Boundary stability of nonlinear hyperbolic systems in different norms

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Abstract

For one-dimensional nonlinear hyperbolic systems, sharp dissipative boundary conditions can be expressed through a family of diagonal-scaling quantities ρ_p, indexed by the norm in which stability is considered. These quantities provide sufficient conditions for local exponential stability in different Sobolev norms. This is particularly important as exponential stability in different norms are not equivalent for nonlinear hyperbolic systems. but their mutual comparison was not known beyond the classical inequality between the Euclidean and supremum cases. In this paper, we prove that the Euclidean case (i.e. $p = 2$) is in fact the least restrictive one among the whole family: for every real square matrix, the quantities ρ_p are nonincreasing with p on $[1, 2]$ and nondecreasing above 2. In particular, the condition associated with $p = 2$ implies all the conditions associated with $p \geq 2$. The proof combines interpolation of diagonally scaled operator norms, a semidefinite characterization of the Euclidean case, and an averaging argument on the Euclidean sphere.

1 Introduction

Boundary stability is a central question in the analysis and control of one-dimensional quasilinear hyperbolic systems. Such systems model conservation and balance laws in which information propagates along characteristic curves, and boundary conditions play a decisive role because incoming characteristic components are prescribed at the endpoints; see for instance [1]. Such nonlinear systems of hyperbolic PDEs can be written

$$\partial_t u + A(u)\partial_x u = 0, \quad \text{on } [0, L], \tag{1}$$

where $A(0) = \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, with $\lambda_i \neq 0$, $u \in \mathbb{R}^n$, $u \equiv 0$ is the steady-state considered and the boundary conditions have the form (see [7, 12, 1, 17])

$$\begin{pmatrix} u^+(t,0) \\ u^-(t,L) \end{pmatrix} = G \begin{pmatrix} u^+(t,L) \\ u^-(t,0) \end{pmatrix}. \tag{2}$$

Here $u^+ \in \mathbb{R}^m$ and $u^- \in \mathbb{R}^{n-m}$ denote respectively the characteristic components associated with positive and negative characteristic speeds at the equilibrium. Specifically, one can assume without loss of generality that there exists $m \in \{1, ..., n\}$ such that $\lambda_i > 0$ for $i \leq m$ and $\lambda_i < 0$ for $i \geq m+1$ and then $u^+ = (u_i)_{i\in\{1,...,m\}}$ and $u^- = (u_i)_{i\in\{m+1,...,n\}}$. The boundary map $G : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1 and satisfies $G(0) = 0$, so that $u \equiv 0$ is indeed a steady-state. When the system is linear with all characteristic speeds of the same sign, that is $A(u) = \Lambda$ with $\lambda_i > 0$ and $G(M) = KM$ for some $K \in \mathbb{R}^{n\times n}$, the boundary dynamics can be rewritten, by the method of characteristics, as a linear time-delay system. In this setting, a criterion due to Hale and Silkowski, and based on the theory of linear delay equations [16, 8, 9], characterizes the robust exponential stability with respect to perturbations of the propagation speeds by the condition

$$\rho_0(K) := \max_{\theta\in\mathbb{R}^n} \rho\Big(\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})K\Big) < 1. \tag{3}$$

This, however, is not a sufficient stability condition for nonlinear systems, depending on the norm considered (see [5] for a counterexample in the C^1 norm). When studying the exponential stability of such a system, a natural quantity that appears is the following function

$$\rho_p : M \mapsto \inf \left\{ \|\Delta M \Delta^{-1}\|_p \;\middle|\; \Delta = \mathrm{diag}(\delta_1, \dots, \delta_n),\ \delta_i > 0,\ \forall i \in \{1, \dots, n\} \right\}, \tag{4}$$

where $\| \cdot \|_p$ denotes the operator norm induced by the vector ℓ^p-norm on $\mathbb{R}^n$. For nonlinear systems, a well-known sufficient condition to ensure stability of the system is given by the following theorem (see [5])

Theorem 1. *Assume that*

$$\rho_p(G'(0)) < 1, \tag{5}$$

then the system (1)–(2) *is (locally) exponentially stable for the* $W^{2,p}$ *norm.*

This condition was progressively obtained in the last decades. The case $p = \infty$ goes back to Greenberg and Li for a 2×2 system [7], and to Qin and Zhao for general first-order quasilinear hyperbolic systems [15, 18] (see also Li's monograph [12]) and a stability in the $W^{1,\infty}$ / C^1 norm. The particular case of $p = 2$ was later shown in [4] by Bastin, Coron and d'Andrea-Novel, while the case $p \in [1, +\infty)$ was shown by Coron and Nguyen in [5]. Interestingly, the approaches and the proof of [7, 4, 5] are quite different: the first one uses a characteristic approach, the second one a Lyapunov approach and the third one a time-delay approach. It was later showed in [3] that, in the particular case $p = \infty$, the Lyapunov function approach yields stability in $W^{p,\infty}$ norm for $p \geq 1$. Further extensions include, for instance, [1, Chapter 6] and [14, 10, 11]. In particular, when $A(u)$ only has positive eigenvalues this stability can even be extended to the weaker norm L^p for $p = 1$ and $p = \infty$ [2].

Because the stability in different norm are not equivalent for nonlinear infinite-dimensional systems [5] (in particular quasilinear systems of hyperbolic PDEs like (1)), it is natural to seek to understand the relationship between the different ρ_p and, in particular, which one is the less restrictive.
It is relatively easy to show ([4, Eq. (3.1)], where it is attributed to [12, Lemma 2.4, p. 146] that

$$\rho_1(M) = \rho_\infty(M),\ \forall\ M \in \mathbb{R}^{n \times n}. \tag{6}$$

A more involved comparison, was provided by Bastin, Coron and d'Andrea Novel in [4] and showed that

$$\rho_2(M) \leq \rho_\infty(M),\ \forall\ M \in \mathbb{R}^{n \times n} \tag{7}$$

and that the inequality can be strict [4, Proposition 3.2][1].
It was also shown in [4] that for $n \leq 5$, $\rho_2(M) = \rho_0(M)$, where ρ_0 is the quantity given by (3) and involved in the criterion for linear systems, making $p = 2$ a good candidate for being the least restrictive condition. However, for $n > 5$ the condition is much less clear as there exist matrices M such that $\rho_2(M) > \rho_0(M)$. Despite further studies, the comparison between the different ρ_p remained open.

In this paper, we answer this question and we prove the following result

Theorem 2. *For any* $M \in \mathbb{R}^{n \times n}$ *one has*

$$\rho_2(M) \leq \rho_3(M) \leq \cdots \leq \rho_\infty(M). \tag{8}$$

This paper is organized as follows: in Section 2 we prove some preliminary lemmas, in Section 3 we prove a weaker version of Theorem 2 as an intermediate result, and in Section 4 we show how this implies Theorem 2.

[1]In the notation of [4], this is written as $\rho_1(K) \leq \rho_2(K)$: their ρ_1 corresponds to the present ρ_2, while their ρ_2 corresponds to the present ρ_∞.

2 Preliminary Results

Throughout, $M \in \mathbb{R}^{n\times n}$ is fixed and $\|\cdot\|_p$ denotes the operator norm induced by the vector ℓ^p-norm on $\mathbb{R}^n$. We write $\mathcal{D}_{++}$ for the set of positive diagonal matrices.

2.1 Log-convexity

Lemma 3 (Interpolation of diagonal scalings)**.** *Let* $1 \le p_0, p_1 \le \infty$, $0 \le \theta \le 1$, *and* $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *For any* $\Delta_0, \Delta_1 \in \mathcal{D}_{++}$, *define* $\Delta_\theta := \Delta_0^{1-\theta}\Delta_1^\theta$ *(entrywise). Then*

$$\|\Delta_\theta M \Delta_\theta^{-1}\|_{p_\theta} \le \|\Delta_0 M \Delta_0^{-1}\|_{p_0}^{1-\theta} \|\Delta_1 M \Delta_1^{-1}\|_{p_1}^{\theta}.$$

Proof. Write $\Delta_j = D(h^{(j)})$ with

$$D(h) = \operatorname{diag}(e^{h_1}, \dots, e^{h_n}), \qquad h^{(j)} = (\log(\Delta_j)_{11}, \dots, \log(\Delta_j)_{nn}).$$

For z in the strip

$$S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\},$$

define

$$T_z = D\big((1-z)h^{(0)} + zh^{(1)}\big)\, M\, D\big(-((1-z)h^{(0)} + zh^{(1)})\big).$$

Each entry of T_z is of the form

$$(T_z)_{ij} = M_{ij} \exp\!\big((1-z)(h_i^{(0)} - h_j^{(0)}) + z(h_i^{(1)} - h_j^{(1)})\big),$$

hence is analytic in the open strip and continuous on the closed strip. Moreover,

$$|(T_z)_{ij}| \le |M_{ij}| \exp\!\Big(\max\{h_i^{(0)} - h_j^{(0)},\, h_i^{(1)} - h_j^{(1)}\}\Big), \qquad z \in S,$$

so the family is uniformly bounded on S.
We now work temporarily on $\mathbb{C}^n$. On the boundary line $\operatorname{Re} z = 0$, writing $z = iy$, we have

$$T_{iy} = U_y\, T_0\, U_y^{-1}, \qquad U_y = D\big(iy(h^{(1)} - h^{(0)})\big).$$

Since U_y is diagonal with unimodular entries, it is an isometry on every complex ℓ^p, and therefore

$$\|T_{iy}\|_{p_0,\mathbb{C}} = \|T_0\|_{p_0,\mathbb{C}}.$$

Similarly, on $\operatorname{Re} z = 1$,

$$T_{1+iy} = V_y\, T_1\, V_y^{-1}$$

with V_y diagonal unimodular, so

$$\|T_{1+iy}\|_{p_1,\mathbb{C}} = \|T_1\|_{p_1,\mathbb{C}}.$$

For a real matrix A, let $\|A\|_{p,\mathbb{R}}$ and $\|A\|_{p,\mathbb{C}}$ denote the induced operator norms on $\ell^p(\mathbb{R}^n)$ and $\ell^p(\mathbb{C}^n)$, respectively. Trivially,

$$\|A\|_{p,\mathbb{R}} \le \|A\|_{p,\mathbb{C}}.$$

We claim that equality holds for every $1 \le p \le \infty$.
If $1 \le p < \infty$, fix $z \in \mathbb{C}^n$ with $\|z\|_p = 1$, write $z_j = r_j e^{i\theta_j}$, and set

$$x_t = \operatorname{Re}(e^{it} z) \in \mathbb{R}^n.$$

Because A is real,

$$(Ax_t)_i = \operatorname{Re}(e^{it}(Az)_i).$$

Hence, with

$$c_p = \frac{1}{2\pi}\int_0^{2\pi} |\cos s|^p \, ds > 0,$$

we obtain, using that cos is 2π-periodic and that $(Az)_j = |(Az)_j| e^{i\phi_j}$ for some $\phi_j \in [-\pi, \pi)$

$$\frac{1}{2\pi}\int_0^{2\pi} |(Ax_t)_i|^p \, dt = c_p |(Az)_i|^p, \qquad \frac{1}{2\pi}\int_0^{2\pi} |(x_t)_j|^p \, dt = c_p r_j^p.$$

Summing over i and j gives

$$c_p \|Az\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} \|Ax_t\|_p^p \, dt \le \|A\|_{p,\mathbb{R}}^p \frac{1}{2\pi}\int_0^{2\pi} \|x_t\|_p^p \, dt = \|A\|_{p,\mathbb{R}}^p \, c_p.$$

Thus $\|Az\|_p \le \|A\|_{p,\mathbb{R}}$, and taking the supremum over z yields

$$\|A\|_{p,\mathbb{C}} \le \|A\|_{p,\mathbb{R}}.$$

If $p = \infty$, fix $z \in \mathbb{C}^n$ with $\|z\|_\infty = 1$. If $Az = 0$, there is nothing to prove. Otherwise choose i_0 such that

$$|(Az)_{i_0}| = \|Az\|_\infty.$$

Let $t = -\arg((Az)_{i_0})$ and set

$$x = \mathrm{Re}(e^{it} z) \in \mathbb{R}^n.$$

Then $\|x\|_\infty \le 1$, and since A is real,

$$(Ax)_{i_0} = \mathrm{Re}(e^{it}(Az)_{i_0}),$$

so

$$\|Az\|_\infty = |(Az)_{i_0}| = \left|\mathrm{Re}(e^{it}(Az)_{i_0})\right| = |(Ax)_{i_0}| \le \|Ax\|_\infty \le \|A\|_{\infty,\mathbb{R}}.$$

Taking the supremum over z gives

$$\|A\|_{\infty,\mathbb{C}} \le \|A\|_{\infty,\mathbb{R}}.$$

Therefore

$$\|A\|_{p,\mathbb{C}} = \|A\|_{p,\mathbb{R}} \qquad \text{for every real matrix } A \text{ and every } 1 \le p \le \infty.$$

The family $\{T_z\}$ now satisfies the hypotheses of Stein's interpolation theorem for analytic families on the strip (see, e.g., [13, Chapter 4, Chapter 5] or [6, Theorem 1.3.7] with $X = Y = \{1, ..., n\}$ with counting measure). Applying it on the complex spaces $\ell^{p_0}(\mathbb{C}^n)$ and $\ell^{p_1}(\mathbb{C}^n)$ yields

$$\|T_\theta\|_{p_\theta,\mathbb{C}} \le \|T_0\|_{p_0,\mathbb{C}}^{1-\theta} \|T_1\|_{p_1,\mathbb{C}}^{\theta}.$$

Since T_0, T_1, and T_θ are real matrices, the preceding equality of real and complex operator norms gives

$$\|T_\theta\|_{p_\theta} \le \|T_0\|_{p_0}^{1-\theta} \|T_1\|_{p_1}^{\theta}.$$

Finally,

$$T_\theta = D\big((1-\theta)h^{(0)} + \theta h^{(1)}\big) \, M \, D\big(-((1-\theta)h^{(0)} + \theta h^{(1)})\big) = \Delta_\theta M \Delta_\theta^{-1},$$

so the lemma follows. □

Proposition 4 (Log-convexity of ρ_p)**.** *For any $M \in \mathbb{R}^{n\times n}$, any $1 \le p_0, p_1 \le \infty$, and any $\theta \in [0,1]$,*

$$\rho_{p_\theta}(M) \le \rho_{p_0}(M)^{1-\theta} \rho_{p_1}(M)^{\theta}, \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Equivalently, with the convention $1/0 = \infty$ and $\log 0 = -\infty$, the map

$$t \longmapsto \log \rho_{1/t}(M), \qquad t \in [0,1],$$

is convex on $[0,1]$ as an extended-real-valued function.

Proof. Fix $\varepsilon > 0$. Choose $\Delta_0, \Delta_1 \in \mathcal{D}_{++}$ such that

$$\|\Delta_j M \Delta_j^{-1}\|_{p_j} \le \rho_{p_j}(M) + \varepsilon, \qquad j = 0, 1.$$

By Lemma 3,

$$\|\Delta_\theta M \Delta_\theta^{-1}\|_{p_\theta} \le (\rho_{p_0}(M) + \varepsilon)^{1-\theta}(\rho_{p_1}(M) + \varepsilon)^{\theta}.$$

Since $\Delta_\theta \in \mathcal{D}_{++}$, the left-hand side is at least $\rho_{p_\theta}(M)$. Therefore

$$\rho_{p_\theta}(M) \le (\rho_{p_0}(M) + \varepsilon)^{1-\theta}(\rho_{p_1}(M) + \varepsilon)^{\theta}.$$

Letting $\varepsilon \downarrow 0$ gives the stated inequality.
For the logarithmic reformulation, if $\rho_p(M) > 0$ for every p, then the preceding inequality is exactly the convexity of $t \mapsto \log \rho_{1/t}(M)$ with $t = 1/p$. If $\rho_{p_*}(M) = 0$ for some $p_* \in [1, \infty]$, then for every $1 \le q \le \infty$ and every matrix A one has

$$\|x\|_q \le n^{\max\{0,\, 1/q - 1/p_*\}} \|x\|_{p_*}, \qquad \|x\|_{p_*} \le n^{\max\{0,\, 1/p_* - 1/q\}} \|x\|_q,$$

hence

$$\|Ax\|_q \le n^{\max\{0,\, 1/q - 1/p_*\}} \|Ax\|_{p_*} \le n^{|1/p_* - 1/q|} \|A\|_{p_*} \|x\|_q.$$

Therefore

$$\|A\|_q \le n^{|1/p_* - 1/q|} \|A\|_{p_*}.$$

Applying this to $A = \Delta M \Delta^{-1}$ and taking infima over $\Delta \in \mathcal{D}_{++}$ gives

$$\rho_q(M) \le n^{|1/p_* - 1/q|} \rho_{p_*}(M) = 0. \tag{9}$$

So $\rho_q(M) = 0$ for every q, and consequently $t \mapsto \log \rho_{1/t}(M)$ is identically $-\infty$, which is convex by convention. □

2.2 Duality

Lemma 5 (Duality)**.** *For $1/p + 1/q = 1$, $\rho_p(M) = \rho_q(M^T)$. In particular, $\rho_2(M) = \rho_2(M^T)$.*

Proof. By the norm duality $\|A\|_p = \|A^T\|_q$, we have

$$\|\Delta M \Delta^{-1}\|_p = \|(\Delta M \Delta^{-1})^T\|_q = \|\Delta^{-1} M^T \Delta\|_q$$

for any $\Delta \in \mathcal{D}_{++}$. Since $\Delta \mapsto \Delta^{-1}$ is a bijection of $\mathcal{D}_{++}$, taking infima gives $\rho_p(M) = \rho_q(M^T)$. □

Corollary 6. *To prove $\rho_2(M) \le \rho_p(M)$ for all M and all $p \ge 1$, it suffices to prove it for $p > 2$.*

Proof. The case $p = 1$ is already known: $\rho_1(M) \ge \rho_2(M)$ by (6). The case $p = 2$ is tautological. If $1 < p < 2$ and $q > 2$ is the conjugate exponent, then Lemma 5 gives

$$\rho_p(M) = \rho_q(M^T) \ge \rho_2(M^T) = \rho_2(M).$$

Thus only the range $p > 2$ remains. □

3 Proof of $\rho_2(M) \le \rho_p(M)$

In this Section, we prove the following Theorem, which is a weaker version of Theorem 2 but, together with the results of Section 2, will imply Theorem 2.

Theorem 7. *For all $n \ge 1$, all $M \in \mathbb{R}^{n \times n}$, and all $p \ge 1$:*

$$\rho_2(M) \le \rho_p(M).$$

The proof proceeds in three steps. First, we reduce the conjecture to a universal matrix inequality (Proposition 8). Then we prove this inequality using an SDP alternative theorem and the probabilistic method on the unit sphere.

3.1 Step 1: Reduction to a universal inequality

Proposition 8. *For every $A \in \mathbb{R}^{n\times n}$ and every $p \in [1,\infty)$:*

$$\rho_2(A) \le \|A\|_p.$$

Before proving Proposition 8, let us show it implies Theorem 7.

Proof that Proposition 8 implies Theorem 7. For $p \in [1,\infty)$: since ρ_2 is invariant under diagonal similarity (if Δ is a positive diagonal matrix, then $\rho_2(\Delta M \Delta^{-1}) = \rho_2(M)$, because $\{\Delta'\Delta M\Delta^{-1}\Delta'^{-1} : \Delta' \text{ pos. diag.}\} = \{\Delta'' M \Delta''^{-1} : \Delta'' \text{ pos. diag.}\}$), Proposition 8 gives

$$\rho_2(M) = \rho_2(\Delta M \Delta^{-1}) \le \|\Delta M \Delta^{-1}\|_p$$

for every positive diagonal Δ. Taking the infimum over Δ yields $\rho_2(M) \le \rho_p(M)$.
For $p = \infty$: the known result (6) gives $\rho_\infty(M) = \rho_1(M) \ge \rho_2(M)$. □

3.2 SDP characterization of ρ_2 and its alternative

Lemma 9 (SDP characterization)**.** *For $A \in \mathbb{R}^{n\times n}$:*

$$\rho_2(A) < 1 \quad \iff \quad \exists\, Q = \operatorname{diag}(q_1,\dots,q_n) \succ 0 \text{ such that } A^T Q A \prec Q.$$

Proof. We have $\rho_2(A) = \inf_\Delta \|\Delta A \Delta^{-1}\|_2$ where Δ ranges over positive diagonal matrices. Setting $Q = \Delta^2$:

$$\|\Delta A\Delta^{-1}\|_2^2 = \lambda_{\max}\big((\Delta A\Delta^{-1})^T(\Delta A\Delta^{-1})\big) = \lambda_{\max}\big(\Delta^{-1}A^T\Delta^2 A\Delta^{-1}\big).$$

Hence $\|\Delta A\Delta^{-1}\|_2 < 1$ iff $\Delta^{-1}A^T\Delta^2A\Delta^{-1} \prec I$, iff $A^TQA \prec Q$ where $Q = \Delta^2$. Since $\Delta \mapsto Q = \Delta^2$ is a bijection of $\mathcal{D}_{++}$ onto itself (with inverse $Q \mapsto \Delta = Q^{1/2}$, the entrywise positive square root), the equivalence between $\inf_\Delta \|\Delta A\Delta^{-1}\|_2 < 1$ and $\exists\, Q \in \mathcal{D}_{++}$ with $A^TQA \prec Q$ follows. □

Lemma 10 (SDP alternative)**.** *For symmetric matrices $B_1,\dots,B_n \in \mathbb{R}^{m\times m}$, exactly one of the following holds:*

(I) *There exist $q_1,\dots,q_n > 0$ such that* $\displaystyle\sum_{i=1}^n q_iB_i \succ 0$.

(II) *There exists $Y \in \mathbb{R}^{m\times m}$, $Y \succeq 0$, $Y \ne 0$, such that $\operatorname{Tr}(YB_i) \le 0$ for all $i = 1,\dots,n$.*

Proof. (I) and (II) cannot both hold. If (I) holds with weights $q_i > 0$ and (II) holds with $Y \succeq 0$, $Y \ne 0$, then

$$0 < \operatorname{Tr}\Big(Y\sum_i q_iB_i\Big) = \sum_i q_i \operatorname{Tr}(YB_i) \le 0,$$

a contradiction (the first inequality uses $Y \succeq 0$, $Y \ne 0$, and $\sum q_iB_i \succ 0$).
Not-(I) implies (II). Suppose (I) fails. Define the convex cone

$$\mathcal{C} = \Big\{\sum_{i=1}^n q_iB_i : q_i \ge 0\Big\} \subset \mathcal{S}^m,$$

where $\mathcal{S}^m$ is the space of real symmetric $m \times m$ matrices equipped with the trace inner product $\langle X, Z\rangle = \operatorname{Tr}(XZ)$. The cone $\mathcal{C}$ is convex (as a conic combination of fixed matrices). The open cone $\mathcal{S}^m_{++}$ of positive definite matrices is convex and open.
We claim $\mathcal{C} \cap \mathcal{S}^m_{++} = \varnothing$. If some $S = \sum q_iB_i \in \mathcal{S}^m_{++}$ with $q \ge 0$, then for any $\varepsilon > 0$, $S' = \sum(q_i+\varepsilon)B_i = S + \varepsilon\sum B_i$ lies in $\mathcal{S}^m_{++}$ for ε small enough (since $\mathcal{S}^m_{++}$ is open) and has all coefficients $q_i + \varepsilon > 0$, contradicting the failure of (I).

By the Hahn–Banach separation theorem (in the form: if K is convex and U is open convex with $K \cap U = \varnothing$ in a finite-dimensional space, there exists a separating hyperplane; no closedness of K is required), there exist a nonzero $Y \in \mathcal{S}^m$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{Tr}(YX) \leq \alpha \quad \text{for all } X \in \mathcal{C}, \qquad \operatorname{Tr}(YZ) \geq \alpha \quad \text{for all } Z \in \mathcal{S}^m_{++}.$$

Since $0 \in \mathcal{C}$, the first condition gives $\alpha \geq 0$. Since both $\mathcal{C}$ and $\mathcal{S}^m_{++}$ are cones (stable under positive scalar multiplication), we can sharpen: for any $X \in \mathcal{C}$ and $t > 0$, $tX \in \mathcal{C}$, so $t \operatorname{Tr}(YX) \leq \alpha$ for all $t > 0$, which forces $\operatorname{Tr}(YX) \leq 0$. Similarly, for any $Z \in \mathcal{S}^m_{++}$ and $t > 0$, $tZ \in \mathcal{S}^m_{++}$, so $t \operatorname{Tr}(YZ) \geq \alpha \geq 0$; if $\operatorname{Tr}(YZ) < 0$ for some Z, taking $t \to \infty$ yields a contradiction. Hence $\operatorname{Tr}(YZ) \geq 0$ for all $Z \in \mathcal{S}^m_{++}$.
By continuity, $\operatorname{Tr}(YZ) \geq 0$ for all $Z \in \overline{\mathcal{S}^m_{++}} = \mathcal{S}^m_{+}$. In particular, taking $Z = vv^T \in \mathcal{S}^m_+$ for any $v \in \mathbb{R}^m$ gives $v^T Y v \geq 0$, so $Y \succeq 0$. Since $Y \neq 0$ (it is a nonzero separating functional), condition (II) holds: taking $X = B_i$ (which corresponds to $q = e_i \in \mathbb{R}^n_{\geq 0}$) gives $\operatorname{Tr}(YB_i) \leq 0$. □

3.3 Proof of Proposition 8

Proof of Proposition 8. We may assume $A \neq 0$ (otherwise $\rho_2(A) = 0$ and the result is trivial). It suffices to show:

$$\rho_2(A) \geq 1 \quad \Longrightarrow \quad \|A\|_p \geq 1. \tag{10}$$

Indeed, once (10) is established, for any $\varepsilon > 0$, applying it to $A/(\|A\|_p + \varepsilon)$ (which has $\|A/(\|A\|_p + \varepsilon)\|_p < 1$, hence $\rho_2(A/(\|A\|_p + \varepsilon)) < 1$ by contrapositive of (10)) gives, by positive homogeneity $\rho_2(cA) = |c|\rho_2(A)$ (which follows directly from $\|\Delta(cA)\Delta^{-1}\|_2 = |c|\|\Delta A \Delta^{-1}\|_2$), that $\rho_2(A) < \|A\|_p + \varepsilon$. Letting $\varepsilon \to 0$ yields $\rho_2(A) \leq \|A\|_p$.

Step 1: Dual certificate from SDP alternative. Assume $\rho_2(A) \geq 1$. By Lemma 9, there is no positive diagonal matrix Q with $A^T Q A \prec Q$. Writing $Q = \operatorname{diag}(q_1, \ldots, q_n)$ and

$$B_i = e_i e_i^T - A^T e_i e_i^T A, \qquad i = 1, \ldots, n,$$

the condition $Q - A^T Q A \succ 0$ is exactly $\sum_i q_i B_i \succ 0$. Since this system has no solution with $q_i > 0$, Lemma 10 provides a *dual certificate*:

$$\exists\, Y \succeq 0,\ Y \neq 0, \quad \text{with} \quad \operatorname{Tr}(YB_i) \leq 0 \ \ \forall i.$$

Expanding:

$$\operatorname{Tr}(YB_i) = \operatorname{Tr}(Y e_i e_i^T) - \operatorname{Tr}(Y A^T e_i e_i^T A) = Y_{ii} - (AYA^T)_{ii}.$$

So the dual certificate satisfies:

$$(AYA^T)_{ii} \geq Y_{ii} \qquad \text{for all } i = 1, \ldots, n. \tag{11}$$

Step 2: Gram factorization. Factor $Y = VV^T$ where $V \in \mathbb{R}^{n \times k}$ with $k = \operatorname{rank}(Y) \geq 1$. Let $v_i \in \mathbb{R}^k$ denote the i-th row of V and $w_i \in \mathbb{R}^k$ denote the i-th row of $W := AV$. Then:

$$Y_{ii} = \|v_i\|_2^2, \qquad (AYA^T)_{ii} = \|w_i\|_2^2.$$

Inequality (11) becomes:

$$\|w_i\|_2 \geq \|v_i\|_2 \qquad \text{for all } i = 1, \ldots, n. \tag{12}$$

Step 3: Spherical averaging (probabilistic method). Let u be uniformly distributed on the unit sphere $S^{k-1} \subset \mathbb{R}^k$. For any fixed $z \in \mathbb{R}^k$, the rotational invariance of the uniform measure on S^{k-1} gives

$$\mathbb{E}\big[|z^T u|^p\big] = c_{k,p}\, \|z\|_2^p, \tag{13}$$

where $c_{k,p} := \mathbb{E}[|u_1|^p] > 0$ depends only on k and p (not on z). Indeed, writing $z = \|z\|_2 \hat{z}$ with $\|\hat{z}\|_2 = 1$, the random variable $\hat{z}^T u$ has the same distribution as u_1 by rotational invariance.

For each i, applying (13) to $z = w_i$ and $z = v_i$, and using (12) together with the monotonicity of $t \mapsto t^{p/2}$ on $[0, \infty)$ (which gives $\|w_i\|_2^p \geq \|v_i\|_2^p$):

$$\mathbb{E}\big[|w_i^T u|^p\big] = c_{k,p}\,\|w_i\|_2^p \geq c_{k,p}\,\|v_i\|_2^p = \mathbb{E}\big[|v_i^T u|^p\big].$$

Summing over $i = 1, \ldots, n$ and setting $x(u) = Vu$:

$$\mathbb{E}\big[\|Ax(u)\|_p^p\big] = \sum_{i=1}^{n} \mathbb{E}\big[|w_i^T u|^p\big] \geq \sum_{i=1}^{n} \mathbb{E}\big[|v_i^T u|^p\big] = \mathbb{E}\big[\|x(u)\|_p^p\big]. \tag{14}$$

Step 4: Extracting a witness. Define $f(u) = \|Ax(u)\|_p^p - \|x(u)\|_p^p$ for $u \in S^{k-1}$. By (14), $\mathbb{E}[f(u)] \geq 0$. Since $V \neq 0$ (as $Y \neq 0$) and $k = \mathrm{rank}(Y) \geq 1$, $\ker(V)$ is a proper linear subspace of $\mathbb{R}^k$ of dimension $d < k$. Its intersection $\ker(V) \cap S^{k-1}$ is either empty (if $d = 0$) or a great subsphere of dimension $d - 1 < k - 1$, which has $(k-1)$-dimensional surface measure zero on S^{k-1} (since a smooth submanifold of dimension $< k-1$ has zero $(k-1)$-dimensional Hausdorff measure). For $u \in \ker(V)$, we have $x(u) = 0$ and $Ax(u) = 0$, so $f(u) = 0$.
If $f(u) < 0$ for (almost) every $u \notin \ker(V)$, then $\mathbb{E}[f(u)] < 0$, contradicting $\mathbb{E}[f(u)] \geq 0$. Therefore, there exists $u_0 \in S^{k-1} \setminus \ker(V)$ with $f(u_0) \geq 0$.
Set $x_0 = Vu_0 \neq 0$. Then $\|Ax_0\|_p^p \geq \|x_0\|_p^p$, so

$$\|A\|_p \geq \frac{\|Ax_0\|_p}{\|x_0\|_p} \geq 1.$$

This completes the proof of (10) and hence of Proposition 8. □

Remark 11. *The proof avoids direct analysis of norming vectors, magnitude matching conditions, or Hessian inequalities—all of which were shown to encounter fundamental obstructions. Instead, the SDP alternative theorem converts the* absence *of a diagonal quadratic Lyapunov certificate into a* dual object Y *carrying row-wise* ℓ^2 *information. The probabilistic method then transfers this* ℓ^2 *information to* ℓ^p *via the rotational invariance of the uniform measure on the sphere, exploiting the fact that the moment identity* (13) *holds uniformly across all directions.*

4 Proof of Theorem 2

We first treat the degenerate case $\rho_p(M) = 0$ for some $p^* \in [1, +\infty]$. In this case, thanks to (9) we deduce that $\rho_p(M) = 0$ for any $p \in [1, +\infty]$ and the theorem holds. Henceforth assume

$$\rho_p(M) > 0 \qquad \text{for all } p \in [1, \infty].$$

Since $\Delta = I$ is admissible, also $\rho_p(M) \leq \|M\|_p < \infty$, so all logarithms below are well defined.
Define

$$f(t) = \log \rho_{1/t}(M), \qquad t \in [0, 1],$$

where $1/0 := \infty$, so that $f(0) = \log \rho_\infty(M)$.
Step 1: f is convex. This is Proposition 4.
Step 2: f attains its global minimum at $t = \frac{1}{2}$. Let $t \in (0, 1]$ and set $p = 1/t \in [1, \infty)$. By Theorem 7,

$$\rho_2(M) \leq \rho_p(M) = \rho_{1/t}(M).$$

Since log is increasing,

$$f\big(\tfrac{1}{2}\big) \leq f(t) \qquad (0 < t \leq 1).$$

For $t = 0$, Theorem 7 at $p = 1$ gives $\rho_2(M) \leq \rho_1(M)$, and the known identity $\rho_1(M) = \rho_\infty(M)$ yields

$$\rho_2(M) \leq \rho_\infty(M),$$

hence

$$f\left(\tfrac{1}{2}\right) \le f(0).$$

Therefore f attains its global minimum on $[0,1]$ at $t = \frac{1}{2}$.

Step 3: Monotonicity of f on each side of $\frac{1}{2}$. A direct consequence of the convexity of f and its global minimum on $[0,1/2]$ is that f is nonincreasing on $[0,\frac{1}{2}]$ and f is nondecreasing on $[\frac{1}{2},1]$.

Step 4: Translate this back to p. If $2 \le p_1 \le p_2 \le \infty$, let $t_i = 1/p_i$ (with $1/\infty = 0$). Then

$$0 \le t_2 \le t_1 \le \tfrac{1}{2}.$$

Since f is nonincreasing on $[0,\frac{1}{2}]$,

$$f(t_1) \le f(t_2).$$

Exponentiating gives

$$\rho_{p_1}(M) \le \rho_{p_2}(M).$$

Thus $p \mapsto \rho_p(M)$ is nondecreasing on $[2,\infty]$. In particular,

$$\rho_2(M) \le \rho_3(M) \le \rho_4(M) \le \cdots \le \rho_\infty(M).$$

If $1 \le p_1 \le p_2 \le 2$, then with $t_i = 1/p_i$ we have

$$\tfrac{1}{2} \le t_2 \le t_1 \le 1.$$

Since f is nondecreasing on $[\frac{1}{2},1]$,

$$f(t_2) \le f(t_1),$$

and therefore

$$\rho_{p_2}(M) \le \rho_{p_1}(M).$$

So $p \mapsto \rho_p(M)$ is nonincreasing on $[1,2]$. Finally, using the known identity $\rho_1(M) = \rho_\infty(M)$, we conclude that $p \mapsto \rho_p(M)$ decreases on $[1,2]$, increases on $[2,\infty]$, and has global minimum at $p=2$.

Conclusion

In this paper we presented a comparison of the diagonal-scaling quantities

$$\rho_p(M) = \inf_{\Delta \in \mathcal{D}_{++}} \|\Delta M \Delta^{-1}\|_p$$

which arise in dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. We proved that, for every real square matrix M, the quantities $\rho_p(M)$ are ordered monotonically away from $p=2$. In particular,

$$\rho_2(M) \le \rho_3(M) \le \rho_4(M) \le \cdots \le \rho_\infty(M).$$

Thus, among the sufficient $W^{2,p}$-stability conditions expressed through $\rho_p(G'(0)) < 1$, the case $p=2$ is the least restrictive within this family. The proof combines several ingredients. The interpolation properties of induced matrix norms together with Stein's interpolation theorem give a log-convexity structure for the map $p \mapsto \rho_p(M)$, once diagonal scalings are incorporated. The central step is then the estimate $\rho_2(M) \le \rho_p(M)$, which follows from the semidefinite characterization of ρ_2 and a spherical averaging argument. This estimate, together with log-convexity and duality, yields the desired monotonicity.

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