The Joint S&P 500/VIX Smile Calibration Puzzle Solved

Julien Guyon

Bloomberg L.P.
Quantitative Research

Bloomberg Quant (BBQ) Seminar
New York, February 19, 2020

jguyon2@bloomberg.net

Preprint available on SSRN

THE JOINT S&P 500/VIX SMILE CALIBRATION PUZZLE SOLVED

JULIEN GUYON
QUANTITATIVE RESEARCH, BLOOMBERG L.P.

ABSTRACT. Since VIX options started trading in 2006, many researchers have tried to build a model that jointly and exactly calibrates to the prices of S&P 500 (SPX) options, VIX futures, and VIX options. So far the best attempts, which used parametric continuous-time jump-diffusion models on the SPX, only produced an approximate fit. In this article we solve this longstanding puzzle using a nonparametric discrete-time model. Given a VIX future maturity T_1, we build a joint probability measure on the SPX at T_1, the VIX at T_1, and the SPX at $T_2 = T_1 + 30$ days which is perfectly calibrated to the SPX smiles at T_1 and T_2, and the VIX future and VIX smile at T_1. Our model satisfies the martingality constraint on the SPX as well as the requirement that the VIX at T_1 is the implied volatility of the 30-day log-contract on the SPX. We prove by duality that the existence of such a model means that the SPX and VIX markets are jointly arbitrage-free.

The joint calibration puzzle is cast as a dispersion-constrained martingale transport problem which is solved using (an extension of) the Sinkhorn algorithm, in the spirit of De March and Henry-Labordère (2019). The algorithm identifies joint SPX/VIX arbitrages should they arise. Our numerical experiments show that the algorithm performs very well in both low and high volatility regimes. Finally we explain how to handle the fact that the VIX future and SPX option monthly maturities do not perfectly coincide, and how to extend the two-maturity model to include all available monthly maturities.

1. INTRODUCTION

Volatility indices, such as the VIX index [10], do not only serve as market-implied indicators of volatility.

Motivation

- Volatility indices, such as the VIX index, are not only used as market-implied indicators of volatility.
- Futures and options on these indices are also widely used as risk-management tools to hedge the volatility exposure of options portfolios.
- Existence of a liquid market for these futures and options $\Longrightarrow$ need for models that jointly calibrate to the prices of options the underlying asset and prices of volatility derivatives.
- Calibration of stochastic volatility models to liquid hedging instruments: e.g., S&P 500 (SPX) options + VIX futures and options.
- Since VIX options started trading in 2006, many researchers and practitioners have tried to build a model that jointly and exactly calibrates to the prices of SPX options, VIX futures and VIX options.
- **Very challenging problem, especially for short maturities**.

Motivation

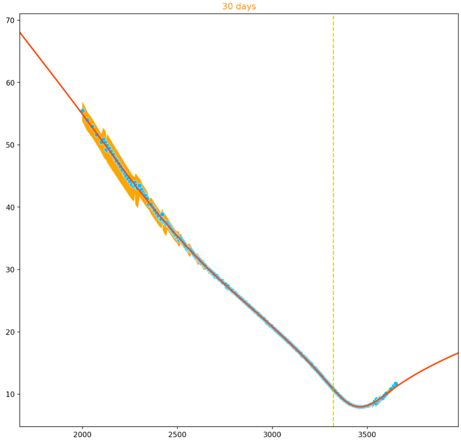


Figure: SPX smile as of January 22, 2020, $T = 30$ days

Motivation

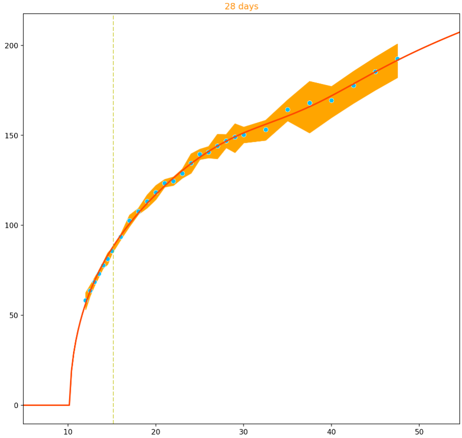


Figure: VIX smile as of January 22, 2020, $T = 28$ days

Motivation

- ATM skew:

$$\begin{array}{rccl} \text{Definition:} & \mathcal{S}_T & = & \left.\dfrac{d\sigma_{\mathrm{BS}}(K,T)}{\frac{dK}{K}}\right|_{K=F_T} \\ \text{SPX, small } T\text{:} & \mathcal{S}_T & \approx & -1.5 \\ \text{Classical one-factor SV model:} & \mathcal{S}_T & \underset{T\to 0}{\longrightarrow} & \dfrac{1}{2} \times \text{spot-vol correl} \times \text{vol-of-vol} \end{array}$$

- Calibration to short-term ATM SPX skew $\Longrightarrow$

$$\text{vol-of-vol} \geq 3 = 300\% \gg \text{short-term ATM VIX implied vol}$$

The **very large negative skew of short-term SPX options**, which in continuous models implies a very large volatility of volatility, **seems inconsistent with the comparatively low levels of VIX implied volatilities**.

Gatheral (2008)

Consistent Modeling of SPX and VIX options

Consistent Modeling of SPX and VIX options

Jim Gatheral

Merrill Lynch

The Fifth World Congress of the Bachelier Finance Society
London, July 18, 2008

Double CEV dynamics

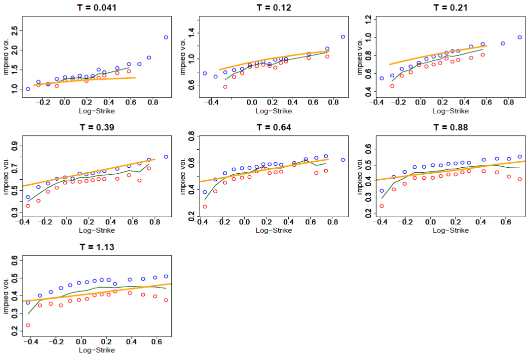
- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned} \frac{dS}{S} &= \sqrt{v}\, dW \\ dv &= -\kappa\,(v - v')\, dt + \eta_1\, v^{\alpha}\, dZ_1 \\ dv' &= -c\,(v' - z_3)\, dt + \eta_2\, v'^{\beta}\, dZ_2 \end{aligned} \tag{2}$$

  for any choice of $\alpha,\, \beta \in [1/2, 1]$.
  - We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
  - the case $\alpha = \beta = 1$ *Double Lognormal*,
  - and the general case *Double CEV*.
- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

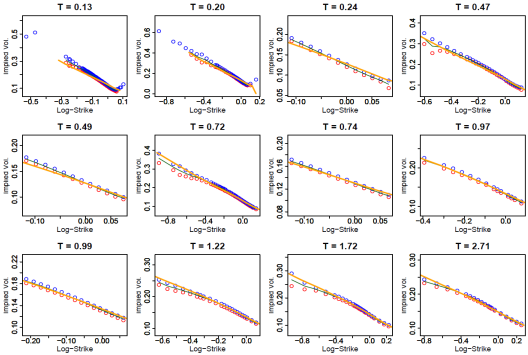
Double CEV fit to VIX options as of 03-Apr-2007

Setting the correlation ρ between volatility factors z_1 and z_2 to its historical average (see later) and iterating on the volatility of volatility parameters ξ_1 and ξ_2 to minimize the differences between model and market VIX option prices, we obtain the following fits (orange lines):

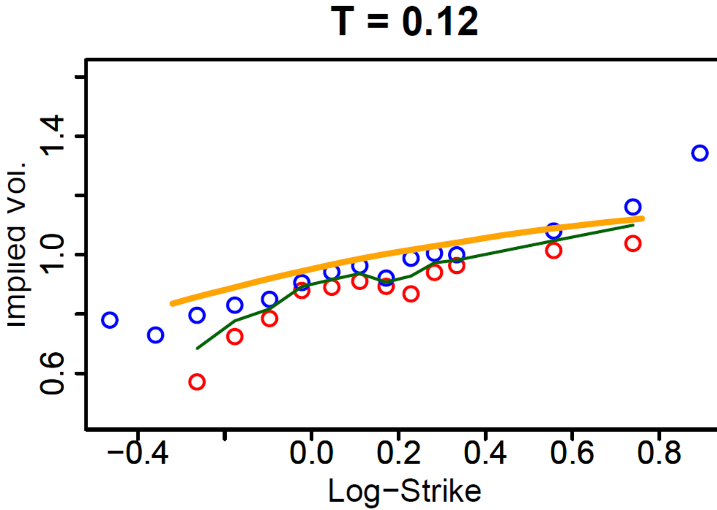


Double CEV fit to SPX options as of 03-Apr-2007

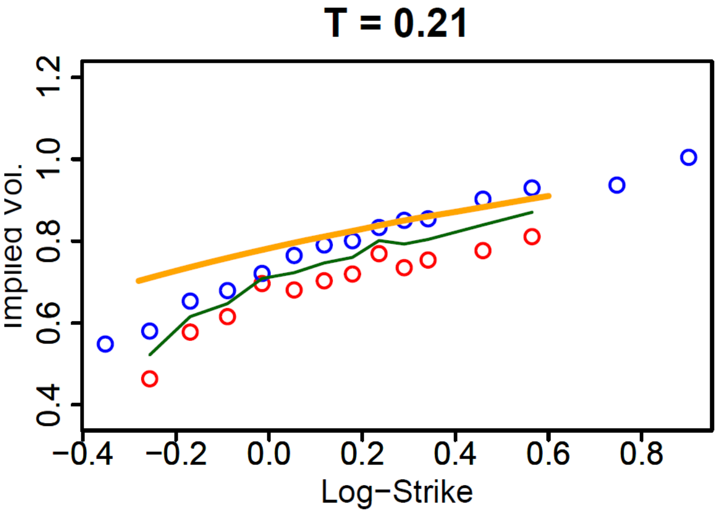
Minimizing the differences between model and market SPX option prices, we find $\rho_1 = -0.9$, $\rho_2 = -0.7$ and obtain the following fits to SPX option prices (orange lines):



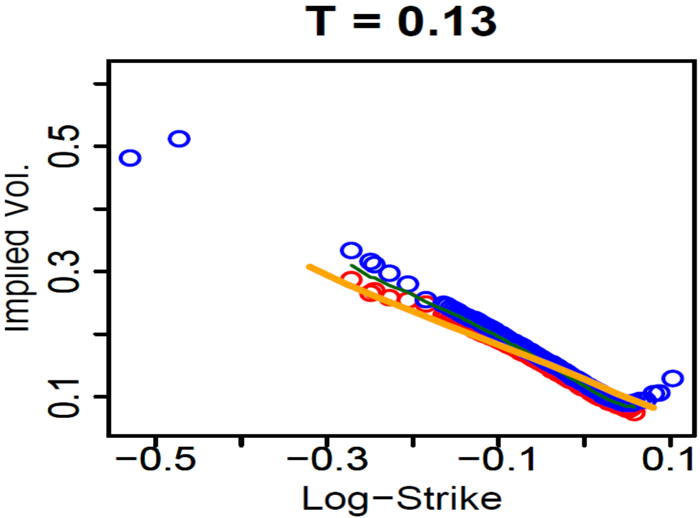
Fit to VIX options



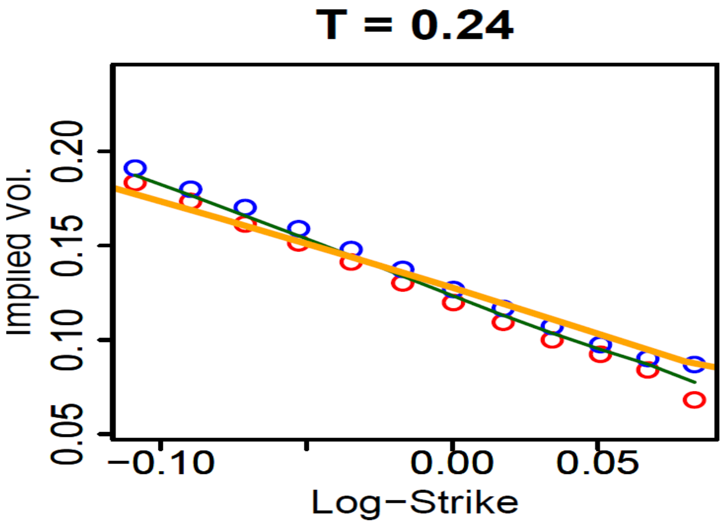
Fit to VIX options



Fit to SPX options



Fit to SPX options



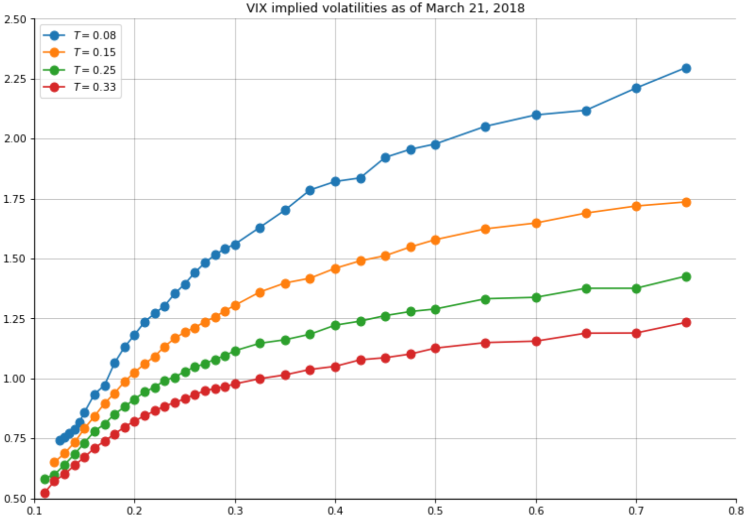
Skewed rough Bergomi model (G., 2018)

- Following Bergomi (2008), we suggested using a linear combination of two lognormal random variables to model the instantaneous variance σ_t^2 so as to generate positive VIX skew (G., 2018):

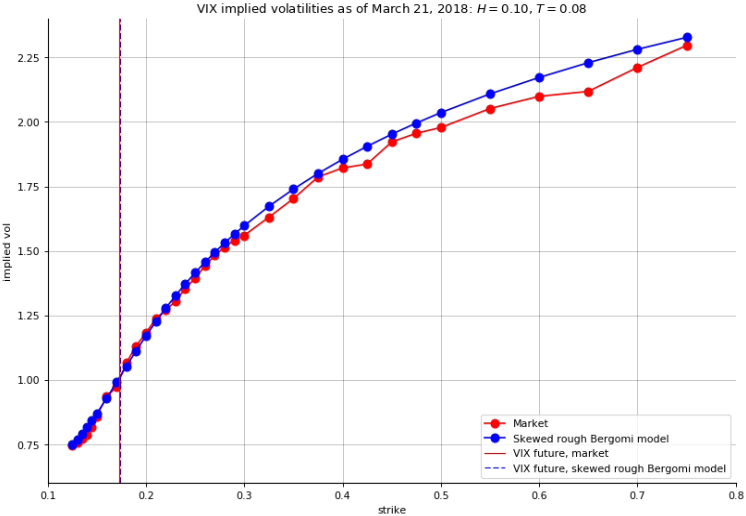
$$\sigma_t^2 = \xi_0^t \left((1-\lambda)\mathcal{E}\left(\nu_0 \int_0^t (t-s)^{H-\frac{1}{2}} dZ_s\right) + \lambda\mathcal{E}\left(\nu_1 \int_0^t (t-s)^{H-1/2} dZ_s\right)\right)$$

  with $\lambda \in [0,1]$.
- $\mathcal{E}(X)$ is simply a shorthand notation for $\exp\left(X - \frac{1}{2}\mathrm{Var}(X)\right)$.
- Also independently proposed by De Marco.

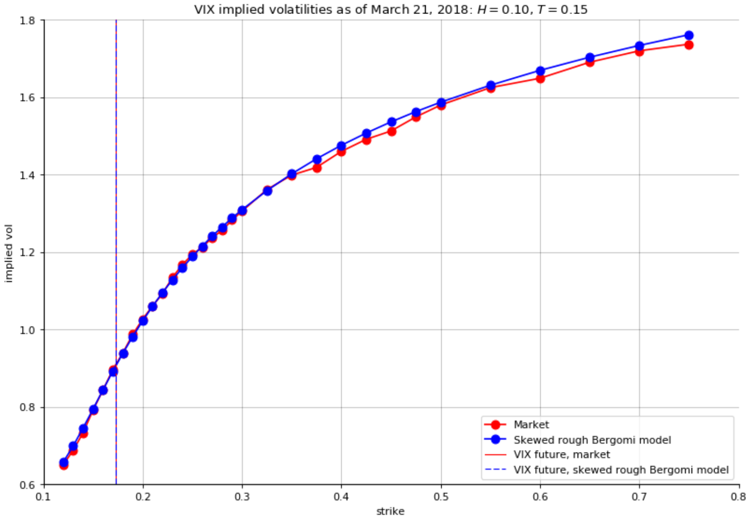
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018)



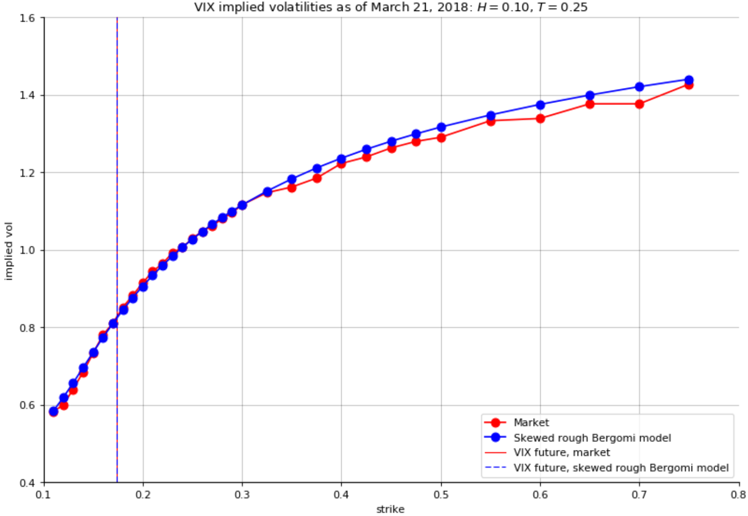
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018)



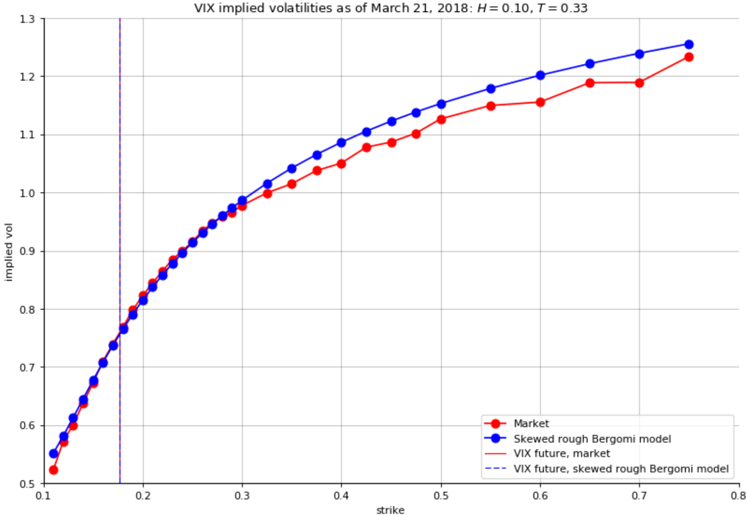
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018)



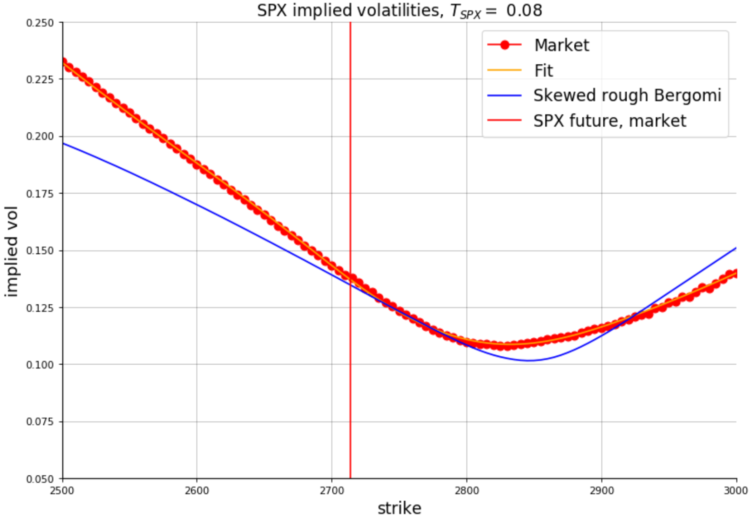
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018)



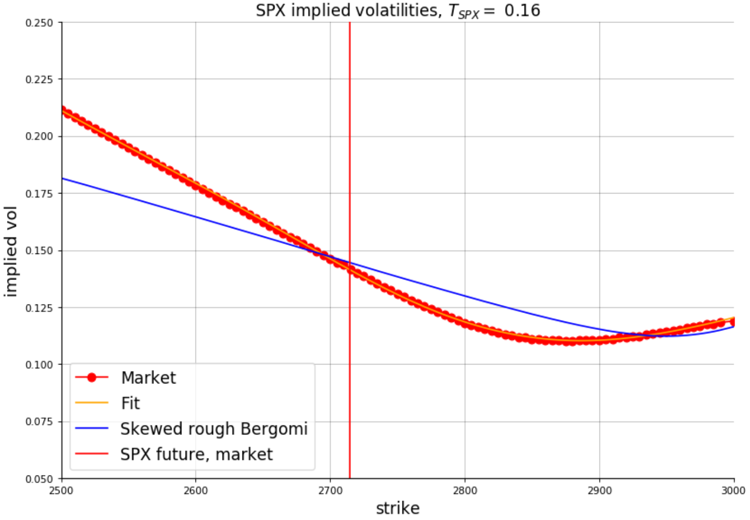
Skewed rough Bergomi: Calibration to VIX future and VIX options (March 21, 2018)



Skewed rough Bergomi calibrated to VIX: SPX smile (March 21, 2018)



Skewed rough Bergomi calibrated to VIX: SPX smile (March 21, 2018)



Skewed rough Bergomi calibrated to VIX: SPX smile

- **Not enough ATM skew for SPX**, despite pushing negative spot-vol correlation as much as possible.
- I get **similar results** when I use the **skewed 2-factor Bergomi model** instead of the skewed rough Bergomi model.

SLV calibrated to SPX: VIX smile (Aug 1, 2018)

- All continuous models on SPX that are calibrated to full SPX smile are of the form:

$$\frac{dS_t}{S_t} = \frac{a_t}{\sqrt{\mathbb{E}[a_t^2|S_t]}}\sigma_{\mathrm{loc}}(t, S_t)\, dW_t.$$

- They are stochastic local volatility (SLV) models

$$\frac{dS_t}{S_t} = a_t \ell(t, S_t)\, dW_t$$

with stochastic volatility (SV) (a_t) and leverage function

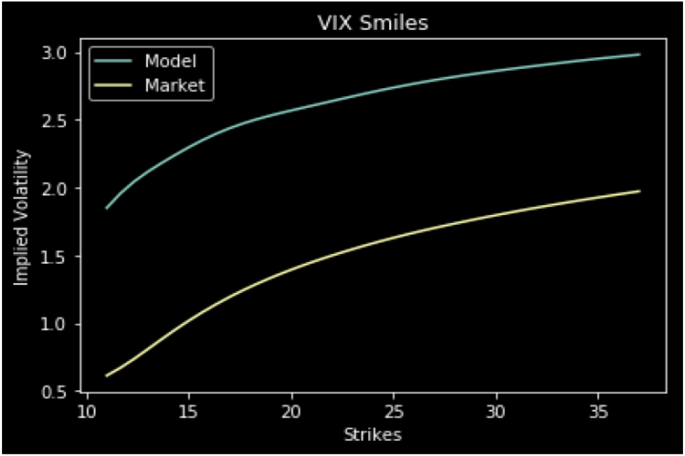
$$\ell(t, S_t) = \frac{\sigma_{\mathrm{loc}}(t, S_t)}{\sqrt{\mathbb{E}[a_t^2|S_t]}}.$$

- In those models ($\tau := 30$ days)

$$\mathrm{VIX}_T^2 \quad = \quad \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\frac{a_t^2}{\mathbb{E}[a_t^2|S_t]}\sigma_{\mathrm{loc}}^2(t, S_t)\middle|\mathcal{F}_T\right] dt.$$

- Optimize SV parameters to fit VIX options.

SLV calibrated to SPX: VIX smile, $T = 21$ days (Aug 1, 2018)



SLV model, SV $=$ skewed 2-factor Bergomi model
SV params optimized to fit VIX smile

Related works with continuous models on the SPX

- Fouque-Saporito (2017), Heston with stochastic vol-of-vol. Problem: their approach does not apply to short maturities (below 4 months), for which VIX derivatives are most liquid and the joint calibration is most difficult.
- Goutte-Ismail-Pham (2017), Heston with parameters driven by a Hidden Markov jump process.
- Jacquier-Martini-Muguruza, *On the VIX futures in the rough Bergomi model* (2017):

  "Interestingly, we observe a 20% difference between the [vol-of-vol] parameter obtained through VIX calibration and the one obtained through SPX. This suggests that the volatility of volatility in the SPX market is 20% higher when compared to VIX, revealing potential data inconsistencies (arbitrage?)."

Motivation

- To try to jointly fit the SPX and VIX smiles, many authors have incorporated **jumps** in the dynamics of the SPX: Sepp, Cont-Kokholm, Papanicolaou-Sircar, Baldeaux-Badran, Pacati et al, Kokholm-Stisen, Bardgett et al...
- Jumps offer extra degrees of freedom to partly decouple the ATM SPX skew and the ATM VIX implied volatility.
- So far all the attempts at solving the joint SPX/VIX smile calibration problem only produced an **approximate fit**.

Our approach

- We solve this puzzle using a **completely different approach**: instead of postulating a parametric continuous-time (jump-)diffusion model on the SPX, we build a **nonparametric discrete-time model**:
  - Decouples SPX skew and VIX implied vol.
  - Perfectly fits the smiles.
- Given a VIX future maturity T_1, we build a **joint probability measure on** (S_1, V, S_2) which is **perfectly calibrated** to the SPX smiles at T_1 and $T_2 = T_1 + 30$ days, and the VIX future and VIX smile at T_1.
- S_1: SPX at T_1, V: VIX at T_1, S_2: SPX at T_2.
- Our model satisfies:
  - **Martingality constraint** on the SPX;
  - **Consistency condition**: the VIX at T_1 is the implied volatility of the 30-day log-contract on the SPX.
- Our model is cast as the solution of a **dispersion-constrained martingale transport problem** which is solved using the **Sinkhorn algorithm**, in the spirit of De March and Henry-Labordère (2019).

Setting and notation

- For simplicity: zero interest rates, repos, and dividends.
- μ_1 = risk-neutral distribution of $S_1 \longleftrightarrow$ market smile of SPX at T_1.
- μ_V = risk-neutral distribution of $V \longleftrightarrow$ market smile of VIX at T_1.
- μ_2 = risk-neutral distribution of $S_2 \longleftrightarrow$ market smile of SPX at T_2.
- F_V: value at time 0 of VIX future maturing at T_1.
- We denote $\mathbb{E}^i := \mathbb{E}^{\mu_i}$, $\mathbb{E}^V := \mathbb{E}^{\mu_V}$ and assume

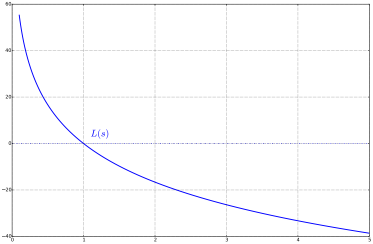
$$\mathbb{E}^i[S_i] = S_0, \quad \mathbb{E}^i[|\ln S_i|] < \infty, \quad i \in \{1,2\}; \qquad \mathbb{E}^V[V] = F_V, \quad \mathbb{E}^V[V^2] < \infty.$$

- No calendar arbitrage $\Longleftrightarrow \mu_1 \leq_c \mu_2$ (convex order)

Setting and notation

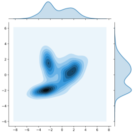
$$V^2 := (\mathrm{VIX}_{T_1})^2 := -\frac{2}{\tau}\mathrm{Price}_{T_1}\left[\ln\left(\frac{S_2}{S_1}\right)\right] = \mathrm{Price}_{T_1}\left[L\left(\frac{S_2}{S_1}\right)\right]$$

- $\tau := 30$ days.
- $L(x) := -\frac{2}{\tau}\ln x$: convex, decreasing.



Superreplication, duality

Superreplication of forward-starting options



- The knowledge of μ_1 and μ_2 gives little information on the prices $\mathbb{E}^\mu[g(S_1, S_2)]$, e.g., prices of forward-starting options $\mathbb{E}^\mu[f(S_2/S_1)]$.
- Computing upper and lower bounds of these prices:
  Optimal transport (Monge, 1781; Kantorovich)
- Adding the no-arbitrage constraint that (S_1, S_2) is a martingale leads to more precise bounds, as this provides information on the conditional average of S_2/S_1 given S_1:
  Martingale optimal transport (Henry-Labordère, 2017)
- When $S =$ SPX: Adding VIX market data information produces even more precise bounds, as it gives information on the conditional dispersion of S_2/S_1, which is controlled by the VIX V:
  Dispersion-constrained martingale optimal transport

Classical optimal transport

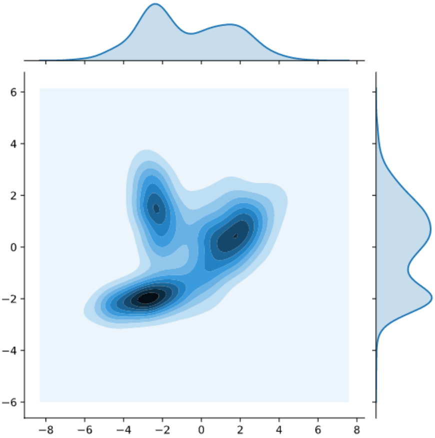


Figure: Example of a transport plan. Source: Wikipedia

Superreplication: primal problem

Fundamental principle: Upper bound for the price of payoff $f(S_1, V, S_2)$ = smallest price at time 0 of a superreplicating portfolio.

Following De Marco-Henry-Labordère (2015), G.-Menegaux-Nutz (2017), the available instruments for superreplication are:

- At time 0:
  - $u_1(S_1)$: SPX vanilla payoff maturity T_1 (including cash)
  - $u_2(S_2)$: SPX vanilla payoff maturity T_2
  - **$u_V(V)$: VIX vanilla payoff maturity T_1**

  Cost: $\text{MktPrice}[u_1(S_1)] + \text{MktPrice}[u_2(S_2)] + \text{MktPrice}[u_V(V)]$
  $= \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^2[u_2(S_2)] + \mathbb{E}^V[u_V(V)]$
- At time T_1:
  - $\Delta_S(S_1, V)(S_2 - S_1)$: delta hedge
  - $\Delta_L(S_1, V)(L(S_2/S_1) - V^2)$: buy $\Delta_L(S_1, V)$ log-contracts

  Cost: 0

Shorthand notation:

$$\Delta^{(S)}(s_1, v, s_2) := \Delta(s_1, v)(s_2 - s_1), \quad \Delta^{(L)}(s_1, v, s_2) := \Delta(s_1, v)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right)$$

Superreplication: primal problem

- The model-independent no-arbitrage upper bound for the derivative with payoff $f(S_1, V, S_2)$ is the smallest price at time 0 of a superreplicating portfolio:

$$P_f := \inf_{\mathcal{U}_f} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \right\}.$$

- $\mathcal{U}_f$: set of superreplicating portfolios, i.e., the set of all functions $(u_1, u_V, u_2, \Delta_S, \Delta_L)$ that satisfy the superreplication constraint:

$$u_1(s_1) + u_V(v) + u_2(s_2) + \Delta_S^{(S)}(s_1, v, s_2) + \Delta_L^{(L)}(s_1, v, s_2) \geq f(s_1, v, s_2).$$

- Linear program.

Superreplication: dual problem

- $\mathcal{P}(\mu_1, \mu_V, \mu_2)$: set of all the probability measures μ on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ such that

$$S_1 \sim \mu_1, \quad V \sim \mu_V, \quad S_2 \sim \mu_2, \quad \mathbb{E}^\mu [S_2 | S_1, V] = S_1, \quad \mathbb{E}^\mu \left[L\left(\frac{S_2}{S_1}\right) \middle| S_1, V \right] = V^2.$$

- Dual problem:

$$D_f := \sup_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} \mathbb{E}^\mu [f(S_1, V, S_2)].$$

- **Dispersion-constrained martingale optimal transport problem**.
- $\mathbb{E}^\mu[S_2|S_1, V] = S_1$: martingality condition of the SPX index, condition on the average of the distribution of S_2 given S_1 and V.
- $\mathbb{E}^\mu[L(S_2/S_1)|S_1, V] = V^2$: consistency condition, condition on dispersion around the average.

Superreplication: absence of a duality gap

Theorem

Let $f : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}$ *be upper semicontinuous and satisfy*

$$|f(s_1, v, s_2)| \leq C\big(1 + s_1 + s_2 + |L(s_1)| + |L(s_2)| + v^2\big)$$

for some constant $C > 0$. *Then*

$$P_f := \inf_{\mathcal{U}_f} \left\{ \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \right\}$$

$$= \sup_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} \mathbb{E}^\mu[f(S_1, V, S_2)] =: D_f.$$

Moreover, $D_f \neq -\infty$ *if and only if* $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$, *and in that case the supremum is attained.*

Superreplication of forward-starting options

- The knowledge of μ_1 and μ_2 gives little information on the prices $\mathbb{E}^{\mu}[g(S_1, S_2)]$, e.g., prices of forward-starting options $\mathbb{E}^{\mu}[f(S_2/S_1)]$.
- Computing upper and lower bounds of these prices:
  Optimal transport (Monge, 1781; Kantorovich)
- Adding the no-arbitrage constraint that (S_1, S_2) is a martingale leads to more precise bounds, as this provides information on the conditional average of S_2/S_1 given S_1:
  Martingale optimal transport (Henry-Labordère, 2017)
- When $S = \text{SPX}$: Adding VIX market data information produces even more precise bounds, as it information on the conditional dispersion of S_2/S_1, which is controlled by the VIX V:
  Dispersion-constrained martingale optimal transport
- **Adding VIX market data may possibly reveal a joint SPX/VIX arbitrage. Corresponds to $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \emptyset$** (see next slides).
- In the limiting case where $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \{\mu_0\}$ is a singleton, the joint SPX/VIX market data information completely specifies the joint distribution of (S_1, S_2), hence the price of forward starting options.

Joint SPX/VIX arbitrage

Joint SPX/VIX arbitrage

- $\mathcal{U}_0$ = the portfolios $(u_1, u_2, u_V, \Delta^S, \Delta^L)$ superreplicating 0:

$$u_1(s_1)+u_2(s_2)+u_V(v)+\Delta^S(s_1,v)(s_2-s_1)+\Delta^L(s_1,v)\left(L\left(\frac{s_2}{s_1}\right)-v^2\right) \geq 0$$

- An (S_1, S_2, V)-arbitrage is an element of $\mathcal{U}_0$ with negative price:

$$\text{MktPrice}[u_1(S_1)] + \text{MktPrice}[u_2(S_2)] + \text{MktPrice}[u_V(V)] < 0$$

- Equivalently, there is an (S_1, S_2, V)-arbitrage if and only if

$$\inf_{\mathcal{U}_0} \{\text{MktPrice}[u_1(S_1)] + \text{MktPrice}[u_2(S_2)] + \text{MktPrice}[u_V(V)]\} = -\infty$$

Consistent extrapolation of SPX and VIX smiles

- If $\mathbb{E}^V[V^2] \neq \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]$, there is a trivial (S_1, S_2, V)-arbitrage. For instance, if $\mathbb{E}^V[V^2] < \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]$, pick

$$u_1(s_1) = L(s_1), \quad u_2(s_2) = -L(s_2), \quad u_V(v) = v^2, \quad \Delta_S(s_1, v) = 0, \quad \Delta_L(s_1, v) = 1.$$

- $\Longrightarrow$ We assume that

$$\mathbb{E}^V[V^2] = \mathbb{E}^2[L(S_2)] - \mathbb{E}^1[L(S_1)]. \tag{3.1}$$

- Violations of (3.1) in the market have been reported, suggesting arbitrage opportunities, see, e.g., Section 7.7.4 in Bergomi (2016).
- However, the quantities in (3.1) do not purely depend on market data. They depend on smile extrapolations.
- The reported violations of (3.1) actually rely on some arbitrary smile extrapolations.
- G. (2018) explains how to build **consistent extrapolations of the VIX and SPX smiles** so that (3.1) holds.

Joint SPX/VIX arbitrage

Theorem (G., 2018)

The following assertions are equivalent:

(i) *The market is free of* (S_1, S_2, V)*-arbitrage,*

(ii) $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$,

(iii) *There exists a coupling* ν *of* μ_1 *and* μ_V *such that* $\mathrm{Law}_\nu(S_1, L(S_1) + V^2)$ *and* $\mathrm{Law}_{\mu_2}(S_2, L(S_2))$ *are in convex order, i.e., for any convex function* $f : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}^\nu[f(S_1, L(S_1) + V^2)] \leq \mathbb{E}^2[f(S_2, L(S_2))].$$

Joint SPX/VIX arbitrage

(i) The market is free of (S_1, S_2, V)-arbitrage,

(ii) $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$,

(iii) There exists a coupling ν of μ_1 and μ_V such that $\mathrm{Law}_\nu(S_1, L(S_1) + V^2)$ and $\mathrm{Law}_{\mu_2}(S_2, L(S_2))$ are in convex order.

- Directly solving the linear problem associated to (i) is not easy as one needs to try all possible $(u_1, u_V, u_2, \Delta_S, \Delta_V)$ and check the superreplication constraints for all $s_1, s_2 > 0$ and $v \geq 0$.
- Checking (iii) numerically is difficult as, in dimension two, the extreme rays of the convex cone of convex functions are dense in the cone (Johansen 1974), contrary to the case of dimension one where the extreme rays are the call and put payoffs (Blaschke-Pick 1916).
- Instead, we verify absence of (S_1, S_2, V)-arbitrage by building – numerically, but with high accuracy – an element of $\mathcal{P}(\mu_1, \mu_V, \mu_2)$, thus checking (ii).

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

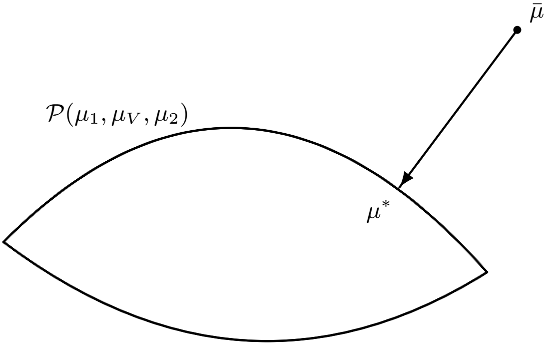
Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- We build a model $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$.
- **We thus solve a longstanding puzzle in derivatives modeling: build an arbitrage-free model that jointly calibrates to the prices of SPX options, VIX futures and VIX options.**
- Our strategy is inspired by the recent work of De March and Henry-Labordère (2019).
- We assume that $\mathcal{P}(\mu_1, \mu_V, \mu_2) \neq \emptyset$ and try to build an element μ in this set. To this end, we fix a **reference probability measure $\bar{\mu}$** on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ and look for the measure $\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$ that **minimizes the relative entropy $H(\mu, \bar{\mu})$** of μ w.r.t. $\bar{\mu}$, also known as the Kullback-Leibler divergence:

$$D_{\bar{\mu}} := \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}), \quad H(\mu, \bar{\mu}) := \begin{cases} \mathbb{E}^{\mu}\left[\ln \frac{d\mu}{d\bar{\mu}}\right] = \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu}{d\bar{\mu}} \ln \frac{d\mu}{d\bar{\mu}}\right] & \text{if } \mu \ll \bar{\mu}, \\ +\infty & \text{otherwise.} \end{cases}$$

- This is a **strictly convex problem that can be solved after dualization using Sinkhorn's fixed point iteration** (Sinkhorn, 1967).

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$



Reminder on Lagrange multipliers

$$\begin{aligned}\inf_{g(x,y)=c} f(x,y) &= \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y)-\lambda(g(x,y)-c)\} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y)-\lambda(g(x,y)-c)\}\end{aligned}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
- Constraint $g(x,y) = c \iff \frac{\partial}{\partial\lambda}\{f(x,y)-\lambda(g(x,y)-c)\} = 0$.

$$\begin{aligned}\inf_{\mu \text{ s.t. } S_1\sim\mu_1} H(\mu,\bar{\mu}) &= \inf_{\mu} \sup_{u_1(\cdot)} \left\{H(\mu,\bar{\mu}) + \mathbb{E}^1[u_1(S_1)] - \mathbb{E}^{\mu}\left[u_1(S_1)\right]\right\} \\ \inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}[S_2|S_1,V]=S_1} H(\mu,\bar{\mu}) &= \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{H(\mu,\bar{\mu}) - \mathbb{E}^{\mu}\left[\Delta_S(S_1,V)(S_2-S_1)\right]\right\}\end{aligned}$$

Reminder on Lagrange multipliers

$$\begin{aligned}\inf_{g(x,y)=c} f(x,y) &= \inf_{x,y} \sup_{\lambda\in\mathbb{R}} \{f(x,y) - \lambda(g(x,y)-c)\} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{x,y} \{f(x,y) - \lambda(g(x,y)-c)\}\end{aligned}$$

- To compute the inner inf over x, y unconstrained, simply solve $\nabla f(x,y) = \lambda \nabla g(x,y)$: easy!
- Then maximize the result over λ unconstrained: easy!
- Constraint $g(x,y) = c \iff \frac{\partial}{\partial\lambda}\{f(x,y) - \lambda(g(x,y)-c)\} = 0$.

$$\inf_{\mu \text{ s.t. } S_1\sim\mu_1} H(\mu,\bar{\mu}) = \inf_{\mu} \sup_{u_1(\cdot)} \left\{H(\mu,\bar{\mu}) + \mathbb{E}^1[u_1(S_1)] - \mathbb{E}^{\mu}\left[u_1(S_1)\right]\right\}$$

$$\inf_{\mu \text{ s.t. } \mathbb{E}^{\mu}[S_2|S_1,V]=S_1} H(\mu,\bar{\mu}) = \inf_{\mu} \sup_{\Delta_S(\cdot,\cdot)} \left\{H(\mu,\bar{\mu}) - \mathbb{E}^{\mu}\left[\Delta_S(S_1,V)(S_2-S_1)\right]\right\}$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$\begin{aligned}
D_{\bar{\mu}} \quad := & \quad \inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu, \bar{\mu}) \\
= & \quad \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
& \quad -\mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
= & \quad \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
& \quad -\mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar\mu} &:= \inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu,\bar\mu) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu,\bar\mu) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad -\mathbb{E}^\mu\Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1,V,S_2) + \Delta_L^{(L)}(S_1,V,S_2)\Big]\Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu,\bar\mu) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad -\mathbb{E}^\mu\Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1,V,S_2) + \Delta_L^{(L)}(S_1,V,S_2)\Big]\Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1, \mu_V, \mu_2)} H(\mu, \bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

- $\mathcal{M}_1$: set of probability measures on $\mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$: **unconstrained**
- $\mathcal{U}$: set of portfolios $u = (u_1, u_V, u_2, \Delta_S, \Delta_L)$: **Lagrange multipliers**

$$
\begin{aligned}
D_{\bar{\mu}} &:= \inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu, \bar{\mu}) \\
&= \inf_{\mu \in \mathcal{M}_1} \sup_{u \in \mathcal{U}} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\} \\
&= \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\
&\quad - \mathbb{E}^\mu \Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2) \Big] \Big\}
\end{aligned}
$$

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

$$D_{\bar{\mu}} = \sup_{u \in \mathcal{U}} \inf_{\mu \in \mathcal{M}_1} \Big\{ H(\mu, \bar{\mu}) + \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] - \mathbb{E}^{\mu}\Big[u_1(S_1) + u_V(V) + u_2(S_2) + \Delta_S^{(S)}(S_1, V, S_2) + \Delta_L^{(L)}(S_1, V, S_2)\Big]\Big\}$$

- **Remarkable fact: The inner infimum can be exactly computed:**

$$\inf_{\mu \in \mathcal{M}_1} \{H(\mu, \bar{\mu}) - \mathbb{E}^{\mu}[X]\} = -\ln \mathbb{E}^{\bar{\mu}}\left[e^X\right]$$

and **the infimum is attained at $\mu = \bar{\mu}_X$ defined by** (Gibbs type)

$$\frac{d\bar{\mu}_X}{d\bar{\mu}} = \frac{e^X}{\mathbb{E}^{\bar{\mu}}[e^X]}.$$

- That is why we like (and chose) the “distance” $H(\mu, \bar{\mu})$!

Build a model in $\mathcal{P}(\mu_1, \mu_V, \mu_2)$

$$D_{\bar\mu} := \inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu, \bar\mu) \;=\; \sup_{u \in \mathcal{U}} \Psi_{\bar\mu}(u) =: P_{\bar\mu}$$

$$\Psi_{\bar\mu}(u) := \mathbb{E}^1[u_1(S_1)] + \mathbb{E}^V[u_V(V)] + \mathbb{E}^2[u_2(S_2)] \\ - \ln \mathbb{E}^{\bar\mu}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S^{(S)}(S_1,V,S_2)+\Delta_L^{(L)}(S_1,V,S_2)}\right].$$

- $\inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)}$: **constrained** optimization, **difficult**.
- $\sup_{u \in \mathcal{U}}$: **unconstrained** optimization, **easy!** To find the optimum $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$, simply cancel the gradient of $\Psi_{\bar\mu}$.
- Most important, $\inf_{\mu \in \mathcal{P}(\mu_1,\mu_V,\mu_2)} H(\mu, \bar\mu)$ is reached at

$$\mu^*(ds_1, dv, ds_2) = \bar\mu(ds_1, dv, ds_2) \frac{e^{u_1^*(s_1)+u_V^*(v)+u_2^*(s_2)+\Delta_S^{*(S)}(s_1,v,s_2)+\Delta_L^{*(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar\mu}\left[e^{u_1^*(S_1)+u_V^*(V)+u_2^*(S_2)+\Delta_S^{*(S)}(S_1,V,S_2)+\Delta_L^{*(L)}(S_1,V,S_2)}\right]}.$$

- **Problem solved: $\mu^* \in \mathcal{P}(\mu_1, \mu_V, \mu_2)$!**

Equations for $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$

$$
\begin{array}{lrcl}
\frac{\partial \Psi_{\bar\mu}}{\partial u_1(s_1)}=0: \quad \forall s_1 > 0, & u_1(s_1) & = & \Phi_1(s_1; u_V, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_V(v)}=0: \quad \forall v \geq 0, & u_V(v) & = & \Phi_V(v; u_1, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_2(s_2)}=0: \quad \forall s_2 > 0, & u_2(s_2) & = & \Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_S(s_1,v)}=0: \quad \forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_S}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v)) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_L(s_1,v)}=0: \quad \forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_L}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v))
\end{array}
$$

- We could have simply postulated a model of the form

$$
\mu(ds_1, dv, ds_2) = \bar\mu(ds_1, dv, ds_2) \frac{e^{u_1(s_1)+u_V(v)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}}{\mathbb{E}^{\bar\mu}\left[e^{u_1(S_1)+u_V(V)+u_2(S_2)+\Delta_S^{(S)}(S_1,V,S_2)+\Delta_L^{(L)}(S_1,V,S_2)}\right]}.
$$

- Then the 5 conditions defining $\mathcal{P}(\mu_1, \mu_V, \mu_2)$ translate into the 5 above equations.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$
\begin{array}{rcl}
\forall s_1 > 0, \quad u_1^{(n+1)}(s_1) & = & \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad u_V^{(n+1)}(v) & = & \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \quad u_2^{(n+1)}(s_2) & = & \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0, \ \forall v \geq 0, \quad 0 & = & \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0, \ \forall v \geq 0, \quad 0 & = & \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{array}
$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer** u^*.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$
\begin{array}{rcl}
\forall s_1 > 0, \quad u_1^{(n+1)}(s_1) & = & \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad u_V^{(n+1)}(v) & = & \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \quad u_2^{(n+1)}(s_2) & = & \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0, \ \forall v \geq 0, \quad 0 & = & \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0, \ \forall v \geq 0, \quad 0 & = & \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{array}
$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$
\begin{aligned}
\forall s_1 > 0, \qquad u_1^{(n+1)}(s_1) &= \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \qquad u_V^{(n+1)}(v) &= \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \qquad u_2^{(n+1)}(s_2) &= \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, \qquad 0 &= \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, \qquad 0 &= \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{aligned}
$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$\begin{aligned}
\forall s_1 > 0, \quad u_1^{(n+1)}(s_1) &= \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad u_V^{(n+1)}(v) &= \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \quad u_2^{(n+1)}(s_2) &= \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad 0 &= \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad 0 &= \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{aligned}$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$\begin{aligned}
\forall s_1 > 0, \quad u_1^{(n+1)}(s_1) &= \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad u_V^{(n+1)}(v) &= \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \quad u_2^{(n+1)}(s_2) &= \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad 0 &= \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad 0 &= \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{aligned}$$

  until convergence.
- Each of the above 5 lines corresponds to a Bregman projection in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$\begin{array}{rrcl}
\forall s_1 > 0, & u_1^{(n+1)}(s_1) & = & \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, & u_V^{(n+1)}(v) & = & \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, & u_2^{(n+1)}(s_2) & = & \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{array}$$

  until convergence.
- Each of the above 5 lines corresponds to a **Bregman projection** in the space of measures.

Sinkhorn's algorithm

- Sinkhorn's algorithm (1967) was first used in the context of optimal transport by Cuturi (2013).
- In our context: **fixed point method that iterates computions of one-dimensional gradients to approximate the optimizer u^***.
- Start from initial guess $u^{(0)} = (u_1^{(0)}, u_V^{(0)}, u_2^{(0)}, \Delta_S^{(0)}, \Delta_L^{(0)})$, recursively define $u^{(n+1)}$ knowing $u^{(n)}$ by

$$
\begin{aligned}
\forall s_1 > 0, \quad & u_1^{(n+1)}(s_1) = \Phi_1(s_1; u_V^{(n)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall v \geq 0, \quad & u_V^{(n+1)}(v) = \Phi_V(v; u_1^{(n+1)}, u_2^{(n)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_2 > 0, \quad & u_2^{(n+1)}(s_2) = \Phi_2(s_2; u_1^{(n+1)}, u_V^{(n+1)}, \Delta_S^{(n)}, \Delta_L^{(n)}) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad & 0 = \Phi_{\Delta_S}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n)}(s_1, v)) \\
\forall s_1 > 0,\ \forall v \geq 0, \quad & 0 = \Phi_{\Delta_L}(s_1, v; u_2^{(n+1)}, \Delta_S^{(n+1)}(s_1, v), \Delta_L^{(n+1)}(s_1, v))
\end{aligned}
$$

- Each of the above five lines corresponds to a **Bregman projection** in the space of measures.
- **If the algorithm diverges**, then $P_{\bar{\mu}} = +\infty$, so $D_{\bar{\mu}} = +\infty$, i.e., $\mathcal{P}(\mu_1, \mu_V, \mu_2) = \emptyset$: **there exists a joint SPX/VIX arbitrage**.

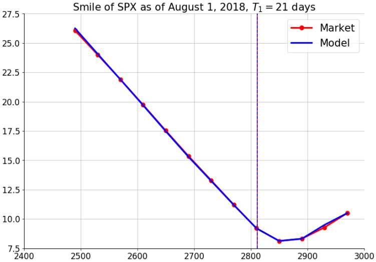
Numerical experiments

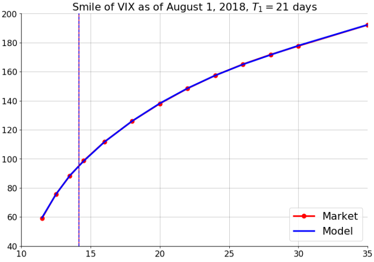
Implementation details

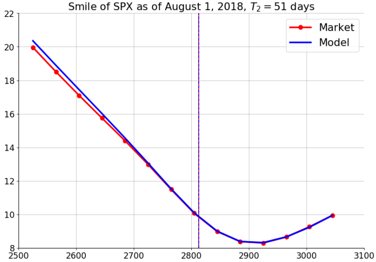
- Choice of $\bar{\mu}$:
  - $S_1 \sim \mu_1$ and $V \sim \mu_V$ independent;
  - Conditional on (S_1, V), S_2 lognormal with mean S_1 and variance V.

  Under $\bar{\mu}$, $S_2 \not\sim \mu_2$.
- Instead of abstract payoffs u_1, u_V, u_2, we work with market strikes and market prices of vanilla options on S_1, V, and S_2.
- Initial guess of the Sinkhorn algorithm: zero.
- Integrals in the expressions of $\Phi_1, \Phi_V, \Phi_2, \Phi_{\Delta_S}, \Phi_{\Delta_L}$ estimated using Gaussian quadrature.
- Enough accuracy is typically reached after ≈ 100 iterations.

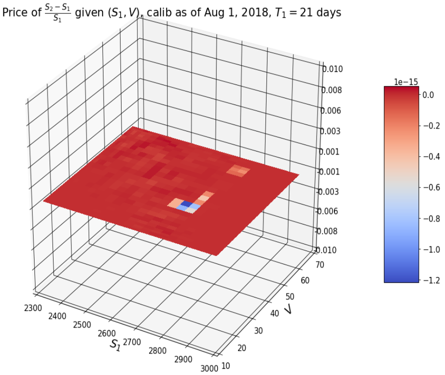
August 1, 2018, $T_1 = 21$ days

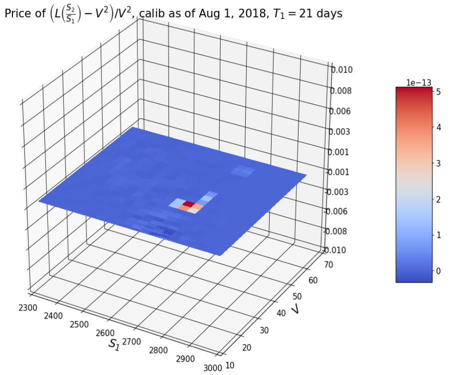






August 1, 2018, $T_1 = 21$ days





August 1, 2018, $T_1 = 21$ days

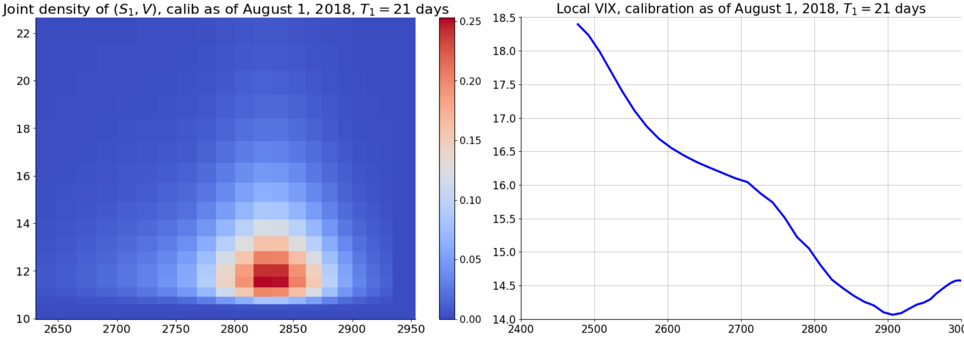
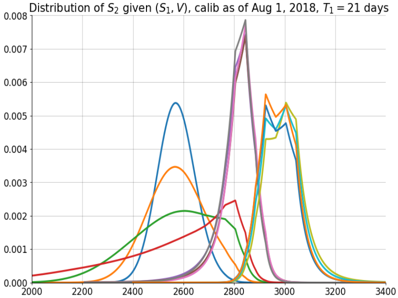


Figure: Joint distribution of (S_1, V) and local VIX function $\text{VIX}_{\text{loc}}(s_1)$

$$\text{VIX}^2_{\text{loc}}(S_1) := \mathbb{E}^{\mu^*}\left[V^2 \middle| S_1\right]$$

August 1, 2018, $T_1 = 21$ days



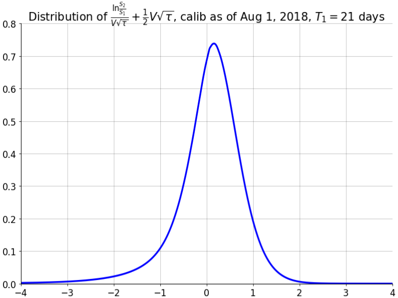


Figure: Conditional distribution of S_2 given (s_1, v) under μ^* for different vales of (s_1, v): $s_1 \in \{2571, 2808, 3000\}$, $v \in \{10.10, 15.30, 23.20, 35.72\}\%$, and distribution of the normalized return $R := \frac{\ln(S_2/S_1)}{V\sqrt{\tau}} + \frac{1}{2}V\sqrt{\tau}$

August 1, 2018, $T_1 = 21$ days

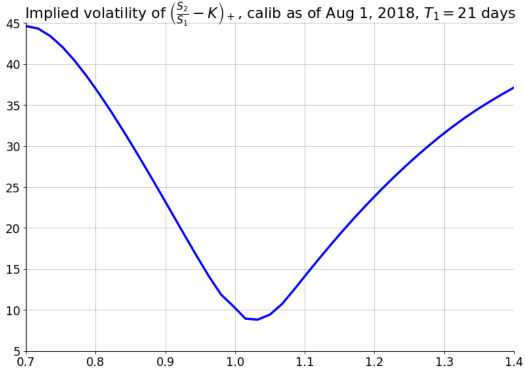
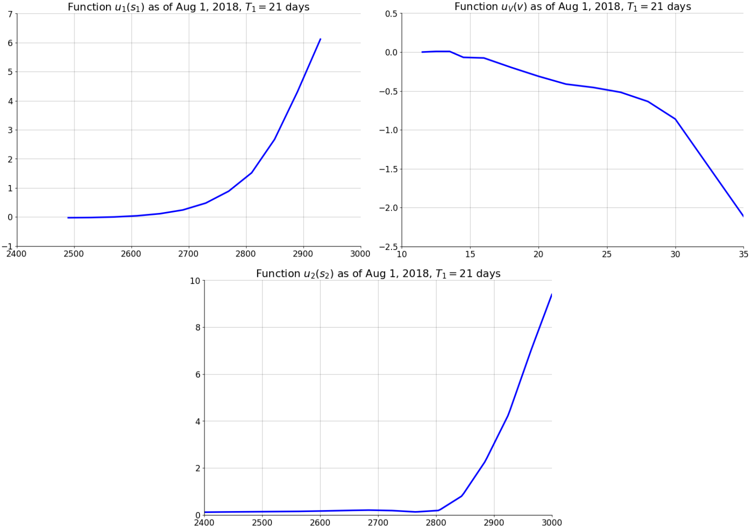


Figure: Smile of forward starting call options $(S_2/S_1 - K)_+$

August 1, 2018, $T_1 = 21$ days



August 1, 2018, $T_1 = 21$ days

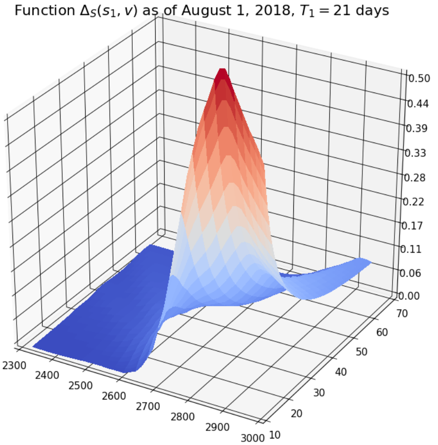
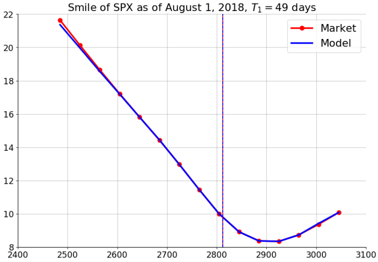
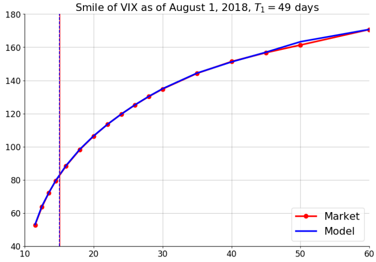
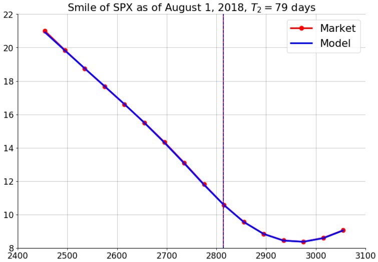


Figure: Optimal functions $\Delta_S^*(s_1, v)$ and $\Delta_L^*(s_1, v)$ for (s_1, v) in the quadrature grid

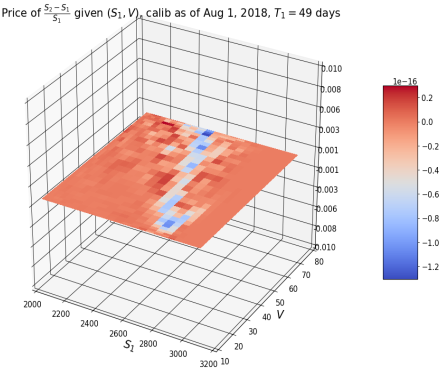
August 1, 2018, $T_1 = 49$ days

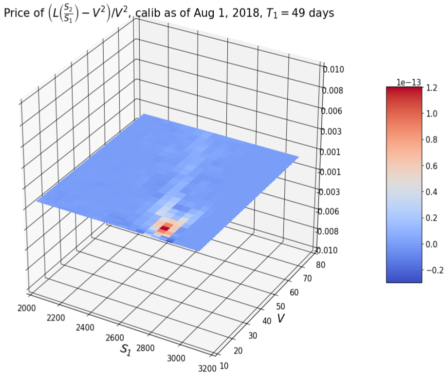






August 1, 2018, $T_1 = 49$ days





August 1, 2018, $T_1 = 49$ days

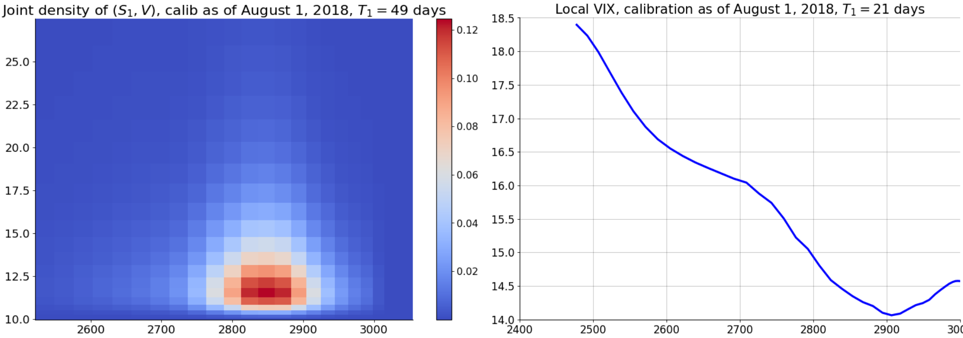


Figure: Joint distribution of (S_1, V) and local VIX function $\mathrm{VIX}_{\mathrm{loc}}(s_1)$

August 1, 2018, $T_1 = 49$ days

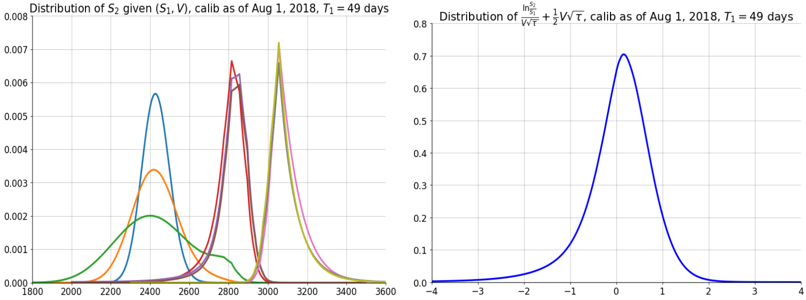


Figure: Conditional distribution of S_2 given (s_1, v) under $\mu^*_{\mathcal{K}}$ for different vales of (s_1, v): $s_1 \in \{2571, 2808, 3000\}$, $v \in \{10.10, 15.30, 23.20, 35.72\}\%$, and distribution of the normalized return $R := \frac{\ln(S_2/S_1)}{V\sqrt{\tau}} + \frac{1}{2}V\sqrt{\tau}$

August 1, 2018, $T_1 = 49$ days

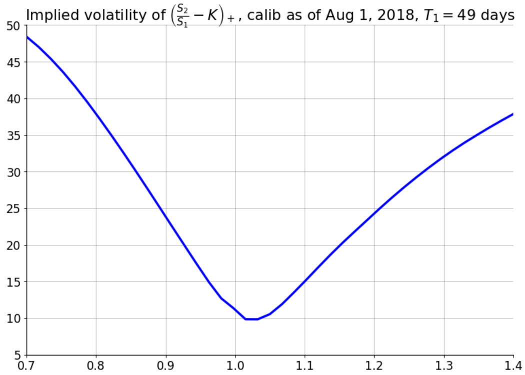
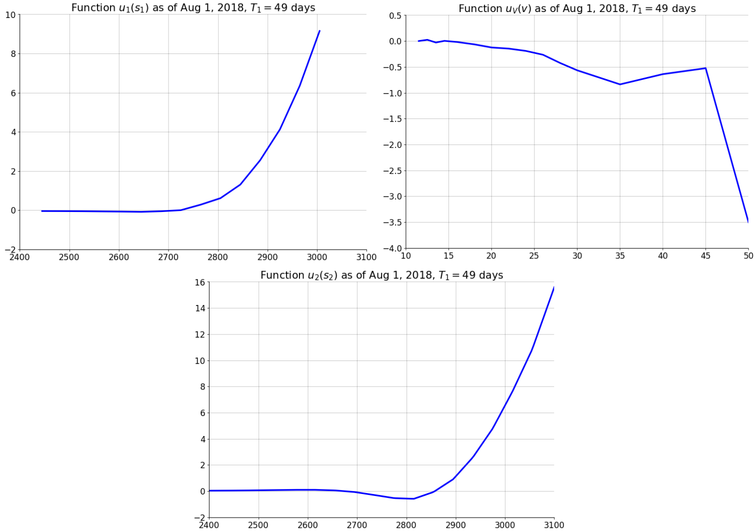


Figure: Smile of forward starting call options $(S_2/S_1 - K)_+$

August 1, 2018, $T_1 = 49$ days



August 1, 2018, $T_1 = 49$ days

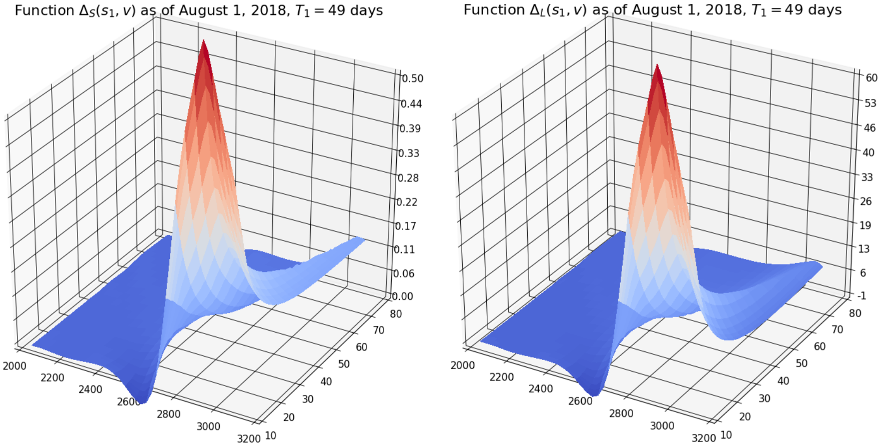
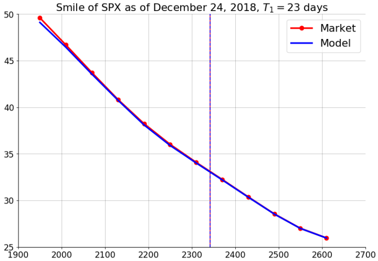
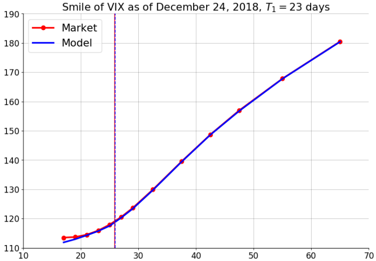
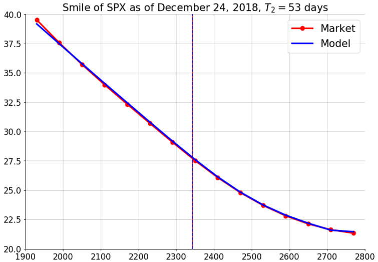


Figure: Optimal functions $\Delta_S^*(s_1, v)$ and $\Delta_L^*(s_1, v)$ for (s_1, v) in the quadrature grid

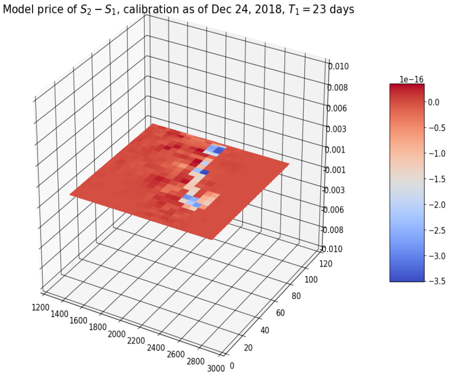
December 24, 2018, $T_1 = 23$ days: large VIX, $F_V \approx 26\%$

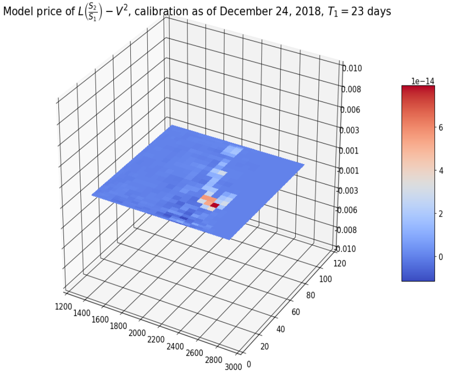




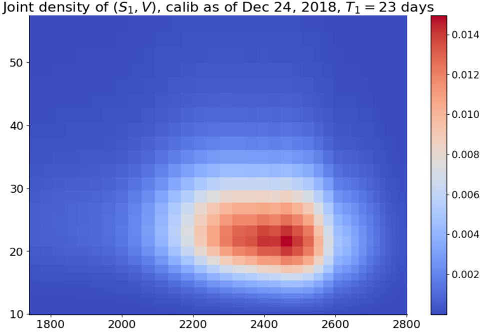


December 24, 2018, $T_1 = 23$ days





December 24, 2018, $T_1 = 23$ days



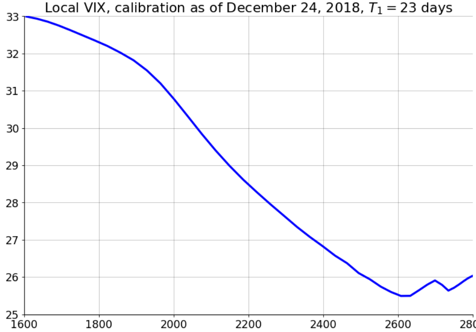


Figure: Joint distribution of (S_1, V) and local VIX function $\mathrm{VIX}_{\mathrm{loc}}(s_1)$

December 24, 2018, $T_1 = 23$ days

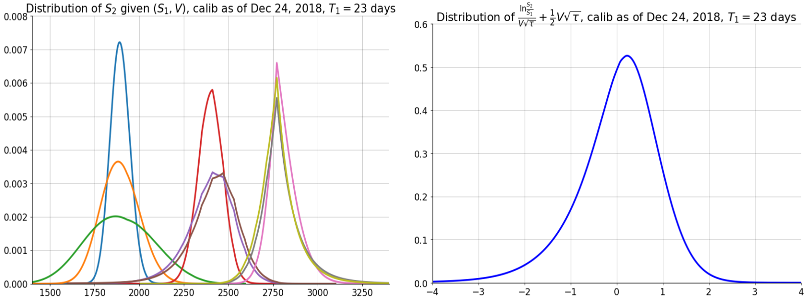


Figure: Conditional distribution of S_2 given (s_1, v) under μ^* for different vales of (s_1, v): $s_1 \in \{2571, 2808, 3000\}$, $v \in \{10.10, 15.30, 23.20, 35.72\}\%$, and distribution of the normalized return $R := \frac{\ln(S_2/S_1)}{V\sqrt{\tau}} + \frac{1}{2}V\sqrt{\tau}$

December 24, 2018, $T_1 = 23$ days

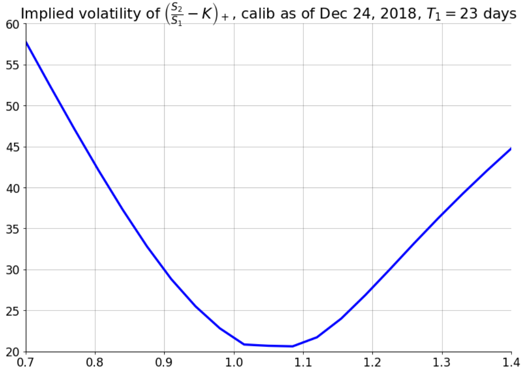
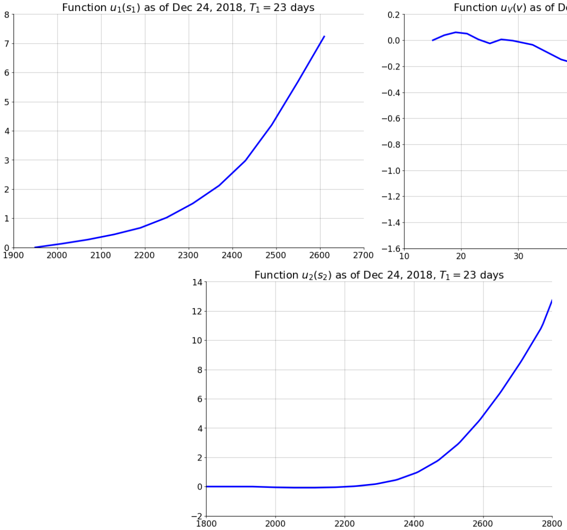
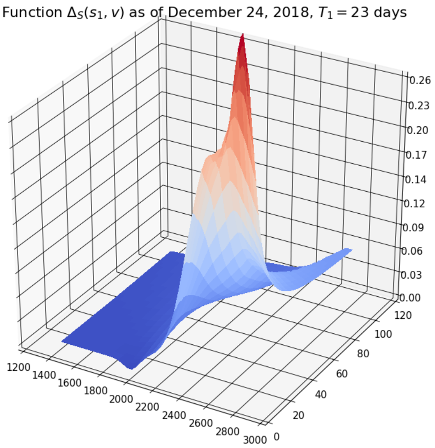


Figure: Smile of forward starting call options $(S_2/S_1 - K)_+$

December 24, 2018, $T_1 = 23$ days



December 24, 2018, $T_1 = 23$ days



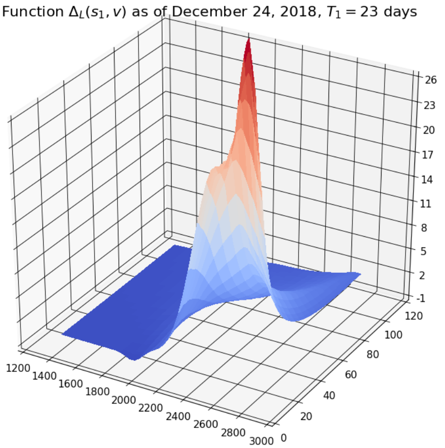


Figure: Optimal functions $\Delta_S^*(s_1, v)$ and $\Delta_L^*(s_1, v)$ for (s_1, v) in the quadrature grid

Extension to the multi-maturity case

Extension to the multi-maturity case

S_i, V_i $\qquad\qquad\qquad\qquad$ S_{i+1}

T_i $\qquad$ $\nu_i = \mathrm{Law}(S_i, V_i, S_{i+1})$ $\qquad$ T_{i+1}

- Assume for simplicity that monthly SPX options and VIX futures maturities T_i perfectly coincide: $T_{i+1} - T_i = \tau$ for all $i \geq 1$.
- For each i we build a jointly calibrating model ν_i using the Sinkhorn algorithm.
- **Glue the models together**: Build a calibrated model on $(S_i, V_i)_{i \geq 1}$ as follows: $(S_1, V_1, S_2) \sim \nu_1$; recursively define the distribution of (V_{i+1}, S_{i+2}) given $(S_1, V_1, S_2, V_2, \ldots, S_i, V_i, S_{i+1})$ as the conditional distribution of (V_{i+1}, S_{i+2}) given S_{i+1} under ν_{i+1}.
- The resulting model ν is arbitrage-free, consistent, and calibrated to all the SPX and VIX monthly market smiles μ_{S_i} and μ_{V_i}: for all $i \geq 1$,

$$S_i \sim \mu_{S_i}, \quad V_i \sim \mu_{V_i}, \quad \mathbb{E}^\nu\left[S_{i+1} \middle| (S_j, V_j)_{1 \leq j \leq i}\right] = S_i, \quad \mathbb{E}^\nu\left[L\left(\frac{S_{i+1}}{S_i}\right) \middle| (S_j, V_j)_{1 \leq j \leq i}\right] = V_i^2.$$

Continuous time

Continuous time

- Same technique: Pick a reference measure $\mathbb{P}_0 \longleftrightarrow$ a particular SV model:

$$\begin{aligned} \frac{dS_t}{S_t} &= a_t\, dW_t^0 \\ da_t &= b(a_t)\, dt + \sigma(a_t) \left(\rho\, dW_t^0 + \sqrt{1-\rho^2} dW_t^{0,\perp}\right) \end{aligned}$$

- We want to prove that $\mathcal{P} \neq \emptyset$ and build $\mathbb{P} \in \mathcal{P}$, where

$$\mathcal{P} := \{\mathbb{P} \ll \mathbb{P}_0 | S_1 \sim \mu_1, S_2 \sim \mu_2, \sqrt{\mathbb{E}^{\mathbb{P}}[L(S_2/S_1)|\mathcal{F}_1]} \sim \mu_V, S \text{ is a } \mathbb{P}\text{-martingale}\}.$$

- We look for $\mathbb{P} \in \mathcal{P}$ that minimizes the relative entropy w.r.t. $\mathbb{P}_0$:

$$D := \inf_{\mathbb{P} \in \mathcal{P}} H(\mathbb{P}, \mathbb{P}_0)$$

- Inspired by Henry-Labordère (2019): *From (Martingale) Schrödinger Bridges to a New Class of Stochastic Volatility Models*.

Continuous time

$$\begin{aligned} \frac{dS_t}{S_t} &= a_t\, dW_t^* \\ da_t &= \left(b(a_t) + (1-\rho^2)\sigma(a_t)^2 \partial_a u^*(t, S_t, a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2} dW_t^{*,\perp}\right) \end{aligned}$$

- Let $P := \sup_{u_1, u_V, u_2} \left\{ \sum_{i \in \{1,2,V\}} (\mu_i, u_i) - u(0, S_0, a_0) \right\}$ where u is solution to a nonlinear Hamilton-Jacobi-Bellman PDE:

$$\begin{aligned} u(T_2, s, a; \delta^L) &= u_2(s) + \delta^L L(s), \\ \partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in (T_1, T_2), \\ \Phi(s,a) = \sup_{v \geq 0} \inf_{\delta^L \in \mathbb{R}} \Big\{ u_V(v) &- \delta^L (L(s) + v^2) + u(T_1, s, a; \delta^L) \Big\}, \\ u(T_1, s, a) &= u_1(s) + \Phi(s,a), \\ \partial_t u + \mathcal{L}^0 u + \frac{1}{2}(1-\rho^2)\sigma(a)^2(\partial_a u)^2 &= 0, \quad t \in [0, T_1). \end{aligned}$$

- Assume $P < +\infty$ and (u_1^*, u_V^*, u_2^*) maximizes $P \longrightarrow u^*$

Continuous time

$$\frac{dS_t}{S_t} = a_t\, dW_t^*$$
$$da_t = \left(b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a u^*(t,S_t,a_t)\right) dt + \sigma(a_t)\left(\rho\, dW_t^* + \sqrt{1-\rho^2}dW_t^{*,\perp}\right)$$

- Optimal delta:

$$\Delta_t^* = -\partial_s u^*(t,S_t,a_t) - \rho\frac{\sigma(a_t)}{a_t S_t}\partial_a u^*(t,S_t,a_t).$$

- The probability $\mathbb{P}^* \in \mathcal{P}$ that minimizes $H(\mathbb{P},\mathbb{P}_0)$ satisfies

$$\frac{d\mathbb{P}^*}{d\mathbb{P}_0} = Ze^{u_1^*(S_1)+u_2^*(S_2)+u_V^*(V^*)+\int_0^{T_2}\Delta_t^* dS_t+\Delta^{*,L}\left(L\left(\frac{S_2}{S_1}\right)-(V^*)^2\right)}.$$

- The drift of (a_t) under $\mathbb{P}^*$ also reads as

$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_1^*(S_1)+\int_t^{T_1}\Delta^*(r,S_r,a_r)dS_r+\Phi^*(S_1,a_1)}|S_t,a_t], \quad t\in[0,T_1],$$
$$b(a_t)+(1-\rho^2)\sigma(a_t)^2\partial_a \ln \mathbb{E}^0[e^{u_2^*(S_2)+\int_t^{T_2}\Delta^*(r,S_r,a_r)dS_r+\Delta^{*,L}L(S_2)}|S_t,a_t], \quad t\in[T_1,T_2].$$

- If $P=+\infty$, then $\mathcal{P}=\emptyset$.

Inversion of convex ordering in the VIX market

Inversion of Convex Ordering in the VIX Market (G., 2019, SSRN)

INVERSION OF CONVEX ORDERING IN THE VIX MARKET

JULIEN GUYON
QUANTITATIVE RESEARCH, BLOOMBERG L.P.

ABSTRACT. We investigate conditions for the existence of a *continuous* model on the S&P 500 index (SPX) that jointly calibrates to a *full surface* of SPX implied volatilities and to the VIX smiles. We present a novel approach based on the SPX smile calibration condition $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\text{lv}}^2(t, S_t)$. In the limiting case of instantaneous VIX, a novel application of martingale transport to finance shows that such model exists if and only if, for each time t, the local variance $\sigma_{\text{lv}}^2(t, S_t)$ is smaller than the instantaneous variance σ_t^2 in convex order. The real case of a 30 day VIX is more involved, as averaging over 30 days and projecting onto a filtration can undo convex ordering.

We show that in usual market conditions, and for reasonable smile extrapolations, the distribution of VIX_T^2 in the market local volatility model is *larger* than the market-implied distribution of VIX_T^2 in convex order for short maturities T, and that the two distributions are not rankable in convex order for intermediate maturities. In particular, a *necessary* condition for continuous models to jointly calibrate to the SPX and VIX markets is the *inversion of convex ordering* property: the fact that, even though associated local variances are smaller than instantaneous variances in convex order, the VIX squared is larger in convex order in the associated local volatility model than in the original model for short maturities. We argue and numerically demonstrate that, when the (typically negative) spot-vol correlation is large enough in absolute value, (a) traditional stochastic volatility models with large mean reversion, and (b) rough volatility models with small Hurst exponent, satisfy the inversion of convex ordering property, and more generally can reproduce the market term-structure of convex ordering of the local and stochastic squared VIX.

Keywords. VIX, convex order, inversion of convex ordering, martingale transport, local volatility, stochastic volatility, mean reversion, rough volatility, smile calibration.

Continuous model on SPX calibrated to SPX options

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t, \qquad S_0 = x. \tag{8.1}$$

- Corresponding local volatility function σ_{loc}: $\sigma^2_{\text{loc}}(t, S_t) := \mathbb{E}[\sigma_t^2 | S_t]$.
- Corresponding local volatility model:

$$\frac{dS_t^{\text{loc}}}{S_t^{\text{loc}}} = \sigma_{\text{loc}}(t, S_t^{\text{loc}}) \, dW_t, \qquad S_0^{\text{loc}} = x.$$

- From Gyöngy (1986): $\forall t \geq 0, \quad S_t^{\text{loc}} \stackrel{(d)}{=} S_t$.
- Using Dupire (1994), we conclude that Model (8.1) is calibrated to the full SPX smile if and only if $\sigma_{\text{loc}} = \sigma_{\text{lv}}$ (market local volatility computed using Dupire's formula).
- Market local volatility model:

$$\frac{dS_t^{\text{lv}}}{S_t^{\text{lv}}} = \sigma_{\text{lv}}(t, S_t^{\text{lv}}) \, dW_t, \qquad S_0^{\text{lv}} = x.$$

VIX

- By definition, the (idealized) VIX at time $T \geq 0$ is the implied volatility of a 30 day log-contract on the SPX index starting at T. For continuous models (8.1), this translates into

$$\mathrm{VIX}_T^2 = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2 \, dt \middle| \mathcal{F}_T\right] = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}\left[\sigma_t^2 \middle| \mathcal{F}_T\right] dt.$$

- Since $\mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|\mathcal{F}_T] = \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}]$, $\mathrm{VIX}_{\mathrm{loc},T}$ satisfies

$$\mathrm{VIX}_{\mathrm{loc},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}})|S_T^{\mathrm{loc}}] \, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{loc}}^2(t, S_t^{\mathrm{loc}}) \, dt \middle| S_T^{\mathrm{loc}}\right].$$

- Similarly,

$$\mathrm{VIX}_{\mathrm{lv},T}^2 = \frac{1}{\tau}\int_T^{T+\tau} \mathbb{E}[\sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}})|S_T^{\mathrm{lv}}] \, dt = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_{\mathrm{lv}}^2(t, S_t^{\mathrm{lv}}) \, dt \middle| S_T^{\mathrm{lv}}\right].$$

Reminder on convex order

- (The distributions of) two random variables X and Y are said to be in convex order if and only if, for any convex function f, $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$.
- Denoted by $X \leq_c Y$.
- Both distributions have same mean, but distribution of Y is more "spread" than that of X.
- **In financial terms: X and Y have the same forward value, but calls (puts) on Y are more expensive than calls (puts) on X** (dimension 1).

The case of instantaneous VIX: $\tau \to 0$

- Assume SV model is calibrated to the SPX smile: $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\mathrm{lv}}^2(t, S_t)$.
- As observed by Dupire (2005), by conditional Jensen, $\sigma_{\mathrm{lv}}^2(t, S_t) \leq_c \sigma_t^2$, i.e.,

$$\text{mkt local var}_t \leq_c \text{instVIX}_t^2.$$

- Conversely, if mkt local var$_t \leq_c$ instVIX$_t^2$, there exists a jointly calibrating SPX/instVIX model (G., 2017).
- $\Longrightarrow$ **Convex order condition is necessary and sufficient for instVIX**.
- Proof uses a **new type of application of martingale transport to finance**: martingality constraint applies to (mkt local var$_t$, instVIX$_t^2$) at a single date, instead of (S_1, S_2).

The real VIX: $\tau = 30$ days

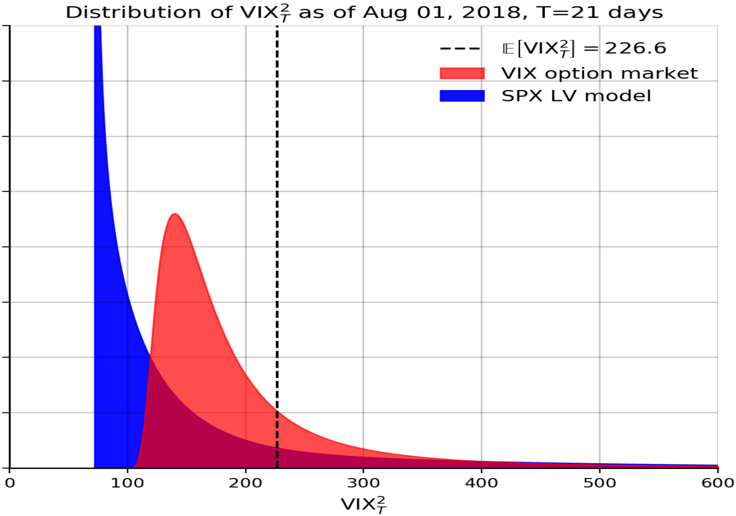
- In reality, squared VIX are not instantaneous variances but the **fair strikes** of **30-day** realized variances.
- Let us look at market data (August 1, 2018). We compare the market distributions of

$$\mathrm{VIX}^2_{\mathrm{lv},T} := \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{lv}}(t, S^{\mathrm{lv}}_t)\,dt \middle| S^{\mathrm{lv}}_T\right]$$

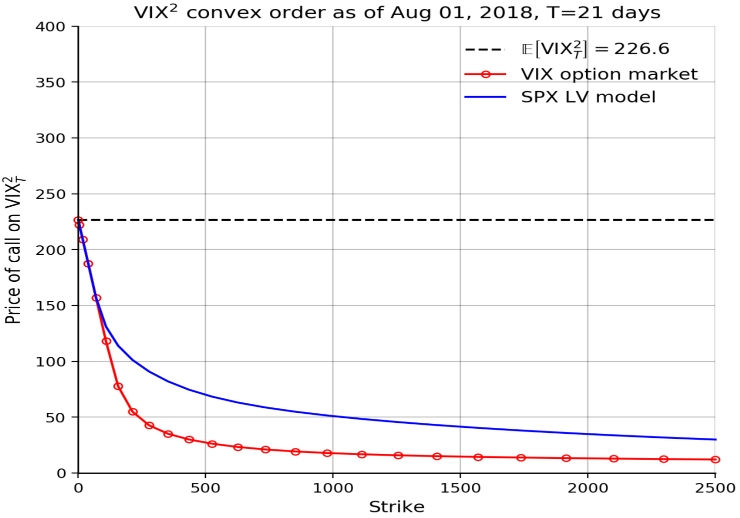
and

$$\mathrm{VIX}^2_{\mathrm{mkt},T} \quad \left(\longleftrightarrow \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\,dt \middle| \mathcal{F}_T\right]\right)$$

$T = 21$ days

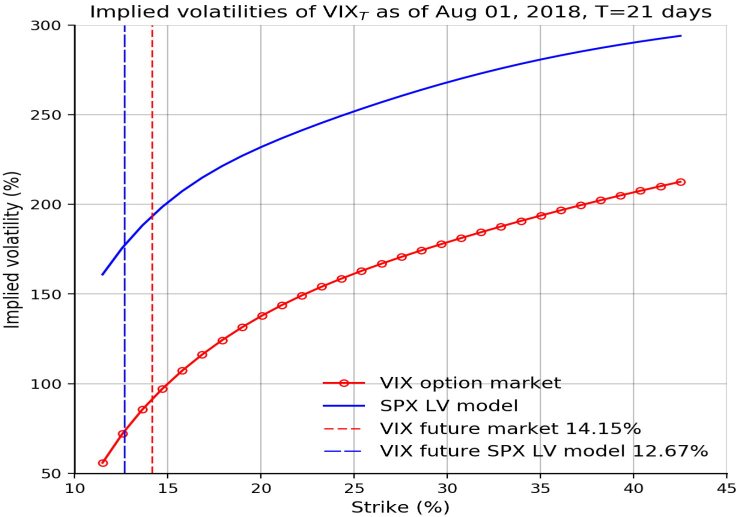

Distribution of VIX_T^2 as of Aug 01, 2018, T=21 days
$\mathbb{E}[\mathrm{VIX}_T^2] = 226.6$
VIX option market
SPX LV model
0
100
200
300
400
500
600
VIX_T^2

$T = 21$ days

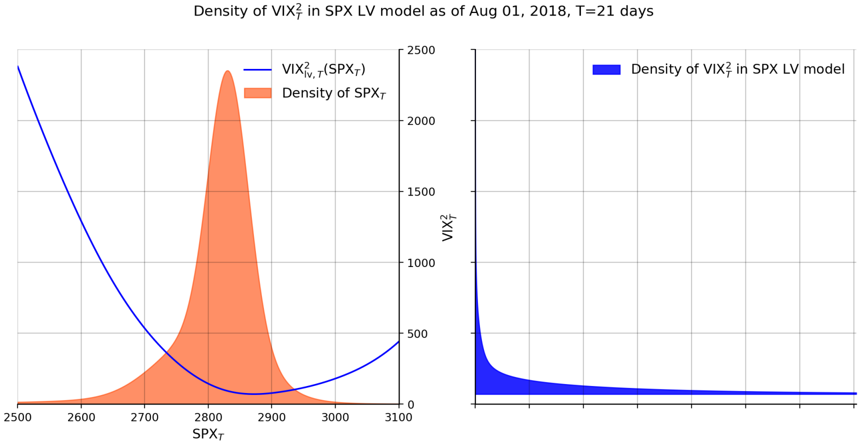


VIX² convex order as of Aug 01, 2018, T=21 days
$\mathbb{E}[\mathrm{VIX}_T^2] = 226.6$
VIX option market
SPX LV model
Price of call on VIX_T^2
Strike
400
350
300
250
200
150
100
50
0
500
1000
1500
2000
2500

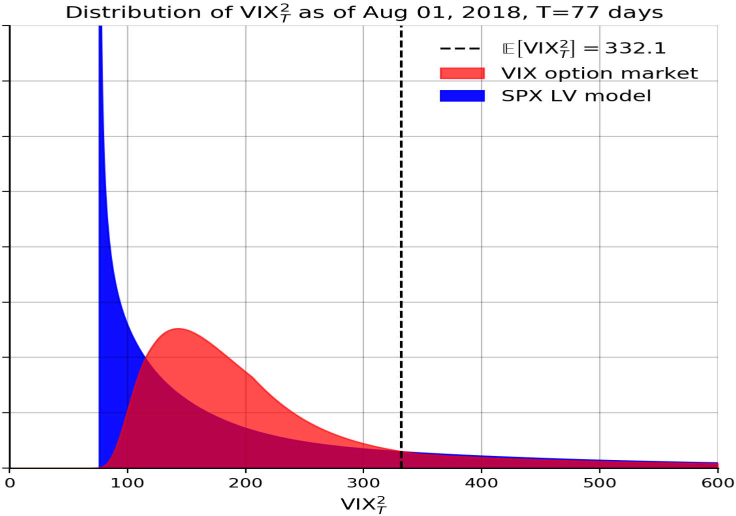
$T = 21$ days


Implied volatilities of VIX_T as of Aug 01, 2018, T=21 days
300
250
200
150
100
50
Implied volatility (%)
10
15
20
25
30
35
40
45
Strike (%)
VIX option market
SPX LV model
VIX future market 14.15%
VIX future SPX LV model 12.67%

$\mathrm{VIX}^2_{\mathrm{lv},T}(S^{\mathrm{lv}}_T)$

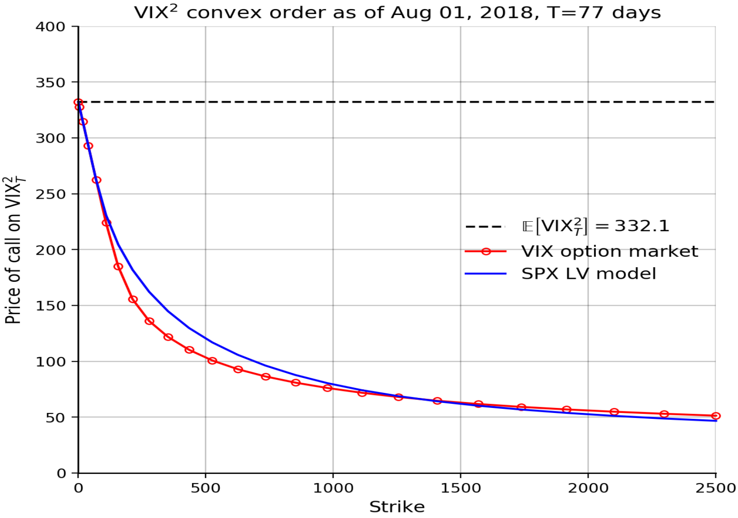


$T = 77$ days

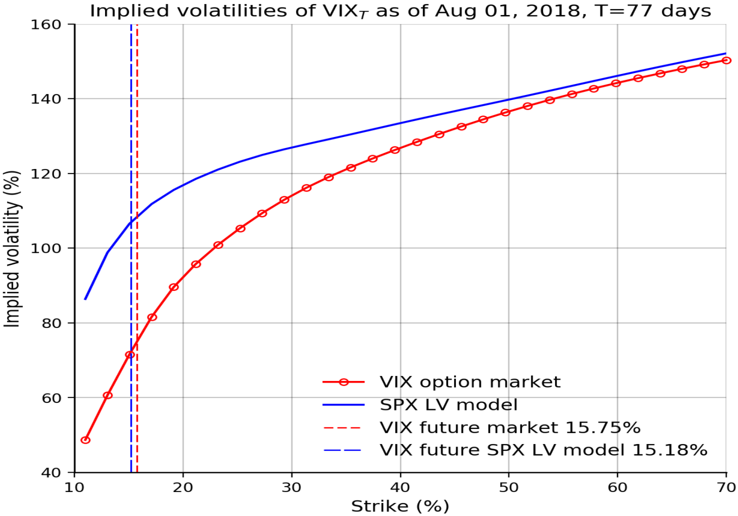


Distribution of VIX_T^2 as of Aug 01, 2018, T=77 days
$\mathbb{E}[\text{VIX}_T^2] = 332.1$
VIX option market
SPX LV model
0
100
200
300
400
500
600
VIX_T^2

$T = 77$ days


VIX² convex order as of Aug 01, 2018, T=77 days
Price of call on VIX²_T
Strike
E[VIX²_T] = 332.1
VIX option market
SPX LV model
0
50
100
150
200
250
300
350
400
0
500
1000
1500
2000
2500

$T = 77$ days


Implied volatilities of VIX_T as of Aug 01, 2018, T=77 days
Implied volatility (%)
Strike (%)
160
140
120
100
80
60
40
10
20
30
40
50
60
70
VIX option market
SPX LV model
VIX future market 15.75%
VIX future SPX LV model 15.18%

Inversion of convex ordering

- **Inversion of convex ordering**: the fact that, for small T, $\mathrm{VIX}^2_{\mathrm{loc},T} \geq_c \mathrm{VIX}^2_T$ **despite the fact that for all** t**,** $\sigma^2_{\mathrm{loc}}(t, S_t) \leq_c \sigma^2_t$.
- A **necessary** condition for continuous models to jointly calibrate to the SPX and VIX markets.
- In the paper, we numerically show that when the spot-vol correlation is large enough in absolute value,
  (a) traditional SV models with **large mean reversion**, and
  (b) rough volatility models with **small Hurst exponent**

  satisfy the inversion of convex ordering property, and more generally can reproduce the market term-structure of convex ordering of the local and stochastic squared VIX.
- Not a sufficient condition though.
- Actually we have proved that **inversion of convex ordering can be produced by a continuous SV model**.
- In such models, for small T, $\mathrm{VIX}^2_{\mathrm{loc},T} >_c \mathrm{VIX}^2_T$ so ($x \mapsto \sqrt{x}$ concave)

$$\mathbb{E}[\mathrm{VIX}_T] > \mathbb{E}\left[\mathrm{VIX}_{\mathrm{loc},T}\right]:$$

Local volatility does NOT maximize the price of VIX futures.

Inversion of Convex ordering: Local Volatility Does Not Maximize the Price of VIX Futures (with B. Acciaio, to appear in SIFIN)

INVERSION OF CONVEX ORDERING: LOCAL VOLATILITY DOES NOT MAXIMIZE THE PRICE OF VIX FUTURES

BEATRICE ACCIAIO AND JULIEN GUYON

Abstract. It has often been stated that, within the class of continuous stochastic volatility models calibrated to vanillas, the price of a VIX future is maximized by the Dupire local volatility model. In this article we prove that this statement is incorrect: we build a continuous stochastic volatility model in which a VIX future is *strictly more expensive* than in its associated local volatility model. More generally, in this model, strictly convex payoffs on a squared VIX are strictly cheaper than in the associated local volatility model. This corresponds to an *inversion of convex ordering* between local and stochastic variances, when moving from instantaneous variances to squared VIX, as convex payoffs on instantaneous variances are always cheaper in the local volatility model. We thus prove that this inversion of convex ordering, which is observed in the SPX market for short VIX maturities, can be produced by a continuous stochastic volatility model. We also prove that the model can be extended so that, as suggested by market data, the convex ordering is preserved for long maturities.

1. Introduction

For simplicity, let us assume zero interest rates, repos, and dividends. Let $\mathcal{F}_t$ denote the market information available up to time t. We consider continuous stochastic volatility models on the SPX index of the form

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t, \qquad S_0 = s_0, \tag{1.1}$$

where $W = (W_t)_{t\geq 0}$ denotes a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion, $\sigma = (\sigma_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-adapted process such that $\int_0^t \sigma_s^2 \, ds < +\infty$ a.s. for all $t \geq 0$, and $s_0 > 0$ is the initial SPX price. By

Acknowledgements

I would like to thank Bruno Dupire, Pierre Henry-Labordère, Stefano de Marco, and Bryan Liang for interesting discussions, and Bryan Liang for providing some graphs.

A few selected references

- Acciaio, B., Guyon, J.: *Inversion of Convex Ordering: Local Volatility Does Not Maximize the Price of VIX Futures*, to appear in SIFIN, 2019.
- Beiglböck, M., Henry-Labordère, P., Penkner, F.: *Model-independent bounds for option prices: A mass-transport approach*, Finance Stoch., 17(3):477–501, 2013.
- Blaschke, W, Pick, G.: *Distanzschätzungen im Funktionenraum II*, Math. Ann. 77:277–302, 1916.
- Cuturi, M.: *Sinkhorn distances: Lightspeed computation of optimal transport*, Advances in neural information processing systems, 2292–2300, 2013.
- De March, A.: *Entropic approximation for multi-dimensional martingale optimal transport*, preprint, arXiv, 2018.
- De March, A., Henry-Labordère, P.: *Building arbitrage-free implied volatility: Sinkhorn's algorithm and variants* , preprint, SSRN, 2019.

A few selected references

- De Marco, S., Henry-Labordere, P.: *Linking vanillas and VIX options: A constrained martingale optimal transport problem*, SIAM J. Finan. Math. 6:1171–1194, 2015.
- Dupire, B.: *Pricing with a smile*, Risk, January, 1994.
- Gatheral, J.: *Consistent modeling of SPX and VIX options*, presentation at Bachelier Congress, 2008.
- Gatheral, J., Jusselin, P., Rosenbaum, M.: *The quadratic rough Heston model and the joint calibration problem*, preprint, 2020.
- Guyon, J., Menegaux, R., Nutz, M.: *Bounds for VIX futures given S&P 500 smiles*, Finan. & Stoch. 21(3):593–630, 2017.
- Guyon, J.: *On the joint calibration of SPX and VIX options*, Conference in honor of Jim Gatheral's 60th birthday, NYU Courant, 2017. And Finance and Stochastics seminar, Imperial College London, 2018.
- Guyon, J.: *The Joint S&P 500/VIX Smile Calibration Puzzle Solved*, SSRN preprint available at ssrn.com/abstract=3397382, 2019.

A few selected references

- Guyon, J.: *Inversion of convex ordering in the VIX market*, SSRN preprint available at ssrn.com/abstract=3504022, 2019.
- Henry-Labordère, P.: *Automated Option Pricing: Numerical Methods*, Intern. Journ. of Theor. and Appl. Finance 16(8), 2013.
- Henry-Labordère, P.: *Model-free Hedging: A Martingale Optimal Transport Viewpoint*, Chapman & Hall/CRC Financial Mathematics Series, 2017.
- Jacquier, A., Martini, C., Muguruza, A.: *On the VIX futures in the rough Bergomi model*, preprint, 2017.
- Johansen, S.: The extremal convex functions. Math. Scand., 34, 61–68 (1974)
- Sinkhorn, R.: *Diagonal equivalence to matrices with prescribed row and column sums*, The American Mathematical Monthly, 74(4):402–405, 1967.
- Strassen, V.: *The existence of probability measures with given marginals*, Ann. Math. Statist., 36:423–439, 1965.

Equations for $u^* = (u_1^*, u_V^*, u_2^*, \Delta_S^*, \Delta_L^*)$

$$\begin{array}{rrcl}
\frac{\partial \Psi_{\bar\mu}}{\partial u_1(s_1)}=0: & \forall s_1 > 0, & u_1(s_1) & = & \Phi_1(s_1; u_V, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_V(v)}=0: & \forall v \geq 0, & u_V(v) & = & \Phi_V(v; u_1, u_2, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial u_2(s_2)}=0: & \forall s_2 > 0, & u_2(s_2) & = & \Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_S(s_1,v)}=0: & \forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_S}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v)) \\
\frac{\partial \Psi_{\bar\mu}}{\partial \Delta_L(s_1,v)}=0: & \forall s_1 > 0,\ \forall v \geq 0, & 0 & = & \Phi_{\Delta_L}(s_1, v; \Delta_S(s_1,v), \Delta_L(s_1,v))
\end{array}$$

$$\begin{array}{rcl}
\Phi_1(s_1; u_V, \Delta_S, \Delta_L) & := & \ln \mu_1(s_1) - \ln\left(\int \bar\mu(s_1, dv, ds_2) e^{u_V(v)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_V(v; u_1, \Delta_S, \Delta_L) & := & \ln \mu_V(v) - \ln\left(\int \bar\mu(ds_1, v, ds_2) e^{u_1(s_1)+u_2(s_2)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_2(s_2; u_1, u_V, \Delta_S, \Delta_L) & := & \ln \mu_2(s_2) - \ln\left(\int \bar\mu(ds_1, dv, s_2) e^{u_1(s_1)+u_V(v)+\Delta_S^{(S)}(s_1,v,s_2)+\Delta_L^{(L)}(s_1,v,s_2)}\right) \\
\Phi_{\Delta_S}(s_1, v; u_2, \delta_S, \delta_L) & := & \int \bar\mu(s_1, v, ds_2)(s_2 - s_1) e^{u_2(s_2)+\delta_S(s_2-s_1)+\delta_L\left(L\left(\frac{s_2}{s_1}\right)-v^2\right)} \\
\Phi_{\Delta_L}(s_1, v; u_2, \delta_S, \delta_L) & := & \int \bar\mu(s_1, v, ds_2)\left(L\left(\frac{s_2}{s_1}\right) - v^2\right) e^{u_2(s_2)+\delta_S(s_2-s_1)+\delta_L\left(L\left(\frac{s_2}{s_1}\right)-v^2\right)}.
\end{array}$$

Implementation details

Practically, we consider market strikes $\mathcal{K} := (\mathcal{K}_1, \mathcal{K}_V, \mathcal{K}_2)$ and market prices (C_K^1, C_K^V, C_K^2) of vanilla options on S_1, V, and S_2, and we build the model

$$\mu_{\mathcal{K}}^*(ds_1, dv, ds_2) = \bar{\mu}(ds_1, dv, ds_2) e^{c^* + \Delta_S^{0*} s_1 + \Delta_V^{0*} v + \sum_{K \in \mathcal{K}_1} a_K^{1*}(s_1 - K)_+} \\ e^{\sum_{K \in \mathcal{K}_V} a_K^{V*}(v-K)_+ + \sum_{K \in \mathcal{K}_2} a_K^{2*}(s_2 - K)_+ + \Delta_S^{*(S)}(s_1, v, s_2) + \Delta_L^{*(L)}(s_1, v, s_2)}$$

where $\theta^* := (c^*, \Delta_S^{0*}, \Delta_V^{0*}, a^{1*}, a^{V*}, a^{2*}, \Delta_S^*, \Delta_L^*)$ maximizes

$$\bar{\Psi}_{\bar{\mu}, \mathcal{K}}(\theta) := c + \Delta_S^0 S_0 + \Delta_V^0 F_V + \sum_{K \in \mathcal{K}_1} a_K^1 C_K^1 + \sum_{K \in \mathcal{K}_V} a_K^V C_K^V + \sum_{K \in \mathcal{K}_2} a_K^2 C_K^2$$
$$- \mathbb{E}^{\bar{\mu}} \left[e^{c + \Delta_S^0 S_1 + \Delta_V^0 V + \sum_{\mathcal{K}_1} a_K^1 (S_1 - K)_+ + \sum_{\mathcal{K}_V} a_K^V (V - K)_+ + \sum_{\mathcal{K}_2} a_K^2 (S_2 - K)_+ + \Delta_S^{(S)}(\ldots) + \Delta_L^{(L)}(\ldots)} \right]$$

over the set Θ of portfolios $\theta := (c, \Delta_S^0, \Delta_V^0, a^1, a^V, a^2, \Delta_S, \Delta_L)$ such that $c, \Delta_S^0, \Delta_V^0 \in \mathbb{R}$, $a^1 \in \mathbb{R}^{\mathcal{K}_1}$, $a^V \in \mathbb{R}^{\mathcal{K}_V}$, $a^2 \in \mathbb{R}^{\mathcal{K}_2}$, and $\Delta_S, \Delta_L : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are measurable functions of (s_1, v).

Implementation details

- This corresponds to solving the entropy minimization problem

$$P_{\bar{\mu},\mathcal{K}} := \inf_{\mu\in\mathcal{P}(\mathcal{K})} H(\mu,\bar{\mu}) = \sup_{\theta\in\Theta} \bar{\Psi}_{\bar{\mu},\mathcal{K}}(\theta) =: D_{\bar{\mu},\mathcal{K}}$$

  where $\mathcal{P}(\mathcal{K})$ denotes the set of probability measures μ on $\mathbb{R}_{>0}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$ such that

$$\begin{gathered}\mathbb{E}^{\mu}[S_1] = S_0, \quad \mathbb{E}^{\mu}[V] = F_V, \quad \forall K\in\mathcal{K}_1,\ \mathbb{E}^{\mu}\left[(S_1-K)_+\right] = C^1_K, \\ \forall K\in\mathcal{K}_V,\ \mathbb{E}^{\mu}\left[(V-K)_+\right] = C^V_K, \quad \forall K\in\mathcal{K}_2,\ \mathbb{E}^{\mu}\left[(S_2-K)_+\right] = C^2_K, \\ \mathbb{E}^{\mu}\left[S_2|S_1,V\right] = S_1, \quad \mathbb{E}^{\mu}\left[L\left(\frac{S_2}{S_1}\right)\middle| S_1,V\right] = V^2.\end{gathered}$$

- One can directly check that model $\mu^*_{\mathcal{K}}$ is an arbitrage-free model that jointly calibrates the prices of SPX futures, options, VIX future, and VIX options. Indeed, if $\bar{\Psi}_{\bar{\mu},\mathcal{K}}$ reaches its maximum at θ^*, then θ^* is solution to $\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\theta_i}(\theta) = 0$:

Implementation details

$$\bar{\Psi}_{\bar{\mu},\mathcal{K}}(\theta) := c + \Delta_S^0 S_0 + \Delta_V^0 F_V + \sum_{K\in\mathcal{K}_1} a_K^1 C_K^1 + \sum_{K\in\mathcal{K}_V} a_K^V C_K^V + \sum_{K\in\mathcal{K}_2} a_K^2 C_K^2$$

$$-\mathbb{E}^{\bar{\mu}}\left[e^{c+\Delta_S^0 S_1+\Delta_V^0 V+\sum_{\mathcal{K}_1} a_K^1 (S_1-K)_+ +\sum_{\mathcal{K}_V} a_K^V (V-K)_+ +\sum_{\mathcal{K}_2} a_K^2 (S_2-K)_+ +\Delta_S^{(S)}(\ldots)+\Delta_L^{(L)}(\ldots)}\right]$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial c} = 0 : \mathbb{E}^{\bar{\mu}}\left[\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = 1 \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_S^0} = 0 : \mathbb{E}^{\bar{\mu}}\left[S_1\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = S_0$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_V^0} = 0 : \mathbb{E}^{\bar{\mu}}\left[V\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = F_V \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^1} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_1-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^1$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^V} = 0 : \mathbb{E}^{\bar{\mu}}\left[(V-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^V \qquad \frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial a_K^2} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-K)_+\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\right] = C_K^2$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_S(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[(S_2-S_1)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$

$$\frac{\partial\bar{\Psi}_{\bar{\mu},\mathcal{K}}}{\partial\Delta_L(s_1,v)} = 0 : \mathbb{E}^{\bar{\mu}}\left[\left(L\left(\frac{S_2}{S_1}\right)-V^2\right)\frac{d\mu^*_{\mathcal{K}}}{d\bar{\mu}}\middle| S_1=s_1, V=v\right] = 0, \quad \forall s_1\geq 0, v>0$$