A game of dices and zombies

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Rules of the game

Zombie dice is a simple boardgame played with specific dices. It is a *push your luck* game, where a player have to choose between stopping and taking his gains, or trying to earn more at the risk of loosing all he had. Player play a turn in succession. The winner is the first to reach 13 points.
In the game you are playing a zombie trying to eat brains. To do so you are going to roll dices, with three different possible output : brains, shotgun or steps. A *turn* of the game is a succession of *throws*. Each turn, you take three dice from the box and roll them. A brain symbol is worth one point at the end of the round, while footsteps means that you will reroll this particular dice in the next throw. Shotgun blasts on the other hand are rather bad, cause if you collect three shotgun blasts during your turn, the turn is over for you and you get no points. After rolling three dice, you may decide if you want to *stop* and score your current brain collection or if you want to *push* your luck by grabbing new dices so you have three again and roll once more. If you cannot grab enough dices to have 3 to roll in the next throw you have to *stop* and earn your current gain. Once you *stop* you add the number of brain to your current points, return all dices in the box, and pass it to the next player.
The box contain 13 dices, 3 red, 4 orange and 6 green. The dices are as follows:

green: 3 brains, 2 steps, 1 shotguns;

orange: 2 brains, 2 steps, 2 shotguns;

red: 1 brains, 2 steps, 3 shotguns.

An example

Player A start the game. He currently have 0 points. He grab 3 dices (2 green, one red), roll them, and get 2 brains and 1 step (on the red). Having less then 3 shotgun he decide to *push*, grab 2 new dices (he has to reroll the red from the previous throw) and roll them, earning 2 brains and a shotgun. During this turn he has earned 4 brains and 1 shotgun (which is less than three) so he decide to stop, earning 4 points.
It's player B's turn. He grab 3 dices, and roll 1 brains and 2 shotgun. He decide to stop, earning only 1 point.
It's player A's turn. He grab 3 dices, and roll 2 brains and a shotgun. He decide to continue, grab 3 new dices, and unluckily roll 1 brain and 2 shotguns. Having reached 3 shotgun during this turn he looses all the brains from this turn. He still have the 4 points earned during the first turn.

A piece of advice

- Sometimes an analytic answer (with close formula) is reachable. This is very uncommon, but might be worth noticing.
- Sometimes an exact answer can be obtained by tedious computation. The computer is better than you at this game: use it!
- Sometimes an exact answer is also out of reach, but can be estimated by simulation. Estimation through simulation is a good idea for quick results, for checking your computations or when an exact answer is out of reach.
- The fact that I ask you to solve the problem does not mean that an exact solution is within your reach. Approximate solution are welcome, as long as you clearly state why and how you obtain this solution. In particular: is it an exact solution of an approximate problem, is it an approximate solution of the original problem, is it an approximate solution of an approximate problem...

1 A simple coin game

Question 1. We start with some simplified version of the problem. Here you throw a single coin (instead of 3 dices), returning a brain with probability p and a shotgun otherwise. In this simplified version, a single shotgun will stun you (hence making you loose all brains from this turn).

(a) Justify that, in this simplified version, it is equivalent to choose a fixed number of throw before stopping.

(b) What is the policy maximizing the expected gain of a turn ? An analytic answer is possible.

(c) Write a code to check by simulation, for various p, that the proposed policy is indeed optimal.

2 Single color game

Question 2. In this question we assume that we have a bag with an infinite number of dices, all of them green. We can still only gain a maximum of 13 brains in one turn. Our objective is to determine a strategy that maximize the expected number of points made in one turn.

(a) Justify that a strategy cannot be reduced to a deterministic number of throws.

(b) Justify that the output of a roll of 3 dices, in this setting, can be summed up by the number of brains and shotgun obtained. Compute the law of the output of a roll.

(c) Write a Dynamic Programming equation to solve this problem. (A first step can be to detail a relevant controlled Markov Chain.)

(d) Numerically solve this problem giving the optimal strategy and expected cost.

(e) Notice that the optimal strategy has a remarkable form and can be summed up with 3 parameters. Numerically check that the three parameters you obtained are optimal.

(f) Now assume that the 13 dices are orange. Compute the optimal strategy and expected cost. Same with 13 red dices.

Question 3. In this question we assume that we have only 13 dices in the bag. They are still green. Our objective is to determine a strategy that maximize the expected number of points made in one turn.

(a) Without any computation can you compare the optimal value of the above problem (with unlimited dices) and of this problem ?

(b) Write a Dynamic Programming equation to solve this problem.

(c) Numerically solve this problem, giving optimal strategy and expected cost.

(d) Can you compare the optimal strategy in this setting and in the previous one ?

3 Single player game - infinite bag

We now consider a bag with an infinite number of dices. When a player grab a dice he has 6/13 chance of it being green, 4/13 chance of it being orange and 3/13 chance of being red.

Question 4. In this question we assume that the player have an "all-in" strategy: it stop only once he has 13 brains (or is stunned)

(a) For each possible color of dice kept in front of you (because you rolled "steps" on them), compute the law of the ouput.

(b) Model the game, under the "all-in" strategy, by a Markov Chain and compute its transition kernel.

(c) (Bonus Question) Compute the probability of getting the 13 brains.

Question 5. We are now looking for the strategy maximizing the expected number of points during a turn.

(a) Before any computation, can you relate the optimal value of this problem to the optimal values computed in question 2 ?

(b) Write a Dynamic Programming equation to solve this problem.

(c) Numerically solve this problem, and give the optimal expected cost.

4 Single player game - 13 dices bag

We now consider the single player game with a bag of 13 dices, 6 green, 4 orange, 3 red. The aim is to maximize the expected number of point during a turn.

Question 6. Justify that we can model the game as a Controlled Markov Chain with (less than) $7 \times 5 \times 4 \times 4^3 \times 14 \times 3 + 1$ states. Write a Dynamic Programming equation associated with this model.

Question 7. This approach is numerically too demanding. Propose an alternative approach and test it.

5 What else ?

Question 8. Discuss the difference between the problems you have solved and the original problem of finding an optimal strategy for the game. Underline foreseen difficulties and ideas you might have to solve the original problem.