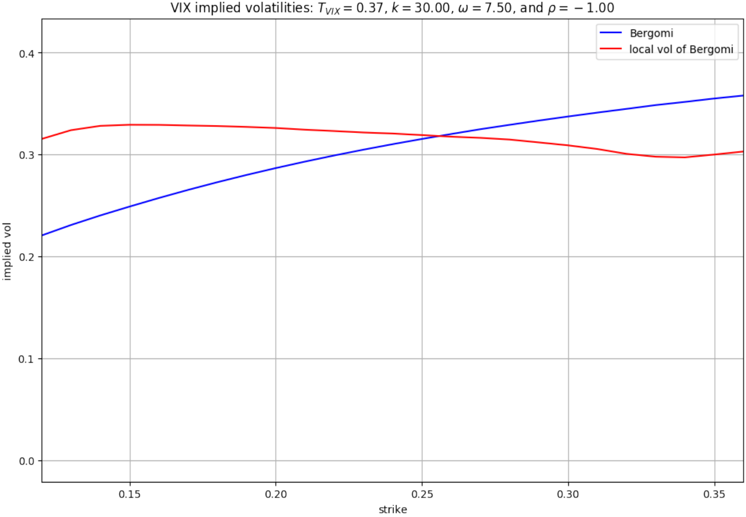


One-factor Bergomi model



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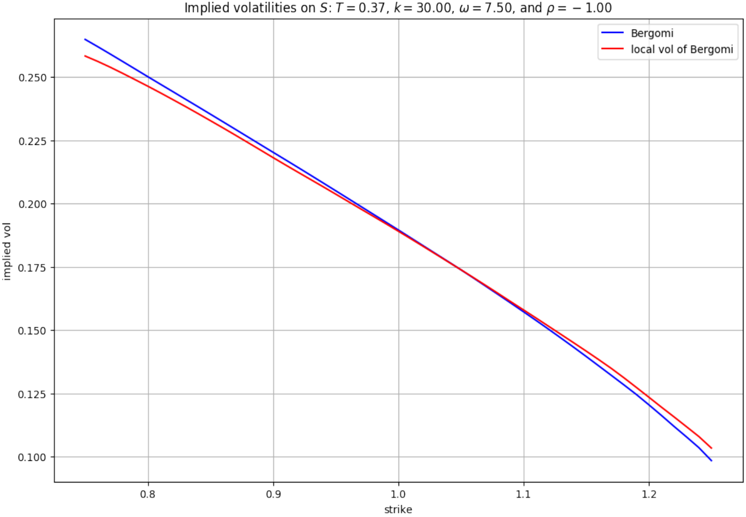


- Similar behavior in the rough Bergomi model

Skewing the models on ξ_t^u

- Following Bergomi (2008), we use a linear combination of two lognormal random variables to model the instantaneous variance σ_t^2 so as to generate positive VIX skew:

$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\omega_0 \int_0^t e^{-k(t-s)} dZ_s \right) + \lambda \mathcal{E}\left(\omega_1 \int_0^t e^{-k(t-s)} dZ_s \right) \right)$$

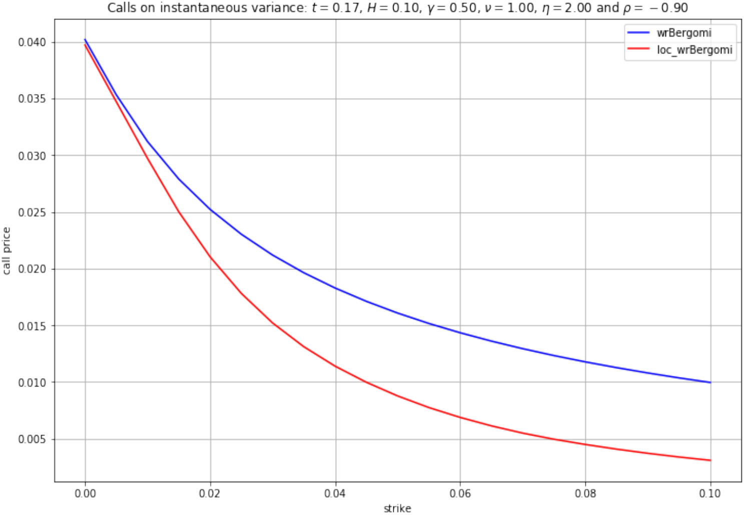
or

$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\nu_0 \int_0^t (t-s)^{H-\frac{1}{2}} dZ_s \right) + \lambda \mathcal{E}\left(\nu_1 \int_0^t (t-s)^{H-1/2} dZ_s \right) \right)$$

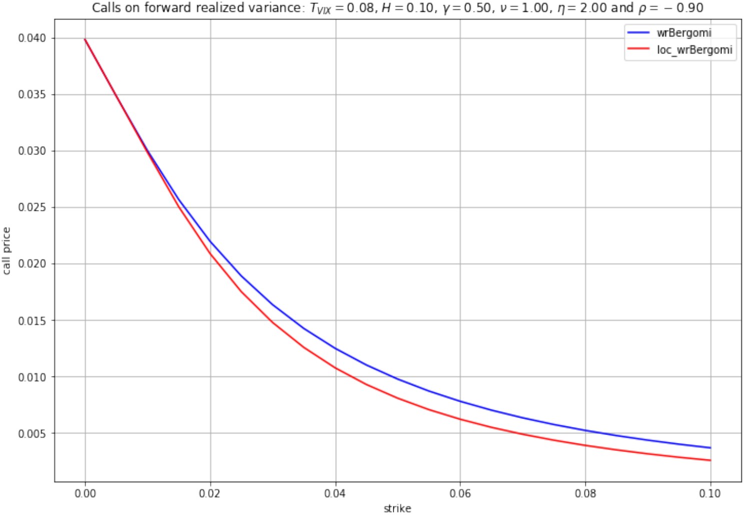
with $\lambda \in [0,1]$.

- $\mathcal{E}(X)$ is simply a shorthand notation for $\exp\left(X - \frac{1}{2}\mathrm{Var}(X)\right)$.
- Similar idea recently used and developed by De Marco.

Skewed rough Bergomi



Skewed rough Bergomi



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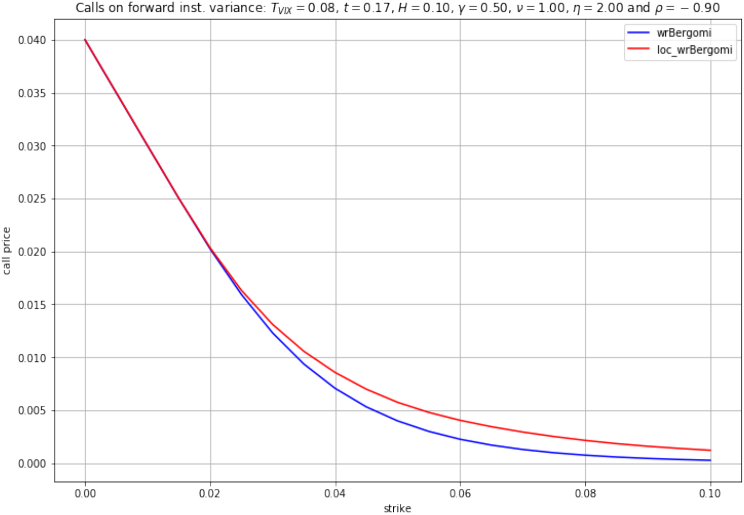

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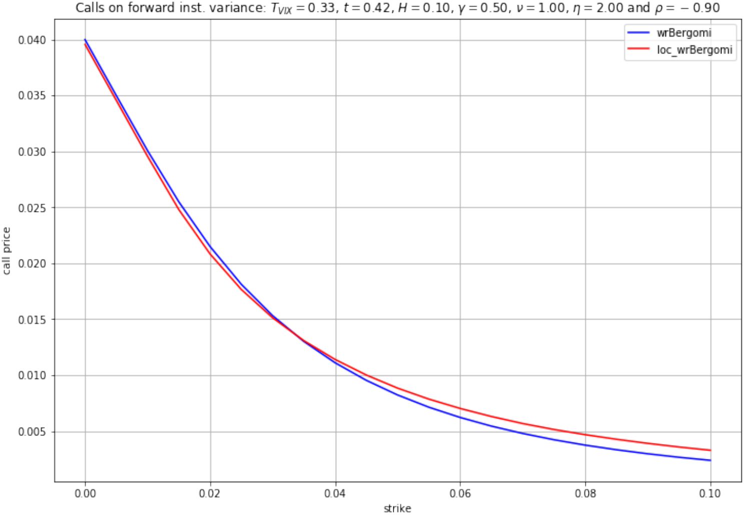

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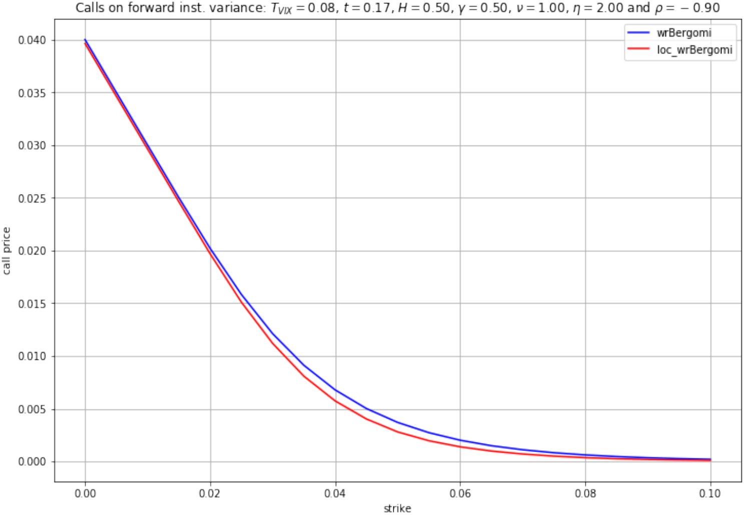

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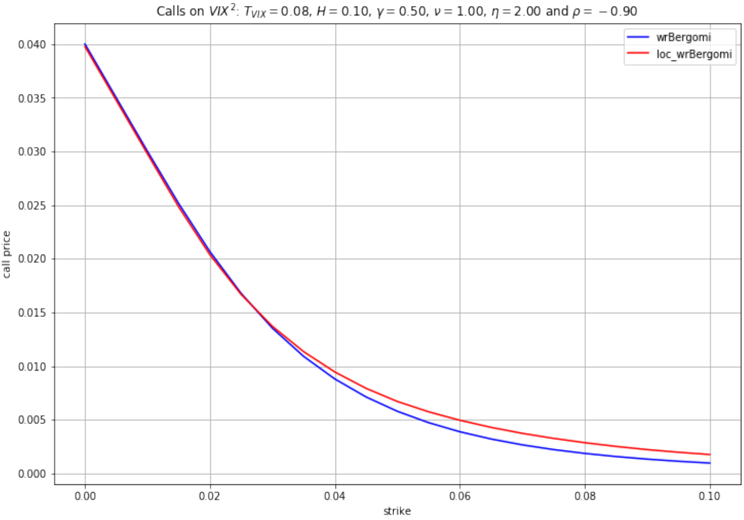

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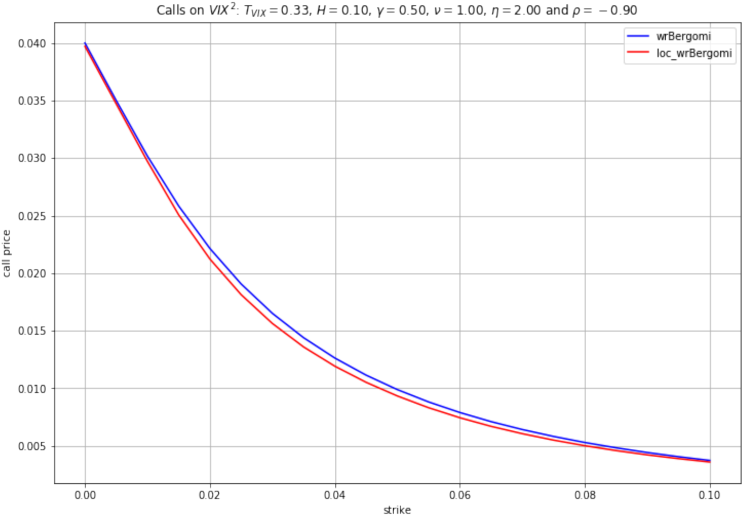

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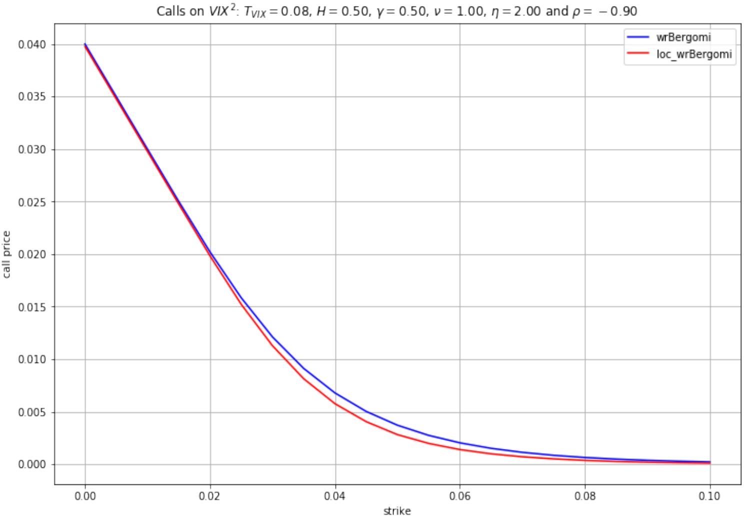

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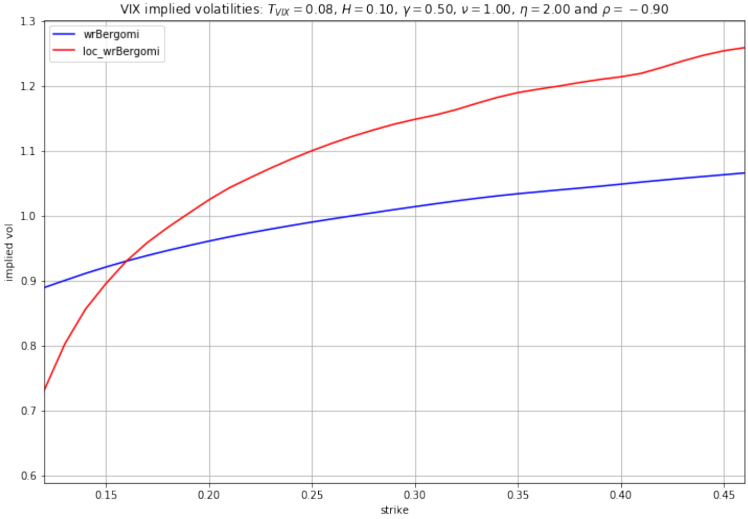

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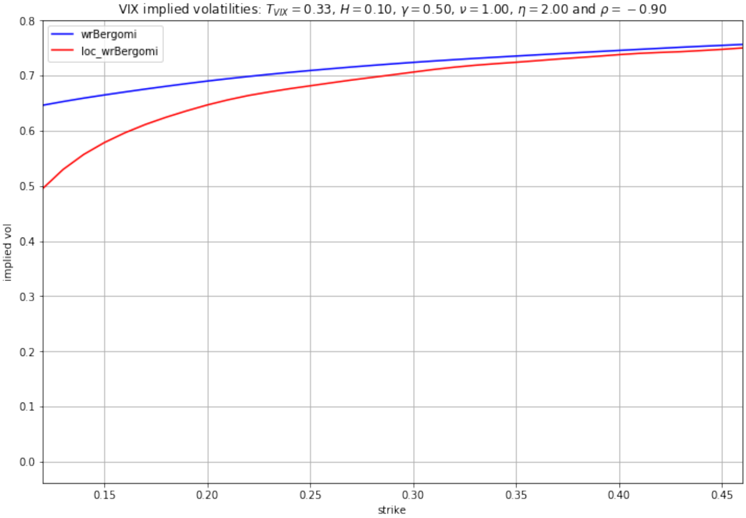

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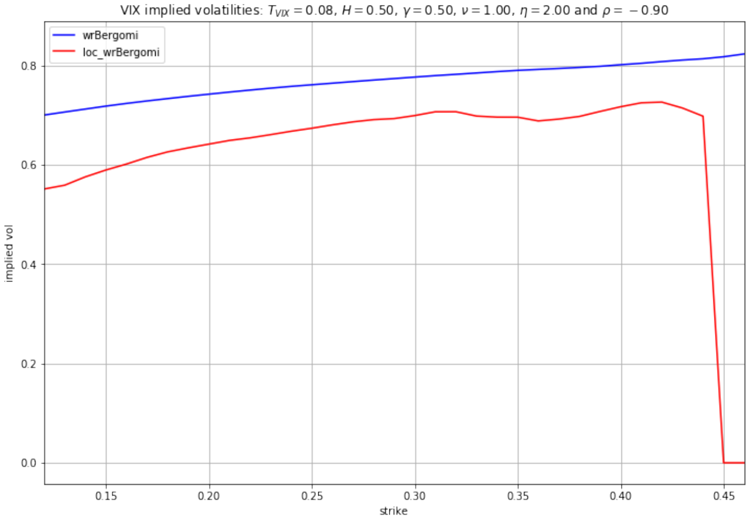
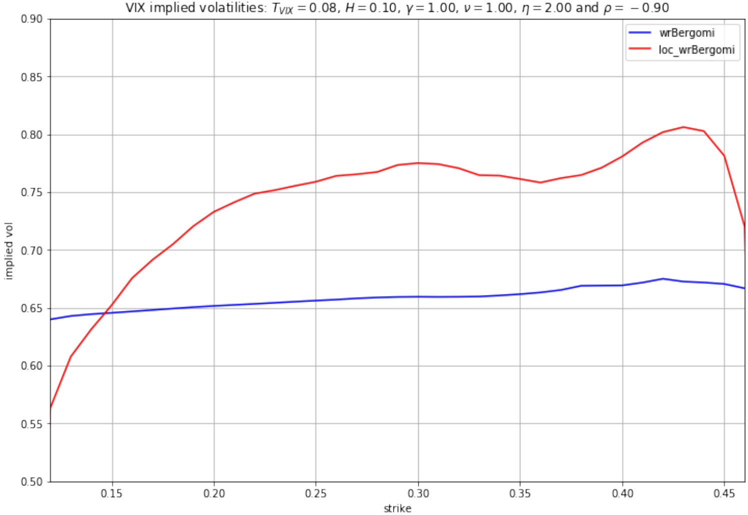
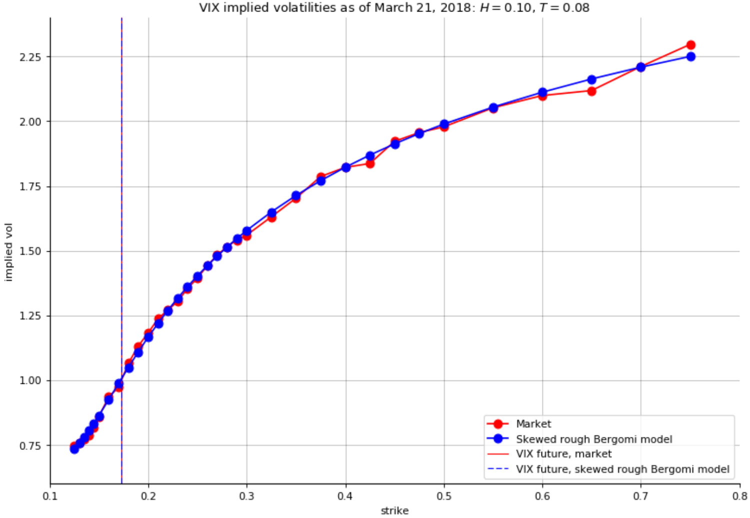


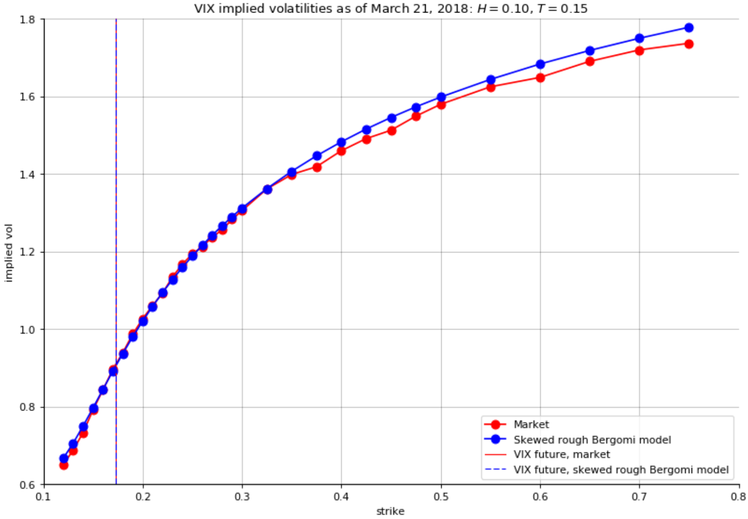
Non-skewed rough Bergomi



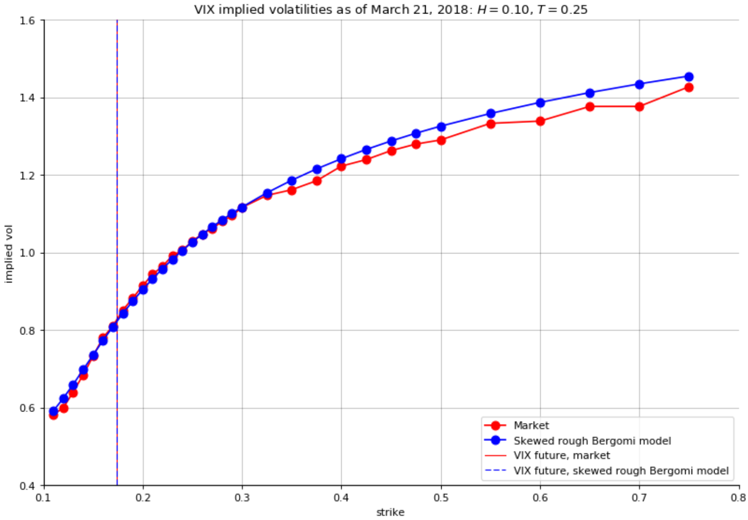
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018



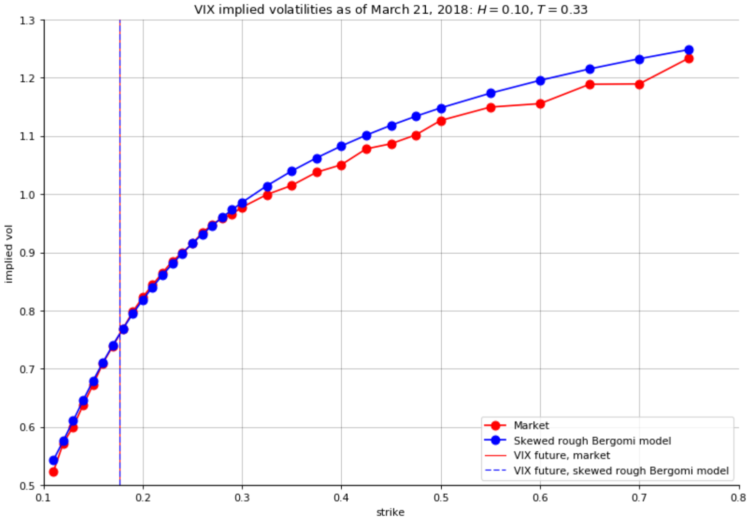
Skewed rough Bergomi: Calibration to VIX future and VIX options



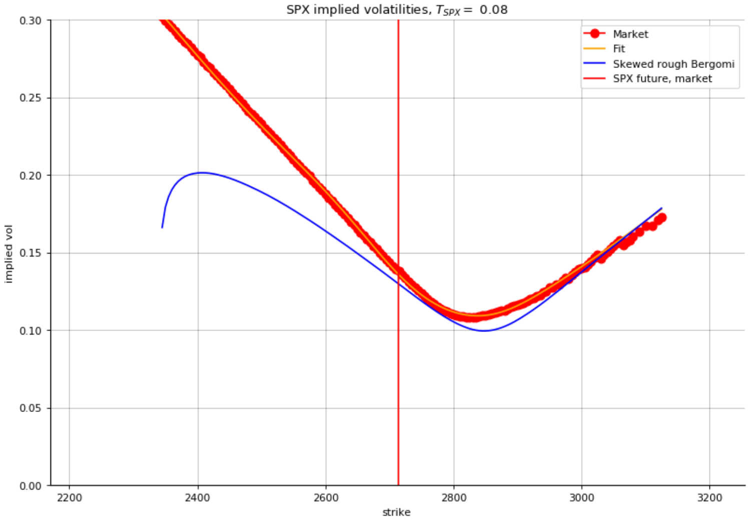
Skewed rough Bergomi: Calibration to VIX future and VIX options



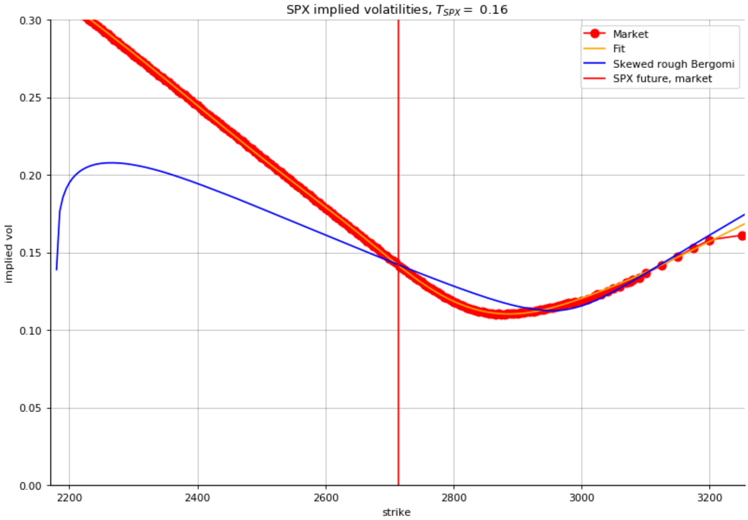
Skewed rough Bergomi: Calibration to VIX future and VIX options



Skewed rough Bergomi calibrated to VIX: SPX smile



Skewed rough Bergomi calibrated to VIX: SPX smile



Skewed rough Bergomi calibrated to VIX: SPX smile

Skewed rough Bergomi calibrated to VIX: SPX smile

Parameters $[1-\lambda, \nu_0, \nu_1]$ at the first five VIX maturities:
[0.68, 0.79, 5.64],
[0.57, 0.80, 5.03],
[0.51 , 0.79, 4.34],
[0.46, 0.70, 3.71],
[0.56, 1.31, 6.27]

Current work

- Absence of arbitrage between SPX options at T and $T + 30$ days, VIX future and VIX options at T using an LP solver (cf De Marco and Henry-Labordère, 2015).
- Application of Bergomi-G. expansion (2012) to rough vol models and extension to the smile of VIX options.
- Consider only continuous models on the SPX that are calibrated to the SPX smile:

$$\frac{dS_t}{S_t} = \frac{a_t}{\sqrt{\mathbb{E}[a_t^2|S_t]}}\sigma_{\mathrm{lv}}(t, S_t)\,dW_t$$

  and optimise on (a_t) so as to match VIX options — or compute the infimum of VIX implied vols within those models (use of neural networks for VIX computation).

Why jumps can help

- For a continuous model to calibrate jointly to SPX and VIX options, the distribution of $\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma_t^2\,dt\middle|\mathcal{F}_T\right]$ should be as narrow as possible, but without killing the SPX skew. The problem of ergodic/stationary (σ_t) is that they produce flat SPX skew.
- Jump-Lévy processes are precisely examples of processes that can generate deterministic realized variance together with a smile on the underlying.
- This explains why jumps have proved useful in this problem.

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