Exercises : Optimality conditions

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Exercise 1. *Solve the following optimization problem*

$$\begin{aligned} \underset{x,y\in\mathbb{R}^2}{\text{Min}} \quad & (x-1)^2 + (y-2)^2 \\ & x \le y \\ & x + 2y \le 2 \end{aligned}$$

Exercise 2 (First order optimality condition)**.** *Consider, for f differentiable,*

$$\begin{aligned} (P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad & f(x) \\ s.t. \quad & x \in X \end{aligned}$$

Recall that

$$T_X(x_0) = \{d \in \mathbb{R}^n \mid \exists d_k \to d, \exists t_k \searrow 0, \; s.t.\ x_0 + t_k d_k \in X\}$$

and $K^+ = \{\lambda \mid \lambda^\top x \ge 0,\ \forall x \in K\}$.
Show that

1. *If x_0 is an optimal solution to (P), then $\nabla f(x_0) \in [T_X(x_0)]^+$.*
2. *If f is convex, X is closed convex, and $\nabla f(x_0) \in [T_X(x_0)]^+$, then x_0 is an optimal solution to (P).*

Exercise 3. *In the following cases, are the KKT conditions necessary / sufficient ?*

1.

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & 12x_1 - 5x_2 + 3x_3 \\ s.t. \quad & x_1 + 2x_2 - x_3 = 5 \\ & x_1 - x_2 \ge -2 \\ & 2x_1 - 4x_2 \le 12 \end{aligned}$$

2.

$$\begin{aligned} \min_{x_1,x_2} \quad & 4x_1^2 - x_1x_2 + x_2^2 - 12x_1 \\ s.t. \quad & x_1 - 2x_2 + x_3 = 5 \\ & x_1^2 + 3x_2^2 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

3.

$$\begin{aligned} \min_{x_1,x_2,x_3} \quad & e_1^x - x_1x_2 + x_3^3 \\ s.t. \quad & \ln(e^{x_1-4x_2} + e^{x_1+x_3}) \le 2x_1 + 3 \\ & 2x_1^2 + x_2^2 \le 2 \end{aligned}$$

4.

$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & -x_2 - (x_1-1)^3 \le 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

5.

$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 \\ s.t. \quad & x_2 - (x_1-1)^3 \le 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Exercise 4. *Solve the following problem using first order optimality conditions*

$$\begin{aligned} \min_{x_1,x_2} \quad & -2(x_1-2)^2 - x_2^2 \\ s.t. \quad & x_1^2 + x_2^2 \le 25 \\ & x_1 \ge 0 \end{aligned}$$