Bilinear local controllability to the trajectories of the Fokker-Planck equation with a localized control

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Abstract

This work is devoted to the control of the Fokker-Planck equation, posed on a smooth bounded domain of $\mathbb{R}^d$ ($d \geqslant 1$). More precisely, the control is the drift force, localized on a small open subset. We prove that this system is locally controllable to regular nonzero trajectories. Moreover, under some conditions on the reference control, we explain how to reduce the number of controls around the reference control. The results are obtained thanks to a linearization method based on a standard inverse mapping procedure and the fictitious control method. The main novelties of the present article are twofold: First, we propose an alternative strategy to the standard fictitious control method: the algebraic solvability is performed and used directly on the adjoint problem. We then prove a new Carleman inequality for the heat equation with a first-order term with non constant coefficients: the right-hand side is the gradient of the solution localized on a subset (rather than the solution itself), and the left-hand side can contain arbitrary high derivatives of the solution. We finally give an example of regular trajectory around which the Fokker-Planck equation is not controllable with a reduced number of controls, to highlight that our conditions are relevant.

*Keywords:*Controllability, Parabolic equations, Carleman estimates, Fictitious control method, Algebraic solvability.

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1 Introduction and main results

1.1 Introduction

Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}^*$), regular enough (for example of class $\mathcal{C}^\infty$). Denote by $Q_T := (0,T) \times \Omega$ and $\Sigma_T := (0,T) \times \partial\Omega$. We consider the following system

$$\begin{cases} \partial_t y & = \Delta y + \operatorname{div}(uy) & \text{in } Q_T, \\ y & = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) & = y^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $y^0 \in L^2(\Omega)$ is the initial data and $u = (u_1, ..., u_d) \in L^\infty((0,T) \times \Omega)^d$ is the control.

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It is well-known (see for instance [22, Theorem and Proposition 3.1]) that for every initial data $y^0 \in L^2(\Omega)$ and every control $u \in L^\infty((0,T)\times\Omega)^d$, there exists a unique solution y to System (1.1) in the space $W(0,T)$, where

$$W(0,T) := L^2((0,T), H_0^1(\Omega)) \cap H^1((0,T), H^{-1}(\Omega)) \hookrightarrow \mathcal{C}^0([0,T]; L^2(\Omega)).$$

Equation (1.1), introduced in [30], is called the Fokker-Planck equation. When the Fokker-Planck equation is posed on the whole space $\mathbb{R}^d$, it is strongly related to the stochastic differential equation (SDE)

$$\begin{cases} dX_t &= \sum_{i=1}^d u_i(X_t)dt + dW_t \quad \text{in } (0,T)\times\mathbb{R}^d, \\ X(0,\cdot) &= X^0 \qquad\qquad\qquad\quad\ \text{in } \mathbb{R}^d, \end{cases} \tag{1.2}$$

where W_t is the standard multi-dimensional Brownian motion starting from 0. System (1.2) describes the movement of a particule of negligible mass, with constant and isotropic diffusion, under the action of a force field $u = (u_1, \dots, u_d)$.

Under some regularity conditions on the drift term U, it is well-known that, by the Itô Lemma, the probability density function p associated to (1.2) verifies

$$\begin{cases} \partial_t p &= \frac{1}{2}\Delta p + \operatorname{div}(up) \quad \text{in } (0,T)\times\mathbb{R}^d, \\ p(0,\cdot) &= p^0 \qquad\qquad\qquad \text{in } \mathbb{R}^d, \end{cases} \tag{1.3}$$

where p^0 is some initial probability density function (see *e.g.* [41, Section 5.3]). By definition of a probability measure, we have $p^0 \geqslant 0$ a.e. and $\int_{\mathbb{R}^d} p^0 = 1$. Then, we can easily prove the preservation of these properties during the time: any solution p of System (1.3) verifies also $p(t,\cdot) \geqslant 0$ a.e. and $\int_{\mathbb{R}^d} p(t,\cdot) = 1$, for any $t \in [0,T]$ and hence remains a probability measure. We refer to [42] for more explanations on the Fokker-Planck equation, notably in the case of nonlinear drift terms or non-constant and anisotropic diffusion.

However, in the case where we impose Dirichlet boundary conditions as in (1.1), the derivation of the Fokker-Planck equation from a SDE is more complicated: the Brownian motion has to be replaced by an "absorbed" or "killed" Brownian motion, see *e.g.* [10, pp. 31-60]. Moreover, the total mass of the initial condition is not conserved anymore, meaning that the probability of remaining inside Ω decreases in time, and the solution to (1.1) is not a probability density function anymore. We refer to [22, Section 2] for a discussion on the relevance of Dirichlet boundary conditions in this context. Neumann boundary conditions (that would restore the conservation of mass) are beyond the scope of the present article (see the last item of Remark 4 for more explanations).

The controllability properties of the scalar linear heat equation in the case of a distributed control on an open subset and Dirichlet boundary condition are now well-understood (see notably [32] and [23]). The bilinear controllability seems to have been less explored. The equation (1.1) has been studied in [7], in the whole space and with controls localized everywhere in space and time. Concerning bilinear control when the bilinear term $\operatorname{div}(uy)$ is replaced by uy with $u \in L^\infty((0,T)\times\Omega)$, we refer to [8, 9, 27, 28, 26, 29, 34, 40, 45, 46].

Optimal bilinear control of parabolic equations has previously been studied. A first result was proved in [1], where a close fourth-order in time model is investigated, with controls depending only on time. This result has been extended to second-order parabolic equations firstly in [4] in the one-dimensional case, then in [5] in the multi-dimensional case, still for time-varying controls. For equation (1.1) (in a slightly more general form), the case of space and time-varying controls is treated in [22]. Notably, for a drift term that is affine in the control, the authors prove the existence of optimal controls for general cost functionals, and derive first-order necessary optimality conditions using an adjoint state. The controllability of the continuity equation, *i.e.* System (1.1) without diffusion, has been investigated in [18, 19].

The paper is organized as follows: in Section 1.2, we present the main results of the article (Theorem 1.1, resp. Theorem 1.2, which provides a result of local controllability to the trajectories with d controls, resp. a reduced number of controls around the reference control) and some remarks. Section 2 is devoted to studying a linearized version of (1.1). In Section 2.1, we prove a new Carleman estimate (Proposition 2.1) for solutions of the linear backward heat equation with first-order terms. The main novelty is that the local observation term is the gradient of the solution of the adjoint problem (2.4), which has already been proved in [16] for constant coefficients. Moreover, we can put as many derivatives as we want in the left-hand side of our Carleman estimate, which will be need for the rest of the proof. In Section 2.2, we explain how to remove some components of the gradient in the Carleman inequality. To demonstrate that, we use we call an argument of "algebraic solvability" (as introduced in [11] in the context of the stabilization of ODEs and in [15] for the study of coupled systems of PDEs), based on ideas developed by Gromov in [24, Section 2.3.8]. This procedure has already been used successfully in [2, 16, 17, 14, 33, 43, 44]. The main novelty compared to the existing literature is that the algebraic solvability is performed directly on the dual problem. Moreover, we can get rid the high order derivatives of the right in order to obtain the final Carleman estimate (2.35). In Section 2.3, we use some arguments coming from optimal control theory in order to derive from our observability inequality the existence of regular enough controls, with a special form, in appropriate weighted spaces. In Section 3, we go back to the nonlinear problem by using a standard strategy coming from [37] together with some adapted inverse mapping Theorem. To finish, in Section 4, we give an example of a trajectory around which the local controllability does not hold with a reduced number of controls.

1.2 Main results

Let $(\overline{y}, \overline{u})$ be a trajectory of (1.1), *i.e.* verifying

$$\left\{\begin{array}{lll} \partial_t \overline{y} & = \Delta \overline{y} + \operatorname{div}(\overline{u}\overline{y}) & \text{in } Q_T, \\ \overline{y} & = 0 & \text{on } \Sigma_T, \\ \overline{y}(0,\cdot) & = \overline{y}^0 \in L^2(\Omega) \setminus \{0\} & \text{in } \Omega. \end{array}\right. \tag{1.4}$$

1.2.1 Controls with d components

We first state a result of local controllability to the trajectories for System (1.1) with a control containing d components:

Theorem 1.1. *Let ω be any nonempty open subset of Ω. Assume that the trajectory $(\overline{y}, \overline{u})$ with $\overline{u} = (\overline{u}_1, ..., \overline{u}_d)$ of System (1.4) is regular enough (for example of class C^∞ on $(0,T) \times \Omega$). Then, System (1.1) is locally controllable with localized controls, in the following sense:*

for every $\varepsilon > 0$ and every $T > 0$, there exists $\eta > 0$ such that for any $y^0 \in L^2(\Omega)$ verifying

$$||y^0 - \overline{y}^0||_{L^2(\Omega)} \leqslant \eta, \tag{1.5}$$

there exists a trajectory (y, u) to System (1.1) such that

$$\begin{cases} y(T) & = \overline{y}(T), \\ u & = \overline{u} + v \text{ for some } v \in L^\infty((0,T)\times\Omega)^d, \\ \text{Supp (v)} & \subset (0,T)\times\omega, \\ ||v||_{L^\infty((0,T)\times\Omega)^d} & \leqslant \varepsilon, \\ ||y-\overline{y}||_{W(0,T)} & \leqslant \varepsilon. \end{cases}$$

Remark 1.

- The regularity assumptions on $(\overline{y}, \overline{u})$ can be improved, notably it is enough that the reference trajectory is C^r for some $r \in \mathbb{N}^*$ large enough, on an open subset of $(0,T)\times\omega$.
- If $y^0 = 0$, the only solution to (1.1) is $y \equiv 0$, whatever u is, so that the only reachable state at time T is 0. As a consequence, $\eta > 0$ has notably to be chosen small enough such that $y^0 \neq 0$.
- From the results given in [6], as soon as $y^0 \geqslant 0$, then any trajectory to System (1.1) remains non-negative (see also [22]). This fact differs from the usual linear heat equation with internal control (see [38]).
- We can also remark that we do not assume any relation between the control domain ω and the support of $\bar{u}$. In particular, they can be disjoint.

1.2.2 Controllability acting through a control operator

In this section, we give a result of local controllability to the trajectories to System (1.4) with a control acting through a control operator $B \in \mathcal{M}_{d,m}(\mathbb{R})$ with $m \in \mathbb{N}^*$ such that $m \leqslant d$.

We first introduce some notations. For $j \in \{1, ..., m\}$, we call $B_j^* \in \mathbb{R}^d$ the j-th line of B^*, and

$$(B_j^* \cdot \nabla) : \psi \in C^\infty(\mathbb{R}^d, \mathbb{R}) \mapsto B_j^*(\nabla\psi) \in C^\infty(\mathbb{R}^d, \mathbb{R}).$$

For $(\alpha_1, \dots \alpha_m) \in \mathbb{N}^m$, we introduce the following operator:

$$(B^* \cdot \nabla)^{\alpha_1, \dots \alpha_m} : \psi \in C^\infty(\mathbb{R}^d, \mathbb{R}) \mapsto \underbrace{(B_1^* \cdot \nabla)\dots(B_1^* \cdot \nabla)}_{\alpha_1 \text{ times}} \dots \underbrace{(B_m^* \cdot \nabla)\dots(B_m^* \cdot \nabla)}_{\alpha_m \text{ times}} \psi \in C^\infty(\mathbb{R}^d, \mathbb{R}),$$

and the family of $\mathbb{R}^d$ given by

$$M(\bar{u})(t,x) = \{B_1^*, ..., B_m^*\} \cup \{((B^* \cdot \nabla)^\alpha \bar{u}_i(t,x))_{i\in\{1,\cdots,d\}}, \alpha \in \mathbb{N}^m, \alpha \neq 0\}.$$

We have the following controllability result.

Theorem 1.2. *Let $m \in \mathbb{N}^*$ (with possibly $m < d$). Under the hypothesis of Theorem 1.1, assume that there exists some $(t_0, x_0) \in (0,T)\times\omega$ such that*

$$rank(M(\overline{u})(t_0,x_0)) = d. \tag{1.6}$$

Then, System (1.1) is locally controllable with localized controls, in the following sense: for every $\varepsilon > 0$ and every $T > 0$, there exists $\eta > 0$ such that for any $y^0 \in L^2(\Omega)$ verifying

$$||y^0 - \overline{y}^0||_{L^2(\Omega)} \leqslant \eta,$$

there exists a trajectory (y, u) to System (1.1) such that

$$\begin{cases} y(T) & = \overline{y}(T), \\ u & = \overline{u} + Bv \textit{ for some } v \in L^\infty((0,T)\times\Omega)^m, \\ \mathrm{Supp}\,(\mathrm{v}) & \subset (0,T)\times\omega, \\ ||v||_{L^\infty((0,T)\times\Omega)^m} & \leqslant \varepsilon, \\ ||y-\overline{y}||_{W(0,T)} & \leqslant \varepsilon. \end{cases}$$

Remark 2.

- Remark that if $B = I_d$ (*i.e.* we control every component of the gradient of u), condition (1.6) is automatically verified for $q = 0$, whatever $\overline{u}$ is. Hence Theorem 1.2 contains the result given in Theorem 1.1. Thus we will only give a proof of Theorem 1.2.
- In Section 4, we give an example of trajectory which does not satisfy condition (1.6) and for which the local controllability to the trajectories does not hold. It highlights that Condition (1.6) is relevant. Even if the authors think that Condition 1.6 is not optimal, finding a necessary and sufficient condition remains on open problem.
- We can also remark that condition (1.6) is local on ω. Notably, contrary to Theorem 1.1, if $m < d$, we necessarily have that the control domain ω and the support of $\bar{u}$ intersect.

Exemple 1.1. We give an explicit example, to explain better condition (1.6). Let us assume that we want to control only the $m(< n)$ first components of the gradient, *i.e.*

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{R}).$$

For $\alpha = (\alpha_1, \dots \alpha_m) \in \mathbb{N}^m$, we have

$$(B^* \cdot \nabla)^{\alpha_1,\dots\alpha_m}(\psi) = \partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m}(\psi).$$

We deduce that

$$M(\overline{u})(t,x) = \{e_1, \dots, e_m\} \cup \{(\partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m} \bar{u}_i(t,x))_{i\in\{1,\cdots,d\}}, \alpha \in \mathbb{N}^m, \alpha \neq 0\},$$

where the vector e_i is the i-th element of the canonical basis of $\mathbb{R}^d$. We observe that there exists $(t_0, x_0) \in (0,T)\times\omega$ such that the rank of the family $M(\overline{u})(t_0,x_0)$ is equal to d if and only if there exists $(t_0,x_0) \in (0,T)\times\omega$ such that the rank of the family

$$\{(\partial_{x_1}^{\alpha_1} \dots \partial_{x_m}^{\alpha_m} \bar{u}_i(t,x))_{i\in\{m+1,\cdots,d\}}, \alpha \in \mathbb{N}^m, \alpha \neq 0\}$$

is equal to $d - m$.

2 Null controllability of the linearized system

In what follows, we always assume that the trajectory $(\overline{y}, \overline{u})$ of (1.4) verifies the hypothesis of Theorem 1.1. Consider the following linear parabolic system

$$\begin{cases} \partial_t y & = \Delta y + \mathrm{div}(\overline{u}y) + \mathrm{div}(\theta u) & \text{in } Q_T, \\ y & = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) & = y^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $y^0 \in L^2(\Omega)$ and $\theta \in \mathcal{C}^\infty(\overline{\Omega})$ is such that

$$\begin{cases} \mathrm{Supp}(\theta) & \subseteq \omega, & \\ \theta & \equiv 1 & \text{in } \omega_0, \\ 0 & \leqslant \theta \leqslant 1 & \text{in } \Omega, \end{cases} \tag{2.2}$$

for some non-empty open subset ω_0 which is strongly included in ω. The goal of this section is to prove the null controllability of System (2.1), with less controls than equations and regular enough controls in a special form.

Remark 3. Notice that the null controllability of (2.1) is equivalent to the null controllability of the "real" linearized version of (1.1) around $(\overline{y}, \overline{u})$ given by

$$\begin{cases} \partial_t y & = \Delta y + \mathrm{div}(\overline{u} y) + \mathrm{div}(\overline{y}\tilde{u}) & \text{in } Q_T, \\ y & = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) & = y^0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

Indeed, since the solution of $(\overline{y}, \overline{u})$ of (1.4) is in $C^\infty((0,T)\times\Omega)$, as soon as $\overline{y}^0 \neq 0$, on $(0,T)\times\omega$, $\overline{y}^{-1}(\{0\})$ is a closed subset of $(0,T)\times\omega$, which cannot be $(0,T)\times\omega$ since it has a finite d-dimensional Hausdorff measure in $\mathbb{R}^{d+1}$ (see [25]). Hence, $(0,T)\times\omega \setminus \overline{y}^{-1}(\{0\})$ contains a nonzero open subset, there exists some subset $(T_1,T_2)\times\tilde{\omega}$ of $(0,T)\times\omega$ such that $|\overline{y}| \geqslant C > 0$ on $(T_1,T_2)\times\tilde{\omega}$, that we can assume to be exactly $(0,T)\times\omega$ without loss of generality. Hence, for any $i \in \{1,\dots,d\}$, one can solve (in $\tilde{u}_i$) the equation $\theta u_i = \overline{y}\tilde{u}_i$ by posing

$$\tilde{u}_i = \frac{\theta u_i}{\overline{y}}.$$

Remark that $\tilde{u}_i$ enjoys the same regularity properties as u_i.

2.1 Carleman estimates

Let us consider the following adjoint system associated to System (2.1)

$$\begin{cases} -\partial_t \psi & = \Delta\psi + \overline{u}\cdot\nabla\psi & \text{in } Q_T, \\ \psi & = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) & = \psi^0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

First of all, we will introduce some notations. We denote by $|\cdot|$ the euclidean norm on $\mathbb{R}^M$, whatever $M \in \mathbb{N}^*$ is. For $s, \lambda > 0$ and $p \geqslant 1$, let us define the two following functions:

$$\alpha(t,x) := \frac{\exp((2p+2)\lambda\|\eta^0\|_\infty) - \exp[\lambda(2p\|\eta^0\|_\infty + \eta^0(x))]}{t^p(T-t)^p} \tag{2.5}$$

and

$$\xi(t,x) := \frac{\exp[\lambda(2p\|\eta^0\|_\infty + \eta^0(x))]}{t^p(T-t)^p}. \tag{2.6}$$

Here, $\eta^0 \in \mathcal{C}^\infty(\overline{\Omega})$ is a function satisfying

$$|\nabla\eta^0| \geqslant \kappa \text{ in } \Omega\backslash\omega_1, \quad \eta^0 > 0 \text{ in } \Omega \quad \text{and} \quad \eta^0 = 0 \text{ on } \partial\Omega,$$

with $\kappa > 0$ and ω_1 some open subset verifying $\omega_1 \subset\subset \omega_0$. The proof of the existence of such a function η^0 can be found in [23, Lemma 1.1, Chap. 1] (see also [12, Lemma 2.68, Chap. 2]). We will use the two notations

$$\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(t,x) \quad \text{and} \quad \xi_*(t) := \min_{x\in\overline{\Omega}} \xi(t,x), \tag{2.7}$$

for all $t \in (0,T)$. Notice that these maximum and minimum are reached at the boundary $\partial\Omega$. For $s, \lambda > 0$, let us define

$$I(s,\lambda;u) := s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3 u^2 dxdt + s\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi|\nabla u|^2 dxdt. \tag{2.8}$$

Let us now give some useful auxiliary results that we will need in our proofs. The first one is a Carleman estimate which holds for solutions of the heat equation with non-homogeneous Neumann boundary conditions:

Lemma 2.1. *There exists a constant $C > 0$ such that for any $u^0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$ and $f_2 \in L^2(\Sigma_T)$, the solution to the system*

$$\begin{cases} -\partial_t u - \Delta u &= f_1 \quad \text{in } Q_T, \\ \frac{\partial u}{\partial n} &= f_2 \quad \text{on } \Sigma_T, \\ u(T,\cdot) &= u^0 \quad \text{in } \Omega \end{cases}$$

satisfies

$$I(s,\lambda;u) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3 u^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi_* f_2^2 d\sigma dt \right.$$
$$\left. + \iint_{Q_T} e^{-2s\alpha} f_1^2 dxdt\right),$$

for all $\lambda \geqslant C$ and $s \geqslant C(T^p + T^{2p})$.

Lemma 2.1 is proved in [21, Theorem 1] in the case $p = 1$. However, following the steps of the proof given in [21], one can prove exactly the same inequality for any $p \in \mathbb{N}^*$.

From Lemma 2.1, one can deduce the following result:

Lemma 2.2. *Let $f \in L^2(\Sigma_T)$, $G = (g_1, \dots g_d) \in L^\infty(Q_T)^d$ and $h \in L^2(Q_T)$. Then, there exists a constant $C > 0$ such that for every $\varphi^T \in L^2(\Omega)$, the solution φ to the system*

$$\begin{cases} -\partial_t \varphi &= \Delta\varphi + G\cdot\nabla\varphi + h \quad \text{in } Q_T, \\ \frac{\partial \varphi}{\partial n} &= f \quad \text{on } \Sigma_T, \\ \varphi(T,\cdot) &= \varphi^T \quad \text{in } \Omega \end{cases}$$

satisfies

$$I(s,\lambda;\varphi) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3 \varphi^2 dxdt + s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*}\xi_* f^2 d\sigma dt + \iint_{Q_T} e^{-2s\alpha} h^2 dxdt\right),$$

for every $\lambda \geqslant C$ and $s \geqslant s_0 = C(T^p + T^{2p})$.

The proof of Lemma 2.2 is standard and is left to the reader (one just has to apply Lemma 2.1 and absorb the remaining lower-order terms thanks to the left-hand side).

We will also need the following estimates.

Lemma 2.3. *Let $r \in \mathbb{R}$. Then, there exists $C := C(r,\omega_1,\Omega) > 0$ such that, for every $T > 0$ and every $u \in L^2((0,T), H^1(\Omega))$,*

$$s^{r+2}\lambda^{r+2} \iint_{Q_T} e^{-2s\alpha}\xi^{r+2} u^2 dxdt \leqslant C\left(s^r\lambda^r \iint_{Q_T} e^{-2s\alpha}\xi^r |\nabla u|^2 dxdt \right.$$
$$\left. + s^{r+2}\lambda^{r+2} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^{r+2} u^2 dxdt\right),$$

for every $\lambda \geqslant C$ and $s \geqslant C(T^{2p})$.

The proof of this lemma can be found for example in [13, Lemma 3] in the case $p = 9$. However, following the steps of the proof given in [13], one can prove exactly the same inequality for any $p \in \mathbb{N}^*$.

To deal with more regular solutions, one needs the following lemma.

Lemma 2.4. *Let $z_0 \in H_0^1(\Omega)$, $G \in C^\infty(Q_T)^d$ and $f \in L^2(Q_T)^m$. Let us denote by $\mathcal{R} := -\Delta - G \cdot \nabla$ and consider the solution z to the system*

$$\begin{cases} \partial_t z &= \Delta z + G \cdot \nabla z + f & \text{in } Q_T, \\ z &= 0 & \text{on } \Sigma_T, \\ z(0,\cdot) &= z_0 & \text{in } \Omega. \end{cases}$$

Let $n \in \mathbb{N}$. Let us assume that $z_0 \in H^{2n+1}(\Omega)$, $f \in L^2((0,T), H^{2n}(\Omega)) \cap H^n((0,T), L^2(\Omega))$ and satisfy the following compatibility conditions:

$$\begin{cases} g_0 := z_0 \in H_0^1(\Omega), \\ g_1 := f(0,\cdot) - \mathcal{R} g_0 \in H_0^1(\Omega), \\ \vdots \\ g_n := \partial_t^{n-1} f(0,\cdot) - \mathcal{R} g_{n-1} \in H_0^1(\Omega). \end{cases} \tag{2.9}$$

Then $z \in L^2((0,T), H^{2n+2}(\Omega)) \cap H^{n+1}((0,T), L^2(\Omega))$ and we have the estimate

$$\|z\|_{L^2((0,T),H^{2n+2}(\Omega))\cap H^{n+1}((0,T),L^2(\Omega))} \leqslant C(\|f\|_{L^2((0,T),H^{2n}(\Omega))\cap H^n((0,T),L^2(\Omega))} + \|z_0\|_{H^{2n+1}(\Omega)}).$$

It is a classical result that can be easily deduced for example from [20, Th. 6, p. 365].

We are now able to prove the following crucial inequality:

Proposition 2.1. *Let $\mu > 0$ and $N \in \mathbb{N}$ with $N \geqslant 3$. Then, there exists $p \geqslant 2$ and $C > 0$ such that for every $\psi^0 \in L^2(\Omega)$, the corresponding solution ψ to System (2.4) satisfies*

$$\begin{aligned} \lambda^2 \iint_{Q_T} e^{-2s\alpha - 2\mu s\alpha^*} (s\xi) |\nabla^{N+1}\psi|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha - 2\mu s\alpha^*} (s\xi)^{2N+1} |\nabla \psi|^2 dxdt \\ + \lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha^* - 2\mu s\alpha^*} (s\xi_*)^{2N+1} |\psi|^2 dxdt \\ \leqslant C \lambda^{2N+2} \iint_{(0,T)\times\omega_0} e^{-2s\alpha - 2\mu s\alpha^*} (s\xi)^{2N+1} |\nabla\psi|^2 dxdt \end{aligned} \tag{2.10}$$

for every $\lambda \geqslant C$ and $s \geqslant s_0 = C(T^p + T^{2p})$.

Such a Carleman inequality seems new to the authors in the context of non-constant coefficients (proved in [16] in the case of constant coefficients). The main improvement comes from the fact that the observation is a gradient of the solution ψ on ω_0 (and not the solution itself). We are also able to introduce as many derivatives of ψ as we want in the left-hand side, as soon as $\overline{u}_i$ is regular enough.

Remark 4.

- Notice that the proof proposed here relies on the fact that the lower-order terms in equation (2.4) are of first order, and would fail in the presence of lower-order terms of order 0. Indeed, in the first step of our proof (inequality (2.13)), some term that cannot be absorbed will appear.
- Notice that inequality (2.10) automatically implies that any solution ψ of (2.4) lives in high order weighted Sobolev spaces. This is not a surprise since we know that away from the final time $t = T$, any solution of (2.4) is regular.

- Remark that the proof provided here would fail for Neumann boundary conditions, since the argument in our last step, based on a Poincaré-like inequality, is not true anymore. It is not clear for the authors how one can adapt it in this case.

Proof of Proposition 2.1.

The proof is inspired by [13] and is quite similar to [16]. Let $\mu > 0$. In all what follows, $C > 0$ is a constant that does not depend on s or λ (but that might depend on the other parameters, notably p, N, η, T, μ) and that might change from inequality to inequality. We assume without loss of generality that N is odd (the case N even can be treated similarly).

Let ψ the solution to System (2.4). We introduce the following auxiliary functions:

$$\rho_1^* := e^{-\mu s \alpha^*}, \ \psi_1 := \rho_1^* \psi. \tag{2.11}$$

Then ψ_1 is solution of

$$\begin{cases} -\partial_t \psi_1 & = \Delta \psi_1 + \overline{u} \cdot \nabla \psi_1 - \partial_t \rho_1^* \psi & \text{in } \ Q_T, \\ \psi_1 & = 0 & \text{on } \ \Sigma_T, \\ \psi_1(T, \cdot) & = 0 & \text{in } \ \Omega. \end{cases} \tag{2.12}$$

We remark that $\phi := \nabla^N \psi_1$ (the operator ∇ applied N times, or in other words, all the derivatives of order N of ψ_1, ordered for example lexicographically) satisfies the system

$$\begin{cases} -\partial_t \phi = \Delta \phi + \sum_{i=1}^{N} G_i \cdot \nabla^i \psi_1 + \overline{u} \cdot \nabla \phi - \partial_t \rho_1^* \nabla^N \psi & \text{in } \ Q_T, \\ \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial n} & \text{on } \ \Sigma_T, \\ \phi(T, \cdot) = 0 & \text{in } \ \Omega, \end{cases}$$

where, for any $i \in \{1, ..., N\}$, G_i is an essentially bounded tensor of appropriate size, whose coefficients are depending only on $\overline{u}_i$ and its derivatives in space up to the order i. Applying Lemma 2.2 to the different components of ϕ, we obtain the following estimate

$$I(s, \lambda; \phi) \leqslant C \Bigg(\underbrace{s\lambda \iint_{\Sigma_T} e^{-2s\alpha^*} \xi_* \left| \frac{\partial \phi}{\partial n} \right|^2 d\sigma dt}_{(I)} + \underbrace{\iint_{Q_T} e^{-2s\alpha} |\partial_t \rho_1^* \nabla^N \psi|^2 dxdt}_{(II)} + \underbrace{\iint_{Q_T} e^{-2s\alpha} \sum_{i=1}^{N} |\nabla^i \psi_1|^2 dxdt}_{(III)} + s^3 \lambda^4 \iint_{(0,T) \times \omega_1} e^{-2s\alpha} \xi^3 |\phi|^2 dxdt \Bigg). \tag{2.13}$$

The rest of the proof is divided into four steps:

- In a first step, we will estimate the boundary term (I) by some global interior term involving ψ_1, which will be absorbed later on (in the last step). We will also absorb the term (II) under some condition on p.
- In a second step, we will estimate the term (III) by some local terms involving $\nabla \psi_1$ and its derivatives on ω_1, and get rid of the third term of the right-hand side.
- In a third step, we will estimate the high-order local terms created at the previous step by some local terms involving only $\nabla \psi_1$ on ω_0.

- In a last step, we will use some Poincaré-like inequality in order to recover the variable ψ in the left-hand side and bound the global interior term of the right-hand side involving ψ_1 by an interior term involving $\nabla\psi$. We will conclude by coming back to the original variable ψ, in order to establish (2.10).

Step 1: Let $\tilde{\theta} \in \mathcal{C}^2(\overline{\Omega})$ a function satisfying

$$\frac{\partial \tilde{\theta}}{\partial n} = \tilde{\theta} = 1 \text{ on } \partial\Omega.$$

An integration by parts of the boundary term leads to

$$\begin{aligned} & s\lambda \int_0^T e^{-2s\alpha^*} \xi_* \int_{\partial\Omega} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt = s\lambda \int_0^T e^{-2s\alpha^*} \xi_* \int_{\partial\Omega} \frac{\partial \phi}{\partial n} \nabla \phi \cdot \nabla \tilde{\theta} d\sigma dt \\ & = s\lambda \int_0^T e^{-2s\alpha^*} \xi_* \int_{\Omega} \Delta\phi \nabla\phi \cdot \nabla\tilde{\theta} dx dt + s\lambda \int_0^T e^{-2s\alpha^*} \xi_* \int_{\Omega} \nabla(\nabla\tilde{\theta}\cdot\nabla\phi)\cdot\nabla\phi dx dt. \end{aligned}$$

Hence

$$s\lambda \int_0^T e^{-2s\alpha^*} \xi_* \int_{\partial\Omega} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt \quad \leqslant \quad C\lambda \int_0^T e^{-2s\alpha^*} s\xi_* \|\psi_1\|_{H^{N+2}(\Omega)} \|\psi_1\|_{H^{N+1}(\Omega)} dt.$$

Using the interpolation inequality

$$\|\psi_1\|_{H^{N+2}(\Omega)} \leqslant C \|\psi_1\|^{1/2}_{H^{N+1}(\Omega)} \|\psi_1\|^{1/2}_{H^{N+3}(\Omega)}$$

and Young's inequality $ab \leqslant \frac{a^q}{q} + \frac{b^{q'}}{q'}$ ($\frac{1}{q} + \frac{1}{q'} = 1$) for $a, b \geqslant 0$ and $q = 4$, we deduce that for any $c \in \mathbb{R}$, we have

$$\begin{aligned} \lambda \int_0^T e^{-2s\alpha^*} s\xi_* \int_{\partial\Omega} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt \quad & \leqslant \quad C\lambda \int_0^T e^{-2s\alpha^*} (s\xi_*)^c \|\psi_1\|^{1/2}_{H^{N+3}(\Omega)} (s\xi_*)^{(1-c)} \|\psi_1\|^{3/2}_{H^{N+1}(\Omega)} dt \\ & \leqslant \quad C\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{4c} \|\psi\|^2_{H^{N+3}(\Omega)} dt \\ & \qquad + C\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}} \|\psi\|^2_{H^{N+1}(\Omega)} dt. \end{aligned} \tag{2.14}$$

Consider the function $\psi_2 := \rho_2^* \psi$, where

$$\rho_2^* := (s\xi_*)^{\frac{2(1-c)}{3}} e^{-(1+\mu)s\alpha^*}. \tag{2.15}$$

The function ψ_2 is solution to the system

$$\left\{ \begin{array}{rll} -\partial_t \psi_2 & = \Delta\psi_2 + \overline{u}\cdot\nabla\psi_2 - \partial_t(\rho_2^*)\psi & \text{in } Q_T, \\ \psi_2 & = 0 & \text{on } \Sigma_T, \\ \psi_2(T,\cdot) & = 0 & \text{in } \Omega. \end{array} \right.$$

Using Lemma 2.4 for ψ_2 (remark that the compatibility conditions (2.9) are verified, since $\psi_2(T,\cdot) = 0$ and $\partial_t^j \rho_2^*(T,\cdot) = 0$ for any $j \in \mathbb{N}$), we deduce that

$$\|\psi_2\|_{L^2((0,T),H^{2n+2}(\Omega))\cap H^{n+1}((0,T),L^2(\Omega))} \leqslant C \|\partial_t(\rho_2^*)\psi\|_{L^2((0,T),H^{2n}(\Omega))\cap H^n((0,T),L^2(\Omega))}, \tag{2.16}$$

for $n = 1, 2, \ldots, (N+1)/2$. The definitions of ξ_* and α^* given in (2.7), the definition of ρ_2^* given in (2.15) lead to

$$|\partial_t^k \rho_2^*| \leqslant C(s\xi_*)^{\frac{2(1-c)}{3}+k+\frac{k}{p}} e^{-s(1+\mu)\alpha^*} \tag{2.17}$$

for $k \in \{1, \ldots, \frac{N+3}{2}\}$ (we recall that C can depend on μ). Remark that for any $k \leqslant l$, we have

$$|\partial_t^k \rho_2^*| \leqslant C|\partial_t^l \rho_2^*|. \tag{2.18}$$

Combining (2.16) for $n = (N-1)/2$, (2.17), (2.18) and the equations satisfied by ψ and ψ_1, we obtain

$$\begin{aligned} \lambda \int_0^T & e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}} \|\psi\|^2_{H^{N+1}(\Omega)} dt \\ & \leqslant C\lambda \left(\int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}+N+1+\frac{N+1}{p}} ||\psi||^2_{L^2(\Omega)} dt \right. \\ & \qquad \left. + \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}+2+\frac{2}{p}} ||\psi||^2_{H^{N-1}(\Omega)} dt \right). \end{aligned} \tag{2.19}$$

In the right-hand side of (2.19), we would like to estimate the term

$$\int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}+2+\frac{2}{p}} ||\psi||^2_{H^{N-1}(\Omega)} dt.$$

This can be done using exactly the same processus by introducing some appropriate auxiliary weight that multiplies ψ or ψ_2 as in (2.15), using Lemma 2.4 successively for $n = (N-1)/2, \ldots, 0$, (2.17) and (2.18). At the end, by gathering all the inequalities, we obtain

$$\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}} \|\psi\|^2_{H^{N+1}(\Omega)} dt \leqslant C\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}+N+1+\frac{N+1}{p}} ||\psi||^2_{L^2(\Omega)} dt. \tag{2.20}$$

Applying the same technique also leads to

$$\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{4c} \|\psi\|^2_{H^{N+3}(\Omega)} dt \leqslant C\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{4c+N+3+\frac{N+3}{p}} ||\psi||^2_{L^2(\Omega)} dt. \tag{2.21}$$

From (2.14), (2.20) and (2.21), we deduce that

$$\begin{aligned} \lambda \int_0^T e^{-2(1+\mu)s\alpha^*} s\xi_* \int_{\partial\Omega} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt & \leqslant C\lambda \left(\int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{4(1-c)}{3}+N+1+\frac{N+1}{p}} ||\psi||^2_{L^2(\Omega)} dt \right. \\ & \qquad \left. + \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{4c+N+3+\frac{N+3}{p}} ||\psi||^2_{L^2(\Omega)} dt \right). \end{aligned} \tag{2.22}$$

Since we would like the powers in the right-hand side to be equal, it is natural to impose that

$$4c + N + 3 + \frac{N+3}{p} = \frac{4(1-c)}{3} + N + 1 + \frac{N+1}{p},$$

i.e.

$$c = \frac{-3-p}{8p}. \tag{2.23}$$

Thus, using (2.22) and (2.23), we deduce that

$$\lambda \int_0^T e^{-2s\alpha^*} s\xi_* \int_{\partial\Omega} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt \leqslant C\lambda \int_0^T e^{-2(1+\mu)s\alpha^*} (s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}} ||\psi||^2_{L^2(\Omega)} dt. \tag{2.24}$$

From (2.13), (2.24), the first line of (2.17) and the definition of ψ_1 given in (2.11), we already deduce that

$$I(s,\lambda;\phi) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3 |\nabla^N \psi_1|^2 dxdt + \lambda \iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}} |\psi_1|^2 dxdt \right.$$
$$\left. + \iint_{Q_T} e^{-2s\alpha} \sum_{i=1}^N |\nabla^i \psi_1|^2 dxdt + \iint_{Q_T} e^{-2s\alpha}(s\xi_*)^{2+\frac{2}{p}} |\nabla^N \psi_1|^2 dxdt \right).$$

By definition the definition of ξ_* given in (2.7), it is clear that $\xi_* \leqslant \xi$. Hence, taking p large enough such that $2+\frac{2}{p} \leqslant 3$ (*i.e.* $p \geqslant 2$), s, λ large enough and using the definition of $I(s,\lambda;\phi)$ given in (2.8), we deduce that we can absorb the last term of the right-hand-side, so that we obtain

$$I(s,\lambda;\phi) \leqslant C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3 |\nabla^N \psi_1|^2 dxdt + \lambda \iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}} |\psi_1|^2 dxdt \right.$$
$$\left. + \iint_{Q_T} e^{-2s\alpha} \sum_{i=1}^N |\nabla^i \psi_1|^2 dxdt \right). \tag{2.25}$$

Step 2: We apply Lemma 2.3 successively with

$$(u,r) = (\nabla^{N-1}\psi_1, 3), \dots, (u,r) = (\nabla\psi_1, 2N-1).$$

We obtain a sequence of inequalities of the form

$$s^5\lambda^6 \iint_{Q_T} e^{-2s\alpha}\xi^5 |\nabla^{N-1}\psi_1|^2 dxdt \leqslant C\left(s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3 |\nabla^N\psi_1|^2 dxdt \right.$$
$$\left. + s^5\lambda^6 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5 |\nabla^{N-1}\psi_1|^2 dxdt \right),$$
$$\dots$$
$$s^{2N+1}\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha}\xi^{2N+1} |\nabla\psi_1|^2 dxdt \leqslant C\left(s^{2N-1}\lambda^{2N} \iint_{Q_T} e^{-2s\alpha}\xi^{2N-1} |\nabla^2\psi_1|^2 dxdt \right.$$
$$\left. + s^{2N+1}\lambda^{2N+2} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^{2N+1} |\nabla\psi_1|^2 dxdt \right).$$

We deduce by starting from the last inequality and using in cascade the other ones that

$$s^5\lambda^6 \iint_{Q_T} e^{-2s\alpha}\xi^5 |\nabla^{N-1}\psi_1|^2 dxdt + \dots + s^{2N+1}\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha}\xi^{2N+1} |\nabla\psi_1|^2 dxdt$$
$$\leqslant C\left(s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha}\xi^3 |\nabla^N\psi_1|^2 dxdt + s^5\lambda^6 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5 |\nabla^{N-1}\psi_1|^2 dxdt \right. \tag{2.26}$$
$$\left. + \dots + s^{2N+1}\lambda^{2N+2} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^{2N+1} |\nabla\psi_1|^2 dxdt \right).$$

Combining (2.25), (2.26) and using the definition of $I(s,\lambda,\phi)$ given in (2.8), we deduce that we can absorb the first term on the right-hand side of (2.26) and obtain

$$
\begin{aligned}
& s\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi|\nabla^{N+1}\psi_1|^2 dxdt + \ldots + s^{2N+1}\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2 dxdt \\
& \leqslant C\left(\lambda \iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}}|\psi_1|^2 dxdt + \iint_{Q_T} e^{-2s\alpha}\sum_{i=1}^{N-1}|\nabla^i\psi_1|^2 dxdt\right. \\
& +s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt + s^5\lambda^6 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^5|\nabla^N\psi_1|^2 dxdt + \ldots \\
& \left.+s^{2N+1}\lambda^{2N+2} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2 dxdt\right).
\end{aligned}
$$

Absorbing the second term of the right-hand side, we deduce that for s, λ large enough, we have

$$
\begin{aligned}
& s\lambda^2 \iint_{Q_T} e^{-2s\alpha}\xi|\nabla^{N+1}\psi_1|^2 dxdt + \ldots + s^{2N+1}\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2 dxdt \\
& \leqslant C\left(\lambda \iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}}|\psi_1|^2 dxdt\right. \\
& \left.+s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt + \ldots + s^{2N+1}\lambda^{2N} \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2 dxdt\right).
\end{aligned}
\tag{2.27}
$$

Step 3: Now, we consider some open subset ω_2 such that $\omega_1 \subset\subset \omega_2 \subset\subset \omega_0$. We consider some function $\tilde{\theta} \in C^\infty(\Omega, \mathbb{R})$ such that:

- $\mathrm{Supp}(\tilde{\theta}) \subset \omega_2$,
- $\tilde{\theta} = 1$ on ω_1,
- $\tilde{\theta} \in [0, 1]$.

Some integrations by parts give

$$
\begin{aligned}
& s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt \leqslant s^3\lambda^4 \iint_{(0,T)\times\omega_2} \theta e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt \\
& \leqslant Cs^3\lambda^4 \iint_{(0,T)\times\omega_2} \left(|\nabla(\theta e^{-2s\alpha}\xi^3)|.|\nabla^N\psi_1|.|\nabla^{N-1}\psi_1| + |\theta e^{-2s\alpha}\xi^3|.|\nabla^{N+1}\psi_1|.|\nabla^{N-1}\psi_1|\right) dxdt.
\end{aligned}
$$

From the definition of ξ and α given in (2.5) and (2.6), we deduce that

$$
|\nabla(\theta e^{-2s\alpha}\xi^3)| \leqslant Cs\lambda e^{-2s\alpha}\xi^4. \tag{2.28}
$$

Combining this estimate with Young's inequality, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for any s and λ large enough, we have

$$
\begin{aligned}
& s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt \leqslant C\left(\varepsilon s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^3|\nabla^N\psi_1|^2 dxdt\right. \\
& \left.+\varepsilon s\lambda^2 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi|\nabla^{N+1}\psi_1|^2 dxdt + C_\varepsilon s^5\lambda^6 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^5|\nabla^{N-1}\psi_1|^2 dxdt\right).
\end{aligned}
\tag{2.29}
$$

Combining (2.27) and (2.29), we can absorb the local terms in $|\nabla^{N+1}\psi_1|^2$ and $|\nabla^N\psi_1|^2$ to deduce

$$\begin{aligned}
&s\lambda^2\iint_{Q_T} e^{-2s\alpha}\xi|\nabla^{N+1}\psi_1|^2dxdt+\ldots+s^{2N+1}\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2dxdt\\
&\leqslant C\left(\lambda\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}}|\psi_1|^2dxdt+s^5\lambda^6\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^5|\nabla^{N-1}\psi_1|^2dxdt\right.\\
&\left.+\ldots+s^{2N+1}\lambda^{2N+2}\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2dxdt\right).
\end{aligned}$$

We can perform exactly the same procedure on the terms

$$s^5\lambda^6\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^5|\nabla^{N-1}\psi_1|^2dxdt,\ldots,s^{2N-1}\lambda^{2N-2}\iint_{(0,T)\times\omega_2} e^{-2s\alpha}\xi^{2N-1}|\nabla^2\psi_1|^2dxdt$$

in order to obtain the following estimate:

$$\begin{aligned}
&s\lambda^2\iint_{Q_T} e^{-2s\alpha}\xi|\nabla^{N+1}\psi_1|^2dxdt+\ldots+s^{2N+1}\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2dxdt\\
&\leqslant C\left(\lambda\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(r+1)+5r+3}{2r}}|\psi_1|^2dxdt\right.\\
&\left.+s^{2N+1}\lambda^{2N+2}\iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2N+1}|\nabla\psi_1|^2dxdt\right).
\end{aligned}\tag{2.30}$$

Step 4: Since the weight $(s\xi_*)^{2N-1}$ does not depend on the space variable, $s\xi_*$ is bounded from below by a positive number, and using the definition of α^* and ξ_* given in (2.7), the following Poincaré's inequality holds:

$$\begin{aligned}
&\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{2N+1}|\psi_1|^2dxdt\leqslant C\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{2N+1}|\nabla\psi_1|^2dxdt\\
&\leqslant C\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha}(s\xi)^{2N+1}|\nabla\psi_1|^2dxdt.
\end{aligned}\tag{2.31}$$

Combining (2.30) and (2.31), we deduce that for s large enough

$$\begin{aligned}
&\lambda^2\iint_{Q_T} e^{-2s\alpha}s\xi|\nabla^{N+1}\psi_1|^2dxdt+\ldots+\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha}(s\xi)^{2N+1}|\nabla\psi_1|^2dxdt\\
&+\lambda^{2N+2}\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{2N+1}|\psi_1|^2dxdt\\
&\leqslant C\left(\lambda\iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}}|\psi_1|^2dxdt+\lambda^{2N+2}\iint_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^{2N+1}|\nabla\psi_1|^2dxdt\right).
\end{aligned}\tag{2.32}$$

We now fix $p\geqslant 2$ large enough such that

$$\frac{2N(p+1)+5p+3}{2p}<2N+1,$$

which is clearly possible since $\frac{2N(p+1)+5p+3}{2p}\to N+\frac{5}{2}$ as $p\to\infty$ and $N\geqslant 3$ (so that $N+5/2<2N+1$).

Using that $e^{-2s\alpha^*}(s\xi_*)^{\frac{2N(p+1)+5p+3}{2p}}\leqslant Ce^{-2s\alpha}(s\xi)^{2N+1}$, we deduce by absorbing the first term of the right-hand side of (2.32) that

$$\lambda^2 \iint_{Q_T} e^{-2s\alpha}(s\xi)|\nabla^{N+1}\psi_1|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha}(s\xi)^{2N+1}|\nabla\psi_1|^2 dxdt$$
$$+\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha^*}(s\xi_*)^{2N+1}|\psi_1|^2 dxdt \leqslant C\lambda^{2N+2} \iint_{(0,T)\times\omega_0} e^{-2s\alpha}(s\xi)^{2N+1}|\nabla\psi_1|^2 dxdt.$$

Going back to ψ thanks to (2.12), we deduce (2.10).

■

2.2 Algebraic resolubility

In this section, we will derive a new Carleman inequality, adapted to the control problem with less controls we want to prove.

Lemma 2.5. *Let $m \in \mathbb{N}^*$ such that $m \leqslant d-1$. Assume that the $\overline{u}$ is regular enough (for example of class $\mathcal{C}^\infty$).*

Consider two partial differential operators $\mathcal{L}_1 : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)^m$ and $\mathcal{L}_2 : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ defined for every $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ by

$$\mathcal{L}_1\varphi := B^*(\nabla\varphi) \text{ and } \mathcal{L}_2\varphi := \partial_t\varphi + \Delta\varphi + (\overline{u}\cdot\nabla)\varphi.$$

Assume that (1.6) holds, and let $q \in \mathbb{N}$ such that

$$rank(\{B_1^*, ..., B_m^*\} \cup \{((B^*\cdot\nabla)^\alpha \overline{u}_i(t,x))_{i\in\{1,\cdots,d\}}, \alpha \in \mathbb{N}^m,\ \alpha \neq 0,\ ||\alpha||_1 \leqslant q\}) = d. \tag{2.33}$$

There exists an open subset $(t_1,t_2)\times\widetilde{\omega}$ of $(0,T)\times\omega$ and there exist two partial differential operators $\mathcal{M}_1 : C^\infty(\mathbb{R}^d)^m \to C^\infty(\mathbb{R}^d)^d$ (of order 1 in time and $q+1$ in space) and $\mathcal{M}_2 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^d)^d$ (of order 0 in time and q in space) such that

$$\mathcal{M}_1 \circ \mathcal{L}_1 + \mathcal{M}_2 \circ \mathcal{L}_2 = \nabla \text{ in } \mathcal{C}^\infty((t_1,t_2)\times\widetilde{\omega}). \tag{2.34}$$

Proof of Lemma 2.5: If $q = 0$, necessarily, by condition (1.6), we have $m = d$ and we can take $\mathcal{M}_1 = (B^*)^{-1}$ and $\mathcal{M}_2 = 0$. We assume from now on that $q \in \mathbb{N}^*$. Let $j \in \{1, ..., m\}$. We call $\mathcal{L}_1^j$ the $j-th$ line of $\mathcal{L}_1$. We remark that

$$\begin{aligned}
(B_j^*\cdot\nabla)\mathcal{L}_2\varphi - (\partial_t+\Delta)\mathcal{L}_1^j\varphi - (\overline{u}\cdot\nabla)\mathcal{L}_1^j\varphi &= (B_j^*\cdot\nabla)(\overline{u}\cdot\nabla)\varphi - (\overline{u}\cdot\nabla)(B_j^*\cdot\nabla)\varphi \\
&= (\overline{u}\cdot\nabla)(B_j^*\cdot\nabla)\varphi + \sum_{k=1}^d ((B_j^*\cdot\nabla)\overline{u}_k)\partial_k\varphi \\
&\qquad - (\overline{u}\cdot\nabla)(B_j^*\cdot\nabla)\varphi \\
&= \sum_{k=1}^d ((B_j^*\cdot\nabla)\overline{u}_k)\partial_k\varphi \\
&=: \mathcal{L}_3^j.
\end{aligned}$$

Now, for some $l \in \{1, ..., m\}$, the same computations easily give

$$(B_l^*\cdot\nabla)\mathcal{L}_3^j\varphi - \sum_{k=1}^d ((B_j^*\cdot\nabla)\overline{u}_k)\partial_k\mathcal{L}_1^l\varphi = \sum_{k=1}^d ((B_l^*\cdot\nabla)(B_j^*\cdot\nabla)\overline{u}_k)\partial_k\varphi =: \mathcal{L}_4^{j,l}\varphi.$$

Continuing this procedure, we can easily create two partial differential operators $\widetilde{\mathcal{M}_1}$ (of order 1 in time and $q+1$ in space) and $\widetilde{\mathcal{M}_2}$ (of order 0 in time and q in space) such that

$$\widetilde{\mathcal{M}_1}(\mathcal{L}_1(\varphi))(t_0,x_0)+\widetilde{\mathcal{M}_2}(\mathcal{L}_2(\varphi))(t_0,x_0)=\widetilde{M}(\overline{u})(\nabla\varphi)(t_0,x_0),$$

where $\widetilde{M}(\overline{u})(t_0,x_0)$ is a matrix composed by d independent vectors of the family $M(\overline{u})(t_0,x_0)$ with $\|\alpha\|_1\leqslant q$ (which is possible since (2.33) is verified). By continuity, there exists an open neighbourhood $(t_1,t_2)\times\widetilde{\omega}$ of (t_0,x_0) in $(0,T)\times\omega$ and $C>0$ such that $|\det(\widetilde{M}(\overline{u}))|>C$ on $(t_1,t_2)\times\widetilde{\omega}$. We call $\widetilde{M}(\overline{u})^{-1}(t,x)$ the inverse of $\widetilde{M}(\overline{u})(t,x)$ for $(t,x)\in(t_1,t_2)\times\widetilde{\omega}$. Then, is is clear that $\mathcal{M}_1:=\widetilde{M}(\overline{u})^{-1}\widetilde{\mathcal{M}_1}$ and $\mathcal{M}_2:=\widetilde{M}(\overline{u})^{-1}\widetilde{\mathcal{M}_2}$ verify (2.34) and have C^∞ coefficients on $(t_1,t_2)\times\widetilde{\omega}$. ■

We now have all the tools to deduce our final Carleman inequality:

Proposition 2.2. *Assume that Condition* (1.6) *and the hypotheses of Proposition 2.1 hold. Then, for all* $\eta\in(0,1)$*, there exists* $p\geqslant 2$*,* $C>0$ *and* $K>0$ *such that for every* $\psi^0\in L^2(\Omega)$*, the corresponding solution* ψ *to System* (2.4) *satisfies*

$$\begin{aligned}\int_\Omega\psi(0)^2dx+\iint_{Q_T}e^{\frac{-2K}{\eta(T-t)^p}}\{\psi^2+|\partial_t\psi|^2+\ldots+|\partial_{t\ldots t}^{\lfloor\frac{N+1}{2}\rfloor}\psi|^2+|\nabla\psi|^2+\ldots+|\nabla^{N+1}\psi|^2\}dxdt\\ \leqslant Ce^{K/T^p}\iint_{(0,T)\times\omega_0}e^{\frac{-2K}{(T-t)^p}}|B^*(\nabla\psi)|^2dxdt.\end{aligned}\tag{2.35}$$

Proof of Proposition 2.2. Let ω_1 some open subset strongly included in ω_0. Combining Proposition 2.1, Lemma 2.5 (that is still true by replacing ω_0 by ω_1), and the fact that any solution ψ of (2.4) verifies by definition $\mathcal{L}_2\psi=0$, we deduce that, for any $\psi^0\in L^2(\Omega)$, the corresponding solution ψ to System (2.4) satisfies

$$\begin{aligned}\lambda^2\iint_{Q_T}e^{-2s\alpha-2\mu s\alpha^*}(s\xi)|\nabla^{N+1}\psi|^2dxdt+\ldots+\lambda^{2N+2}\iint_{Q_T}e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\nabla\psi|^2dxdt\\ +\lambda^{2N+2}\iint_{Q_T}e^{-2s\alpha^*-2\mu s\alpha^*}(s\xi_*)^{2N+1}|\psi|^2dxdt\\ \leqslant C\lambda^{2N+2}\iint_{Q_T}\tilde{\theta}e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\mathcal{M}_1B^*(\nabla\psi)|^2dxdt,\end{aligned}$$

where $\mathcal{M}_1$ is a linear partial differential operator of order 1 in time and $q+1$ in space, and $\tilde{\theta}\in C^\infty(\Omega,\mathbb{R})$ such that:

- $\tilde{\theta}=1$ on ω_1,
- $\mathrm{Supp}(\tilde{\theta})\subset\omega_0$,
- $\tilde{\theta}\in[0,1]$.

We first remark that

$$\begin{aligned}\lambda^{2N+2}\iint_{Q_T}\theta e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\mathcal{M}_1B^*(\nabla\psi)|^2dxdt\\ \leqslant C\lambda^{2N+2}\iint_{Q_T}\theta e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}\left(\sum_{i=0}^{q+1}\left(|\nabla^iB^*\nabla\psi|^2+|\partial_t\nabla^iB^*\nabla\psi|^2\right)\right)dxdt.\end{aligned}$$

Using that ψ verifies (2.4), we can deduce that

$$\begin{aligned}\lambda^{2N+2}\iint_{Q_T}\theta e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\mathcal{M}_1B^*(\nabla\psi)|^2dxdt\\ \leqslant C\lambda^{2N+2}\iint_{Q_T}\theta e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}\left(\sum_{i=0}^{q+3}|\nabla^iB^*\nabla\psi|^2\right)dxdt.\end{aligned}$$

Some integrations by parts give

$$\begin{aligned}
&\lambda^{2N+2} \iint_{Q_T} \tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1} |\nabla B^*(\nabla\psi)|^2 dxdt \\
&\leqslant C\lambda^{2N+2} \iint_{Q_T} \tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1} |B^*(\nabla\psi)||\nabla^3\psi| dxdt \\
&+C\lambda^{2N+2} \iint_{Q_T} |\nabla(\tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1})||B^*(\nabla\psi)||\nabla^2\psi| dxdt.
\end{aligned}$$

Let $\varepsilon > 0$. Young's inequality gives

$$\begin{aligned}
&\lambda^{2N+2} \iint_{Q_T} \tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1} |B^*(\nabla\psi)||\nabla^3\psi| dxdt \\
&\leqslant C_\varepsilon \lambda^{2N+6} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+5} |B^*(\nabla\psi)|^2 dxdt \\
&+\varepsilon\lambda^{2N-2} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N-3} |\nabla^3\psi|^2 dxdt
\end{aligned}$$

and also, by (2.28),

$$\begin{aligned}
&\lambda^{2N+2} \iint_{Q_T} |\nabla(\tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1})||B^*(\nabla\psi)||\nabla^2\psi| dxdt \\
&\leqslant C\lambda^{2N+3} \iint_{Q_T} \tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+2} |B^*(\nabla\psi)||\nabla^2\psi| dxdt \\
&\leqslant C_\varepsilon \lambda^{2N+6} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+5} |B^*(\nabla\psi)|^2 dxdt \\
&+\varepsilon\lambda^{2N} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N-1} |\nabla^2\psi|^2 dxdt.
\end{aligned}$$

Thus, by taking ε small enough, we deduce that

$$\begin{aligned}
&\lambda^2 \iint_{Q_T} e^{-2s\alpha-2\mu s\alpha^*}(s\xi)|\nabla^{N+1}\psi|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\nabla\psi|^2 dxdt \\
&+\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha^*-2\mu s\alpha^*}(s\xi_*)^{2N+1}|\psi|^2 dxdt \\
&\leqslant C\lambda^{2N+6} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+5} |B^*(\nabla\psi)|^2 dxdt \\
&+C\lambda^{2N+2} \iint_{Q_T} \tilde{\theta} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1} \left(\sum_{i=2}^{q+3} |\nabla^i B^*\nabla\psi|^2\right) dxdt.
\end{aligned}$$

By iterating this process for $i = 2, \ldots, q+3$, we can get rid of the sum in the right-hand side and obtain

$$\begin{aligned}
&\lambda^2 \iint_{Q_T} e^{-2s\alpha-2\mu s\alpha^*}(s\xi)|\nabla^{N+1}\psi|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha-2\mu s\alpha^*}(s\xi)^{2N+1}|\nabla\psi|^2 dxdt \\
&+\lambda^{2N+2} \iint_{Q_T} e^{-2s\alpha^*-2\mu s\alpha^*}(s\xi_*)^{2N+1}|\psi|^2 dxdt \\
&\leqslant C\lambda^{2N+2+4(q+2)} \iint_{(0,T)\times\omega_0} e^{-2s\alpha-2\mu s\alpha^*} (s\xi)^{2N+1+4(q+2)} |B^*(\nabla\psi)|^2 dxdt.
\end{aligned}$$

We deduce that

$$\lambda^2 \iint_{Q_T} e^{-2(1+\mu)s\alpha^*}(s\xi_*)|\nabla^{N+1}\psi|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2(1+\mu)s\alpha^*}(s\xi_*)^{2N+1}|\nabla\psi|^2 dxdt$$
$$+\lambda^{2N+2} \iint_{Q_T} e^{-2(1+\mu)\mu s\alpha^*}(s\xi_*)^{2N+1}|\psi|^2 dxdt$$
$$\leqslant C\lambda^{2N+2+4(q+2)} \iint_{(0,T)\times\omega_0} e^{-2\mu s\alpha^*}\,(s\xi^*)^{2N+1+4(q+2)}\,|B^*(\nabla\psi)|^2 dxdt,$$

where $\xi^* = \max_{\Omega} \xi$. Defining

$$\tilde{\alpha}^* = \begin{cases} \alpha^*(T/2) & \text{on } (0,T/2), \\ \alpha^* & \text{on } (T/2,T), \end{cases} \quad \tilde{\xi}_* = \begin{cases} \xi_*(T/2) & \text{on } (0,T/2), \\ \xi_* & \text{on } (T/2,T), \end{cases} \quad \tilde{\xi}^* = \begin{cases} \xi^*(T/2) & \text{on } (0,T/2), \\ \xi^* & \text{on } (T/2,T), \end{cases}$$

then, for s and λ large enough, using usual energy estimates,

$$\lambda^2 \iint_{Q_T} e^{-2(1+\mu)s\tilde{\alpha}^*}(s\tilde{\xi}_*)|\nabla^{N+1}\psi|^2 dxdt + \ldots + \lambda^{2N+2} \iint_{Q_T} e^{-2(1+\mu)s\tilde{\alpha}^*}(s\tilde{\xi}_*)^{2N+1}|\nabla\psi|^2 dxdt$$
$$+\tilde{\lambda}^{2N+2} \iint_{Q_T} e^{-2(1+\mu)\mu s\tilde{\alpha}^*}(s\tilde{\xi}_*)^{2N+1}|\psi|^2 dxdt$$
$$\leqslant C\lambda^{2N+2+4(q+2)} \iint_{(0,T)\times\omega_0} e^{-2\mu s\tilde{\alpha}^*}\left(s\tilde{\xi}^*\right)^{2N+1+4(q+2)} |B^*(\nabla\psi)|^2 dxdt.$$

Fixing s and λ, using (2.5) and (2.6), and remarking that $\tilde{\xi}^*$ does not depend on μ, we deduce that there exists $R > 0$ such that for any $\mu > 0$ large enough,

$$\iint_{Q_T} e^{-\frac{2(2+\mu)R}{(T-t)^p}}\{|\nabla^{N+1}\psi|^2 + \ldots + |\psi|^2\}dxdt \leqslant C \iint_{(0,T)\times\omega_0} e^{-\frac{2(\mu-1)R}{(T-t)^p}}|B^*(\nabla\psi)|^2 dxdt.$$

We remark that the fact that ψ verifies (2.4) enables us to add all the derivatives in time on the left-hand side. Hence, we can conclude by fixing $\eta \in (0,1)$, introducing $K = (\mu-1)R$ (for $\mu > 1$), and taking $\mu > 0$ large enough so that

$$\frac{(\mu-1)R}{\eta} > (2+\mu)R,$$

which is always possible since the ratio $(\mu-1)/(2+\mu)$ tends to 1 as $\mu \to \infty$.

■

2.3 Regular control

Our goal in this section is to construct regular enough controls. Remind that θ is defined in (2.2).

Proposition 2.3. *Let $r \in \mathbb{N}$. Assume that Condition* (1.6) *holds.*

Under the hypotheses of Proposition 2.1, System

$$\begin{cases} \partial_t y = \Delta y + \operatorname{div}(\overline{u}y) + \operatorname{div}(\theta B v) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{2.36}$$

is null controllable at time T, i.e. for every $y^0 \in L^2(\Omega)$, there exists a control $v \in L^2(Q_T)^m$ such that the solution z to System (2.36) *satisfies $z(T) \equiv 0$ in Ω. Moreover, we can choose $v \in L^2((0,T),$ $H^{2r+2}(\Omega))^m \cap H^{r+1}((0,T),L^2(\Omega))^m$ with*

$$\|v\|_{L^2((0,T),H^{2r+2}(\Omega))^m\cap H^{r+2}((0,T),L^2(\Omega))^m} \leqslant Ce^{K/T^p}\|y^0\|_{L^2(\Omega)},$$

where K is the constant in (2.35).

Proof of Proposition 2.3. Let $k \in \mathbb{N}^*$ and let us consider the following optimal control problem

$$\left\{ \begin{array}{l} \text{minimize } J_k(v) := \dfrac{1}{2}\|\widetilde{\rho}^{-1/2}v\|^2_{L^2(Q_T)^m} + \dfrac{k}{2}\displaystyle\int_\Omega |z(T)|^2 dx, \\ v \in \mathcal{U} := \{w \in L^2(Q_T)^m : \widetilde{\rho}^{-1/2}w \in L^2(Q_T)^m\}, \end{array} \right. \tag{2.37}$$

where $\widetilde{\rho} := e^{\frac{-2K}{(T-t)^p}}$ (for the $K > 0$ given by Proposition 2.2 with N an even number to be chosen later and some fixed $\eta \in (1/2, 1)$) and z is the solution in $W(0,T)$ to

$$\left\{ \begin{array}{ll} \partial_t z = \mathcal{A}z + \mathcal{B}v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0 & \text{in } \Omega, \end{array} \right.$$

where

$$\left\{ \begin{array}{l} \mathcal{A} := \Delta + \operatorname{div}(\overline{u}\ \cdot\ ), \\ \mathcal{B} := \operatorname{div}(B\theta\ \cdot\ ). \end{array} \right. \tag{2.38}$$

Here, $\mathcal{U}$ is endowed with its natural weighted L^2-norm.

The functional $J_k : \mathcal{U} \to \mathbb{R}^+$ is differentiable, coercive and strictly convex on the space $\mathcal{U}$. Therefore, following [35, [p. 116], there exists a unique solution to the optimal control problem (2.37) and the optimal control v_k is characterized thanks to the solution z_k of the primal system by

$$\left\{ \begin{array}{ll} \partial_t z_k = \mathcal{A}z_k + \mathcal{B}v_k & \text{in } Q_T, \\ z_k = 0 & \text{on } \Sigma_T, \\ z_k(0,\cdot) = y^0 & \text{in } \Omega, \end{array} \right. \tag{2.39}$$

the solution φ_k to the dual system

$$\left\{ \begin{array}{ll} -\partial_t \varphi_k = \mathcal{A}^*\varphi_k & \text{in } Q_T, \\ \varphi_k = 0 & \text{on } \Sigma_T, \\ \varphi_k(T,\cdot) = kz_k(T,\cdot) & \text{in } \Omega \end{array} \right. \tag{2.40}$$

and the relation

$$\left\{ \begin{array}{l} v_k = -\widetilde{\rho}\mathcal{B}^*\varphi_k \text{ in } Q_T, \\ v_k \in \mathcal{U}. \end{array} \right. \tag{2.41}$$

The characterization (2.39), (2.40) and (2.41) of the minimizer v_k of J_k in $\mathcal{U}$ leads to the following computations

$$\begin{array}{rcl} J_k(v_k) & = & -\dfrac{1}{2}\langle \mathcal{B}^*\varphi_k, v_k\rangle_{L^2(Q_T)^m} + \dfrac{1}{2}\langle z_k(T), \varphi_k(T)\rangle_{L^2(\Omega)} \\ & = & -\dfrac{1}{2}\displaystyle\int_0^T \langle \varphi_k, \mathcal{B}v_k\rangle_{L^2(\Omega)} dt + \dfrac{1}{2}\int_0^T \{\langle z_k, \partial_t\varphi_k\rangle_{L^2(\Omega)} + \langle \partial_t z_k, \varphi_k\rangle_{L^2(\Omega)}\} dt \\ & & \qquad + \dfrac{1}{2}\langle y^0, \varphi_k(0,\cdot)\rangle_{L^2(\Omega)} \\ & = & \dfrac{1}{2}\langle y^0, \varphi_k(0,\cdot)\rangle_{L^2(\Omega)}. \end{array} \tag{2.42}$$

Moreover, using (2.35) and the expression of $\widetilde{\rho}$, we infer

$$\|\varphi_k(0,\cdot)\|_{L^2(\Omega)} \leqslant Ce^{K/T^p}\|\widetilde{\rho}^{-1/2}v_k\|_{L^2(Q_T)^m}. \tag{2.43}$$

Now, using the definition of J_k, the expression (2.42), the inequality (2.43) and the Cauchy-Schwartz inequality, we infer

$$\|\varphi_k(0,\cdot)\|^2_{L^2(\Omega)} \leqslant Ce^{2K/T^p} J_k(v_k) \leqslant Ce^{2K/T^p}\|\varphi_k(0,\cdot)\|_{L^2(\Omega)}\|y^0\|_{L^2(\Omega)},$$

from which we deduce

$$\|\varphi_k(0,\cdot)\|_{L^2(\Omega)} \leqslant Ce^{2K/T^p}\|y^0\|_{L^2(\Omega)}. \tag{2.44}$$

Then, using (2.42) and (2.44), we deduce

$$J_k(v_k) \leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}. \tag{2.45}$$

Furthermore, we have (see [35, p. 116])

$$\begin{aligned} \|z_k\|_{W(0,T)} &\leqslant C\left(\|\mathcal{B}v_k\|_{L^2((0,T),H^{-1}(\Omega))} + \|y^0\|_{L^2(\Omega)}\right), \\ &\leqslant C\left(\|\widetilde{\rho}^{-1/2}v_k\|_{L^2(Q_T)^m} + \|y^0\|_{L^2(\Omega)}\right), \\ &\leqslant C(1+Ce^{K/T^p})\|y^0\|_{L^2(\Omega)}, \end{aligned} \tag{2.46}$$

where C does not depend on y^0 and k. Then, using inequalities (2.45) and (2.46), we deduce that there exist subsequences, which are still denoted v_k, z_k, such that the following weak convergences hold:

$$\begin{cases} v_k \rightharpoonup v & \text{in } \mathcal{U}, \\ z_k \rightharpoonup z & \text{in } W(0,T), \\ z_k(T) \rightharpoonup 0 & \text{in } L^2(\Omega). \end{cases}$$

Passing to the limit in k, z is solution to System (2.38). Moreover, using the expression of J_k given in (2.37) and inequality (2.45), we deduce by letting k going to ∞ that $z(T) \equiv 0$ in Ω. Thus the solution z to System (2.38) with control $v \in \mathcal{U}$ satisfies $z(T) \equiv 0$ in Ω and using (2.45), we obtain the inequality

$$\|v\|^2_{\mathcal{U}} \leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}.$$

Since $\tilde{\rho}^{-1} \geqslant 1$, using the definition of the norm on $\mathcal{U}$, we also deduce that

$$\|v\|^2_{L^2(Q_T)^m} \leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}.$$

Now, let us explain why the controls are more regular. First of all, using the fact that φ_k verifies (2.40), we deduce that for any $j \in \mathbb{N}$,

$$\|\mathcal{B}^*\partial_t^j\varphi_k(t,.)\|^2_{L^2(\Omega)} \leqslant C\|\partial_t^{j+1}\varphi_k(t,.)\|^2_{L^2(\Omega)}, \forall t \in (0,T).$$

Hence, for each $i \in \{1,...,\frac{N}{2}-1\}$ and $k \in \mathbb{N}$, using inequalities similar to (2.17) and (2.18), we deduce that for any $\varepsilon > 0$, there exists $C > 0$ such that

$$\begin{aligned} \|\partial_t^i v_k\|^2_{L^2(Q_T)^m} &= \iint_{Q_T} \partial_t^i\left(|-\tilde{\rho}\mathcal{B}^*\varphi_k|\right)^2 \\ &\leqslant C\iint_{Q_T} \tilde{\rho}^{2-2\varepsilon}\{|\varphi_k|^2 + \cdots + |\partial_t^{i+1}\varphi_k|^2\} \\ &\leqslant C\iint_{Q_T} \tilde{\rho}^{2-2\varepsilon-\frac{1}{\eta}}\tilde{\rho}^{\frac{1}{\eta}}\{|\varphi_k|^2 + \cdots + |\partial_t^{i+1}\varphi_k|^2\}. \end{aligned} \tag{2.47}$$

Now, we fix $\varepsilon > 0$ small enough (with respect to η) such that $2 - 2\varepsilon - \frac{1}{\eta} \geqslant 0$. With this choice of ε, we infer that $\tilde{\rho}^{2-2\varepsilon-\frac{1}{\eta}} \leqslant 1$. Hence, using (2.47) together with (2.35) and (2.45), we deduce that, for each $i \in \{0,...,\frac{N}{2}-1\}$, $\|\partial_t^i v_k\| \in L^2(Q_T)$ and

$$\begin{aligned} \|\partial_t^i v_k\|^2_{L^2(Q_T)^m} &\leqslant C\iint_{Q_T} e^{\frac{-2K}{\eta(T-t)^p}}\{|\varphi_k|^2 + \cdots + |\partial_t^{i+1}\varphi_k|^2\} \\ &\leqslant C\iint_{Q_T} e^{\frac{-2K}{(T-t)^p}}|\theta\mathcal{B}^*(\varphi_k)|^2 \\ &\leqslant C\|v_k\|^2_{\mathcal{U}} \\ &\leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}. \end{aligned}$$

$$\|\mathcal{B}^*\partial_t^j\phi_k(t,.)\|^2_{L^2(\Omega)} \leqslant C\|\partial_t^{j+1}\phi_k(t,.)\|^2_{L^2(\Omega)}, \forall t\in(0,T),$$

Thus, extracting one more time a subsequence if necessary and letting k go to $+\infty$, we deduce that for each $i\in\{1,...,\frac{N}{2}-1\}$,

$$\|\partial_t^i v\|_{L^2(Q_T)^m} \leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}.$$

We similarly deduce that, for each $i\in\{1,...,N-2\}$,

$$\|\nabla^i v\|_{L^2(Q_T)^{m\times i\times d}} \leqslant Ce^{2K/T^p}\|y^0\|^2_{L^2(\Omega)}.$$

The proof is completed by setting $r=\frac{N}{2}+1$. ■

3 Controllability to the trajectories

Let $r\in\mathbb{N}$. We use the strategy developed in [37], modifying it slightly to fit our case. Usual interpolation estimates (see [36, Section 13.2, p. 96]) show that

$$L^2((0,T),H^{2r+2}(\Omega))\cap H^{r+1}((0,T),L^2(\Omega))\hookrightarrow L^2((0,T),H^{2r+2}(\Omega))\cap H^1((0,T),H^{2r}(\Omega)),$$

from which we deduce

$$L^2((0,T),H^{2r+2}(\Omega))\cap H^{r+1}((0,T),L^2(\Omega))\hookrightarrow L^\infty((0,T),H^{2r}(\Omega)).$$

Now, there exists $R>0$ large enough such that by Sobolev embeddings, we have

$$L^2((0,T),H^{2R+2}(\Omega))\cap H^{R+1}((0,T),L^2(\Omega))\hookrightarrow L^\infty((0,T),W^{1,\infty}(\Omega)).$$

Hence, from Proposition 2.3 and Remark 3, for any $y^0\in L^2(\Omega)$, there exists a control $v\in L^\infty((0,T),W^{1,\infty}(\Omega))^m$ such that the solution y to System (2.3) satisfies $y(T)\equiv 0$ in Ω and

$$\|v\|_{L^\infty((0,T),W^{1,\infty}(\Omega))^m} \leqslant Ce^{K/T^p}\|y^0\|_{L^2(\Omega)},$$

where $K>0$ is the constant given by Proposition 2.2 with $N=2R$ and $p\geqslant 2$ is given in Proposition 2.1.

Letting the system evolve freely a little bit if needed, we may assume without loss of generality that $y^0-\overline{y}^0\in H^1_0(\Omega)$. Indeed, by the regularizing effect, it is very easy to deduce that for any solution $(\overline{y},\overline{u})$ to (1.4), there exists some $C(T)>0$ such that for any solution $(y,0)$ to (1.1) on $[0,\frac{T}{2}]$, we have $y\left(\frac{T}{2}\right)-\overline{y}\left(\frac{T}{2}\right)\in H^1_0(\Omega)$ and

$$||y\left(\frac{T}{2}\right)-\overline{y}\left(\frac{T}{2}\right)||_{H^1(\Omega)} \leqslant C(T)||y^0-\overline{y}^0||_{L^2(\Omega)}.$$

Hence, if $||y^0-\overline{y}^0||_{L^2(\Omega)}$ is small, so is $||y\left(\frac{T}{2}\right)-\overline{y}\left(\frac{T}{2}\right)||_{H^1(\Omega)}$, so that the condition (1.5) is sufficient for our argument to be valid.

Following [37, p. 24], we introduce the cost of controllability given by

$$\gamma(t)=Ce^{K/t^p},\ \ t\in(0,T),$$

and the following weight functions

$$\rho_F(t)=e^{-\frac{\alpha}{(T-t)^{p+1}}}, t\in[0,T]$$

and

$$\rho_0(t) = e^{\frac{K}{((q-1)(T-t))^p} - \frac{\alpha}{q^{2p+2}(T-t)^{p+1}}},\ t \in \left[T\left(1-\frac{1}{q^2}\right), T\right],$$

extended on $[0, T\left(1-\frac{1}{q^2}\right)]$ by

$$\rho_0(t) = \rho_0\left(T\left(1-\frac{1}{q^2}\right)\right),\ t \in [0, T\left(1-\frac{1}{q^2}\right)],$$

for some parameters $q > 1$ and $\alpha > 0$ to be chosen later on.

We remark that ρ_F and ρ_0 are non-increasing, verify $\rho_F(T) = \rho_0(T) = 0$ and are related by the relation

$$\rho_0(t) = \rho_F(q^2(T-t)+T)\gamma((q-1)(T-t)),\ t \in \left[T\left(1-\frac{1}{q^2}\right), T\right].$$

We introduce for some $\beta > 0$ the weight function

$$\rho(t) = e^{-\frac{\beta}{(T-t)^{p+1}}}.$$

We remark that

$$\rho_F \leqslant C\rho,\ \rho_0 \leqslant C\rho,\ |\rho'|\rho_0 \leqslant C\rho^2,$$

as soon as $\beta > 0$ is chosen small enough, precisely

$$\beta < \frac{\alpha}{q^{2p+2}}. \tag{3.1}$$

We introduce the following spaces:

$$\mathcal{F} = \{f \in L^2((0,T)\times\Omega), \frac{f}{\rho_F} \in L^2((0,T)\times\Omega)\},$$

$$\mathcal{U} = \{u \in L^2((0,T)\times\Omega), \frac{u}{\rho_0} \in L^\infty((0,T), W^{1,\infty})\}$$

and

$$\mathcal{Z} = \{z \in L^2((0,T)\times\Omega), \frac{z}{\rho} \in H^1((0,T), L^2) \cap L^2((0,T), H^2 \cap H_0^1)\},$$

endowed with the weighted Sobolev norms naturally induced by the definition of these spaces.

Following [37, Proofs of Propositions 2.5, 2.8] in the spirit of [31, Section 7.2 and Appendix 5], it is easy to obtain the following result.

Proposition 3.1. *For any $z^0 \in H_0^1(\Omega)$ and any $f \in \mathcal{F}$, there exists $v \in \mathcal{U}$ such that the solution z of*

$$\begin{cases} \partial_t z &= \Delta z + \operatorname{div}(\overline{u}z) + \operatorname{div}(\theta\overline{y}Bv) + f & \text{in } Q_T, \\ z &= 0 & \text{on } \Sigma_T, \\ z(0,\cdot) &= z^0 & \text{in } \Omega, \end{cases}$$

verifies $z \in \mathcal{Z}$ (and hence $z(T) = 0$).

To conclude, we use the following inverse mapping theorem:

Theorem 3.1 (see [3]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $M : \mathcal{X} \mapsto \mathcal{Y}$ be a $\mathcal{C}^1$ mapping. Consider $x_0 \in \mathcal{X}$ and $y_0 := M(x_0) \in \mathcal{Y}$. Assume that the derivative $M'(x_0) : \mathcal{X} \mapsto \mathcal{Y}$ is onto. Then, there exist $\eta > 0$, a mapping $W : B_\eta(y_0) \subset \mathcal{Y} \mapsto \mathcal{X}$ and a constant $K > 0$ satisfying:*

$$\begin{cases} W(z) \in \mathcal{X} \text{ and } M(W(z)) = z \ \forall z \in B_\eta(y_0), \\ \|W(z) - x_0\|_{\mathcal{X}} \leqslant K\|z - y_0\|_{\mathcal{Y}} \ \forall z \in B_\eta(y_0). \end{cases}$$

Proof of Theorem 1.2. We are looking for a solution in the form

$$y(x,t) = \overline{y}(x,t) + w(x,t),\ u(x,t) = \overline{u}(x,t) + \theta(x)Br(x,t),$$

where (y,u) and $(\overline{y},\overline{u})$ are solution to the Systems (1.1) and (1.4), respectively. Then (w,r) is solution to

$$\begin{cases} N(w,r) &:= & \partial_t w - \Delta w - \operatorname{div}(\overline{u}w + \theta Br\overline{y} + \theta Brw) = 0 & \text{in } Q_T, \\ w &= & 0 & \text{on } \Sigma_T, \\ w(0,\cdot) &= & y^0 - \overline{y}^0 & \text{in } \Omega. \end{cases}$$

We introduce the following spaces:

$$\mathcal{X} := \{(w,r) \in \mathcal{Z} \times \mathcal{U} \text{ such that } \partial_t w - \Delta w - \operatorname{div}(\overline{u}w + \theta Br\overline{y}) \in \mathcal{F}\},$$

endowed with the norm

$$||(w,r)||_{\mathcal{X}} = ||w||_{\mathcal{Z}} + ||r||_{\mathcal{U}} + ||\partial_t w - \Delta w - \operatorname{div}(\overline{u}w + \theta Br\overline{y})||_{\mathcal{F}},$$

and the space

$$\mathcal{Y} = \mathcal{F} \times H^1_0(\Omega),$$

endowed with the norm

$$||(f,z^0)||_{\mathcal{Y}} := ||f||_{\mathcal{F}} + ||z^0||_{H^1(\Omega)}.$$

Introduce the mapping M given by

$$\begin{array}{rccc} M: & \mathcal{X} & \to & \mathcal{Y} \\ & (w,r) & \mapsto & (N(w,r), w(0,\cdot)). \end{array}$$

Let us determine what are the conditions on q, α, β ensuring that M is well-defined. It is clear that

$$||w(0,\cdot)||_{H^1(\Omega)} \leqslant ||w||_{C^0([0,T],H^1_0(\Omega))} \leqslant C\left\|\frac{w}{\rho}\right\|_{C^0([0,T],H^1_0(\Omega))} \leqslant ||(w,r)||_{\mathcal{X}}.$$

Now, we remark that by definition of the space $\mathcal{X}$, we have

$$||\partial_t w - \Delta w - \operatorname{div}(\overline{u}w + \theta Br\overline{y})||_{\mathcal{F}} \leqslant ||(w,r)||_{\mathcal{X}}.$$

Hence, the only difficulty is to treat the bilinear part $\operatorname{div}(\theta wBr)$. We remark that

$$\left\|\frac{\operatorname{div}(\theta wBr)}{\rho_F}\right\|_{L^2((0,T)\times\Omega)} \leqslant C\left\|\frac{r}{\rho_F^{\frac{1}{2}}}\right\|_{L^\infty((0,T),W^{1,\infty}(\Omega))}\left\|\frac{w}{\rho_F^{\frac{1}{2}}}\right\|_{L^2((0,T),H^1(\Omega))}.$$

We can impose that $\rho^2 \leqslant C\rho_F$ and $\rho_0^2 \leqslant C\rho_F$ as soon as

$$\alpha < 2\beta \ \text{ and } q^{2p+2} < 2. \tag{3.2}$$

Remark that these conditions are compatible with condition (3.1).

Hence, under conditions (3.1) and (3.2), we deduce that

$$\left\|\frac{\operatorname{div}(\theta wBr)}{\rho_F}\right\|_{L^2((0,T)\times\Omega)} \leqslant C\left\|\frac{r}{\rho_0}\right\|_{L^\infty((0,T),W^{1,\infty}(\Omega))}\left\|\frac{w}{\rho}\right\|_{L^2((0,T),H^1(\Omega))} \leqslant C||(w,r)||^2_{\mathcal{X}}.$$

We conclude that under these conditions, M is indeed well-defined and continuous. Moreover, we remark that $M(0,0)=(0,0)$ and M is of class $\mathcal{C}^1$ as a sum of a continuous linear function and a continuous quadratic function. Furthermore, Proposition 3.1 exactly means that $M'(0,0)$ is onto (see Remark 3), when

$$\frac{\alpha}{q^{2p+p}} < 1 \tag{3.3}$$

and $\eta \in (0,1)$ is chosen as

$$\eta := \frac{\alpha}{q^{2p+p}}.$$

Conditions (3.1), (3.2) and (3.3) can be summarized as follows:

$$\frac{\alpha}{2} < \beta < \frac{\alpha}{q^{2p+p}} < 1 \text{ and } q^{2p+2} < 2,$$

which is satisfied for $q=(3/2)^{1/(2p+2)} > 1$ (remind that $p \geqslant 2$, so that $2p+p \geqslant 2p+2 \geqslant 1$), $\alpha = 1$ and $\beta = 7/12$. Theorem 3.1 leads to the conclusion.

■

4 Example of a non-controllable trajectory with a reduced number of controls

In this section, we give the example of a trajectory which does not satisfy condition (1.6) and for which the local controllability to the trajectories does not hold.

Consider $\overline{u} \in L^\infty(Q_T)^m$ which is independent of the time variable and will be determined later on. Assume that for each $\overline{y}^0 \in L^2(\Omega)\setminus\{0\}$ the following system is locally controllable to the trajectories with a control operator B to be chosen later on:

$$\begin{cases} \partial_t \overline{y} &= \Delta\overline{y} + \operatorname{div}(\overline{u}\overline{y}) & \text{in } Q_T, \\ \overline{y} &= 0 & \text{on } \Sigma_T, \\ \overline{y}(0,\cdot) &= \overline{y}^0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Then, for any $\varepsilon \in (0,1)$, there exists $u \in L^\infty(Q_T)^m$ such that

$$\begin{cases} \partial_t y &= \Delta y + \operatorname{div}(uy) & \text{in } Q_T, \\ y &= 0 & \text{on } \Sigma_T, \\ y(0,\cdot) &= (1-\varepsilon)\overline{y}^0 & \text{in } \Omega, \\ y(T,\cdot) &= \overline{y}(T) & \text{in } \Omega, \end{cases}$$

where $u = \overline{u} + Bv$ with $\operatorname{Supp}(v) \subset (0,T)\times\omega$. We remark that $(z,w) := (y-\overline{y}, yv)$ is solution to

$$\begin{cases} \partial_t z &= \Delta z + \operatorname{div}(\overline{u}z) + \operatorname{div}(Bw) & \text{in } Q_T, \\ z &= 0 & \text{on } \Sigma_T, \\ z(0,\cdot) &= \varepsilon\overline{y}^0 & \text{in } \Omega, \\ z(T,\cdot) &= 0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

We deduce that the linear control system (4.2) is null controllable at time $T > 0$, then approximately controllable at time $T > 0$. It is well known that the approximate controllability of System (4.2) on $(0, T)$ implies the following property, called the Fattorini-Hautus test (see *e.g.* [39]) : for every $s \in \mathbb{C}$ and every $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\left.\begin{aligned} -\Delta\varphi - \overline{u}\cdot\nabla\varphi &= s\varphi \text{ in } \Omega \\ B^*\nabla\varphi &= 0 \text{ in } \omega \end{aligned}\right\} \Rightarrow \varphi = 0. \tag{4.3}$$

Now, we give an explicit situation in contradiction with (4.3). Consider $\Omega = (0,\pi)^2$, $B^* = (0,1)$ and $s = 25$ (ω and $\overline{u}$ will be chosen later on). The goal is to find a nontrivial solution $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ of

$$\begin{cases} -\Delta\varphi + \overline{u}\cdot\nabla\varphi &= 25\varphi \text{ in } \Omega, \\ \partial_{x_2}\varphi &= 0 \text{ in } \omega. \end{cases} \tag{4.4}$$

We introduce two functions f and g defined on Ω and given by

$$f(x_1,x_2) = \sin(3x_1)\sin(4x_2) \quad \text{and} \quad g(x_1,x_2) = -\frac{2\sqrt{2}}{5}\sin(5x_2) + \frac{2\sqrt{2}}{5}\cos(5x_2).$$

Remark that $f \in C^\infty(\Omega)$ and $g \in C^\infty(\Omega)$ are chosen in such a way that

$$\begin{cases} -\Delta f &= 25f \text{ on } \Omega, \\ -\Delta g &= 25g \text{ on } \Omega, \\ \partial_{x_1} g &= 0 \text{ on } \Omega, \\ f &= 0 \text{ on } \partial\Omega, \\ f\left(\frac{\pi}{2},\frac{\pi}{4}\right) &= g\left(\frac{\pi}{2},\frac{\pi}{4}\right), \\ \nabla f\left(\frac{\pi}{2},\frac{\pi}{4}\right) &= \nabla g\left(\frac{\pi}{2},\frac{\pi}{4}\right). \end{cases} \tag{4.5}$$

Now, let us consider some cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi \in [0,1]$, $\chi \equiv 0$ on $\mathbb{R}\setminus[\frac{\pi}{4},\frac{3\pi}{4}] \times \mathbb{R}\setminus[\frac{\pi}{8},\frac{3\pi}{8}]$ and $\chi = 1$ on $[\frac{3\pi}{8},\frac{5\pi}{8}] \times [\frac{3\pi}{16},\frac{5\pi}{16}]$. For a parameter $h \in (0,1)$, we call

$$\chi_h(x_1,x_2) := \chi\left(\frac{\pi}{2} + \frac{x_1 - \pi/2}{h}, \frac{\pi}{4} + \frac{x_2 - \pi/4}{h}\right).$$

Note that χ_h is supported in

$$V_h := \left[\frac{\pi}{2} - \frac{h\pi}{4}, \frac{\pi}{2} + \frac{h\pi}{4}\right] \times \left[\frac{\pi}{4} - \frac{h\pi}{8}, \frac{\pi}{4} + \frac{h\pi}{8}\right],$$

verifies

$$\chi_h = 1 \quad \text{on} \quad \left[\frac{\pi}{2} - \frac{h\pi}{8}, \frac{\pi}{2} + \frac{h\pi}{8}\right] \times \left[\frac{\pi}{4} - \frac{h\pi}{16}, \frac{\pi}{4} + \frac{h\pi}{16}\right] := W_h \tag{4.6}$$

and

$$|\chi_h| \leqslant 1,\ |\partial_{x_2}\chi_h| \leqslant \frac{C}{h} \text{ on } \mathbb{R}^2, \tag{4.7}$$

for some $C > 0$ independent on h. Now, we introduce

$$\varphi_h = \chi_h g + (1-\chi_h) f. \tag{4.8}$$

We remark that for any $h \in (0,1)$, $\varphi_h \not\equiv 0$ since it coincides with f outside $V_h \neq \emptyset$ and with g on $W_h \neq \emptyset$. Moreover, one has

$$\partial_{x_2}\varphi_h = \partial_{x_2}\chi_h(g - f) + \chi_h\partial_{x_2} g + (1-\chi_h)\partial_{x_2} f. \tag{4.9}$$

By the two last lines of (4.5) and Taylor expansions, for $h \in (0,1)$, we have

$$|f-g| \leqslant Ch^2,\ |\partial_{x_2} f - \partial_{x_2} g| \leqslant Ch \text{ on } V_h, \tag{4.10}$$

for some $C>0$ independent on h. From (4.7), (4.9) and (4.10), we deduce that

$$|\partial_{x_2}\varphi_h - \partial_{x_2} g| \leqslant |\partial_{x_2}\chi_h|\,|g-f| + |1-\chi_h|\,|\partial_{x_2} g - \partial_{x_2} f| \leqslant Ch \text{ on } V_h,$$

for some $C>0$ independent on h. Since

$$\partial_{x_2} g\left(\frac{\pi}{2},\frac{\pi}{4}\right) = 4 > 0,$$

we deduce that there exists $h_0>0$ small enough such that, $\partial_{x_2}\varphi_{h_0} \geqslant C$ on V_{h_0} for some $C>0$. Accordingly to (4.6), we choose $\omega = W_{h_0}$, and

$$\overline{u} := \begin{cases} (0,0) & \text{in } (0,\pi)^2 \setminus (V_{h_0}\setminus W_{h_0}),\\ (0, -\frac{25\varphi_{h_0}+\Delta\varphi_{h_0}}{\partial_{x_2}\varphi_{h_0}}) & \text{otherwise.}\end{cases}$$

Remark that $\overline{u}$ is well defined: by construction, $\partial_{x_2}\varphi_{h_0} \geqslant C$ where $-\Delta\varphi_{h_0} - 25\varphi_{h_0} \neq 0$ (which is included in $V_{h_0}\setminus W_{h_0}$).

Moreover, $\overline{u}$ is of class C^∞ on Ω. To conclude, we remark that by (4.5) and (4.8),

$$\begin{cases} -\Delta\varphi_{h_0} - \overline{u}\cdot\nabla\varphi_{h_0} & = 25\varphi_{h_0} \text{ in } \Omega,\\ \varphi_{h_0} & = 0 \text{ in } \partial\Omega,\\ \partial_{x_1}\varphi_{h_0} & = 0 \text{ in } \omega,\\ \varphi_{h_0} & \neq 0.\end{cases}$$

Hence, we obtain a contradiction with the Fattorini-Hautus test, which concludes our proof.

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