RAPID STABILIZATION OF A DEGENERATE PARABOLIC EQUATION USING A BACKSTEPPING APPROACH: THE CASE OF A BOUNDARY CONTROL ACTING AT THE DEGENERACY

PIERRE LISSY[1] AND CLAUDIA MORENO[2]

Abstract. The rapid exponential stabilization of a degenerate parabolic equation on a bounded interval with a left Dirichlet control is studied in this paper. Our approach introduces a backstepping transformation of Fredholm type, in order to force the solution of the closed-loop system to decay exponentially to zero with an arbitrary decay rate. The existence, continuity and invertibility of the transformation are obtained thanks to the study of Bessel functions of the first and the second kind, together with abstract results on Riesz bases and some specific properties satisfied by Fredholm operators.

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1. Introduction

In the last years, a lot of works have been devoted to the rapid stabilization of one-dimensional nondegenerate parabolic equations by a boundary control (see *e.g.* [2, 5, 21, 24, 14, 20] and the references therein). Amongst the methods that were developed to prove the boundary stabilization of one-dimensional partial differential equations (PDEs), one of particular interest is the backstepping method, studied intensively notably by Krstic and its collaborators, see *e.g.* [19]. The main idea of the method is to exhibit an invertible integral operator that transforms the original boundary control problem into an exponentially stable system at a desired fixed rate, called in what follows the target system. The original backstepping method for PDEs relied on an integral transformation of Volterra type, which has the advantage of being automatically invertible, but has the drawback that the integral kernel appearing in the transformation is a solution of a PDE posed

[1]CEREMADE, Université Paris-Dauphine & CNRS UMR 7534, Université PSL, 75016 Paris, France (lissy@ceremade.dauphine.fr).

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[2]CEREMADE, Université Paris-Dauphine & CNRS UMR 7534, Université PSL, 75016 Paris, France (claudia.moreno-grueso@dauphine.psl.eu).

on a triangular problem, so that the existence of the kernel might be difficult to obtain*. Here, we will rather consider an integral transformation of a Fredholm type, that was introduced in [3] in the context of stabilization of PDEs. It has exactly the opposite advantages and drawbacks: the kernel appearing in the transformation is now a solution of a PDE posed on a rectangular domain, but the invertibility of the transformation is more difficult to obtain. Here, we will follow the abstract approach developed in [11].

Let us now present the model under study. Our goal is to obtain a rapid stabilization result for the following degenerate parabolic equation

$$\begin{cases} \partial_t u &= (x^\alpha \partial_x u)_x, & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u(t,0) &= U(t),\ u(t,1) = 0, & t \in \mathbb{R}^+, \\ u(0,x) &= u_0(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where U denotes the control. The initial condition u_0 will be chosen in $L^2(0,1)$. Here, we consider the case where $\alpha \in (0,1)$, so that our equation corresponds to what is called a weakly degenerate case according to the terminology of [8]. Notice that the stabilization by the backstepping method has been proved in [17] in the case where the control U is localized at point $x = 1$ (see also [7] for a result of finite-time stabilization with a distributed control, by a totally different technique relying on spectral tools). Here, the situation is different since we impose a Dirichlet control at point $x = 0$, where the degeneracy of the elliptic operator $(x^\alpha \partial_x)_x$ holds, which turns out to be a more difficult question. From the point of view of controllability, null-controllability for (1.1) has been proved in [18] by an indirect method based on the auxiliary study of the corresponding wave equation and a transmutation method. A study of the cost of control in this context, based on the moment method, has been proposed in [10]. For more information about controllability properties of degenerate parabolic equations, we refer to [9] and the references therein.

To study the stabilization of the system (1.1), as already mentioned, we will use the backstepping method. We focus on the construction of an appropriate Fredholm transformation, in the spirit of [11], which has emerged as an alternative to the Volterra transformation during the last years (see also [12, 13, 28, 17, 15, 16]). Let us emphasize that our main stabilization result can be obtained from the abstract theorem given in [1, Theorem 1.6], but the backstepping method has some additional advantages (see [17, Remark 2]) that make relevant the present study.

Let us now emphasize the main difficulties that arise compared to [17] (where the control was localized at point $x = 1$), that are of technical nature but make the present situation different and somehow trickier. Of course, the

*Notably, in the present paper, using a Volterra transformation would lead to solve a wave equation in two dimensions, that is degenerate, on a triangular domain, where both the functional setting and the qualitative properties of the equation are unclear spectral properties of the underlying elliptic operator are the same in both cases, but the behaviour of the control operator changes: the asymptotic behaviour of the normal derivative of the eigenvectors at point $x = 0$ (which is related to the adjoint operator of our control operator) is different from the one at point $x = 1$ (see (1.24) and (1.25)). This difference requires to work in different fractional weighted Sobolev spaces (see notably (2.14) and Proposition 3.4). Another important difference comes from the kernel introduced in (2.7): in [17], the corresponding ψ_n was expressed only in terms of Bessel functions of the first kind. Here, ψ_n is expressed as an appropriate linear combination of Bessel function of the first and the second kind (see the formula given in (2.12)). Bessel functions of the second kind are less standard than Bessel functions of the first kind, so that our study notably requires to prove appropriate estimates for these functions, that differ from the corresponding ones for Bessel functions of the first kind (see notably Lemmas 1.8 and 1.9). These estimates are crucial to obtain the Riesz basis property given in Proposition 3.4, that constitutes the core of our proof and is more delicate to prove than in [17].

Now, we present our main result.

Theorem 1.1. *For any $\lambda > 0$, there exists $C(\lambda) > 0$ and a feedback law $U(t) = K(u(t))$, where $K \in L^2(0,1)'$, such that for any $u_0 \in L^2(0,1)$, there exists a unique solution u of* (1.1) *that verifies, for any $t \geq 0$:*

$$||u(t,\cdot)||_{L^2(0,1)} \leq C(\lambda)||u_0||_{L^2(0,1)}e^{-\lambda t}.$$

Remark 1. *In fact, the backstepping transformation we will use can only be constructed up to a discrete set $\mathscr{S} \subset \mathbb{R}$. This is not a problem in order to obtain Theorem 1.1: if $\lambda \in S$, consider any $\lambda' > \lambda$ such that $\lambda' \notin S$. Then, using the backstepping transformation leads to the rapid stabilization at exponential rate λ', so we also have rapid stabilization at the lower rate λ.*

The precise generic condition on $\lambda > 0$ to construct the backstepping transformation is given in (1.27).

The plan of the paper is as follows: in Section 2, we introduce some basic properties of the parabolic operator associated with the degenerate equation (1.1), and technical results about Bessel Function of the first and second kind. In Section 3, we present three linear operators in order to rewrite the degenerate parabolic equation (1.1) in an abstract form and we prove the existence of the kernel associated to the Fredholm transformation together with some additional properties. Then, in Section 4, we show that the transformation mapping the degenerate parabolic equation into the target system is a continuous and invertible Fredholm operator. At last, in Section 5, we prove Theorem 1.1.

1.1. **Some properties on the degenerate operator** A**.** Here, we summarize the presentation of [17] concerning the degenerate elliptic operator appearing in (1.1). Remind that $\alpha \in (0,1)$. We first define

$$H^1_\alpha(0,1) := \{f \in L^2(0,1) \,|\, x^{\frac{\alpha}{2}} f_x \in L^2(0,1)\},$$

endowed with the natural scalar product

$$\langle f, g\rangle_{H^1_\alpha} := \int_0^1 (fg + x^\alpha f_x g_x)\, dx, \qquad f, g \in H^1_\alpha(0,1), \tag{1.2}$$

that makes H^1_α a Hilbert space.
We also define

$$H^1_{\alpha,0}(0,1) := \{f \in H^1_\alpha(0,1) \,|\, f(0) = f(1) = 0\},$$

which is a closed subspace of $H^1_\alpha(0,1)$ that can be endowed with the scalar product

$$\langle f, g\rangle_{H^1_{\alpha,0}} := \int_0^1 (x^\alpha f_x g_x)\, dx, \qquad f, g \in H^1_{\alpha,0}(0,1), \tag{1.3}$$

which defines an equivalent norm to the norm coming from the scalar product (1.2).
We define the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ by:

$$Au := (x^\alpha u_x)_x,$$
$$D(A) := \{u \in H^1_{\alpha,0}(0,1) \,|\, x^\alpha u_x \in H^1(0,1)\},$$

where $D(A)$ endowed with the scalar product

$$\langle f, g\rangle_{\widetilde{D(A)}} := \int_0^1 (fg + x^\alpha f_x g_x + (x^\alpha f_x)_x\,(x^\alpha g_x)_x)\, dx, \qquad f, g \in D(A), \tag{1.4}$$

that makes $D(A)$ a Hilbert space.
Then, there exists a Hilbert basis $\{\phi_n\}_{n\in\mathbb{N}^*}$ of $L^2(0,1)$ and an increasing sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ of real positive numbers such that $\lambda_n \to +\infty$ and

$$-A\phi_n = \lambda_n \phi_n. \tag{1.5}$$

Moreover, we have

$$\lambda_n := (\kappa j_{\nu,n})^2, \quad n \in \mathbb{N}^*, \tag{1.6}$$

and

$$\phi_n(x) = \frac{\sqrt{2\kappa}}{J'_\nu(j_{\nu,n})} x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,n} x^\kappa), \quad x \in (0,1), \quad n \in \mathbb{N}^*, \tag{1.7}$$

where ν and κ are two parameters given by

$$\nu := \frac{1-\alpha}{2-\alpha} \text{ and } \kappa := \frac{2-\alpha}{2}. \tag{1.8}$$

Here, J_ν is the Bessel function of the first kind of order ν and of the first kind, and $(j_{\nu,n})_{n\in\mathbb{N}^*}$ is the strictly increasing sequence of the zeros of J_ν, that are all simple and positive. For more information about Bessel functions, we refer to the monographs [26, 22, 6]. Remind that we have the expansion

$$J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}, \tag{1.9}$$

where the Γ function is defined for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ by

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

Let us finally remark that we can characterize $D(A)$ and $H^1_{\alpha,0}(0,1)$ as follows:

$$D(A) = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in L^2(0,1) \,\middle|\, \sum_{n=1}^{\infty} \lambda_n^2 a_n^2 < \infty \right\},$$

so that

$$D(A)' = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in \mathscr{D}'(0,1) \,\middle|\, \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2} < \infty \right\}, \tag{1.10}$$

and

$$H^1_{\alpha,0}(0,1) = D(A^{\frac{1}{2}}) = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in L^2(0,1) \,\middle|\, \sum_{n=1}^{\infty} \lambda_n a_n^2 < \infty \right\},$$

so that

$$H^1_{\alpha,0}(0,1)' = D(A^{\frac{1}{2}})' = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in \mathscr{D}'(0,1) \,\middle|\, \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n} < \infty \right\}. \tag{1.11}$$

To conclude, for $s \in [0,1]$, we introduce

$$D(A^s) = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in L^2(0,1) \,\middle|\, \sum_{n=1}^{\infty} \lambda_n^{2s} a_n^2 < \infty \right\} \tag{1.12}$$

and

$$D(A^s)' = \left\{ f = \sum_{n=1}^{\infty} a_n \phi_n \in \mathscr{D}'(0,1) \,\middle|\, \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^{2s}} < \infty \right\}. \tag{1.13}$$

All these spaces are endowed with the natural scalar product induced by their definitions, and the associated norm is equivalent to the norm associated with the scalar product given in (1.3) for $s = 1/2$ and (1.4) for $s = 1$. We also emphasize that for any $s \in [0,1]$, the operator A can be uniquely extended from $D(A^s)$ to $D(A^s)'$ (see [25, Section 3.4]). We will still denote by A these extensions.

1.2. **Some technical Lemmas.** In what follows, we will need various technical results about Bessel functions, the eigenvectors ϕ_n and the eigenvalues λ_n introduced in the previous Section. The goal of this Section is to gather all these results. In what follows, we will denote by Y_ν the Bessel function of the second kind of order ν ($\nu \in (0,1)$) and of the second kind, and $(y_{\nu,n})_{n\in\mathbb{N}^*}$ the strictly increasing sequence of the zeros of Y_ν, that are all simple and positive. By definition,

$$Y_\nu(x) := \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}. \tag{1.14}$$

Remind that the following orthogonality property holds (see *e.g.* [18, Section 4.3.1].

Lemma 1.2.

$$\int_0^1 x^{1-\alpha} J_\nu(j_{\nu,n}x^\kappa) J_\nu(j_{\nu,m}x^\kappa)dx = \frac{\delta_{nm}}{2\kappa} J_\nu'(j_{\nu,n})^2. \tag{1.15}$$

The two following formulas come respectively from [22, Section 11.2, (2) and (4)].

Lemma 1.3. *For all* $(a,b) \in \mathbb{R}$ *such that* $a \neq b$*,*

$$\int_0^1 xJ_\nu(ax)J_\nu(bx)dx = \frac{1}{a^2-b^2}\left[bJ_\nu(a)J_\nu'(b) - aJ_\nu'(a)J\nu(b)\right], \tag{1.16}$$

$$\begin{aligned}\int_0^1 xJ_\nu(ax)Y_\nu(bx)dx = &\frac{1}{a^2-b^2}\left[bJ_\nu(a)Y_\nu'(b) - aJ_\nu'(a)Y_\nu(b)\right] \\ &- \lim_{x\to 0^+} \frac{x}{a^2-b^2}\left[bJ_\nu(ax)Y_\nu'(bx) - aJ_\nu'(ax)Y_\nu(bx)\right].\end{aligned} \tag{1.17}$$

Let us prove the following Lemma.

Lemma 1.4. *For all* $(a,b) \in \mathbb{R}$ *such that* $a \neq b$*,*

$$\lim_{x\to 0^+} \frac{x}{a^2-b^2}\left[bJ_\nu(ax)Y_\nu'(bx) - aJ_\nu'(ax)Y_\nu(bx)\right] = \frac{2\left(\frac{a}{b}\right)^\nu}{\pi\left(a^2-b^2\right)}. \tag{1.18}$$

Proof of Lemma 1.4.

From the Taylor expansion given in (1.9), we observe that

$$J_\nu(x) \sim \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \text{ as } x \to 0^+,\ J_\nu'(x) \sim \frac{\nu x^{\nu-1}}{2^\nu\Gamma(\nu+1)} \text{ as } x \to 0^+. \tag{1.19}$$

This formula is also valid if we replace ν by $-\nu$. Hence, using (1.14), we deduce that

$$Y_\nu(x) \sim -\frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)} \text{ as } x \to 0^+ \tag{1.20}$$

and

$$Y_\nu'(x) \sim \frac{\nu x^{-\nu-1}}{2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)} \text{ as } x \to 0^+.$$

Combining these expressions, we deduce that

$$\begin{aligned}&\frac{x}{a^2-b^2}\left[bJ_\nu(ax)Y_\nu'(bx) - aJ_\nu'(ax)Y_\nu(bx)\right] \\ &\sim \frac{x}{a^2-b^2}\left[b\frac{a^\nu x^\nu}{2^\nu\Gamma(\nu+1)}\frac{\nu b^{-\nu-1}x^{-\nu-1}}{2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)}\right. \\ &\left.+a\frac{\nu a^{\nu-1}x^{\nu-1}}{2^\nu\Gamma(\nu+1)}\frac{b^{-\nu}x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)}\right] \\ &\sim \left(\frac{a}{b}\right)^\nu \frac{2\nu}{(a^2-b^2)}\left[\frac{1}{\sin(\pi\nu)\Gamma(1-\nu)\Gamma(1+\nu)}\right] \text{ as } x \to 0^+.\end{aligned}$$

From the formula $\Gamma(\nu+1) = \nu\Gamma(\nu)$ and the complement formula for the Euler Gamma function (valid because $\nu \in (0,1)$), we have

$$\Gamma(\nu+1)\Gamma(1-\nu) = \frac{\nu\pi}{\sin(\pi\nu)}.$$

We deduce that (1.18) is verified. ∎

The following expansions of the zeros of the Bessel functions and of the derivative at the origin of the Bessel functions is classical (see *e.g.* [23, (1.14) and (1.24), Pages 436-437]).

Lemma 1.5. *As $n \to \infty$, we have*

$$j_{\nu,n} = \pi\left(n + \frac{\nu}{2} - \frac{1}{4}\right) - \frac{4\nu^2 - 1}{8n} + \mathscr{O}\left(\frac{1}{n^3}\right) \tag{1.21}$$

and

$$\left|J_\nu'(j_{\nu,n})\right| \sim \sqrt{\frac{2}{\pi j_{\nu,n}}}. \tag{1.22}$$

From (1.21) (see [17, (45) and (46), page 3839]), we notably obtain the following estimate (here and in all this paper, $\sqrt{\cdot}$ has to be understood as the principal determination of the complex square root).

Lemma 1.6. *As $n \to \infty$, we have*

$$j_{\nu,n} - \frac{\sqrt{\lambda_n - \lambda}}{\kappa} = \frac{\lambda}{2 j_{\nu,n}\kappa^2} + \mathscr{O}\left(\frac{1}{n^3}\right) = \mathscr{O}\left(\frac{1}{n}\right). \tag{1.23}$$

We will also need estimates of the derivative of ϕ_n at the endpoints $x = 0$ and $x = 1$.

Lemma 1.7. *We have*

$$\phi_n'(1) = \sqrt{2\kappa}\,\kappa j_{\nu,n}, \tag{1.24}$$

and

$$\lim_{y \mapsto 0^+} y^\alpha \phi_n'(y) = \frac{(1-\alpha)\sqrt{2\kappa}\, j_{\nu,n}^\nu}{2^\nu J_\nu'(j_{\nu,n})\Gamma(1+\nu)}. \tag{1.25}$$

Proof of Lemma 1.7.

Differentiating expression (1.7) leads to

$$\phi_n'(y) = \frac{\sqrt{2\kappa}}{J_\nu'(j_{\nu,n})}\left[\frac{1-\alpha}{2} y^{-\frac{1+\alpha}{2}} J_\nu(j_{\nu,n} y^\kappa) + y^{\frac{1-\alpha}{2}} \kappa j_{\nu,n} J_\nu'(j_{\nu,n} y^\kappa) y^{\kappa-1}\right]. \tag{1.26}$$

Taking $x = 1$ and using that $j_{\nu,n}$ is a zero of J_ν immediately leads to (1.24), whereas using (1.19) together with (1.8) and (1.9) gives immediately (1.25). ∎

In all what follows, we will assume that $\lambda > 0$ is chosen is such a way that

$$\lambda_n - \lambda \neq 0,\ \lambda_n - \lambda \neq \lambda_k,\ \ \lambda_n - \lambda \neq \kappa^2 y_{\nu,k}^2,\ \ \text{for any } k, n \in \mathbb{N}^*, \quad (1.27)$$

the last condition being new compared to [17]. This generic condition is justified by the fact that we do not want $\sqrt{(\lambda_n - \lambda)/\kappa}$ to be equal to a root of J_ν or Y_ν, for technical reasons (see for instance the expression of B_n given in (2.11), and also the proofs of Lemmas 1.8 and 1.9). Under these conditions, one can obtain the following result.

Lemma 1.8. *As $n \to \infty$, we have*

$$J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) = -\frac{\lambda J'(j_{\nu,n})}{2 j_{\nu,n} \kappa^2} + \mathscr{O}\left(\frac{1}{n^{\frac{7}{2}}}\right). \quad (1.28)$$

Notably, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that, for any $n \in \mathbb{N}^*$, we have

$$\frac{C_1}{n^{\frac{3}{2}}} \leq \left| J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)\right| \leq \frac{C_2}{n^{\frac{3}{2}}}. \quad (1.29)$$

Proof of Lemma 1.8. Assume that n is large enough such that $\lambda_n > \lambda$. Let us introduce

$$\varepsilon_n = j_{\nu,n} - \frac{\sqrt{\lambda_n - \lambda}}{\kappa}. \quad (1.30)$$

A Taylor expansion gives that there exists $e_n \in [j_{\nu,n} - \varepsilon_n, j_{\nu,n}]$ such that

$$J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) = -\varepsilon_n J_\nu'(j_{\nu,n}) + \frac{\varepsilon_n^2}{2} J_\nu''(e_n). \quad (1.31)$$

From [17, (50), Page 3840], we have

$$J_\nu''(e_n) \sim -\frac{J_\nu'(j_{\nu,n})}{j_{\nu,n}} \text{ as } n \to \infty.$$

Taking into account (1.30) and (1.23), we deduce that (1.28) holds. To conclude, (1.29) comes from (1.28) together with (1.21) and (1.22), since (1.29) has to be proved only for n large enough by (1.27). ■

In the same spirit, we will also prove the following property.

Lemma 1.9.

$$Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) = -\frac{2}{J_\nu'(j_{\nu,n}) \pi j_{\nu,n}} + \mathscr{O}\left(\frac{1}{n}\right). \quad (1.32)$$

Notably, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that, for any $n \in \mathbb{N}^*$, we have

$$\frac{C_1}{\sqrt{n}} \leq \left| Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)\right| \leq \frac{C_2}{\sqrt{n}}. \quad (1.33)$$

Proof of Lemma 1.9.

Assume that n is large enough such that $\lambda_n > \lambda$. We have

$$Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) = Y_\nu\left(j_{\nu,n}\right) + Y_\nu'(c)\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa} - j_{\nu,n}\right),$$

for some $c \in \left[\frac{\sqrt{\lambda_n-\lambda}}{\kappa}, j_{\nu,n}\right]$. Moreover, usual recurrence relations on Bessel functions give

$$Y_\nu'(c) = \frac{\nu Y_\nu(c)}{c} - Y_{\nu+1}(c).$$

Since $Y_{\nu+1}$ and $\frac{\nu Y_\nu(\cdot)}{\cdot}$ are bounded on $[1,\infty)$, we deduce that when $n \to +\infty$,

$$Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right) = Y_\nu\left(j_{\nu,n}\right) + \mathscr{O}\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa} - j_{\nu,n}\right).$$

From (1.6) and (1.23), we have

$$Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right) = Y_\nu\left(j_{\nu,n}\right) + \mathscr{O}\left(\frac{1}{n}\right). \tag{1.34}$$

It remains to investigate the behaviour of $Y_\nu\left(j_{\nu,n}\right)$. From the Wronskian identity given in [22, Page 29, (2)], we have, for any $x > 0$,

$$J_\nu(x)J_{-\nu}'(x) - J_{-\nu}(x)J_\nu'(x) = -\frac{2\sin(\nu\pi)}{\pi x}.$$

Applying this identity at point $x = j_{\nu,n}$, we deduce that

$$J_{-\nu}(j_{\nu,n}) = \frac{2\sin(\nu\pi)}{J_\nu'(j_{\nu,n})\pi j_{\nu,n}}.$$

Using identity (1.14), we deduce that

$$Y_\nu(j_{\nu,n}) = -\frac{2}{J_\nu'(j_{\nu,n})\pi j_{\nu,n}}.$$

Gathering the previous computations gives (1.32). To conclude, (1.33) comes from (1.28) together with (1.21) and (1.22), since (1.33) has to be proved only for n large enough by (1.27). ■

2. Abstract setting: the operators T, B and K

Let us now present the target system that we will consider in this paper: for $\lambda > 0$ (that will also satisfy additional constraints, see (1.27)), we introduce

$$\begin{cases} \partial_t v & = (x^\alpha \partial_x v)_x - \lambda v, \quad (t,x) \in \mathbb{R}^+ \times (0,1), \\ v(t,0) & = v(t,1) = 0, \qquad t \in \mathbb{R}^+, \\ v(0,x) & = v_0(x), \qquad x \in (0,1). \end{cases} \tag{2.1}$$

Basic semigroup theory (see *e.g.* [25, Proposition 2.3.5]) implies that for any $v_0 \in D(A)$, there exists a unique solution v to (2.1) verifying

$$v \in C^1([0,\infty), L^2(0,1)) \cap C^0([0,\infty), D(A)),$$

and for any $v_0 \in L^2(0,1)$, there exists a unique solution v to (2.1) (the first line being verified in $D(A)'$ for any $t \in [0,\infty)$) verifying

$$v \in C^1([0,\infty), D(A)') \cap C^0([0,\infty), L^2(0,1)).$$

Basic energy estimates (see [17, Page 3629]) show that for any $t \geq 0$,

$$V(t) \leq V(0)e^{-2\lambda t}, \tag{2.2}$$

where $V(t) := \int_0^1 v(t,x)^2 dx$, at least formally. Then, (2.2) can be proved rigorously by an easy density argument, for any $v_0 \in L^2(0,1)$.

Our general goal is to exhibit an adequate invertible transformation in order to reduce the original question of rapid stabilization of (1.1) to the stability of (2.1).

The goal of the present Section is to introduce some linear operators that enable us to write (1.1) in an abstract form, with an appropriate feedback operator at the boundary $x = 0$. Let us introduce the linear operator given by

$$Tf : x \longmapsto f(x) - \int_0^1 k(x,y)f(y)dy, \tag{2.3}$$

where k is chosen in an appropriate functional space, in such a way that if u is the solution of (1.1), then $v = Tu$ verifies the target system (2.1), with initial condition

$$v_0(x) = u_0(x) - \int_0^1 k(x,y)u_0(y)dy. \tag{2.4}$$

Let us derive formally the equation that the kernel k should verify. Using (2.3) with $f = u$ and (1.1), we have

$$\begin{aligned} \partial_t v(t,x) &= \partial_t u(t,x) - \int_0^1 k(x,y)\partial_t u(t,y)dy \\ &= (x^\alpha \partial_x u)_x - \int_0^1 k(x,y)(y^\alpha \partial_y u)_y dy. \end{aligned}$$

From the first line of (2.1), we deduce that

$$(x^\alpha \partial_x v)_x - \lambda v = (x^\alpha \partial_x u)_x - \int_0^1 k(x,y)(y^\alpha \partial_y u)_y dy. \tag{2.5}$$

Using one more time (2.3) with $f = u$, we obtain

$$\begin{aligned} &(x^\alpha \partial_x u)_x - \int_0^1 u(t,y)[(x^\alpha k_x(x,y))_x - \lambda k(x,y)]dy - \lambda u(t,x) \\ &= (x^\alpha \partial_x u)_x - \int_0^1 k(x,y)(y^\alpha u_y(t,y))_y dy. \end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned} &- \int_0^1 u(t,y)[(x^\alpha k_x(x,y))_x - \lambda k(x,y)]dy - \lambda u(t,x) \\ &= - \int_0^1 (y^\alpha k_y(x,y))_y u(t,y)dy \\ &- k(x,y)y^\alpha u_y(t,y)|_0^1 + k_y(x,y)y^\alpha u(t,y)|_0^1. \end{aligned}$$

Since u verifies Dirichlet boundary conditions and taking into account the relations given in (1.24) and (1.25), it is reasonable to cancel the boundary terms by imposing the conditions $k(x,1) = k(x,0) = k_y(x,0) = 0$ and $\lim_{y\to 0^+} y^\alpha k_y(x,y) = 0$. Hence, the kernel k has to satisfy the following system:

$$\begin{cases} -(y^\alpha k_y(x,y))_y + (x^\alpha k_x(x,y))_x - \lambda k(x,y) & = -\lambda\delta_{x=y}, \quad (x,y)\in(0,1)^2, \\ \lim_{y\to 0^+} y^\alpha k_y(x,y) & = 0, \quad (x,y)\in(0,1)^2, \\ k(x,0) & = 0, \quad x\in(0,1), \\ k(x,1) & = 0, \quad x\in(0,1), \\ k(1,y) & = 0, \quad y\in(0,1). \end{cases} \tag{2.6}$$

Let us decompose the kernel k formally as

$$k(x,y) = \sum_{n\in\mathbb{N}^*} \psi_n(x)\phi_n(y). \tag{2.7}$$

Formally, the system (2.6) is equivalent to solving for any $n\in\mathbb{N}^*$ the system

$$\begin{cases} \lambda_n\psi_n(x) + (x^\alpha\partial_x\psi_n(x))_x - \lambda\psi_n(x) & = -\lambda\phi_n(x), \quad x\in(0,1), \\ \psi_n(1) & = 0, \\ \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \psi_n(x)\phi_n'(y) & = 0, \quad x\in(0,1). \end{cases} \tag{2.8}$$

Using the change of unknowns

$$\psi_n = \phi_n - \xi_n,$$

and taking into account the fact that $T\phi_n = \phi_n - \psi_n$ (where T is defined in (2.3)) and (1.5), this is equivalent to solving

$$\begin{cases} -(\lambda_n-\lambda)\xi_n(x) - (x^\alpha\partial_x\xi_n(x))_x & = 0, \quad x\in(0,1), \\ \xi_n(1) & = 0, \\ \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \xi_k(x)\phi_k'(y) & = \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_k(x)\phi_k'(y), \quad x\in(0,1). \end{cases} \tag{2.9}$$

We introduce the Sturm-Liouville problem

$$\begin{cases} -(x^\alpha y'(x))' & = \mu y(x), \quad x\in(0,1),\ \mu\in\mathbb{R}, \\ y(0) & = 0, \\ y(1) & = 0. \end{cases} \tag{2.10}$$

According to [18, Section 4.3.2], the solution of the first line of (2.10) can be written as

$$y(x,\mu) = x^{\frac{1}{2}(1-\alpha)} Z_\nu\left(\frac{\sqrt{\mu}x^\kappa}{\kappa}\right),\ x\in(0,\infty),$$

where Z_ν is any Bessel function of order ν. The solution to the first line of system (2.9) can be written under the form

$$\xi_n(x) = A_n x^{\frac{1}{2}(1-\alpha)} J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right) + B_n x^{\frac{1}{2}(1-\alpha)} Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right),$$

for some real numbers A_n, B_n.

Since we impose $\xi_n(1) = 0$, we choose A_n, B_n such that

$$A_n J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) + B_n Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right) = 0.$$

From Assumption (1.27) and (1.6), we have that $Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right) \neq 0$, so that we can solve this equation as

$$B_n = -\frac{A_n J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}. \tag{2.11}$$

Hence, by comparison with the expression (1.7), since we would like $\widetilde{\xi}_n$ to be "close" in some sense to ϕ_n, it is reasonable to consider

$$\widetilde{\xi}_n(x) = \frac{\sqrt{2\kappa}x^{\frac{1-\alpha}{2}}}{J'(j_{\nu,n})}\left(J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right) - \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right)\right), \tag{2.12}$$

and to write ψ_n under the form

$$\psi_n = \phi_n - c_n\tilde{\xi}_n, \tag{2.13}$$

where c_n is a number to be determined.
Let us now us justify rigorously that in this case, a duality argument enables us to define a control operator $B : \mathbb{R} \to D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$. This will be a consequence of abstract interpolation theorems between Banach Spaces (see *e.g.* [4, Section 4.1]). Let us consider the operator

$$C : g \mapsto x^\alpha g_x.$$

Then, C is linear continuous from $D(A^{1/2})$ to $L^2(0,1)$ (because for any $g \in H^1_{\alpha,0}(0,1) = D(A^{1/2})$, we have $||x^\alpha g_x||_{L^2(0,1)} \leq ||x^{\alpha/2}g_x||_{L^2(0,1)} \leq ||g||_{H^1_{\alpha,0}(0,1)}$ by (1.3)) and is also linear continuous from $D(A)$ to $H^1(0,1)$ (because for any $g \in D(A)$, we have $||x^\alpha g_x||_{H^1(0,1)} \leq ||g||_{D(A)}$ by a similar argument and (1.4)). Hence, for any $s \in (1/2, 1)$, C is also linear continuous from $D(A^s)$ to $H^{2s-1}(0,1)$, since for $\theta = 2s-1 \in (0,1)$, the interpolation space $[D(A^{1/2}), D(A)]_\theta$ is $D(A^s)$ and the interpolation space $[L^2(0,1), H^1(0,1)]_\theta$ is $H^{2s-1}(0,1)$. Here, since $\nu \in (0,1/2)$ by (1.8), for $s = \frac{\nu}{2} + \frac{3}{4} \in (3/4, 1)$, we obtain that C is continuous from $D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)$ to $H^{\nu+\frac{1}{2}}(0,1)$, which is also continuously embedded in $C^0([0,1])$ because $\nu > 0$ by (1.8).
We deduce that $\varphi \in D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right) \mapsto \lim\limits_{x\to 0^+} x^\alpha\varphi'(x)$ is well-defined and continuous. Hence, for any $z \in \mathbb{R}$, we can define by duality the linear continuous

operator $B : \mathbb{R} \to D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$ as follows:

$$\langle Bz, \varphi \rangle_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)', D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)} = z \lim_{x \to 0^+} x^\alpha \varphi'(x), \; z \in \mathbb{R}, \; \varphi \in D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right). \tag{2.14}$$

We also introduce the linear operator K given by

$$Kf : x \longmapsto \int_0^1 k(0, y) f(y) dy. \tag{2.15}$$

The existence of K will be proved later on. With these notations and setting $U = Ku$, equation (1.1) can be rewritten in an abstract way as

$$\begin{array}{lll} \partial_t u & = (A + BK)u, & t \in (0, T), \\ u(0, \cdot) & = u_0, & x \in (0, 1). \end{array}$$

To conclude, let us introduce an appropriate functional setting for $A + BK$. Let us call

$$H^1_{\alpha,R}(0, 1) := \{f \in H^1_\alpha(0, 1) \,|\, f(1) = 0\}$$

and

$$D(A)_R := \{f \in H^1_{\alpha,R}(0, 1) \,|\, x^\alpha f_x \in H^1(0, 1)\}.$$

Finally, we define

$$D(A + BK) = \{f \in D(A)_R \,|\, f(0) = Kf\},$$

where Kf has been defined in (2.15).

Remark 2. *Notice that the computations leading to the expression of the kernel k are purely formal at this stage. However, in some sense, these computations will be made more rigorous in Section 3.4; this is exactly the purpose of the functional identities $T(A + BK) = (A - \lambda I)T$ and $TB = B$.*

3. Proof of Theorem 1.1

One of the main ingredient is the following Theorem, that will be proved in the next subsections.

Theorem 3.1. *Assume that* (1.27) *holds. There exists a unique sequence $(c_n)_{n \in \mathbb{N}^*}$ such that*

$$c_n - 1 \in l^2(\mathbb{N}^*) \tag{3.1}$$

and such that for any $n \in \mathbb{N}^*$, the corresponding ψ_n defined in (2.13) *verifies* (2.8). *Moreover, the corresponding kernel k defined by the formula* (2.7) *verifies $k \in L^2((0, 1)^2)$.*

3.1. **A Riesz basis property.** The first ingredient of our proof is a Riesz basis property for the family $\{\widetilde{\xi}_n\}_{n\in\mathbb{N}^*}$. Let us recall the following definition.

Definition 3.2. *Let H be a Hilbert space and $\{g_n\}\subset H$. The sequence $\{g_n\}$ is said to be ω-independent if for any sequence $(a_n)_{n\in\mathbb{N}^*}$ of real numbers,*

$$\sum_{n\in\mathbb{N}^*} a_n g_n = 0 \text{ and } \sum_{n\in\mathbb{N}^*} |a_n|^2 ||g_n||_H^2 < \infty \Rightarrow a_n = 0,\ \forall n\in\mathbb{N}^*. \tag{3.2}$$

Then, one can prove the following result (see [27, Theorem 15] and [11, Theorem 3.3].

Proposition 3.3. *Let H be an infinite dimensional separable Hilbert space and let $\{e_n\}_{n\in\mathbb{N}^*}$ be a Hilbert basis of H. If $\{g_n\}_{n\in\mathbb{N}^*}$ is quadratically close to $\{e_n\}_{n\in\mathbb{N}^*}$, i.e. $\sum_{n\in\mathbb{N}^*}\|e_n-g_n\|_H^2<+\infty$, and $\{g_n\}_{n\in\mathbb{N}^*}$ is either a ω-independent sequence in the sense of Definition 3.2 or a complete sequence in H, then $\{g_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis of H.*

Proposition 3.3 enables to prove the following result.

Proposition 3.4. *The family $\{(\lambda_n)^{\frac{\nu}{2}+\frac14}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis in $D(A^{\frac{\nu}{2}+\frac14})'$.*

Proof of Proposition 3.4.

First step:$\{(\lambda_n)^{\frac{\nu}{2}+\frac14}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ **is quadratically close to** $\{(\lambda_n)^{\frac{\nu}{2}+\frac14}\phi_{n\in\mathbb{N}^*}\}$**.** We study the scalar product

$$\begin{aligned}\langle \phi_n-\widetilde{\xi}_n,\phi_k\rangle &= \int_0^1 \frac{(2\kappa)x^{1-\alpha}}{J'(j_{\nu,n})J'(j_{\nu,k})} J_\nu(j_{\nu,n}x^\kappa)J_\nu(j_{\nu,k}x^\kappa)dx\\ &\quad - \int_0^1 \frac{(2\kappa)x^{1-\alpha}}{J'(j_{\nu,n})J'(j_{\nu,k})} J_\nu(j_{\nu,k}x^\kappa)\left(J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right)\right.\\ &\quad\left. -\frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)} Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right)\right)dx.\end{aligned} \tag{3.3}$$

By using the change of variable $y=x^\kappa$, then $dx=\frac{dyx^{1-\kappa}}{\kappa}$ and we obtain, using (1.15) for the first term and (1.8),

$$\begin{aligned}\langle \phi_n-\widetilde{\xi}_n,\phi_k\rangle &= \delta_{nk} - \int_0^1 \frac{2y}{J_\nu'(j_{\nu,n})J_\nu'(j_{\nu,k})} J_\nu(j_{\nu,k}y)J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}y\right)dy\\ &\quad + \int_0^1 \frac{2y}{J_\nu'(j_{\nu,n})J_\nu'(j_{\nu,k})}\frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)} J_\nu(j_{\nu,k}y)Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}y\right)dy.\end{aligned} \tag{3.4}$$

By equation (1.16), the fact that $j_{\nu,k}$ is a root of J_ν and (1.6), we obtain

$$\int_0^1 yJ_\nu(j_{\nu,k}y)J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}y\right)dy = -\frac{\kappa^2 j_{\nu,k}}{(\lambda_k-\lambda_n+\lambda)}J_\nu'(j_{\nu,k})J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right). \tag{3.5}$$

From equations (1.17) and (1.18), the fact that $j_{\nu,k}$ is a root of J_ν and (1.6), we obtain that

$$\begin{aligned}&\int_0^1 yJ_\nu(j_{\nu,k}y)Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}y\right)dy\\&=-\frac{\kappa^2 j_{\nu,k}}{(\lambda_k-\lambda_n+\lambda)}J_\nu'(j_{\nu,k})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)-\frac{2\kappa^2\left(\frac{j_{\nu,k}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu}{\pi\left(\lambda_k-\lambda_n+\lambda\right)}.\end{aligned}\tag{3.6}$$

From (3.4), (3.5) and (3.6), we deduce that

$$\langle\phi_n-\widetilde{\xi}_n,\phi_k\rangle=\delta_{kn}-\frac{4\kappa^2\left(\frac{j_{\nu,k}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\pi\left(\lambda_k-\lambda_n+\lambda\right)J_\nu'(j_{\nu,n})J_\nu'(j_{\nu,k})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}.\tag{3.7}$$

In the case $n=k$, from (1.23), (1.28) and (1.32), we deduce that

$$\begin{aligned}\frac{4\kappa^2\left(\frac{j_{\nu,n}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\pi\lambda(J_\nu'(j_{\nu,n}))^2)Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}&=\left(\frac{4\kappa^2\left(\frac{j_{\nu,n}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu}{\pi\lambda(J_\nu'(j_{\nu,n}))^2}\right)\left(\frac{-\frac{\lambda J'(j_{\nu,n})}{2j_{\nu,n}\kappa^2}+\mathscr{O}\left(\frac{1}{n^{\frac{7}{2}}}\right)}{-\frac{2}{J_\nu'(j_{\nu,n})\pi j_{\nu,n}}+\mathscr{O}\left(\frac{1}{n}\right)}\right)\\&=1+\mathscr{O}\left(\frac{1}{n^{\frac{5}{2}}}\right)\\&=1+\mathscr{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Putting this estimate in (3.7) leads to

$$\langle\phi_n-\widetilde{\xi}_n,\phi_n\rangle=\mathscr{O}\left(\frac{1}{n^2}\right).\tag{3.8}$$

In the case $n\neq k$, from (1.6), (1.21), (1.22), (1.29) and (1.33), we deduce that for some $C,C'>0$ independent of n and k, we have

$$\begin{aligned}&\left|\left\langle\frac{{\lambda_n}^{\frac{\nu}{2}+\frac14}}{{\lambda_k}^{\frac{\nu}{2}+\frac14}}\left(\phi_n-\widetilde{\xi}_n\right),\phi_k\right\rangle\right|\\&=\frac{{\lambda_n}^{\frac{\nu}{2}+\frac14}}{{\lambda_k}^{\frac{\nu}{2}+\frac14}}\frac{4\kappa^2\left(\frac{j_{\nu,k}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\sqrt{\lambda_k}\pi\left(\lambda_k-\lambda_n+\lambda\right)J'(j_{\nu,n})J'(j_{\nu,k})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}\\&\leq C\frac{{\lambda_n}^{\frac{\nu}{2}+\frac14}}{{\lambda_k}^{\frac{\nu}{2}+\frac14}}\frac{k^\nu n^{-\nu}n^{-3/2}k^{1/2}n^{1/2}n^{1/2}}{|k^2-n^2|}\\&\leq\frac{C'}{|k^2-n^2|}.\end{aligned}\tag{3.9}$$

From (3.8) and (3.9), we deduce that there exists positive constants C_1, C_2 such that

$$\begin{aligned}||\phi_n - \widetilde{\xi}_n||^2_{D\left(A^{\frac{\nu}{2}+\frac{1}{4}}\right)'} &= \sum_{k\in\mathbb{N}^*} |\langle \phi_n - \widetilde{\xi}_n, \phi_k\rangle|^2 \\ &\leqslant C_1 \sum_{k\in\mathbb{N}^*, k\neq n} \leq \frac{1}{|k^2-n^2|^2} \\ &\leqslant \frac{C_2}{n^2},\end{aligned}$$

where the last inequality comes from [17, (55), page 3841]. This ends the proof of our first step.

Second step: $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_{n\in\mathbb{N}^*}\}$ is complete or ω-independent in $D\left(A^{\frac{\nu}{2}+\frac{1}{4}}\right)'$.

Let us consider

$$\begin{cases}(x^\alpha \partial_x b)_x &= 0, \;\; x\in(0,1),\\ b(0) &= 1,\\ b(1) &= 0.\end{cases} \tag{3.10}$$

One can solve explicitly (3.10) and deduce that there exists a unique solution given by $b(x) = 1 - x^{1-\alpha}$. Notably, $b \in L^2(0,1)$ and we can decompose b as $b = \sum_{n\in\mathbb{N}^*} b_n\phi_n$, with $(b_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. The coefficients b_n can be expressed as

$$b_n = \int_0^1 (1-x^{1-\alpha})\phi_n(x)\,dx = -\frac{1}{\lambda_n}\int_0^1 (1-x^{1-\alpha})(x^\alpha\phi_n')'\,dx. \tag{3.11}$$

An integration by parts gives

$$b_n = -\frac{1}{\lambda_n}\left([(x^\alpha - x)\phi_n']_0^1 - \int_0^1 (1-\alpha)\phi_n'\,dx\right) = -\frac{\lim_{x\to 0^+}(x^\alpha - x)\phi_n'(x)}{\lambda_n}. \tag{3.12}$$

From (1.25) together with the fact that $\alpha\in(0,1)$, we deduce that for all $n\in\mathbb{N}^*$,

$$b_n = -\frac{\lim_{x\to 0^+} x^\alpha\phi_n'(x)}{\lambda_n} \neq 0. \tag{3.13}$$

Now, let us investigate the behaviour of $\widetilde{\xi}_n$ at $x=0$. From (1.9) and (1.8), we have that

$$\lim_{x\to 0^+} \frac{x^{\frac{1-\alpha}{2}}}{J'(j_{\nu,n})} J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right) = 0. \tag{3.14}$$

By (1.14) and (1.9), we obtain that

$$Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right) \sim -\frac{(\frac{\sqrt{\lambda_n-\lambda}}{\kappa})^{-\nu}x^{-\kappa\nu}}{2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)} \text{ as } x\to 0. \tag{3.15}$$

From (3.14) and (3.15) together with (1.8) and the definition of $\tilde{\xi}_n$ given in (2.12), we obtain that

$$\lim_{x\to 0^+}\tilde{\xi}_n(x) = \lim_{x\to 0^+}\left[\frac{\sqrt{2\kappa}x^{\frac{1-\alpha}{2}}}{J'(j_{\nu,n})}\right]\left(-\frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}x^\kappa\right)\right)$$
$$= \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)^{-\nu}\left(\frac{\sqrt{2\kappa}}{J'(j_{\nu,n})2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)}\right). \tag{3.16}$$

Let us introduce

$$\beta_n := \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)^{-\nu}\left(\frac{\sqrt{2\kappa}}{J'(j_{\nu,n})2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)}\right), \tag{3.17}$$

so that $\beta_n = \widetilde{\xi_n}(0)$. We introduce the auxiliary function

$$\eta_n = \widetilde{\xi_n} - \beta_n b. \tag{3.18}$$

From (2.9) and (3.10), we deduce that η_n verifies

$$\begin{cases} \lambda_n\eta_n(x) + (x^\alpha\partial_x\eta_n(x))_x - \lambda\eta_n(x) = -(\lambda_n-\lambda)\beta_n b, \quad x\in(0,1),\\ \eta_n(0) = 0,\\ \eta_n(1) = 0. \end{cases}$$

Hence, $\eta_n \in D(A)$ and η_n verifies

$$-A\eta_n = (\lambda_n-\lambda)\eta_n + (\lambda_n-\lambda)\beta_n b. \tag{3.19}$$

As already explained in [17, Pages 3845-3846], it is in fact enough to prove that $\{\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is either complete, or ω-independent in $L^2(0,1)$. In order to prove that, we introduce a sequence of real numbers $(a_n)_{n\in\mathbb{N}^*}$ such that

$$\sum_{n\in\mathbb{N}^*}|a_n|^2||\tilde{\xi}_n||^2_{L^2(0,1)} < \infty$$

and

$$\sum_{n\in\mathbb{N}^*}a_n\tilde{\xi}_n = 0. \tag{3.20}$$

We remark that thanks to (3.7) and the fact that $\{\phi_k\}_{k\in\mathbb{N}^*}$ is a Hilbert basis of $L^2(0,1)$, we have

$$||\widetilde{\xi_n}||^2_{L^2(0,1)} = \sum_{k=1}^{+\infty}\left|\frac{4\kappa^2\left(\frac{j_{\nu,k}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\pi\left(\lambda_k-\lambda_n+\lambda\right)J'_\nu(j_{\nu,n})J'_\nu(j_{\nu,k})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}\right|^2.$$

Isolating the term $k=n$ and using (1.6) together with (1.21), (1.22), (1.29) and (1.33) gives that

$$||\widetilde{\xi_n}||^2_{L^2(0,1)} \geq \left| \frac{4\kappa^2 \left(\frac{j_{\nu,n}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\pi\lambda J'_\nu(j_{\nu,n})J'_\nu(j_{\nu,n})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}\right|^2 \geq C.$$

Notably, (3.20) implies that $(a_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Let us also remark that from (3.17) together with (1.22), (1.21) and (1.29), we have $\beta_n = \mathscr{O}\left(\frac{1}{n}\right)$.
The rest of the proof is very similar to [17, Pages 3843-3846], so that we only give a sketch of the proof. The first step is to remark that by an easy induction argument together with (3.18), (3.19) and (3.20), for any $p\in\mathbb{N}$, we have

$$\sum_{n\in\mathbb{N}^*} \frac{a_n}{(\lambda_n-\lambda)^p}\widetilde{\xi_n} = \sum_{j=1}^{p}\left(\sum_{n\in\mathbb{N}^*}\frac{a_n\beta_n}{(\lambda_n-\lambda)^j}\right)A^{j-p}b. \tag{3.21}$$

Then, we distinguish two cases.

- If one of the coefficients in the right-hand side of (3.21) does not vanish, let us denote by $j_0\in\mathbb{N}^*$ the first integer such that
$$\sum_{n\in\mathbb{N}^*}\frac{a_n\beta_n}{(\lambda_n-\lambda)^{j_0}} \neq 0. \tag{3.22}$$
Then, using (3.21) with $p=j_0$ and (3.22), we deduce that $b\in \operatorname{span}\{\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$, so that applying A^{-1} successively on (3.21) with $p=j_0$ implies that $\{A^{-k}b\}_{k\in\mathbb{N}} \in \operatorname{span}\{\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$. Following the arguments (based on complex analysis) given in [17, First case, 3844-3845], we can deduce that $\{\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is complete in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$. Since $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is also quadratically close to $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\phi_{n\in\mathbb{N}^*}\}$ in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$, so that Proposition 3.3 implies that $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$.
- On the contrary, if for any $p\in\mathbb{N}$, we have
$$\sum_{n\in\mathbb{N}^*}\frac{a_n\beta_n}{(\lambda_n-\lambda)^{p}} = 0,$$
following now the arguments (based on complex analysis) given in [17, Second case, 3845-3846], we can deduce that for any $n\in\mathbb{N}^*$, we have $a_n=0$, so that $\{\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is now ω-independent in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$. Since it is also quadratically close to $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\phi_{n\in\mathbb{N}^*}\}$ in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$, Proposition 3.3 implies that $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is a Riesz basis in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$.

■

3.2. **Existence of k.** Our goal is now to deduce the existence of $(c_n)_{n\in\mathbb{N}^*}$ satisfying the last equation of (2.9). As soon as the existence of the sequence $(c_n)_{n\in\mathbb{N}^*}$ is verified, using formula (2.13) (remind that $\tilde{\xi}_n$ is defined in (2.12)), we can deduce the existence of a solution of equation (2.6) and then we will obtain the kernel k thanks to equation (2.7). In order to use Riesz basis results, we rewrite the last equation of (2.9) as

$$\begin{aligned}\lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} c_n\widetilde{\xi}_n(x)\phi_n'(y) &= \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_n(x)\phi_n'(y)\\ &= \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(y)[\widetilde{\xi}_n(x)+\phi_n(x)-\widetilde{\xi}_n(x)].\end{aligned} \tag{3.23}$$

Using (1.25), (1.22) and (1.21), we deduce that for some $C_1, C_2 > 0$, we have

$$C_1 n^{\nu+\frac{1}{2}} \leq |\lim_{y\to 0^+} y^\alpha \phi_n'(y)| \leq C_2 n^{\nu+\frac{1}{2}}. \tag{3.24}$$

Since $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\tilde{\xi}_n\}_{n\in\mathbb{N}^*}$ is quadratically close to $\{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\phi_{n\in\mathbb{N}^*}\}$ in $D(A^{\frac{\nu}{2}+\frac{1}{4}})'$, from (1.6) and (1.21), we deduce that

$$\lim_{y\to 0^+} y^{2\alpha} \sum_{n\in\mathbb{N}^*} |\phi_n'(y)|^2 \| \phi_n(x)-\widetilde{\xi}_n(x)\|^2_{D(A^{\frac{\nu}{2}+\frac{1}{4}})'} < \infty. \tag{3.25}$$

Now, using (3.23) and introducing $d_n = c_n - 1$, we rewrite (3.23) as

$$\lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(y) d_n \widetilde{\xi}_n(x) = \lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} |\phi_n'(y)| [\phi_n(x)-\widetilde{\xi}_n(x)]. \tag{3.26}$$

Using (3.24), (3.25), (1.6), (1.21), and Proposition 3.4, we obtain that there exists $\{\widetilde{d}_n\}_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$ such that

$$\lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(y)(\phi_n(x)-\widetilde{\xi}_n(x)) = \sum_{n\in\mathbb{N}^*} (\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}} \tilde{\xi}_n \widetilde{d}_n,$$

which is equivalent to

$$\begin{aligned}\lim_{y\to 0^+} y^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(y)(\phi_n(x)-\widetilde{\xi}_n(x)) &= \sum_{n\in\mathbb{N}^*} \lim_{y\to 0^+} y^\alpha \phi_n'(y) \left(\frac{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\widetilde{d}_n}{\lim\limits_{y\to 0^+} y^\alpha \phi_n'(y)} \right) \widetilde{\xi}_n\\ &= \sum_{n\in\mathbb{N}^*} \lim_{y\to 0^+} y^\alpha \phi_n'(y) d_n \widetilde{\xi}_n,\end{aligned}$$

where the sequence d_n is given by

$$d_n = \frac{(\lambda_n)^{\frac{\nu}{2}+\frac{1}{4}}\widetilde{d}_n}{\lim\limits_{y\to 0^+} y^\alpha \phi_n'(y)}. \tag{3.27}$$

Notice that by (1.6), (1.21) and (3.24), we have $(d_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Since $c_n = 1 + d_n$, using (2.13) and (2.7), we rewrite the kernel k as

$$k(x,y) = \sum_{n\in\mathbb{N}^*} (\phi_n(x) - \widetilde{\xi}_n(x) - d_n\widetilde{\xi}_n(x))\phi_n(y). \tag{3.28}$$

In order to prove Theorem 3.1, we are left to prove that $||k||^2_{L^2((0,1)^2)} < \infty$. We begin by proving that

$$\sum_{n\in\mathbb{N}^*} || \, \phi_n - \widetilde{\xi}_n \, ||^2_{L^2(0,1)} < \infty. \tag{3.29}$$

First of all, from (3.9) together with (1.6) and (1.21), the following estimate holds for some $C > 0$ and for $k \neq n$,

$$\left|\langle \phi_n - \widetilde{\xi}_n, \phi_k\rangle\right| \leq C\frac{|k^{\nu+\frac{1}{2}} n^{-\nu-1/2}|}{|k^2-n^2|^2}. \tag{3.30}$$

Our goal is to get an estimate for the following sum:

$$\sum_{k\in\mathbb{N}^*, k\neq n} \frac{k^{2\nu+1}}{n^{2\nu+1}|k^2-n^2|^2}. \tag{3.31}$$

Note that we can decompose this sum into three sums. The first one is

$$\begin{aligned} \sum_{1\leq k<n} \frac{k^{2\nu+1}}{n^{2\nu+1}|k^2-n^2|^2} &\leq \sum_{1\leq k<n} \frac{1}{|k+n|^2|k-n|^2} \\ &\leq \frac{1}{n^2}\sum_{1\leq k<n}\frac{1}{|k-n|^2} \\ &\leq \frac{C}{n^2}, \end{aligned} \tag{3.32}$$

for some positive C. The second one is

$$\begin{aligned} \sum_{n<k\leq 2n} \frac{k^{2\nu+1}}{n^{2\nu+1}|k^2-n^2|^2} &\leq \sum_{n<k\leq 2n} \frac{2^{2\nu+1}}{|k+n|^2|k-n|^2} \\ &\leq \frac{2^{2\nu+1}}{n^2}\sum_{n<k\leq 2n}\frac{1}{|k-n|^2} \\ &\leq \frac{C}{n^2}, \end{aligned} \tag{3.33}$$

for some positive C. The last one is

$$\begin{aligned}\sum_{k>2n} \frac{k^{2\nu+1}}{n^{2\nu+1}|k^2-n^2|^2} &= \frac{1}{n^{2\nu+1}} \sum_{k>2n} \frac{k^{2\nu+1}}{|k-n|^2|k+n|^2} \\ &\leq \frac{4}{n^{2\nu+1}} \sum_{k>2n} \frac{k^{2\nu+1}}{k^2|k+n|^2} \\ &\leq \frac{4}{n^{2\nu+1}} \sum_{k>2n} \frac{1}{k^{3-2\nu}} \\ &\leq \frac{C}{n^3},\end{aligned} \tag{3.34}$$

for some positive C. Using this last inequality together with (3.8) (for $k = n$) and the fact that $\nu > 0$, we deduce that

$$\sum_{n\in\mathbb{N}^*} \sum_{k\in\mathbb{N}^*} \left|\langle \phi_n - \widetilde{\xi}_n, \phi_k\rangle\right|^2 < \infty. \tag{3.35}$$

It follows from (3.35) that (3.29) is verified. From the previous computations, we can infer that there exists a constant $C > 0$ such that for any $n \in \mathbb{N}^*$

$$\| \phi_n - \widetilde{\xi}_n \|^2_{L^2(0,1)} \leq \frac{C}{n^2}. \tag{3.36}$$

Since $\{\phi_n\}_{n\in\mathbb{N}}$ is a Hilbert basis on $L^2(0,1)$, we obtain

$$||\phi_n - \widetilde{\xi}_n - d_n\widetilde{\xi}_n\phi_n||^2_{L^2(0,1)} \leq 2\left(\| \phi_n - \widetilde{\xi}_n\|^2_{L^2(0,1)} + |\, d_n\,|^2 \| \,\widetilde{\xi}_n\, \|^2_{L^2(0,1)}\right).$$

The last term in the right-hand side can be bounded by

$$2|d_n|^2||\widetilde{\xi}_n - \phi_n||^2_{L^2(0,1)} + 2|d_n|^2||\phi_n||^2_{L^2(0,1)},$$

so that by using (3.36), we have

$$||\phi_n - \widetilde{\xi}_n - d_n\widetilde{\xi}_n\phi_n||^2_{L^2(0,1)} \leq 2\left(\frac{C}{n^2} + 2C\frac{|d_n|^2}{n^2} + 2|d_n|^2\right).$$

Thus, we obtain that there exists a constant $C_1 > 0$ such that

$$||\phi_n - \widetilde{\xi}_n - d_n\widetilde{\xi}_n\phi_n||^2_{L^2(0,1)} \leq C_1\left(\frac{1}{n^2} + d_n^2\right).$$

Hence,

$$||\psi_n||^2_{L^2(0,1)} = ||\phi_n - \widetilde{\xi}_n - d_n\widetilde{\xi}_n\phi_n||^2_{L^2(0,1)} \leq C_1\left(\frac{1}{n^2} + d_n^2\right).$$

Finally, from (3.28) and the fact that $(d_n)_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$, we obtain that

$$||k||^2_{L^2(0,1)^2} = \sum_{n\in\mathbb{N}} ||\phi_n - \widetilde{\xi}_n - d_n\widetilde{\xi}_n\phi_n||^2_{L^2(0,1)} < \infty,$$

which concludes the proof of Theorem 3.1. ∎

3.3. **Continuity.**

Proposition 3.5. *The transformation T belongs to $\mathscr{L}_c(L^2(0,1))$.*

Proof of Proposition 3.5. Let $f \in L^2(0,1)$. By the definition of T given in (2.3), we have, for every $x \in (0,1)$,

$$Tf(x) = f(x) - \int_0^1 k(x,y)f(y)dy = I_{L^2(0,1)}f(x) - \int_0^1 k(x,y)f(y)dy, \quad (3.37)$$

where $I_{L^2(0,1)}$ is the identity operator. Note that the operator T is linear from $L^2(0,1)$ into itself. On the other hand, by Theorem 3.1 we obtain that the linear operator $f \in L^2(0,1) \longmapsto \int_0^1 k(x,y)f(y)dy$ is continuous. This concludes the proof of the continuity. ■

Lemma 3.6. *The operator*

$$K : f \in L^2(0,1) \mapsto \int_0^1 k(0,y)f(y)dy = \sum_{n\in\mathbb{N}^*} \psi_n(0) < f, \phi_n >$$

is well-defined and belongs to $\left(L^2(0,1)\right)'$.

Remark 3. *Contrarily to [15], we are able to prove here the continuity of K in L^2. This comes from the fact that we are working on a boundary control problem, so that we need to estimate the kernel k at $x = 0$, where it enjoys nice properties.*

Proof of Lemma 3.6. In order to prove Lemma (3.6), it suffices to prove that the sequence $(\psi_n(0))_{n\in\mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*)$ and to use the Cauchy-Schwarz inequality.

From (2.7), (2.12) and (2.13), we obtain that

$$\psi_n(x) = \phi_n(x) - c_n \frac{\sqrt{2\kappa}x^{\frac{1-\alpha}{2}}}{J'(j_{\nu,n})} \left(J_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right) - \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)} Y_\nu\left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}x^\kappa\right)\right).$$

From (2.13), (3.16) and the fact that $\phi_n(0) = 0$, we obtain that

$$\psi_n(0) = -c_n \frac{J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)} \left(\frac{\sqrt{\lambda_n - \lambda}}{\kappa}\right)^{-\nu} \left(\frac{\sqrt{2\kappa}}{J'(j_{\nu,n})2^{-\nu}\Gamma(-\nu+1)\sin(\nu\pi)}\right).$$

Using (1.29), (1.33), (1.6), (1.21) and (1.22), we obtain that for some constant $C > 0$,

$$|\psi_n(0)| \leq C \frac{c_n}{n^{1/2+\nu}},$$

where C is a positive constant. Since $(c_n)_{n\in\mathbb{N}}$ belongs to $l^\infty(\mathbb{N}^*)$ and since $\nu > 0$, we have that $(\psi_n(0))_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Therefore, we deduce by the Cauchy-Schwarz inequality that K is well-defined and belongs to $\left(L^2(0,1)\right)'$. This concludes the proof of Lemma 3.6. ∎

3.4. **Functional identities.**

Proposition 3.7. *The operators defined by* (2.3) *and* (2.7) *can be uniquely extended as a linear continuous operator from* $D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$ *to* $D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$ *verifying the functional identity* $TB = B$ *in* $D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$.

Proof of Proposition 3.7. Let us write $f = \sum_{n\in\mathbb{N}} a_n\phi_n \in L^2(0,1)$. Then, using equations (2.3) and (2.7), we have

$$\begin{aligned} Tf(x) &= f(x) - \int_0^1 \left(\sum_{n\in\mathbb{N}^*} \psi_n(x)\phi_n(y)\right)\left(\sum_{n\in\mathbb{N}} a_n\phi_n(y)\right) dy \\ &= \sum_{n\in\mathbb{N}^*} a_n\left(\phi_n(x) - \psi_n(x)\right). \end{aligned} \tag{3.38}$$

Hence, if $f = \sum_{n\in\mathbb{N}} a_n\phi_n \in D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$, we define

$$Tf(x) = \sum_{n\in\mathbb{N}^*} a_n\left(\phi_n(x) - \psi_n(x)\right). \tag{3.39}$$

We have

$$\left\|\sum_{n\in\mathbb{N}^*} \psi_n(x)a_n\right\|^2_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'} = \sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_k^{\nu+3/2}}\left(\sum_{n\in\mathbb{N}} a_n < \phi_k, \psi_n >_{L^2(0,1)}\right)^2, \tag{3.40}$$

where ψ_n is defined in (2.13). By (3.7), we obtain that

$$\begin{aligned} < \phi_k, \psi_n >_{L^2(0,1)} &=< \phi_k, \phi_n - (1+d_n)\widetilde{\xi}_n >_{L^2(0,1)} \\ &= \delta_{kn} - \frac{4\kappa^2(1+d_n)\left(\frac{j_{\nu,k}\kappa}{\sqrt{\lambda_n-\lambda}}\right)^\nu J_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}{\pi\left(\lambda_k-\lambda_n+\lambda\right)J'_\nu(j_{\nu,n})J'_\nu(j_{\nu,k})Y_\nu\left(\frac{\sqrt{\lambda_n-\lambda}}{\kappa}\right)}. \end{aligned} \tag{3.41}$$

In the case $n = k$, by (3.41), (1.21), (1.29), (1.33), (1.22) and the fact that $c_n = 1 + d_n \in l^\infty(\mathbb{N}^*)$, we have that

$$< \phi_n, \psi_n >_{L^2(0,1)}= O(1). \tag{3.42}$$

Hence, we deduce that

$$\sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_n^{\nu+3/2}} a_n^2 < \phi_n, \psi_n >^2_{L^2(0,1)} \leq C \sum_{n\in\mathbb{N}^*} \frac{a_n^2}{\lambda_n^{\nu+3/2}} \leq C||f||^2_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'} < \infty.$$

If $k \neq n$, the same argument leads to

$$| < \phi_n, \psi_k >_{L^2(0,1)} | \leqslant C \frac{k^{\nu+\frac{1}{2}}}{n^{\nu+\frac{1}{2}}|k^2 - n^2|}. \tag{3.43}$$

Hence, using the Cauchy-Schwartz inequality, (3.40), (3.42), (3.43), (1.6), (1.21), (1.29), (1.33), (1.22) and the fact that $c_n = 1 + d_n \in l^\infty(\mathbb{N}^*)$, we have that for some $C > 0$,

$$\begin{aligned} &\left|\left|\sum_{n\in\mathbb{N}^*} \psi_n(x) a_n\right|\right|^2_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'} \\ &\leq C \sum_{k\in\mathbb{N}^*} \sum_{n\neq k} \left(\frac{n^2}{k^2|n^2-k^2|^2}\right) \left(\sum_{n\neq k} \frac{a_n^2}{n^{2\nu+3}}\right) \\ &\leq C \sum_{k\in\mathbb{N}^*} \sum_{n\neq k} \left(\frac{n^2}{k^2|n^2-k^2|^2}\right) ||f||^2_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'}. \end{aligned}$$

From the computations of [17, Proof of Proposition 3], we obtain that

$$\sum_{k\in\mathbb{N}^*} \sum_{n\neq k} \left(\frac{n^2}{k^2|n^2-k^2|^2}\right) < +\infty.$$

Hence, we deduce that $\sum_{n\in\mathbb{N}^*} a_n \psi_n \in D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$, so that by (3.39), $Tf \in D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$ and T is a linear continuous operator from $D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)'$ to itself.

It remains to prove that $TB = B$. Using the last line of (2.8), we obtain, for any $n \in \mathbb{N}^*$,

$$\lim_{y\to 0^+} x^\alpha \sum_{k\in\mathbb{N}^*} < \psi_k, \phi_k > \phi_n'(x) = 0. \tag{3.44}$$

Then, for any $n \in \mathbb{N}^*$, thanks to (3.44), (2.14) and (2.7), we have

$$\begin{aligned} \langle TB, \phi_n \rangle_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)', D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)} &= \langle B, T^*\phi_n \rangle_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)', D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)} \\ &= \lim_{x\to 0^+} x^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(x) \\ &- \lim_{x\to 0^+} y^\alpha \sum_{k\in\mathbb{N}^*} < \psi_k(x), \phi_n > \phi_k'(y) \\ &= \lim_{x\to 0^+} x^\alpha \sum_{n\in\mathbb{N}^*} \phi_n'(x) \\ &= \langle B, \phi_n \rangle_{D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)', D\left(A^{\frac{\nu}{2}+\frac{3}{4}}\right)}, \end{aligned}$$

which concludes our proof. ■

Our next step is to establish that for any $f \in D(A+BK)$, the following identity

$$T(A+BK)f = (A-\lambda I)Tf$$

holds in $L^2(0,1)$. We remark that the relation $(A+BK)f = g$, for $g \in L^2(0,1)$, holds, if and only if the equation

$$\begin{cases} (x^\alpha \partial_x f)_x = g, & x \in (0,1), \\ f(1) = 0, & \\ f(0) = K(f), & \end{cases} \tag{3.45}$$

is satisfied. We have the following proposition:

Proposition 3.8. *For $f \in D(A+BK)$, we have*

$$T(A+BK)f = (A-\lambda I)Tf \text{ in } L^2(0,1).$$

Proof: Let $f \in D(A+BK)$. From (1.5), the two last lines of (3.45) and some integration by parts, we have, for any $n \in \mathbb{N}^*$,

$$\begin{aligned} \langle (A+BK)f, \phi_n \rangle_{L^2(0,1)} &= \langle (x^\alpha \partial_x f)_x, \phi_n \rangle_{L^2(0,1)} \\ &= \langle f, (x^\alpha \partial_x \phi_n)_x \rangle_{L^2(0,1)} + \lim_{x\to 0^+} x^\alpha \phi_k'(x) K(f) \\ &= -\lambda_n \langle f, \phi_n \rangle_{L^2(0,1)} + \lim_{x\to 0^+} x^\alpha \phi_n'(x) K(f), \end{aligned} \tag{3.46}$$

where the limit exists by (1.25). This implies that

$$(\lambda_n \langle f, \phi_n \rangle_{L^2(0,1)} - \lim_{x\to 0^+} x^\alpha \phi_n'(x) K(f))_{n\in\mathbb{N}^*} \in l^2(\mathbb{N}^*).$$

Now, using (2.3), (2.7), (3.46) and (3.44), we also have

$$\begin{aligned} &\langle T(A+BK)f, \phi_n \rangle_{L^2(0,1)} \\ &= \langle (A+BK)f, \phi_n \rangle_{L^2(0,1)} - \sum_{k\in\mathbb{N}^*} \langle \psi_k, \phi_n \rangle \langle (A+BK)f, \phi_k \rangle_{L^2(0,1)} \\ &= -\lambda_n \langle f, \phi_n \rangle_{L^2(0,1)} + \lim_{x\to 0^+} x^\alpha \phi_n'(x) K(f) \\ &+ \sum_{k\in\mathbb{N}^*} \langle \psi_k, \phi_n \rangle_{L^2(0,1)} \lambda_k \langle f, \phi_k \rangle_{L^2(0,1)}. \end{aligned}$$

On the other hand, from system (2.8) and the definition of Kf given by (2.15), we have $Tf \in D(A)$. Integrating (2.8) on $(0,1)$, and integrating by parts, we obtain

$$\langle A\psi_k, \phi_n \rangle_{L^2(0,1)} = -\lambda_n \langle \psi_k, \phi_n \rangle_{L^2(0,1)} + \lim_{x\to 0^+} x^\alpha \psi_k(0) \phi_n'(x). \tag{3.47}$$

On the other hand, equation (2.8) leads to

$$\langle A\psi_k, \phi_n \rangle_{L^2(0,1)} = (\lambda - \lambda_n) \langle \psi_k, \phi_n \rangle_{L^2(0,1)} - \lambda \delta_{nk}. \tag{3.48}$$

Finally, from (2.15), (2.7), together with (3.47) and (3.48), we see that, for any $f \in D(A+BK)$ and any $n \in \mathbb{N}^*$,

$$\begin{aligned}
&\langle (A-\lambda I)Tf, \phi_n \rangle_{L^2(0,1)} \\
&= -\langle Tf, (\lambda_n+\lambda)\phi_n \rangle_{L^2(0,1)} \\
&= -(\lambda_n+\lambda)\langle f, \phi_n \rangle_{L^2(0,1)} + \sum_{k\in\mathbb{N}^*} (\lambda_n+\lambda)\langle \psi_k, \phi_n \rangle_{L^2(0,1)} \langle f, \phi_k \rangle_{L^2(0,1)} \\
&= -(\lambda_n+\lambda)\langle f, \phi_n \rangle_{L^2(0,1)} + \sum_{k\in\mathbb{N}^*} \Big(\lambda_n \langle \psi_k, \phi_n \rangle_{L^2(0,1)} \\
&+ \lim_{x\to 0^+} x^\alpha \psi_k(0)\phi_n'(x) + \lambda\delta_{nk}\Big) \langle f, \phi_k \rangle \\
&= -\lambda_n \langle f, \phi_n \rangle_{L^2(0,1)} + \lim_{x\to 0^+} x^\alpha \phi_n'(x) K(f) \\
&+ \sum_{k\in\mathbb{N}^*} \langle \psi_k, \phi_n \rangle_{L^2(0,1)} \lambda_k \langle f, \phi_k \rangle_{L^2(0,1)} \\
&= \langle T(A+BK)f, \phi_k \rangle_{L^2(0,1)} .
\end{aligned}$$

This concludes our proof. ∎

3.5. **Invertibility of the transformation.** The next step of our proof is to obtain that the transformation T is invertible. The first step is to prove that T can be written as the sum of a compact operator and an invertible operator, so that it is a Fredholm operator of order 0.

Lemma 3.9. *The operator $T : L^2(0,1) \to L^2(0,1)$ can be decomposed as $T = \tilde{T} + \tilde{C}$ where $\tilde{T}$ is an invertible operator and $\tilde{C}$ is a compact operator. Consequently, T is a Fredholm operator of order 0.*

The proof of Lemma 3.9 is totally similar to the proof of [17, Proof of Lemma 4] and is skipped. It is mainly based on the decomposition obtained in Theorem 3.1, which enables to write

$$Tf = \sum_{n\in\mathbb{N}^*} (1+d_n)\widetilde{\xi}_n < f, \phi_n >_{L^2(0,1)},$$

for some $\{d_n\}_{n\in\mathbb{N}^*} \subset \ell^2(\mathbb{N})$, so that we can write

$$\tilde{T}f := \sum_{n\in\mathbb{N}^*} \widetilde{\xi}_n(x) < f, \phi_n >_{L^2(0,1)}, \quad \tilde{C}f := \sum_{n\in\mathbb{N}^*} d_n\widetilde{\xi}_n(x) < f, \phi_n >_{L^2(0,1)},$$

that turn out to verify the desired properties. We are now able to state the invertibility of T.

Proposition 3.10. *The transformation T is invertible from $L^2(0,1)$ to itself.*

The proof is also totally similar to [17, Proof of Proposition 5]. The only difference in the proof is that now, from (2.14), we have $B^*\phi_k = \lim_{y\to 0^+} y^\alpha \phi_k'(y)$, which is always nonzero thanks to (1.25).

4. Proof of Theorem 1.1.

In this section we prove the Theorem 1.1, which is the main result of the present article.

Proof of Theorem 1.1.The proof is then divided into three steps:

- Step 1: we show that $A+BK$ is dissipative.
- Step 2: we show that $A+BK$ is maximal.
- Step 3: we prove the exponential stability of system (1.1).

Step 1: the operator $A+BK$ is dissipative

Since T is continuous and invertible on $L^2(0,1)$, the norm $||\cdot||_T := ||T\cdot||$ (issued from the scalar product $\langle\cdot,\cdot\rangle_T = \langle T\cdot, T\cdot\rangle_{L^2(0,1)}$) is equivalent to the norm $||\cdot||_{L^2(0,1)}$. We want to prove that the operator $A+BK$ is dissipative for this norm, which means that

$$\langle (A+BK)u, u\rangle_T \ \leq\ 0, \quad \forall u \in D(A+BK).$$

Indeed, since A is dissipative and applying Proposition 3.8, we obtain that for any $u \in D(A+BK)$, we have

$$\begin{aligned}\langle (A+BK)u, u\rangle_T =& \langle (A-\lambda I_{L^2(0,1)})Tu, Tu\rangle_{L^2(0,1)}\\ =& \langle ATu, Tu\rangle_{L^2(0,1)} - \lambda ||u||_T^2 \leq 0.\end{aligned}$$

Step 2: the operator $A+BK$ is maximal

We aim to prove that $\mathrm{Ran}(\sigma I_{L^2(0,1)} - (A+BK)) = L^2(0,1)$, for some $\sigma > 0$. It is clear that for any $\sigma > 0$, $\mathrm{Ran}(\sigma I_{L^2(0,1)} - (A+BK)) \subset L^2(0,1)$. Now, to show the other inclusion, it is sufficient to prove that, for any $u \in L^2(0,1)$, there exists $\sigma > 0$ and $\bar{u} \in D(A+BK)$ such that

$$T^{-1}T(A+BK-\sigma I_{L^2(0,1)})\bar{u} = u. \tag{4.1}$$

Applying Proposition (3.8), we have

$$(\sigma I_{L^2(0,1)} - (A-\lambda I_{L^2(0,1)}))T\bar{u} = -Tu. \tag{4.2}$$

Since $A - \lambda I_{L^2(0,1)}$ is maximal, from (4.2), we deduce that, for any $u \in L^2(0,1)$, there exists $\sigma > 0$ and $\bar{u}$ such that (4.1) is verified.
This, together with the fact that $A+BK$ is dissipative, implies that $A+BK$ is $m-$dissipative and by [23, Proposition 3.1.6], $A+BK$ is the infinitesimal generator of a semigroup of contractions. Notably, from [23, Proposition 2.3.5] we obtain the existence of a unique solution of (1.1).

Step 3: exponential stability of system (1.1)

From the continuity and the invertibility of the transformation T, the fact that $v = Tu$ and (2.2), we have, for all $t \geq 0$,

$$\begin{aligned}||u(t,\cdot)||_{L^2(0,1)} =& ||T^{-1}Tu(t,\cdot)||_{L^2(0,1)}\\ &\leq ||T^{-1}||_{L^2(0,1)}||Tu(t,\cdot)||_{L^2(0,1)}\\ &\leq ||T^{-1}||_{\mathscr{L}_c(L^2(0,1))}e^{-\lambda t}||Tu_0||_{(L^2(0,1)}\\ &\leq ||T^{-1}||_{\mathscr{L}_c(L^2(0,1))}||T||_{\mathscr{L}_c(L^2(0,1))}e^{-\lambda t}||u_0||_{L^2(0,1)}.\end{aligned}$$

This concludes Step 3 and the proof of Theorem 1.1. ■

5. Conclusion

The goal of the present paper was to prove a boundary rapid stabilization result for a particular degenerate parabolic equation, in the case where the boundary control acts at the point where the degeneracy occurs. This required to exhibit an adequate integral transformation of a Fredholm type. The existence and uniqueness of the kernel involved in the integral transformation is proved by using properties on Riesz bases. Then, the continuity and invertibility of the integral transformation is proved by means of tools from functional analysis and operator theory. To finish, the exponential stability of the original system is easily deduced by using the exponential stability of the target system and the properties above-mentioned.
Taking into account the present study and [17], Some natural questions arise.

- **Adding lower order terms** It would be interesting to investigate the case where the operator $(x^\alpha \partial_x)_x$ is replaced by an operator of the form $(x^\alpha \partial_x)_x + b(x)\partial_x + c(x)Id$, for some smooth enough functions b and c. Using a spectral approach as in the present paper might be possible, by using perturbation arguments.
- **More general weakly degenerate operators** Another possible generalization would be to consider general weakly degenerate operators in the sense of [8], *i.e.* to replace x^α by some $a(x)$ verifying $a \in C^0([0,1])$, $a \in C^1((0,1])$, $a > 0$ on $(0,1]$, $a(0) = 0$ and $a^{-1} \in L^1(0,1)$. It is not clear that our method, that relies strongly on the fact that we know explicitly many properties of the eigenfunctions of the operator $(x^\alpha \partial_x)_x$. Treating the general case $a(x)$ would require to obtain precise informations on the behaviour of the eigenfunctions, that might be difficult to obtain, or that might widely differ from the particular case x^α, so that it might also happen that our method cannot be applied without imposing stronger restrictions on $a(x)$.
- **Towards more general models** One question that naturally arises is to find some general setting that encompasses for instance well-posed parabolic-like abstract control systems under the form $y' = Ay + Bu$, firstly with diagonalizable operator A, or more generally

with A having generalized eigenvectors that form a Riesz basis, under reasonnable assumptions on the control operator B.

- **Finite-time stabilization** Another challenging open problem would be to understand if the backstepping method with Fredholm transformation enables to obtain some finite-time stabilization results in the spirit of [14], where the authors were able to prove a finite-time stabilization result for the usual heat equation, thanks to the backstepping method with Volterra transformation. If we look carefully to [14], one important point is to obtain some precise estimates on the kernel involved in the transformation, which is not possible here, because Theorem 3.3 (and its proof) does not provide explicit bounds on the Riesz basis $\{\sqrt{\lambda_n}\tilde{\xi}_{n\in\mathbb{N}^*}\}$. It would be very interesting to understand if it is possible to change our approach in order to obtain finite-time stabilization.

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