CONTROLLABILITY OF A COUPLED WAVE SYSTEM WITH A SINGLE CONTROL AND DIFFERENT SPEEDS

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Abstract. We consider an exact controllability problem in a smooth bounded domain Ω of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, for a coupled wave system, with two different speeds and a single control acting on an open subset ω satisfying the Geometric Control Condition and acting on one speed only. Actions for the wave equations with the second speed are obtained through a coupling term. Firstly, we construct appropriate state spaces with compatibility conditions associated with the coupling structure. Secondly, in these well-prepared spaces, we prove that the coupled wave system is exactly controllable if and only if the coupling structure satisfies an operator Kalman rank condition.

Contents

1. Introduction and Main Results

1.1. **General setting.** Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, be a bounded and smooth domain. We use Δ to denote the canonical Laplace operator on Ω, and Δ_D to denote the Laplace operator with domain $H^2(\Omega) \cap H^1_0(\Omega)$. Let $\Box_1 = \partial_t^2 - d_1\Delta$ and $\Box_2 = \partial_t^2 - d_2\Delta$ be two d'Alembert operators with different constant speeds $d_1 \neq d_2$. Let n_1, n_2 be two integers and $n = n_1 + n_2$. We assume that ω is a nonempty open subset of Ω and that $T > 0$ is a final time horizon. In this article, we aim to deal with some

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controllability properties of the following type of coupled wave systems:

$$\left\{\begin{array}{lll}\Box_1 U_1 + A_1 U_2 & = 0 & \text{in } (0,T)\times\Omega,\\ \Box_2 U_2 + A_2 U_2 & = bf\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\ U_1 = U_2 & = 0 & \text{on } (0,T)\times\partial\Omega,\\ (U_1,U_2)|_{t=0} & = (U_1^0,U_2^0) & \text{in } \Omega,\\ (\partial_t U_1,\partial_t U_2)|_{t=0} & = (U_1^1,U_2^1) & \text{in } \Omega.\end{array}\right. \tag{1.1}$$

For $j=1,2$, we use $U_j = \begin{pmatrix} u_1^j \\ \vdots \\ u_{n_j}^j \end{pmatrix}$ to denote the solutions corresponding to the speed d_j. $f\in L^2((0,T)\times\omega)$ is the control function, which is a scalar control and acts on $(0,T)\times\omega$. $A_1\in\mathcal{M}_{n_1,n_2}(\mathbb{R})$ and $A_2\in\mathcal{M}_{n_2}(\mathbb{R})$ are two given coupling matrices and $b\in\mathbb{R}^{n_2}$. Note that System (1.1) is a particular case of systems of the form

$$\left\{\begin{array}{lll}(\partial_t^2 - D\Delta)U + AU & = \hat{b}f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\ U & = 0 & \text{on } (0,T)\times\partial\Omega,\\ (U,\partial_t U)|_{t=0} & = (U^0,U^1) & \text{in } \Omega,\end{array}\right. \tag{1.2}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0\\ 0 & d_2 Id_{n_2}\end{pmatrix}_{n\times n},\ A = \begin{pmatrix} 0 & A_1\\ 0 & A_2\end{pmatrix}_{n\times n},\ \text{and } \hat{b} = \begin{pmatrix} 0\\ b\end{pmatrix}_{n\times 1}, \tag{1.3}$$

where $n = n_1 + n_2$. Let us emphasize the following important and crucial properties of System (1.1): all coefficients are constant, the coupling is in a block-cascade structure (notably, the control f is only acting directly on U_2, which itself acts on U_1 through the matrix A_1), and we restrict to the case of a scalar control (*i.e.* $f\in L^2((0,T),\mathbb{R}^m)$ with $m=1$). We will explain in conclusion the difficulties to treat more general cases.

1.2. **Geometric assumptions.** For our concerned domain Ω, we assume that Ω has no infinite order of tangential contact with the boundary. This assumption will be made more precise in Subsection 2.3. In fact, this assumption is sufficient to ensure the existence and uniqueness of the general bicharateristics passing through a given point in the phase space. Furthermore, for the control set ω, we assume the Geometric Control Condition (GCC).

Definition 1.1. *For $\omega\subset\Omega$ and $T>0$, we shall say that the triple (ω,T,p) satisfies GCC if every generalized bicharacteristic of p meets ω in a time $t<T$, where p is the principal symbol of $\Box$.*

We shall give a precise definition of the generalized bicharacteristics in Subsection 2.3. In the case of an internal control, GCC was firstly raised in [29] as a necessary condition for the controllability of the scalar wave equation from ω, and was proved to be sufficient in [6]. The case of a boundary control was studied in [7, 9].

1.3. **Kalman conditions.** In this part, we recall some Kalman rank conditions introduced in the literature of coupled parabolic systems and the link between them. First of all, we recall the usual Kalman rank condition for the controllability of linear autonomous ordinary differential equations (see *e.g.* [16]).

Definition 1.2 (Usual algebraic Kalman rank condition). *Let m, n be two positive integers. Assume $X \in \mathcal{M}_n(\mathbb{R})$ and $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. We introduce the Kalman matrix associated with X and Y given by $[X|Y] = [X^{n-1}Y|\cdots|XY|Y] \in \mathcal{M}_{n,nm}(\mathbb{R})$. We say that (X,Y) satisfies the Kalman rank condition if $[X|Y]$ has full rank.*

In order to generalize this usual algebraic Kalman rank condition, we introduce the Kalman operator (see [5]).

Definition 1.3 (Kalman operator). *Assume that $X \in \mathcal{M}_n(\mathbb{R})$ and $Y \in \mathcal{M}_{n,m}(\mathbb{R})$. Moreover, let $D \in \mathcal{M}_n(\mathbb{R})$ be a diagonal matrix. Then, the Kalman operator associated with $(-D\Delta_D + X, Y)$ is the matrix operator $\mathscr{K} = [-D\Delta_D + X|Y] : \mathcal{D}(\mathscr{K}) \subset (L^2)^{nm} \to (L^2)^n)$, where the domain of the Kalman operator is given by $\mathcal{D}(\mathscr{K}) = \{u \in (L^2(\Omega))^{nm} : \mathscr{K}u \in (L^2(\Omega))^n\}$.*

Definition 1.4 (Operator Kalman rank condition). *We say that the Kalman operator $\mathscr{K}$ satisfies the operator Kalman rank condition if $Ker(\mathscr{K}^*) = \{0\}$.*

The operator Kalman rank condition can be reformulated as follows.

Proposition 1.5. [5, Proposition 2.2] *The operator Kalman rank condition $Ker(\mathscr{K}^*) = \{0\}$ is equivalent to the following spectral Kalman rank condition:*

$$rank[(\lambda D + X)|Y] = n, \forall \lambda \in \sigma(-\Delta_D).$$

In particular, let $C > 0$ be a constant and $D = CId_n$. Then, the operator Kalman rank condition is equivalent to the usual algebraic Kalman rank conditionon (X,Y) given in Definition 1.2 (see [5, Remark 1.2]).

In the following proposition, we give an equivalent statement of the operator Kalman rank condition associated with System (1.1), which is very specific to our particular coupling structure and the fact that we have a single control.

Proposition 1.6. *We use the same notations $(D, A, \hat{b})$ as in (1.3). We denote by $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ the Kalman operator associated with the System (1.2). Then, $Ker(\mathcal{K}^*) = \{0\}$ is equivalent to satisfying all the following conditions:*

(1) $n_1 = 1$;
(2) (A_2, b) satisfies the usual Kalman rank condition (See Definition 1.2);
(3) Assume that $A_1 = \alpha = (\alpha_1, \cdots, \alpha_{n_2})$. Then, $\forall \lambda \in \sigma(-\Delta_D)$, α satisfies

$$\alpha \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} Id_{n_2} \right) b \neq 0, \tag{1.4}$$

where $(a_j)_{0 \le j \le n_2}$ are the coefficients of the characteristic polynomial $\chi(X)$ of the matrix A_2, i.e. $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$, with the convention that $a_{n_2} = 1$.

We shall give the proof in Appendix A.

Since we consider the control problem in a domain Ω with boundary, it is natural for us to introduce the following Hilbert spaces $H^s_\Omega(\Delta_D)$.

Definition 1.7. *We denote by $(\beta_j^2)_{j \in \mathbb{N}^*}$ the non-decreasing sequence of (positive) eigenvalues of $-\Delta_D$, repeated with multiplicity, and $(e_j)_{j \in \mathbb{N}^*}$ an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions associated with $(\beta_j^2)_{j \in \mathbb{N}^*}$:*

$$-\Delta e_j = \beta_j^2 e_j, \ e_j(x) = 0, x \in \partial\Omega, \quad ||e_j||_{L^2(\Omega)} = 1.$$

For any $s \in \mathbb{R}$, we denote by $H^s_\Omega(\Delta_D)$ the Hilbert space defined by

$$H^s_\Omega(\Delta_D) = \{u = \sum_{j\in\mathbb{N}^*} a_j e_j; \sum_{j\in\mathbb{N}^*} \beta_j^{2s}|a_j|^2 < \infty\}.$$

For convenience, we also denote

$$\mathscr{L}^k_s = (H^s_\Omega(\Delta_D))^k \text{ for any } s \in \mathbb{R}, \text{ and } k \in \mathbb{N}. \tag{1.5}$$

First, we give a necessary condition for the controllability of System (1.1).

Proposition 1.8. *We denote by $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ the Kalman operator associated with the System* (1.2)*. If $\mathcal{K}$ does not satisfy the operator Kalman rank condition, then System* (1.1) *is not null-controllable, in the following sense: there exists a quadruple*

$$(U_1^0, U_2^0, U_1^1, U_2^1) \in \bigcap_{s=1}^{+\infty} \left(\mathscr{L}^n_s \times \mathscr{L}^n_{s-1}\right)$$

such that for any control $f \in L^2(\omega)$, we necessarily have

$$(U(T,\cdot), \partial_t U(T,\cdot)) \neq (0,0).$$

We shall give the proof later in the Subsection 2.1.

From now on, we always assume that $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ satisfies the operator Kalman rank condition, so that we notably have $n_1 = 1$. Before we give a precise definition of the exact controllability property of System (1.1), we first investigate a simpler system. For a fixed $1 \le s \le n_2$, we consider the following system

$$\begin{cases} \Box_1 u_1^1 + \sum_{j=1}^s \alpha_s u_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 u_{n_2-1}^2 + u_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ u_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1\le j\le n_2, \\ (u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) & \text{in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) & \text{in } \Omega. \end{cases} \tag{1.6}$$

Here, we have, $A_1 = (\alpha_1, \cdots, \alpha_s, 0, \cdots, 0)$ and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The control is $f \in L^2((0,T)\times\omega)$. For this simpler system (1.6), taking zero initial conditions (that belong to any linear subspace and hence to any potential state space) together with a forcing term f in the space $L^2((0,T)\times\omega)$, which kind of target spaces will the solutions of System (1.6) arrive in? That is the first question we need to answer in order to be able to obtain an exact controllability result in an appropriate state space. Under this particular structure of coupling, we introduce appropriate compatibility conditions for System (1.6). For $r = 0, 1$, and

$(u,v_1,\cdots,v_{n_2})\in H_\Omega^{n_2-s+2+r}(\Delta_D)\times H_\Omega^{n_2-1+r}(\Delta_D)\times\cdots\times H_\Omega^{r}(\Delta_D)$, let us define a special function U^r_{comp} by

$$U^r_{comp}=\Big((-d_1\Delta)^{n_2-s+1}u + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}v_{j+l} + \sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}v_{j+k+l}\Big). \tag{1.7}$$

Using this special function U^r_{comp}, let us denote by $\mathcal{H}^s_r$ the following space:

$$\mathcal{H}^s_r=\{(u,v_1,\cdots,v_{n_2})\in H_\Omega^{n_2-s+2+r}(\Delta_D)\times H_\Omega^{n_2-1+r}(\Delta_D)\times\cdots\times H_\Omega^{r}(\Delta_D) \ \text{s.t. } U^r_{comp}\in H^r_\Omega(\Delta_D)\}. \tag{1.8}$$

Definition 1.9 (State space)**.** *The state space for System* (4.1) *is defined by*

$$\mathcal{H}^s_1\times\mathcal{H}^s_0.$$

The two conditions

$$U^1_{comp}(u_1^{1,0},u_1^{2,0},\cdots,u_{n_2}^{2,0})\in H^1_\Omega(\Delta_D),$$
$$U^0_{comp}(u_1^{1,1},u_1^{2,1},\cdots,u_{n_2}^{2,1})\in H^0_\Omega(\Delta_D)$$

are called the compatibility conditions for the controllability of System (4.1)*.*

Remark 1.10. *If $s=n_2$, the compatibility conditions reduce to*

$$-d_1\Delta u_1^{1,0}\in H^1_\Omega(\Delta_D),$$
$$-d_1\Delta u_1^{1,1}\in H^0_\Omega(\Delta_D),$$

which is an empty condition since we already know that $(u_0^1,u_1^1)\in H^3_\Omega(\Delta_D)\times H^2_\Omega(\Delta_D)$.

Remark 1.11. *As we will see later on, the solutions of System* (1.6) *will stay in $\mathcal{H}^s_1\times\mathcal{H}^s_0$ if the initial states are in this space. Because of the linearity and the time reversibility of the system, exact controllability is equivalent to null controllability or reachability from 0 for System* (1.6)*. Since the equilibrium 0 is of course in the spaces $\mathcal{H}^s_1\times\mathcal{H}^s_0$, this is the appropriate state space.*

Remark 1.12. *Since we consider a system with a cascade coupling structure, it is natural that there is a gain of regularity for the uncontrolled states $u_j^2\,(2\le j\le n_2)$ (this phenomena has already been observed notably in* [13, Theorem 1.4]*). We shall explain the gain of two derivatives of regularity for the state u_1^1 in Subsection 2.2. We call it "additional regularity".*

Now, we give the definition of the exact controllability of System (1.1).

Definition 1.13. *We say that System* (1.1) *is exactly controllable in time $T>0$ if there exists $1\leqslant s\leqslant n_2$ and $\mathcal{T}\in GL_n(\mathbb{R})$ such that for any initial data $(U_0,U_1)\in\mathcal{T}^{-1}(\mathcal{H}^s_1)\times\mathcal{T}^{-1}(\mathcal{H}^s_0)$ and any target $(\tilde{U}_0,\tilde{U}_1)\in\mathcal{T}^{-1}(\mathcal{H}^s_1)\times\mathcal{T}^{-1}(\mathcal{H}^s_0)$, there exists a control function $f\in L^2((0,T)\times\omega)$ such that the solution U of* (1.1) *satisfies $(U,\partial_tU)|_{t=0}=(U_0,U_1)$ and $(U,\partial_tU)|_{t=T}=(\tilde{U}_0,\tilde{U}_1)$, and $\mathcal{T}(U)$ is a solution of the associated System* (1.6) *with an appropriate control $\tilde{f}$.*

Remark 1.14. *By the definition above, in order to prove the controllability of System* (1.1)*, we first look for an invertible transform to change the system into the simpler but equivalent System* (1.6)*. Then, we prove the result for the simpler System* (1.6) *to conclude the exact controllability of the general System* (1.1)*.*

Remark 1.15.
We shall see later that the transform $\mathcal{T}$ is just

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix},$$

where $P \in GL_{n_2}(\mathbb{R})$ is the transform associated with the Brunovský normal form defined in Theorem 3.1*. Here we can give an example of the transform $\mathcal{T}$ under a simple setting. If we consider a particular case of System* (1.1) *given by*

$$\begin{cases} \Box_1 u_1^1 - 2u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + \frac{3}{2}u_1^2 - \frac{1}{2}u_2^2 & = 2f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \Box_2 u_2^2 + \frac{9}{2}u_1^2 - \frac{3}{2}u_2^2 & = 4f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ u_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega \\ u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2, \\ (u_1^1, u_1^2, u_2^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, u_2^{2,0}) & \text{in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \partial_t u_2^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, u_2^{2,1}) & \text{in } \Omega, \end{cases}$$

we have that

$$A = \begin{pmatrix} 0 & A_1 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{9}{2} & -\frac{3}{2} \end{pmatrix}, \text{ and } \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}.$$

According to the Brunovský normal form, we obtain $\mathcal{T}_2 = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ *such that*

$$\mathcal{T}_2(A_2 b, b) = \mathcal{T}_2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the transform is given by $\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{T}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$*. And moreover, this transform $\mathcal{T}$ satisfies*

$$\mathcal{T}\hat{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \mathcal{T} A \mathcal{T}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There is a large literature on the controllability and observability of the wave equations. This paper is mainly devoted to multi-speed coupled wave systems. We list some of the existing results and references:

- For a single wave equation posed on a smooth bounded domain of $\mathbb{R}^d$ and with an internal control, one can use microlocal analysis to prove the observability inequality as done by Bardos, Lebeau and Rauch in [6]. We have two approaches to define the microlocal defect measures. We can introduce the microlocal defect measures based on the article by Gérard and Leichtnam [14] for Helmoltz equation and Burq [8] for the wave equation, using the extension by 0 across the boundary. On the other hand, we can also use the Melrose cotangent compressed bundle to construct the measure, based on the article by Lebeau [17] and Burq-Lebeau [10] in the setting of systems.

- Although we now have a better picture on the controllability of a single wave equation, the controllability of systems of wave equations is still not totally understood. To our knowledge, most of the references concern the case of systems with the same principal symbol $\Box$ on each equation of the system, which will be discussed in the present paragraph. Notably, Alabau-Boussouira and Léautaud [4] studied the indirect controllability of two coupled wave equations, in which their controllability result was established using a multi-level energy method introduced in [1], and also used in [2, 3]. Liard and Lissy [23] studied the observability and controllability for coupled wave systems with constant coefficients under Kalman type rank conditions. In the case of space-varying coefficients, Cui, Laurent, and Wang [11] studied the observability of wave equations coupled by space-varying first or zero order terms, on a compact manifold. Their results are extended to the case of manifold with boundaries in [12].
- Concerning the multi-speed case, Dehman, Le Roussau, and Léautaud considered two coupled wave equations on a compact manifold in [13]. Lissy and Zuazua [25] proved some general weak observability estimates for wave systems with constant or time-dependant coupling terms. Niu [28] investigated the case of the simultaneous controllability of wave systems, with different speeds and coupling terms involving only the controls, under various conditions on the speeds. Notably, in the case of constant speeds, a necessary and sufficient condition involving a Kalman rank condition was obtained, in the same spirit as in the present article.
- Concerning the boundary controllability of the coupled wave systems, we would like to refer to the works by Tatsien Li and Bopeng Rao, especially their work on the synchronisation of waves. In [18] and [19], Li and Rao for the first time studied the synchronization for systems described by PDEs. Taking a coupled system of wave equations with Dirichlet boundary controls as an example, they proposed the concept of exact boundary synchronization by boundary controls. After that, they and their collaborators successively got quite a lot of results (for instance, see [20, 22]). In particular, in [21], the authors obtain necessary conditions, presented as a criteria of Kalman's type, to the approximate null controllability, the approximate synchronization, respectively, for a coupled system of wave equations with Dirichlet boundary controls, which also show the link between the controllability of coupled wave systems and some appropriate Kalman conditions.

1.4. **Main result.** Our main result is the following one.

Theorem 1.16. *Given $T > 0$, suppose that:*

(i) (ω, T, p_{d_i}) *satisfies GCC, where* p_{d_i} *is the principal symbol of* $\Box_i, i = 1, 2$.
(ii) Ω *has no infinite order of tangential contact with the boundary.*
(iii) The Kalman operator $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ *associated with System* (1.1) *satisfies the operator Kalman rank condition, i.e.* $Ker(\mathcal{K}^*) = \{0\}$.

Then System (1.1) *is exactly controllable in the sense of Definition* 1.13.

Remark 1.17.
- *We will explain the concept of order of contact in the next section.*
- *Assume that conditions (i) and (ii) are verified. Then, condition (iii) is also necessary to have exact controllability in the sense of Definition* 1.13*. Indeed, if (iii) is not verified, Proposition* 1.8 *provides a smooth initial condition (that is notably in the state space introduced in Definition* 1.13*) that is not null-controllable.*
- *In fact, our proof also provides a controllability result for systems of wave equations with a single speed, of the form*

$$\begin{cases} \Box_2 U_2 + A_2 U_2 & = bf\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ U_2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ U_2|_{t=0} & = (U_1^0, U_2^0) & \text{in } \Omega, \\ \partial_t U_2|_{t=0} & = (U_1^1, U_2^1) & \text{in } \Omega. \end{cases} \tag{1.9}$$

If (A_2, b) does not verify the usual Kalman rank condition given in Definition 1.2, then this system is not exactly controllable in the same sense as in Proposition 1.8, with the same proof. If (A_2, b) verifies the usual Kalman rank condition, the state state space is

$$P^{-1}(\tilde{\mathcal{H}}_1) \times P^{-1}(\tilde{\mathcal{H}}_0),$$

where P is the transform associated with the Brunovský normal form defined in Theorem 3.1 *and $\tilde{\mathcal{H}}_r$ $(r = 1, 2)$ is given by*

$$\tilde{\mathcal{H}}_r = \{(v_1, \cdots, v_{n_2}) \in H_\Omega^{n_2-1+r}(\Delta_D) \times \cdots \times H_\Omega^r(\Delta_D)\}.$$

Then, System (1.9) *is exactly controllable under this Kalman rank condition. This is a very particular case of the more general result proved in* [12]*, where space-varying coefficients, multi-dimensional controls and also one-order coupling terms are considered.*

1.5. **Outline of the paper.** The outline of this paper is the following.

Section 2 is devoted to introducing some preliminaries. In Subsection 2.1, we present the necessity of the operator Kalman rank condition by giving the proof of Proposition 1.8. Then Subsection 2.2 is devoted to the "additional regularity" property for coupled wave equations. Subsection 2.3 includes the description of the boundary points, and give the precise definition of general bicharacteristics and the order of tangential contact with the boundary. Subsection 2.4 introduces the microlocal defect measures, which is the basic tool for our proof.

In Section 3, we focus on the special case $n_2 = 2$ to show the whole procedure of the proof of the controllability of the coupled wave system. Subsection 3.1 is devoted to reformulating the system with the help of the Brunovský normal form. Then in Subsection 3.2 we introduce the simpler system with one of the parameters being 0. We demonstrate the proof under this simple setting. In the following Subsection 3.3, we present the result of the general systems in a way very similar to the simpler case.

In Section 4, we plan to deal with any number of equations. Subsection 4.1 provides the corresponding simpler system in analogue with the Subsection 3.2 and gives the clear meaning of the compatibility conditions under the general setting. Then, with the help of the compatibility conditions, we are able to present the proof of the controllability result of Theorem 4.8. In the Subsection 4.2, we give the reformulation procedure of the general system.

In the concluding Section 5, we give some open problems related to our work, and explanations on the difficulties to solve them.

2. Preliminaries

We divide this section into four parts. The first part is devoted to proving the necessity of the operator Kalman rank condition. Then, we consider the regularities of the solutions of two coupled wave equations with different speeds. The third part aims to introduce the geometric preliminaries including the conceptions of general bicharacteristics and order of contact. The final part mainly contains the definition and some properties of the microlocal defect measures.

2.1. **On the necessity of the operator Kalman rank condition.** In this section, we are going to give the proof of Proposition 1.8. At first, we introduce the following proposition for the ordinary differential systems of second order.

Proposition 2.1. *If (A, b) does not satisfy the usual algebraic Kalman rank condition (see Definition 1.2), for any nonzero initial data $(y^0, y^1) \neq (0, 0)$, the ordinary differential system*

$$\begin{cases} \frac{d^2y}{dt^2} & = A^*y \qquad in\ (0,T), \\ (y, \frac{dy}{dt})|_{t=0} & = (y^0, y^1), \end{cases} \tag{2.1}$$

has a nonzero solution satisfying $b^*y(t) = 0$ for every $t \in (0, T)$.

Proof. Define $z = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}$. Then, we are able to rewrite System (2.1) into a first-order system:

$$\begin{cases} \frac{dz}{dt} & = \tilde{A}^*z \qquad \text{in } (0,T) \\ z|_{t=0} = & {}^t(y^0, y^1), \end{cases} \tag{2.2}$$

where $\tilde{A} = \begin{pmatrix} 0 & A \\ Id_n & 0 \end{pmatrix}_{2n\times 2n}$. Let $\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}_{2n\times 1}$. Easy computations give that

$$\tilde{A}^{2k} = \begin{pmatrix} A^k & 0 \\ 0 & A^k \end{pmatrix} \text{ and } \tilde{A}^{2k+1} = \begin{pmatrix} 0 & A^{k+1} \\ A^k & 0 \end{pmatrix} \text{ for } k = 0, 1, \cdots.$$

Therefore, we obtain

$$[\tilde{A}|\tilde{b}] = (\tilde{A}^{2n-1}\tilde{b}|\cdots|\tilde{A}\tilde{b}|\tilde{b}) = \begin{pmatrix} 0 & A^{n-1}b & \cdots & 0 & b \\ A^{n-1}b & 0 & \cdots & b & 0 \end{pmatrix}.$$

As a consequence, we know that $rank[\tilde{A}|\tilde{b}] = 2rank[A|b]$. Since (A, b) does not satisfy the usual algebraic Kalman rank condition, *i.e.*, $rank[A|b] < n$, we deduce that $rank[\tilde{A}|\tilde{b}] < 2n$, which implies that $(\tilde{A}, \tilde{b})$ does not satisfy the usual algebraic Kalman rank condition. By duality, this means that (2.2) is not observable through $\tilde{b}$.

Thus, there exists a nonzero solution $\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \end{pmatrix} \in \mathbb{R}^{2n}$ to the associated adjoint system $\frac{dz}{dt} = \tilde{A}^*z$ satisfying that $\tilde{b}^*\zeta(t) = 0$ for every $t \in (0, T)$. Then, setting $y(t) = \zeta_1(t)$, we derive a nonzero solution $y(t)$ of System (2.1) satisfying that $b^*y(t) = b^*\zeta_1(t) = \tilde{b}^*\zeta(t) = 0$ for every $t \in (0, T)$. □

Now, we go back to the proof of Proposition 1.8.

Proof of Proposition 1.8. According to Proposition 1.5, since $\mathcal{K} = [-D\Delta_D + A|\hat{b}]$ does not satisfy the operator Kalman rank condition, there exists $\lambda_0 \in \sigma(-\Delta_D)$ such that $rank[(\lambda_0 D - A)|\hat{b}] < n$. As a consequence of Proposition 2.1, there exists a nonzero solution $\chi_{\lambda_0}(t) \in \mathbb{R}^n$ to the following ordinary differential system:

$$\begin{cases} \frac{d^2\chi}{dt^2} & = (\lambda_0 D - A^*)\chi \qquad \text{in } (0,T), \\ (\chi, \frac{d\chi}{dt})|_{t=0} & = (\chi^0, \chi^1) \neq (0,0), \end{cases}$$

satisfying $\hat{b}^*\chi_{\lambda_0}(t) = 0$ for every $t \in (0,T)$. Then, let $\Phi(t,x) = \chi_{\lambda_0}(t)\varphi_{\lambda_0}(x)$, where φ_{λ_0} is an eigenfunction of $-\Delta_D$ associated with λ_0. Therefore, Φ satisfies the following system:

$$\begin{cases} (\partial_t^2 - D\Delta + A^*)\Phi & = 0 & \text{in } \Omega, \\ \hat{b}^*\Phi & = 0 & \text{for every } t \in (0,T), \\ \Phi|_{\partial\Omega} & = 0, & \\ (\Phi, \partial_t\Phi)|_{t=0} & = (\chi^0\varphi_{\lambda_0}, \chi^1\varphi_{\lambda_0}) & \text{in } \Omega. \end{cases} \tag{2.3}$$

Suppose that there exists $f \in L^2((0,T)\times\omega)$ such that the corresponding solution U to (1.2) with initial state (U_0, U_1) satisfies

$$(U, \partial_t U)|_{t=T} = (0,0). \tag{2.4}$$

Then, by (1.2), we have

$$((\partial_t^2 - D\Delta_D + A)U, \Phi)_{L^2((0,T)\times\Omega)} = (\hat{b}f\mathbb{1}_\omega, \Phi)_{L^2((0,T)\times\Omega)}.$$

Integrating by parts on the left-hand side and using (2.3) together with (2.4) leads to

$$(U^0, \chi^1\varphi_{\lambda_0})_{L^2(\Omega)} - (U^1, -\chi^0\varphi_{\lambda_0})_{L^2(\Omega)} = (\hat{b}f\mathbb{1}_\omega, \Phi)_{L^2((0,T)\times\Omega)}.$$

Since $\widehat{b}^*\Phi = 0$ for every $t \in (0,T)$, we obtain that

$$(U^0, \chi^1\varphi_{\lambda_0})_{L^2(\Omega)} - (U^1, \chi^0\varphi_{\lambda_0})_{L^2(\Omega)} = 0.$$

Choosing $(U_0, U_1) = (\chi^1\varphi_{\lambda_0}, -\chi^0\varphi_{\lambda_0})$ leads to $(|\chi^1|^2 + |\chi^0|^2)\,||\varphi_{\lambda_0}||^2_{L^2(\Omega)} = 0$, which is a contradiction with $(\chi^0, \chi^1) \neq 0$. □

2.2. **On the regularity of coupled wave equations.** Before investigating more complicated situations, let us concentrate on the regularity properties of the following simple system:

$$\begin{cases} \Box_1 u_1 + u_2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_2 & = f & \text{in } (0,T)\times\Omega, \\ u_1 = 0, u_2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)|_{t=0} & = (u_1^0, u_1^1, u_2^0, u_2^1) & \text{in } \Omega. \end{cases} \tag{2.5}$$

Our next result gives a property of regularity for the solution of System (2.5). Such kind of extra regularity result was also observed in [13, Theorem 1.4], in which the authors stated the corresponding result in the case of a compact manifold without boundary. Here we will present a different (and more elementary) proof.

Lemma 2.2. *Assume that the initial conditions satisfy*

$$(u_1^0, u_1^1, u_2^0, u_2^1) \in H_\Omega^{\sigma+3}(\Delta_D) \times H_\Omega^{\sigma+2}(\Delta_D) \times H_\Omega^{\sigma+1}(\Delta_D) \times H_\Omega^{\sigma}(\Delta_D). \tag{2.6}$$

Then, there exists a unique solution to System (2.5) *satisfying*

$$u_1 \in C^1([0,T], H_\Omega^{\sigma+2}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+3}(\Delta_D)),$$
$$u_2 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D)). \tag{2.7}$$

Proof. Since u_2 satisfies a wave equation with a source term $f \in L^1((0,T), H_\Omega^\sigma(\Delta_D))$, it is classical that there exists a unique solution

$$u_2 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$$

to the second line of System (2.5). Now, let us consider the first equation

$$\Box_1 u_1 = -u_2 \tag{2.8}$$

as a wave equation with a source term $u_2 \in L^1((0,T), H_\Omega^{\sigma+1}(\Delta_D))$. Thus, we know that there exists a unique solution $u_1 \in C^1([0,T], H_\Omega^{\sigma+1}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+2}(\Delta_D))$. Now, we need to state an extra regularity property for u_1. Applying the d'Alembert operator $\Box_2$ on both sides of (2.8), we obtain that

$$\Box_2\Box_1 u_1 = -\Box_2 u_2.$$

Since $\Box_2 u_2 = f$, we know that $\Box_1(\Box_2 u_1) = -f$. We decompose $\Box_2 u_1$ into two parts $\Box_2 u_1 = \Box_1 u_1 + (d_1 - d_2)\Delta_D u_1$. Hence, we obtain that

$$\Box_2 u_1 = -u_2 + (d_1 - d_2)\Delta_D u_1. \tag{2.9}$$

Now, by using (2.6), we remark that the initial condition for $\Box_2 u_1$ verifies:

$$\begin{aligned}\Box_2 u_1|_{t=0} &= -u_2|_{t=0} + (d_1 - d_2)\Delta_D u_1|_{t=0}\\ &= -u_2^0 + (d_1 - d_2)\Delta_D u_1^0 \in H_\Omega^{\sigma+1}(\Delta_D),\end{aligned}$$

$$\begin{aligned}\partial_t(\Box_2 u_1)|_{t=0} &= -\partial_t u_2|_{t=0} + (d_1 - d_2)\Delta_D \partial_t u_1|_{t=0}\\ &= -u_2^1 + (d_1 - d_2)\Delta_D u_1^1 \in H_\Omega^{\sigma}(\Delta_D).\end{aligned}$$

So, we know that $\Box_2 u_1 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$. In addition, we also know that $-\Box_1 u_1 = u_2 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D))$. Hence, we obtain that

$$\Delta_D u_1 = \frac{1}{d_1 - d_2}(\Box_2 - \Box_1)u_1 \in C^1([0,T], H_\Omega^{\sigma}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+1}(\Delta_D)).$$

We conclude that $u_1 \in C^1([0,T], H_\Omega^{\sigma+2}(\Delta_D)) \cap C^0([0,T], H_\Omega^{\sigma+3}(\Delta_D))$. □

2.3. **Generalized bicharacteristics.** As usual, for a variable y, we denote $D_y = i\partial_y$. Let $B = \{y \in \mathbb{R}^d : |y| < 1\}$ be the unit euclidean ball in $\mathbb{R}^d$. In a tubular neighbourhood of the boundary, we can identify $M = \mathbb{R} \times \Omega$ locally as $X = (0,1) \times B$ and $\partial M = \mathbb{R} \times \partial\Omega$ locally as $\{0\} \times B$. Now, we consider $R = R(x, y, D_y)$ which is a second order scalar, self-adjoint, classical, tangential and smooth pseudo-differential operator, defined in a neighbourhood of $[0,1) \times B$ with a real principal symbol $r(x, y, \eta)$, such that

$$\frac{\partial r}{\partial \eta} \neq 0 \text{ for } (x,y) \in [0,1) \times B \text{ and } \eta \neq 0. \tag{2.10}$$

Let $Q_0(x, y, D_y)$, $Q_1(x, y, D_y)$ be smooth classical tangential pseudo-differential operators defined in a neighbourhood of $[0,1) \times B$, of order 0 and 1, and principal symbols $q_0(x,y,\eta)$, $q_1(x,y,\eta)$, respectively. Denote $P = (\partial_x^2 + R)Id + Q_0\partial_x + Q_1$. The principal symbol of P is

$$p = -\xi^2 + r(x,y,\eta). \tag{2.11}$$

We use the usual notations TM and T^*M to denote the tangent bundle and cotangent bundle corresponding to M, with the canonical projection π

$$\pi : TM(\text{ or } T^*M) \to M.$$

Denote $r_0(y,\eta) = r(0,y,\eta)$. Then, we can decompose $T^*\partial M$ into the disjoint union $\mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$, where

$$\mathcal{E} = \{r_0 < 0\}, \quad \mathcal{G} = \{r_0 = 0\}, \quad \mathcal{H} = \{r_0 > 0\}. \tag{2.12}$$

The sets $\mathcal{E}$, $\mathcal{G}$, $\mathcal{H}$ are called elliptic, glancing, and hyperbolic set, respectively. Define

$$\mathrm{Char}(P) = \{(x,y,\xi,\eta) \in T^*\mathbb{R}^{d+1}|_{\overline{M}} : \xi^2 = r(x,y,\eta)\} \tag{2.13}$$

to be the characteristic manifold of P. For more details, one can refer to [10] and [28]. Notice that in [8], one can see another characterization for these sets $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$.

To describe the different phenomena when a bicharacteristic approaches the boundary, we need a more accurate decomposition of the glancing set $\mathcal{G}$. Let $r_1 = \partial_x r|_{x=0}$. Then, we can define the decomposition $\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j$, with

$$\begin{aligned}
\mathcal{G}^2 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) \neq 0\}, \\
\mathcal{G}^3 &= \{(y,\eta) : r_0(y,\eta) = 0, r_1(y,\eta) = 0, H_{r_0}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{k+3} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j \leq k, H_{r_0}^{k+1}(r_1) \neq 0\}, \\
&\vdots \\
\mathcal{G}^{\infty} &= \{(y,\eta) : r_0(y,\eta) = 0, H_{r_0}^j(r_1) = 0, \forall j\}.
\end{aligned}$$

Here $H_{r_0}^j$ is just the Hamiltonian vector field H_{r_0} associated to r_0 composed j times. Moreover, for $\mathcal{G}^2$, we can define $\mathcal{G}^{2,\pm} = \{(y,\eta) : r_0(y,\eta) = 0, \pm r_1(y,\eta) > 0\}$. Thus $\mathcal{G}^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}$. For $\rho \in \mathcal{G}^{2,+}$, we say that ρ is a gliding point and for $\rho \in \mathcal{G}^{2,-}$, we say that ρ is a diffractive point. For $\rho \in \mathcal{G}^j$, $j \geq 2$, we say that a bicharacteristic of p tangentially contacts the boundary $\{x = 0\} \times B$ with order j at the point ρ.

We have the definition of the generalized bicharacteristics (See [15, Section 24.3] for more details):

Definition 2.3. *A generalized bicharacteristic of p is a map:*

$$s \in I \backslash D \mapsto \gamma(s) \in T^*M \cup \mathcal{G},$$

where I is an interval on $\mathbb{R}$ and D is a discrete subset I, such that $p \circ \gamma = 0$ and the following properties hold:

(1) $\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ or $\gamma(s) \in \mathcal{G}^{2,+}$.

(2) Every $s \in D$ is isolated ,i.e., there exists $\epsilon > 0$ such that $\gamma(s) \in T^*\overline{M} \backslash T^*\partial M$ if $0 < |s - t| < \epsilon$, and the limits $\gamma(s^{\pm})$ are different points in the same fiber of $T^*\partial M$.

(3) $\gamma(s)$ is differentiable and $\frac{\mathrm{d}\gamma}{\mathrm{d}s} = H_{-r_0}(\gamma(s))$ if $\gamma(s) \in \mathcal{G}\backslash\mathcal{G}^{2,+}$.

Remark 2.4. *We denote the Melrose cotangent compressed bundle by ${}^bT^*\overline{M}$ and the associated canonical map by $j : T^*\overline{M} \mapsto {}^bT^*\overline{M}$. j is defined by*

$$j(x, y, \xi, \eta) = (x, y, x\xi, \eta). \tag{2.14}$$

Under this map j, one can see $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\overline{M}$. This is the so-called Melrose-Sjöstrand flow (see [10] *for more details).*

From now on we always assume that there is no infinite tangential contact between the bicharacteristic of p and the boundary. This is in the meaning of the following definition:

Definition 2.5. *We say that there is no infinite contact between the bicharacteristics of p and the boundary if there exists $N \in \mathbb{N}$ such that the gliding set $\mathcal{G}$ satisfies*

$$\mathcal{G} = \bigcup_{j=2}^{N} \mathcal{G}^j.$$

It is well-known that under this hypothesis, there exists a unique generalized bicharacteristic passing through any point. This means that the Melrose-Sjöstrand flow is globally well-defined. One can refer to [26] and [27] for the proof.

2.4. **Microlocal defect measures.** In this section, we will give two approaches to construct the microlocal defect measures. The first one is based on the article by Gérard and Leichtnam [14] for Helmoltz equation and Burq [8] for wave equations. The other one follows the idea in the article [17] by Lebeau and we rely on the article [10] by Burq and Lebeau for the setting of wave systems. In the first approach, we can compare two different measures, especially the supports of two different measures. In the later proof, it is crucial to distinguish the measures with different speeds based on this idea. On the other hand, we use the second approach to describe the way the polarization of one measure is turning.

Let $(u^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\left(L^2_{loc}(\mathbb{R}; L^2(\Omega))\right)^n$, converging weakly to 0 and such that

$$\begin{cases} Pu^k &= o(1)_{H^{-1}}, \\ u^k|_{x=0} &= 0. \end{cases}$$

Let $\underline{u}_k$ be the extension by 0 across $\{x = 0\}$. Then the sequence $\underline{u}_k$ is bounded in $\left(L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^d))\right)^n$. Let $\underline{\mathcal{A}}$ be the space of $n\times n$ matrices of classical polyhomogeneous pseudo-differential operators of order 0 with compact support in $\mathbb{R}\times\mathbb{R}^d$ (i.e, $A = \varphi A\varphi$ for some $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$). Let us denote by $\underline{\mathcal{M}}^+$ the set of nonnegative Radon measures on $T^*(\mathbb{R} \times \mathbb{R}^d)$. Following [8, Section 1], we have the existence of the microlocal defect measure as follows:

Proposition 2.6 (Existence of the microlocal defect measure-1)**.** *There exists a subsequence of $(\underline{u}^k)$ (still denoted by $(\underline{u}^k)$) and $\underline{\mu} \in \underline{\mathcal{M}}^+$ such that*

$$\forall A \in \underline{\mathcal{A}}, \quad \lim_{k\to\infty} (A\underline{u}^k, \underline{u}^k)_{L^2(\mathbb{R}\times\Omega)} = \langle \underline{\mu}, \sigma(A)\rangle, \tag{2.15}$$

where $\sigma(A)$ is the principal symbol of the operator A (which is a matrix of smooth functions, homogeneous of order 0 in the variable ξ).

From [8, Théorème 15], we have the following proposition.

Proposition 2.7. *For the microlocal defect measure $\underline{\mu}$ defined above, we have the following properties.*

- *The measure $\underline{\mu}$ is supported Char$(P)\cap(\mathbb{R}\times\overline{\Omega})$, where Char$(P)$ is defined in* (2.13).
- *The measure $\underline{\mu}$ does not charge the hyperbolic points in ∂M :*

$$\underline{\mu}(\mathcal{H})=0.$$

- *In particular, if $n=1$, the scalar measure $\underline{\mu}$ is invariant along the generalized bicharacteristic flow.*

On the other hand, let $\mathcal{A}$ be the space of $n\times n$ matrices of pseudo-differential operators of order 0, in the form of $A=A_i+A_t$, with A_i a classical pseudo-differential operator with compact support in M(*i.e*, $A_i=\varphi A_i\varphi$ for some $\varphi\in C_0^\infty(M)$) and A_t a classical tangential pseudo-differential operator in $\overline{M}$(*i.e.*, $A_t=\varphi A_t\varphi$ for some $\varphi\in C^\infty(\overline{M})$). Then denote

$$Z=j(\mathrm{Char}(P)),\quad \hat{Z}=Z\cup j(T^*\overline{M}|_{x=0}),$$

where j is defined in (2.14) and

$$S\hat{Z}=(\hat{Z}\backslash\overline{M})/\mathbb{R}_+^*,\quad SZ=(Z\backslash\overline{M})/\mathbb{R}_+^*.$$

$S\hat{Z}$ and SZ are the quotient spherical spaces of $\hat{Z}$ and Z and they are locally compact metric spaces. Here, we identify the zero section $\overline{M}\times\{0\}\subset^b T^*\overline{M}$ with $\overline{M}$ itself.

For $A\in\mathcal{A}$, with principal symbol $a=\sigma(A)$, define

$$\kappa(a)(\rho)=a(j^{-1}(\rho)),\forall\rho\in{}^bT^*\overline{M}.$$

Now, we have that $\mathcal{K}=\{\kappa(a):a=\sigma(A),A\in\mathcal{A}\}\subset C^0(S\hat{Z};End(\mathbb{C}^n))$. Define $\mathcal{M}^+$ to be the space of all positive Borel measures on $S\hat{Z}$. By duality, we know that $\mathcal{M}^+$ is the dual space of $C_0^0(S\hat{Z};End(\mathbb{C}^n))$, which verifies the property:

$$\langle\mu,a\rangle\geq 0,\forall a\in C^0(S\hat{Z};End^+(\mathbb{C}^n)),\forall\mu\in\mathcal{M}^+,$$

where $End^+(\mathbb{C}^n)$ denotes the space of $n\times n$ positive hermitian matrices. Following the article [10] by Burq and Lebeau, we obtain the existence of the microlocal defect measure and some properties as follows:

Proposition 2.8 (Existence of the microlocal defect measure-2)**.** *There exists a subsequence of (u^k) (still noted by (u^k)) and $\mu\in\mathcal{M}^+$ such that*

$$\forall A\in\mathcal{A},\quad \lim_{k\to\infty}(Au^k,u^k)_{L^2(\mathbb{R}\times\Omega)}=\langle\mu,\kappa(\sigma(A))\rangle. \tag{2.16}$$

Lemma 2.9. *The microlocal defect measure μ defined in Proposition* 2.8 *satisfies that $\mu\mathbb{1}_{\mathcal{H}\cup\mathcal{E}}=0$, where $\mathcal{H}$ is the set of hyperbolic points and $\mathcal{E}$ is the set of elliptic points as defined in Subsection* 2.3.

In the following, suppose that there is no infinite contact between the bicharacteristic of p and the boundary. This hypothesis implies the existence and uniqueness of the generalized bicharacteristic passing through any point, which ensures that the Melrose-Sjöstrand flow is globally well-defined. By a suitable change of parameter along this flow, we obtain a flow on SZ. Consider S a hypersurface tranverse to the flow. Then locally, $SZ=\mathbb{R}_s\times S$, where s is the well-chosen parameter along the flow. We have the following propagation lemma for the microlocal defect measure.

Lemma 2.10. *Assume that the microlocal defect measure* μ *is defined in Proposition* 2.8*. Then* μ *is supported in* SZ *and there exists a function*

$$(s,z) \in \mathbb{R}_s \times S \mapsto M(s,z) \in \mathbb{C}^n$$

$\mu-$*almost everywhere continuous such that the pullback of the measure* μ *by* M*(i.e., the measure* $\mathcal{P}^*\mu = M^*\mu M$ *defined for* $a \in C^0(SZ)$*) by*

$$\langle M^*\mu M, a\rangle = \langle \mu, MaM^*\rangle$$

satisfies

$$\frac{d}{ds}\mathcal{P}^*\mu = 0.$$

We say that the measure μ *is invariant along the flow associated with* M*. Furthermore, the function* M *is continuous, and along any generalized bicharacteristic, the matrix* M *is solution to a differential equation whose coefficients can be explicitly computed in terms of the geometry and the different terms in the operator* P*.*

For the differential equation that M satisfies, one can refer to [10, Section 3.2] for more details.

Remark 2.11. *Roughly speaking, in the result above, the Frobenius norm of* M *describes the damping of the measure* μ*, whereas the rotation component of* M *(i.e. the orthogonal part of the polar decomposition) describes the way the polarization of the measure (asymptotic polarization of the sequence* (u^k)*) is turning.*

Remark 2.12. *Notice that in* [8, Section 3]*, the author considered the case of solutions to the wave equation at the energy level (bounded in* H^1_{loc}*), and hence was considering second order operators. However, it is easy to change the energy level into* L^2*, one can see* [28, Remark 4.4] *for more details.*

Remark 2.13. *From Proposition* 2.7*, we know that supp*$(\underline{\mu}) \subset$ *Char*(P)*. Notice that in the interior of* M*, the two definitions coincide, i.e., for any pseudo-differential operator* A *of order* 0 *with principal symbokl* $\sigma(A)$ *satisfying supp*$(\sigma(A)) \subset$ *Char*$(P)|_M$*, we have* $\langle \underline{\mu}, \sigma(A)\rangle = \langle \mu, \kappa(\sigma(A))$*, simply by their definitions. At the boundary, since both measures* $\underline{\mu}$ *and* μ *do not not charge the hyperbolic points in* ∂M*, we know that* $\underline{\mu}|_{S\hat{Z}} = \mu$ *holds* μ *almost surely and* $\underline{\mu}$ *almost surely. Under this sense, we can identify the two measures.*

3. Proof of the sufficient part of Theorem 1.16 in the case $n_2 = 2$

In this section, we shall present the sufficient part of the proof of Theorem 1.16 in the case $n_2 = 2$ (and of course $n_1 = 1$). We divide the proof into three steps. Firstly, we give a reformulation of System (3.1). Then we study a simpler problem and obtain a compatibility condition for it. At last, we present the proof for the general case.

3.1. **Reformulation of the system in symmetric spaces.** In the case $n_2 = 2$, we write System (1.1) as follows:

$$(3.1)\qquad . \begin{cases} \partial_t^2 u_1^1 - d_1\Delta u_1^1 + \alpha_1 u_1^2 + \alpha_2 u_2^2 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_1^2 - d_2\Delta u_1^2 + a_{11} u_1^2 + a_{12} u_2^2 & = b_1 f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega, \\ \partial_t^2 u_2^2 - d_2\Delta u_2^2 + a_{21} u_1^2 + a_{22} u_2^2 & = b_2 f \mathbb{1}_\omega \quad \text{in } (0,T)\times\Omega, \\ u_1^1 = 0, u_j^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega,\ j=1,2, \end{cases}$$

with initial conditions

$$(u_1^1(0,x), u_1^2(0,x), u_2^2(0,x), \partial_t u_1^1(0,x), \partial_t u_1^2(0,x), \partial_t u_2^2(0,x))$$

belonging to a space that will be detailed later on.

Before we reformulate the system, we introduce the Brunovský normal form.

Theorem 3.1 (Brunovský Normal Form)**.** *Assume that A is a square matrix of size $n \times n$, B is a matrix of size $n \times 1$ and (A, B) satisfies the Kalman rank condition. Then, there exists an invertible matrix P such that $A = P^{-1}JP$ and $B = P^{-1}e_n$, where*

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{pmatrix}, \text{ and } e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{3.2}$$

and the coefficients $(a_j)_{1 \le j \le n}$ are defined by the characteristic polynomial of A, i.e. $\chi_A(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X + a_n$.

One can find for instance the proof in [30, Théorème 2.2.7] for this theorem. Now, we set $\tilde{A}$, $\tilde{B}$, and α by

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tilde{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ and } \alpha = (\alpha_1, \alpha_2).$$

Then, we obtain $A = \begin{pmatrix} 0 & \alpha \\ 0 & \tilde{A} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix}$. As a consequence of (1.6), we know that $(\tilde{A}, \tilde{B})$ satisfies the Kalman rank condition. Hence, by the Brunovský normal form, there exists an invertible matrix $\tilde{P}$ such that

$$\tilde{A} = \tilde{P} \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix} \tilde{P}^{-1}, \tilde{B} = \tilde{P} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and } \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) = \alpha \tilde{P}^{-1}.$$

Furthermore, according to the third statement of Proposition 1.6, we know that

$$\tilde{\alpha}_2(d_1 - d_2)\lambda + \tilde{\alpha}_1 \neq 0, \forall \lambda \in \sigma(-\Delta_D). \tag{3.3}$$

Using the change of unknowns

$$\begin{pmatrix} \tilde{u}_1^1 \\ \tilde{u}_1^2 \\ \tilde{u}_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{P} \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix}, \tag{3.4}$$

we obtain a simplified system

$$\begin{cases} \Box_1 \tilde{u}_1^1 + \tilde{\alpha}_1 \tilde{u}_1^2 + \tilde{\alpha}_2 \tilde{u}_2^2 & = 0 & \text{in } (0,T) \times \Omega, \\ \Box_2 \tilde{u}_1^2 + \tilde{u}_2^2 & = 0 & \text{in } (0,T) \times \Omega, \\ \Box_2 \tilde{u}_2^2 - a_1 \tilde{u}_1^2 - a_2 \tilde{u}_2^2 & = f \mathbb{1}_\omega & \text{in } (0,T) \times \Omega, \\ \tilde{u}_1^1 = 0, \tilde{u}_1^2 = 0, \tilde{u}_2^2 & = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\tilde{u}_1^1(0,x), \tilde{u}_1^2(0,x), \tilde{u}_2^2(0,x))|_{t=0} & = (\tilde{u}_1^{1,0}, \tilde{u}_1^{2,0}, \tilde{u}_2^{2,0}) & \text{in } \Omega, \\ (\partial_t \tilde{u}_1^1(0,x), \partial_t \tilde{u}_1^2(0,x), \partial_t \tilde{u}_2^2(0,x))|_{t=0} & = (\tilde{u}_1^{1,1}, \tilde{u}_1^{2,1}, \tilde{u}_2^{2,1}) & \text{in } \Omega. \end{cases} \tag{3.5}$$

Therefore, the exact controllability of System (3.1) is equivalent to the exact controllability of System (3.5). Classically, given the initial conditions

$$(\tilde{u}_1^{2,0}, \tilde{u}_2^{2,0}, \tilde{u}_1^{2,1}, \tilde{u}_2^{2,1}) \in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),$$

the solutions $\tilde{u}_1^2$ and $\tilde{u}_2^2$ satisfy

$$\tilde{u}_1^2 \in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)),$$
$$\tilde{u}_2^2 \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)),$$

As for the regularity of the solution $\tilde{u}_1^1$, it depends on the coupling term $\tilde{\alpha}_1\tilde{u}_1^2 + \tilde{\alpha}_2\tilde{u}_2^2$. Thus, it is natural to discuss in two different cases, *i.e.* $\tilde{\alpha}_2 \neq 0$ and $\tilde{\alpha}_2 = 0$.

3.2. **The case $\tilde{\alpha}_2 = 0$.** In what follows, we will present into details the proof of Theorem 1.16 firstly in the case $n_2 = 2$ (and $n_1 = 1$ by Proposition 1.6), and $A_1 = (\alpha_1, 0)$. Here, for the sake of simplicity we remove the ˜ in our notations and we investigate the system

$$\begin{cases} \Box_1 u_1^1 + \alpha_1 u_1^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 u_2^2 - a_1 u_1^2 - a_2 u_2^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ u_1^1 = 0, u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2, \\ (u_1^1, u_1^2, u_2^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, u_2^{2,0}) & \text{in } \Omega, \\ (\partial_t u_1^1, \partial_t u_1^2, \partial_t u_2^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, u_2^{2,1}) & \text{in } \Omega. \end{cases} \tag{3.6}$$

For this system, we have the following well-posedness property.

Proposition 3.2. *Assume that the initial conditions satisfy*

$$(u_1^{2,0}, u_2^{2,0}, u_1^{2,1}, u_2^{2,1}) \in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),$$
$$(u_1^{1,0}, u_1^{1,1}) \in H^4_\Omega(\Delta_D) \times H^3_\Omega(\Delta_D).$$

Additionally, assume that

$$(-\Delta_D)^2 u_1^{1,0} - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^{2,0} \in H^1_\Omega(\Delta_D),\ \ (-\Delta_D)^2 u_1^{1,1} - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^{2,1} \in H^0_\Omega(\Delta_D). \tag{3.7}$$

Then, the solutions u_1^1, u_1^2 and u_2^2 satisfy

$$u_1^1 \in C^0([0,T], H^4_\Omega(\Delta_D)) \cap C^1([0,T], H^3_\Omega(\Delta_D)),$$
$$u_1^2 \in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)),$$
$$u_2^2 \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)),$$
$$(-\Delta_D)^2 u_1^1 - \frac{\alpha_1}{d_1 - d_2}\Delta_D u_1^2 \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

Proof of Proposition 3.2. Classically, given the initial conditions

$$(u_1^{2,0}, u_2^{2,0}, u_1^{2,1}, u_2^{2,1}) \in H^2_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D) \times H^0_\Omega(\Delta_D),$$

the solutions u_1^2 and u_2^2 satisfy

$$\begin{aligned} u_1^2 &\in C^0([0,T], H^2_\Omega(\Delta_D)) \cap C^1([0,T], H^1_\Omega(\Delta_D)), \\ u_2^2 &\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)). \end{aligned} \tag{3.8}$$

According to Lemma 2.2, given the initial condition

$$u_1^{1,0}, u_1^{1,1} \in H^4_\Omega(\Delta_D) \times H^3_\Omega(\Delta_D),$$

the solution u_1^1 satisfies

$$u_1^1 \in C^0([0,T], H^4_\Omega(\Delta_D)) \cap C^1([0,T], H^3_\Omega(\Delta_D)). \tag{3.9}$$

Let us first do some reformulation for the system. Define the transform $\mathcal{S}_0$ by

$$\mathcal{S}_0\begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix}, \tag{3.10}$$

where

$$\begin{cases} v_1^1 = D_t^3 u_1^1, \\ v_1^2 = D_t u_1^2, \\ v_2^2 = u_2^2. \end{cases} \tag{3.11}$$

We need to invert the previous relations by expressing u_1^1, u_1^2, u_2^2 in terms of v_1^1, v_1^2, v_2^2. Firstly, for the term $u_2^2 = v_2^2$, there is nothing to do. Then, we look at the term u_1^2. We need to "invert" in some sense the operator D_t. We use the second equation of System (3.6). We apply D_t on the second equation of System (3.11), and we obtain

$$\begin{aligned} D_t v_1^2 &= D_t^2 u_1^2 \\ &= u_2^2 - d_2\Delta u_1^2 \\ &= v_2^2 - d_2\Delta u_1^2. \end{aligned}$$

Hence, we obtain that

$$u_1^2 = \frac{(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2). \tag{3.12}$$

For the last term u_1^1, we apply D_t on the first equation of System (3.11), then we use the first equation of System (3.6), the second equation of System (3.6) and the last equation of System (3.11) to obtain

$$\begin{aligned} D_t v_1^1 &= D_t^2(D_t^2 u_1^1) \\ &= \alpha_1 D_t^2 u_1^2 - d_1\Delta D_t^2 u_1^1 \\ &= \alpha_1(u_2^2 - d_2\Delta u_1^2) - d_1\Delta_D(\alpha_1 u_1^2 - d_1\Delta u_1^1) \\ &= (-d_1\Delta)^2 u_1^1 - \alpha_1(d_1+d_2)\Delta u_1^2 + \alpha_1 v_2^2. \end{aligned}$$

Therefore, from the above computations, (3.11), and (3.12), an inverse transform is the following:

$$\begin{cases} u_1^1 = \frac{(-\Delta_D)^{-2}}{d_1^2}(D_t v_1^1 + \alpha_1\frac{d_1+d_2}{d_2}D_t v_1^2 + \alpha_1\frac{d_1}{d_2}v_2^2), \\ u_1^2 = \frac{(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2), \\ u_2^2 = v_2^2. \end{cases} \tag{3.13}$$

From the regularity results given in (3.8), (3.9) and the relations (3.13), we obtain that

$$\begin{aligned} v_1^1 &\in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)), \\ v_j^2 &\in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)),\ j=1,2. \end{aligned} \tag{3.14}$$

Moreover, from (3.6) and (3.13), (v_1^1, v_1^2, v_2^2) satisfies the following system:

$$\begin{cases} \Box_1 v_1^1 + \alpha_1 D_t^2 v_1^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_2^2 - \frac{a_1(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2) - a_2 v_2^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ v_1^1 = 0, v_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega,\ j=1,2, \end{cases} \tag{3.15}$$

with appropriate initial conditions. Using the identity

$$-D_t^2 = \frac{1}{d_2-d_1}\left(d_2\Box_1 - d_1\Box_2\right), \tag{3.16}$$

we obtain that

$$D_t^2 v_1^2 = -\frac{1}{d_2-d_1}(d_2\Box_1 - d_1\Box_2)v_1^2. \tag{3.17}$$

Using (3.17) in the first equation of (3.15), we also deduce that

$$\Box_1\left(v_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}v_1^2\right) - \frac{\alpha_1 d_1}{d_2-d_1}D_t v_2^2 = 0. \tag{3.18}$$

Now, let us define

$$y = D_t v_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}D_t v_1^2. \tag{3.19}$$

Then, by (3.19) and (3.18), we obtain that

$$\Box_1 y - \frac{\alpha_1 d_1}{d_2-d_1}D_t^2 v_2^2 = 0. \tag{3.20}$$

We also remark that by using (3.16),

$$-D_t^2 v_2^2 = \frac{1}{d_2-d_1}(d_2\Box_1 - d_1\Box_2)v_2^2. \tag{3.21}$$

Using the last equation of (3.15) together with (3.20) and (3.21), we deduce that

$$\Box_1\left(y + \frac{\alpha_1 d_1 d_2}{(d_2-d_1)^2}v_2^2\right) = \frac{\alpha_1 d_1^2}{(d_2-d_1)^2}f + \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}(D_t v_1^2 - v_2^2) + \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2}v_2^2. \tag{3.22}$$

Let us now express y with respect to the original variables u_1^1, u_1^2, u_2^2. From (3.19), (3.11) and the first equation of (3.6), we obtain that

$$\begin{aligned} y &= D_t v_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}D_t v_1^2 \\ &= D_t^4 u_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}D_t^2 u_1^2 \\ &= D_t^2\left(D_t^2 u_1^1 - \frac{\alpha_1 d_2}{d_2-d_1}u_1^2\right) \\ &= D_t^2\left(-d_1\Delta u_1^1 + \alpha_1 u_1^2 - \frac{\alpha_1 d_2}{d_2-d_1}u_1^2\right) \\ &= D_t^2\left(-d_1\Delta u_1^1 - \frac{\alpha_1 d_1}{d_2-d_1}u_1^2\right). \end{aligned} \tag{3.23}$$

Combining with the second equation of (3.6), we obtain

$$y = (-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta u_1^2 + \frac{\alpha_1 d_1}{d_1-d_2}u_2^2.$$

Hence, we obtain

$$y + \frac{\alpha_1 d_2 d_1}{(d_1-d_2)^2}u_2^2 = (-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1-d_2}\Delta u_1^2 + \frac{\alpha_1 d_1^2}{(d_2-d_1)^2}u_2^2.$$

Now, we define

$$\tilde{y} = y + \frac{\alpha_1 d_2 d_1}{(d_1 - d_2)^2} u_2^2.$$

Then, $\tilde{y}$ satisfies

$$\Box_1 \tilde{y} = \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} f + \frac{\alpha_1 a_1 d_1^2 (-\Delta_D)^{-1}}{d_2 (d_2 - d_1)^2} (D_t v_1^2 - v_2^2) + \frac{\alpha_1 a_2 d_1^2}{(d_2 - d_1)^2} v_2^2. \tag{3.24}$$

The initial condition associated with $\tilde{y}$ is given by

$$\begin{aligned}
\tilde{y}|_{t=0} &= \left((-d_1\Delta)^2 u_1^1 - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta u_1^2 + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^2 \right) |_{t=0} \\
&= (-d_1\Delta)^2 u_1^{1,0} - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta u_1^{2,0} + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^{2,0} \\
&= d_1^2 \left((-\Delta)^2 u_1^{1,0} - \frac{\alpha_1}{d_1 - d_2} \Delta u_1^{2,0} \right) + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^{2,0} \\
\partial_t \tilde{y}|_{t=0} &= \left((-d_1\Delta)^2 \partial_t u_1^1 - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta \partial_t u_1^2 + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} \partial_t u_2^2 \right) |_{t=0} \\
&= (-d_1\Delta)^2 u_1^{1,1} - \frac{\alpha_1 d_1^2}{d_1 - d_2} \Delta u_1^{2,1} + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^{2,1} \\
&= d_1^2 \left((-\Delta)^2 u_1^{1,1} - \frac{\alpha_1}{d_1 - d_2} \Delta u_1^{2,1} \right) + \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} u_2^{2,1}.
\end{aligned}$$

Hence, from our Hypothesis (3.7) together with (3.8) and (3.9), we deduce that

$$\tilde{y}|_{t=0} \in H^1_\Omega(\Delta_D), \quad \partial_t \tilde{y}|_{t=0} \in H^0_\Omega(\Delta_D). \tag{3.25}$$

By (3.24) and (3.14), $\tilde{y}$ satisfies a wave equation with a source term in the space $L^1((0,T), H^0_\Omega(\Delta_D))$ and initial condition in $H^1_\Omega(\Delta_D) \times H^1_\Omega(\Delta_D)$ by (3.25). We deduce that

$$\tilde{y} \in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)).$$

Hence, from (3.24) and (3.23), we deduce that

$$(-\Delta)^2 u_1^1 - \frac{\alpha_1}{d_1 - d_2} \Delta u_1^2 + \frac{\alpha_1}{(d_2 - d_1)^2} u_2^2 \in C^0([0,T]; H^1_\Omega(\Delta_D)) \cap C^1([0,T]; H^0_\Omega(\Delta_D)).$$

Taking into account the last line of (3.8), this implies that

$$(-\Delta)^2 u_1^1 - \frac{\alpha_1}{d_1 - d_2} \Delta u_1^2 \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

■

Remark 3.3. *Let us define the transform $\mathcal{S}$ associated with the system* (3.6) *and* (3.26) *by*

$$\mathcal{S} \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix},$$

where

$$\mathcal{S} = \begin{pmatrix} (-d_1\Delta_D)^2 & -\frac{\alpha_1 d_1^2}{d_1 - d_2}\Delta_D & \frac{\alpha_1 d_1^2}{(d_2 - d_1)^2} \\ 0 & D_t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and its "inverse"

$$\mathcal{S}^{-1} = \begin{pmatrix} \frac{(-\Delta_D)^{-2}}{d_1^2} & -\frac{\alpha_1(-\Delta_D)^{-2}}{d_2(d_1-d_2)}D_t & \frac{\alpha_1(d_1-2d_2)(-\Delta_D)^{-2}}{d_2(d_1-d_2)^2}) \\ 0 & \frac{(-\Delta_D)^{-1}}{d_2}D_t & -\frac{(-\Delta_D)^{-1}}{d_2} \\ 0 & 0 & 1 \end{pmatrix}.$$

The previous computations show that we have a bijection between the solutions of (3.6) *and* (3.26). *Notably, if* $U = \begin{pmatrix} u_1^1 \\ u_1^2 \\ u_2^2 \end{pmatrix}$ *and* $V = \begin{pmatrix} v_1^1 \\ v_1^2 \\ v_2^2 \end{pmatrix}$, *then* $(\mathcal{S} \circ \mathcal{S}^{-1})\, V = V$ *and* $(\mathcal{S}^{-1} \circ \mathcal{S})\, U = U$.

Notably, (3.6) *can be rewritten as*

$$(\partial_t^2 - D\Delta + A)(\mathcal{S}^{-1} \circ \mathcal{S}U) = \hat{b}f.$$

Therefore, since $\mathcal{S}(U) = V$ *we are able to rewrite the system* (3.26) *as follows:*

$$(\partial_t^2 - \mathcal{S}D\mathcal{S}^{-1}\Delta + \mathcal{S}A\mathcal{S}^{-1})V = \mathcal{S}\hat{b}f,$$

where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_2 \end{pmatrix},\ A = \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_1 & -a_2 \end{pmatrix},\ \mathcal{S}\hat{b}f = \begin{pmatrix} \frac{\alpha_s d_1^2}{(d_1-d_2)^2} f \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}$ *and* $\mathcal{S}^{-1}$ *only involve* D_t *and* $(-\Delta_D)^k, k \in \mathbb{Z}$. *This abstract point of view will be useful in the proof of the general case given in Section* 4.

Now, we consider the exact controllability of System (3.6) in the space $\mathcal{H}_1^1 \times \mathcal{H}_0^0$, according to Proposition 3.2.

We have the following result:

Theorem 3.4. *Given* $T > 0$, *suppose that:*

(1) (ω, T, p_{d_i}) *satisfies GCC,* $i = 1, 2$.

(2) Ω *has no infinite order of tangential contact with the boundary.*

Then System (3.6) *is exactly controllable in* $\mathcal{H}_1^1 \times \mathcal{H}_0^0$.

Recall that here the state space $\mathcal{H}_1^1 \times \mathcal{H}_0^0$ is given by

$$\mathcal{H}_1^1 = \{(u, v_1, v_2) \in H_\Omega^4(\Delta_D) \times H_\Omega^2(\Delta_D) \times H_\Omega^1(\Delta_D),$$
$$(-d_1\Delta)^2 u - \frac{\alpha_1 d_1^2}{d_1 - d_2}\Delta v_1 \in H_\Omega^1(\Delta_D)\},$$
$$\mathcal{H}_0^0 = \{(u, v_1, v_2) \in H_\Omega^3(\Delta_D) \times H_\Omega^1(\Delta_D) \times H_\Omega^0(\Delta_D),$$
$$(-d_1\Delta)^2 u - \frac{\alpha_1 d_1^2}{d_1 - d_2}\Delta v_1 \in H_\Omega^0(\Delta_D)\}.$$

Proof of Theorem 3.4.

By the computations of Proposition 3.2, proving Theorem 3.4 is equivalent to proving the exact controllability of the following system:
(3.26)

$$\begin{cases} \Box_1 v_1^1 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}(D_t v_1^2 - v_2^2) - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} v_2^2 & = \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} f \mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_2^2 - \frac{a_1(-\Delta_D)^{-1}}{d_2}(D_t v_1^2 - v_2^2) - a_2 v_2^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ v_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ v_1^2 = v_2^2 & = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

with initial conditions

$$(v_1^1, v_1^2, v_2^2)|_{t=0} \in (H_0^1(\Omega))^3 = \mathscr{L}_1^3,$$
$$(\partial_t v_1^1, \partial_t v_1^2, \partial_t v_2^2)|_{t=0} \in (L^2(\Omega))^3 = \mathscr{L}_0^3,$$

in the state space $\mathscr{L}_1^3 \times \mathscr{L}_0^3$. Recall that we defined $\mathscr{L}_s^k = (H_\Omega^s(\Delta_D))^k$ in (1.5). According to the Hilbert Uniqueness Method of J.-L. Lions [24], the exact controllability of System (3.26) is equivalent to proving the following observability inequality: there exists $C > 0$ such that for any solution of the adjoint system:

$$\begin{cases} \Box_1 w_1^1 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 w_1^2 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} D_t w_1^1 - \frac{a_1(-\Delta_D)^{-1}}{d_2} D_t w_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 w_2^2 + D_t w_1^2 - a_2 w_2^2 + \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} w_1^1 & & \\ -\frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} w_1^1 + \frac{a_1(-\Delta_D)^{-1}}{d_2} w_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ w_1^1 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ w_1^2 = w_2^2 & = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{3.27}$$

with initial conditions

$$(w_1^1, w_1^2, w_2^2)|_{t=0} \in \mathscr{L}_0^3, \tag{3.28}$$
$$(\partial_t w_1^1, \partial_t w_1^2, \partial_t w_2^2)|_{t=0} \in \mathscr{L}_{-1}^3, \tag{3.29}$$

we have the following observability inequality:

$$C\int_0^T \int_\omega \left| \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^1 + w_2^2 \right|^2 dxdt \geq ||W(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3}, \tag{3.30}$$

where $W = (w_1^1, w_1^2, w_2^2)$.

Remark 3.5. *As we showed in Remark* 3.3*, we are able to rewrite the system* (3.27) *as follows:*

$$(\partial_t^2 - (\mathcal{S}')^{-1} D\mathcal{S}'\Delta + (\mathcal{S}')^{-1} A^* \mathcal{S}')W = 0.$$

However, we should pay attention to this $\mathcal{S}'$, which is defined as the invertible transform between two adjoint systems. $\mathcal{S}'$ could be seen as the "adjoint" operator of $\mathcal{S}$. To be more specific, we write the original adjoint system as follows:

$$\begin{cases} \Box_1 z_1^1 & = 0 & \textit{in } (0,T)\times\Omega, \\ \Box_2 z_1^2 + \alpha_1 z_1^1 - a_1 z_2^2 & = 0 & \textit{in } (0,T)\times\Omega, \\ \Box_2 z_2^2 + z_1^2 - a_2 z_2^2 & = 0 & \textit{in } (0,T)\times\Omega, \\ z_1^1 = 0, z_j^2 & = 0 & \textit{on } (0,T)\times\partial\Omega,\ j=1,2. \end{cases} \tag{3.31}$$

The transform $\mathcal{S}'$ associated with the system (3.27) *and* (3.31) *is defined by*

$$\mathcal{S}'\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2^2 \end{pmatrix} = \begin{pmatrix} z_1^1 \\ z_1^2 \\ z_2^2 \end{pmatrix},$$

where

$$\mathcal{S}' = \begin{pmatrix} (-d_1\Delta_D)^2 & 0 & 0 \\ -\frac{\alpha_1 d_1^2}{d_1-d_2}\Delta_D + \frac{a_1\alpha_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_1-d_2)^2} & D_t & \frac{a_1(-\Delta_D)^{-1}}{d_2} \\ \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} & 0 & 1 \end{pmatrix}, \tag{3.32}$$

and its "inverse" by

$$(\mathcal{S}')^{-1} = \begin{pmatrix} (-d_1\Delta_D)^{-2} & 0 & 0 \\ -\frac{\alpha_1(-\Delta_D)^{-2}}{d_2(d_1-d_2)}D_t & (-d_2\Delta_D)^{-1}D_t & 0 \\ -\frac{\alpha_1(-\Delta_D)^{-2}}{(d_2-d_1)^2} & 0 & 1 \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}'$ and $(\mathcal{S}')^{-1}$ only involve D_t and $(-\Delta_D)^k, k \in \mathbb{Z}$. As already written, this point of view will be useful in the proof of the general case given in Section 4*.*

We divide the proof of the observability inequality (3.30) into two steps.

3.2.1. *Step 1: establish a relaxed observability inequality.* Firstly, we establish the following relaxed observability inequality for the adjoint System (3.27).

Proposition 3.6. *For solutions of System* (3.27)*, there exists a constant $C > 0$ such that for any solution of* (3.27) *with initial conditions verifying* (3.28)*, we have*

$$||W(0)||^2_{\mathscr{L}_0^3\times\mathscr{L}_{-1}^3} \leq C\left(\int_0^T\int_\omega\left|\frac{\alpha_1 d_1^2}{(d_2-d_1)^2}w_1^1 + w_2^2\right|^2 dxdt + ||W(0)||^2_{\mathscr{L}_{-1}^3\times\mathscr{L}_{-2}^3}\right). \tag{3.33}$$

Proof of Proposition 3.6. We argue by contradiction. Suppose that the observability inequality (3.33) is not satisfied. Thus, there exists a sequence $(W^k)_{k\in\mathbb{N}}$ of solutions of System (3.27) such that

$$||W^k(0)||^2_{\mathscr{L}_0^3\times\mathscr{L}_{-1}^3} = 1, \tag{3.34}$$

$$\int_0^T\int_\omega\left|\frac{\alpha_1 d_1^2}{(d_2-d_1)^2}w_1^{1,k} + w_2^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty, \tag{3.35}$$

$$||W^k(0)||^2_{\mathscr{L}_{-1}^3\times\mathscr{L}_{-2}^3} \to 0 \text{ as } k\to\infty. \tag{3.36}$$

By the continuity of the solution with respect to the initial data of System (3.27), we know that the sequence $(W^k)_{k\in\mathbb{N}}$ is bounded in $(L^2((0,T)\times\Omega))^3$ and moreover, $W^k \rightharpoonup 0$ in $(L^2((0,T)\times\Omega))^3$. W^k satisfies the following system:

$$\begin{cases} \Box_1 w_1^{1,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_1^{2,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H^{-1}} \quad \text{in } (0,T)\times\Omega,\ k\to\infty, \end{cases} \tag{3.37}$$

where the first equation is decoupled from the two last equations.

Remark 3.7. *We say* $f^k = o(1)_{H^{-1}}$ *if* $\lim_{k\to\infty} ||f^k||_{H^{-1}((0,T)\times\Omega)} = 0$. *Let us explain briefly how to obtain* (3.37). *We take the term* $\frac{(-\Delta_D)^{-1}}{d_2} D_t w_2^{2,k}$ *for instance. Other terms can be treated similarly. For* $\frac{(-\Delta_D)^{-1}}{d_2} D_t w_2^{2,k}$, *we know that* $\frac{(-\Delta_D)^{-1}}{d_2} D_t w_2^{2,k} \in L^2((0,T); H^2_\Omega) \cap H^{-1}((0,T); H^1_\Omega)$ *is a bounded sequence and converges weakly to* 0. *Since the injection from* $L^2((0,T); H^2_\Omega) \cap H^{-1}((0,T); H^1_\Omega)$ *to* $H^{-1}((0,T)\times\Omega)$ *is compact, we obtain that* $\frac{(-\Delta_D)^{-1}}{d_2} D_t w_2^{2,k} = o(1)_{H^{-1}}$.

Hence, we obtain two microlocal defect measures $\underline{\mu}_1$ and $\underline{\mu}_2$ associated with $(w_1^{1,k})_{k\in\mathbb{N}}$ and $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ respectively. From the definition in Proposition 2.6, we know that

$$\begin{aligned} \forall A \in \underline{\mathcal{A}}, \quad \langle \underline{\mu}_1, \sigma(A)\rangle &= \lim_{k\to\infty} (A\underline{w}_1^{1,k}, \underline{w}_1^{1,k})_{L^2}, \\ \langle \underline{\mu}_2(i,j), \sigma(A)\rangle &= \lim_{k\to\infty} (A\underline{w}_i^{2,k}, \underline{w}_j^{2,k})_{L^2}, 1 \le i,j \le 2. \end{aligned} \tag{3.38}$$

Here $\underline{\mu}_2 = (\underline{\mu}_2(i,j))_{1\le i,j\le 2}$ is the matrix measure associated with the sequence $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ and $\underline{w}_i^{j,k}$ is the extension by 0 across the boundary of Ω $(1 \le i,j \le 2)$. Moreover, since the two characteristic manifolds $\mathrm{Char}(p_{d_1})$ and $\mathrm{Char}(p_{d_2})$ are compact and disjoint, $\underline{\mu}_1$ and $\underline{\mu}_2$ are mutually singular in $(0,T)\times\Omega$, from the first point of Proposition 2.7. Therefore, we obtain the following property:

Lemma 3.8. *For* $A \in \underline{\mathcal{A}}$ *with compact support in* $(0,T)\times\Omega$ *and for* $1 \le i \le 2$, *we have*

$$\limsup_{k\to\infty} |(A\underline{w}_1^{1,k}, \underline{w}_i^{2,k})_{L^2(\mathbb{R}\times\Omega)}| = 0. \tag{3.39}$$

Proof. We follow the same strategy as for the proof of [28, Lemma 4.10]. Since $\mathrm{Char}(p_{d_1})$ and $\mathrm{Char}(p_{d_2})$ are disjoint, we choose a cut-off function $\beta \in C^\infty(T^*\mathbb{R}\times\mathbb{R}^d)$ homogeneous of degree 0 for $|(\tau,\xi)| \ge 1$, with compact support in $(0,T)\times\Omega$ such that

$$\beta|_{\mathrm{Char}(p_{d_1})} = 1, \beta|_{\mathrm{Char}(p_{d_2})} = 0, \text{ and } 0 \le \beta \le 1.$$

Since $A \in \underline{\mathcal{A}}$ with compact support in $(0,T)\times\Omega$, for some $\varphi \in C_0^\infty((0,T)\times\omega)$, we have that $A = \varphi A \varphi$. We introduce $\tilde{\varphi} \in C_0^\infty((0,T)\times\omega)$ such that $\tilde{\varphi}|_{\mathrm{supp}(\varphi)} = 1$ i.e, $\tilde{\varphi}\varphi = \varphi$. Now, let us consider $(A\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2}$. First, we have that

$$\begin{aligned} (A\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2} &= (\varphi A\varphi\underline{w}_1^{1,k}, \underline{w}_2^{2,k})_{L^2} \\ &= (\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} \\ &= ((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} + (\mathrm{Op}(\beta)\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}. \end{aligned}$$

For the first term $((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}$, by the Cauchy-Schwarz inequality, we obtain that

$$|((1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}| \le ||(1-\mathrm{Op}(\beta))\varphi A\varphi\underline{w}_1^{1,k}||_{L^2} ||\tilde{\varphi}\underline{w}_2^{2,k}||_{L^2}. \tag{3.40}$$

As we know that $\{\underline{w}_2^{2,k}\}$ is bounded in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^d)$, there exists a constant C such that

$$||\tilde{\varphi}\underline{w}_2^{2,k}||^2_{L^2} = (\tilde{\varphi}\underline{w}_2^{2,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2} \le C. \tag{3.41}$$

From the definition of the measure $\underline{\mu}_1$, we obtain

(3.42)
$$\begin{aligned}\lim_{k\to\infty} \|(1-\mathrm{Op}(\beta))\varphi A\varphi \underline{w}_1^{1,k}\|_{L^2}^2 &= \lim_{k\to\infty} ((1-\mathrm{Op}(\beta))\varphi A\varphi \underline{w}_1^{1,k}, (1-\mathrm{Op}(\beta))\varphi A\varphi \underline{w}_1^{1,k})_{L^2} \\ &= \langle \underline{\mu}_1, (1-\beta)^2\varphi^4|\sigma(A)|^2\rangle.\end{aligned}$$

From Proposition 2.7, we have that supp $(\underline{\mu}_1) \subset \mathrm{Char}(p_{d_1})$. In addition, by the choice of β, we know that $1-\beta \equiv 0$ on supp $(\underline{\mu}_1)$, which implies that $\langle \underline{\mu}_1, (1-\beta)^2\varphi^4|\sigma(A)|^2\rangle = 0$. Combining (3.40), (3.41) and (3.42), we obtain

$$\limsup_{k\to\infty} |((1-\mathrm{Op}(\beta))\varphi A\varphi \underline{w}_1^{1,k}, \tilde{\varphi}\underline{w}_2^{2,k})_{L^2}| = 0. \tag{3.43}$$

The other term is dealt with similarly. One can refer to [28, Lemma 4.10] for more details. □

Let us go back to the proof of Proposition 3.6. We know that

$$\int_0^T \int_\omega \left| \frac{d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 dxdt \to 0 \text{ as } k\to\infty.$$

For $\chi \in C_0^\infty(\omega\times(0,T))$, by expending the above expression,

$$\begin{aligned}&2\left(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \chi w_2^{2,k}\right)_{L^2(\mathbb{R}\times\Omega)} \\ &+\left(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}\right)_{L^2(\mathbb{R}\times\Omega)} + \left(\chi w_2^{2,k}, \chi w_2^{2,k}\right)_{L^2(\mathbb{R}\times\Omega)} \to 0, \text{ as } k\to\infty.\end{aligned}$$

By Lemma 3.8, we know that

$$\limsup_{k\to\infty} \left| \left(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \chi w_2^{2,k}\right)_{L^2(\mathbb{R}\times\Omega)} \right| = 0.$$

As a consequence, since we know that

$$\left| \frac{d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 \geqslant 0,$$

we deduce that

$$\begin{aligned}\left(\frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}, \frac{d_1^2}{(d_2-d_1)^2}\chi w_1^{1,k}\right)_{L^2(\mathbb{R}\times\Omega)} &\to 0, \\ \left(\chi w_2^{2,k}, \chi w_2^{2,k}\right)_{L^2(\mathbb{R}\times\Omega)} &\to 0, \text{ as } k\to\infty.\end{aligned}$$

Thus, using (3.38), we know that (here $\mu_2 = (\mu_2(i,j))_{1\le i,j\le 2}$ is a matrix measure)

$$\underline{\mu}_1|_{(0,T)\times\omega} = 0, \text{ and } \underline{\mu}_2(2,2)|_{(0,T)\times\omega} = 0.$$

For $\underline{\mu}_1$, since $\underline{\mu}_1$ is invariant along the general bicharacteristics of p_{d_1}, combining with GCC, we obtain as usual that $\underline{\mu}_1 \equiv 0$. For $\underline{\mu}_2$, we consider the other definition of the microlocal defect measure. From the definition in Proposition 2.8, we know that there exists a measure μ_2 such that

$$\forall A\in\mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A))\rangle = \lim_{k\to\infty} (AW^{2,k}, W^{2,k})_{L^2}. \tag{3.44}$$

Since $\underline{\mu}_2|_{Char(p_{d_2})} = \mu_2$ μ_2-almost surely by Remark 2.13, we obtain that $\mu_2(2,2)|_{(0,T)\times\omega}$ $= 0$. In the following part, we aim to prove that $\mu_2 = 0$. The basic idea is to use Lemma 2.10. Here we recall this lemma under our setting of this adjoint system.

Lemma 3.9. *Assume that μ_2 is the corresponding microlocal defect measure defined by (3.44) for the sequence $(w_1^{2,k}, w_2^{2,k})_{k\in\mathbb{N}}$ which satisfies the following system (according to (3.27)):*

$$\begin{cases} \Box_2 w_1^{2,k} & = o(1)_{H^{-1}} \quad in\ (0,T)\times\Omega,\ k\to\infty \\ \Box_2 w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H^{-1}} \quad in\ (0,T)\times\Omega,\ k\to\infty. \end{cases} \tag{3.45}$$

If we denote the general bicharacteristic by $s\mapsto\gamma(s)$, then along $\gamma(s)$ there exists a continuous function $s\mapsto M(s)$ such that M satisfies the differential equation:

$$\frac{d}{ds}(M(s)) = iE(\tau)M(s), M(0) = Id,$$

and μ_2 is invariant along the flow associated with M, which means that

$$\frac{d}{ds}(M^*\mu_2 M) = 0.$$

Here we denote by $E(\tau)$ the matrix $\begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix}$.

Remark 3.10. *For the differential equation which M satisfies and the explicit form of the matrix E which we use here, one can refer to* [10, Section 3.2] *for more details.*

Remark 3.11. *In our setting, we can compute explicitly the form of the matrix*

$$M(s) = \begin{pmatrix} 1 & i\tau s \\ 0 & 1 \end{pmatrix}$$

and τ is a constant with respect to s along the generalized bicharacteristic by the explicit form of $Char(P)$ given in (2.13).

Now we use this Lemma 3.9 to prove that $\mu_2 = 0$. First, we would like to show that $supp(\mu_2)\cap\pi^{-1}((0,T)\times\omega) = \emptyset$. Let us fix some point $\rho_0\in\pi^{-1}((0,T)\times\omega)$. Then, there exists a unique bicharacteristic $s\mapsto\gamma_0(s)$ such that $\gamma_0(0) = \rho_0$. Moreover, there exists $\epsilon > 0$, which is sufficiently small, such that $\gamma_0((-2\epsilon, 2\epsilon))\subset\pi^{-1}((0,T)\times\omega)$. Since μ_2 is invariant along the flow associated with M, we obtain $\mu_2(0) = M(\epsilon)^*\mu_2(\epsilon)M(\epsilon)$. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By a straightforward computation using the special form of M, we have

$$M(\epsilon)e_2 = i\tau\epsilon M(\epsilon)e_1 + e_2.$$

Hence, we obtain

$$\begin{aligned} \mu_2(0)e_2 &= M(\epsilon)^*\mu_2(\epsilon)M(\epsilon)e_2 \\ &= M(\epsilon)^*\mu_2(\epsilon)(i\tau\epsilon M(\epsilon)e_1 + e_2) \\ &= i\tau\epsilon\mu_2(0)e_1 + M(\epsilon)^*\mu_2(\epsilon)e_2. \end{aligned} \tag{3.46}$$

We know that $\mu_2(2,2)\equiv 0$ on $(0,T)\times\omega$, which means that $w_{2,2}^k\to 0$ strongly in $L^2((0,T)\times\omega)$. Hence, by (3.38), we also have that $\mu_2(\epsilon)e_2 = 0$. Hence, we obtain $\mu_2(0)e_2 = -i\tau\epsilon\mu(0)e_1$. But by the choice of ρ_0, we know that $\mu_2(0)e_2$ also vanishes, which gives that $-i\tau\epsilon\mu_2(0)e_1 = 0$, *i.e.* $\mu_2(0)e_1 = 0$. Hence, $\mu_2(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since ρ_0 is arbitrary, we deduce that $\text{supp}(\mu_2) \cap \pi^{-1}((0,T)\times\omega) = \emptyset$.

Now, let us go back to prove that $\mu_2 = 0$. For any point $\rho_1 \in \text{supp}(\mu_2)$, there exists a unique bicharacteristic $s \mapsto \gamma_1(s)$ such that $\gamma_1(0) = \rho_1$. Using the GCC (see Definition 1.1), we know that there exists a time t_0 such that $\gamma_1(t_0) \in \pi^{-1}((0,T)\times\omega)$. Since μ_2 is invariant along the flow associated with M, we obtain

$$\mu_2(0) = M(t_0)^*\mu_2(t_0)M(t_0). \tag{3.47}$$

We already know that $supp(\mu_2)\cap\pi^{-1}((0,T)\times\omega) = \emptyset$, which means that $\mu_2(t_0) = 0$. By (3.47), we deduce that $\mu_2(0) = 0$. Due to the arbitrary choice of ρ_1, we obtain that $supp(\mu_2) = \emptyset$, *i.e.* $\mu_2 \equiv 0$, which leads to a contradiction with (3.34) (See [28, Section 4.2] for more details). We conclude that the relaxed observability inequality (3.33) holds for all the solutions of System (3.27). ∎

3.2.2. *Step 2: analysis of the invisible solutions.* With the relaxed observability inequality (3.33) in Proposition 3.6, we are now able to handle the low-frequencies and conclude the proof of the observability (3.30). The main point here is a unique continuation result for solutions of the elliptic problem associated with System (3.27). The idea of reducing the observability for the low frequencies to an elliptic unique continuation result and associated technology are due to [7]. First, let us write for the sake of simplicity the initial conditions as

$$\mathscr{W} = (w_1^{1,0}, w_1^{2,0}, w_2^{2,0}, w_1^{1,1}, w_1^{2,1}, w_2^{2,1})^t \ (\in \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3), \tag{3.48}$$

and define for any $T > 0$ the set of invisible solutions (see [7]) from $(0,T)\times\omega$

$$\mathscr{N}_3(T) = \{\mathscr{W} \in \mathscr{L}_0^3 \times \mathscr{L}_{-1}^3 \text{ such that the associated solution of System (3.27)}$$
$$\text{satisfies } \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^1(x,t) + w_2^2(x,t) = 0, \forall (x,t) \in (0,T)\times\omega\}.$$

We have the following key lemma, which is proved at the end of this section.

Lemma 3.12. $\mathscr{N}_3(T) = \{0\}$.

Assume for the moment that Lemma 3.12 holds. As for the proof of the observability inequality (3.30), we proceed by contradiction. If the observability inequality (3.30) were false, we could find a sequence $(W^k)_{k\in\mathbb{N}}$ of solutions to System (3.27) which satisfy

$$||W^k(0)||^2_{\mathscr{L}_0^3\times\mathscr{L}_{-1}^3} = 1, \tag{3.49}$$

$$\int_0^T \int_\omega \left| \frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^{1,k} + w_2^{2,k} \right|^2 dxdt \to 0 \text{ as } k \to \infty. \tag{3.50}$$

By the well-posedness, we know that $(W^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,T)\times\Omega)$. Hence, there exists a subsequence (also denoted by W^k) weakly converging in $L^2((0,T)\times\Omega)$, towards $W \in L^2((0,T)\times\Omega)$, which is also a solution of System (3.27) (since what we consider is a linear system) and satisfies that $\frac{\alpha_1 d_1^2}{(d_2-d_1)^2} w_1^1 + w_2^2 = 0$ in $(0,T)\times\omega$. Thus, we know that $W(0) \in \mathscr{N}(T) = \{0\}$, which implies that $W(0) = 0$. Since the embedding $L^2 \times H_\Omega^{-1}(\Delta_D) \hookrightarrow H_\Omega^{-1}(\Delta_D) \times H_\Omega^{-2}(\Delta_D)$ is compact, we obtain that

$||W^k(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}} \to ||W(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}}$. From the relaxed observability inequality (3.33), we know that

$$1 \le C||W(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}},$$

which contradicts to the fact that $W(0) = 0$. Then we can conclude the observability inequality (3.30).

It only remains to prove Lemma 3.12.

Proof of Lemma 3.12. According to the relaxed observability inequality (3.33), for $\mathscr{W} \in \mathscr{N}(T)$, we obtain that

$$||W(0)||^2_{\mathscr{L}^3_0\times\mathscr{L}^3_{-1}} \le C||W(0)||^2_{\mathscr{L}^3_{-1}\times\mathscr{L}^3_{-2}}. \tag{3.51}$$

We know that $\mathcal{N}(T)$ is a closed subspace of $\mathscr{L}^3_0 \times \mathscr{L}^3_{-1}$. By the compact embedding $L^2(\Omega) \times H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega) \times H^{-2}(\Omega)$, we know that $\mathcal{N}(T)$ has a finite dimension. Then, we define the operator $\mathscr{A}$ as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -d_1\Delta_D & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_2\Delta_D & 0 & -\frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} & 0 & -\frac{a_1(-\Delta_D)^{-1}}{d_2} \\ \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2} - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2} & 0 & -d_2\Delta_D - a_2 + \frac{a_1(-\Delta_D)^{-1}}{d_2} & 0 & 1 & 0 \end{pmatrix}.$$

We know that the solution $(w_1^1, w_1^2, w_2^2, D_t w_1^1, D_t w_1^2, D_t w_2^2)^t$ can be written as

$$\begin{pmatrix} w_1^1 \\ w_1^2 \\ w_2^2 \\ D_t w_1^1 \\ D_t w_1^2 \\ D_t w_2^2 \end{pmatrix} = e^{-t\mathscr{A}}\mathscr{W},$$

where $\mathscr{W}$ is defined in (3.48). Let $\delta \in (0,T)$, we know that (3.51) is still true for $\mathscr{W} \in \mathscr{N}(T-\delta)$. Taking $\mathscr{W} \in \mathscr{N}(T)$, for $\epsilon \in]0,\delta[$, we have $e^{-\epsilon\mathscr{A}}\mathscr{W} \in \mathscr{N}(T-\delta)$. For α large enough, as $\epsilon \to 0^+$,

$$(\alpha+\mathscr{A})^{-1}\frac{1}{\epsilon}(Id - e^{-\epsilon\mathscr{A}})\mathscr{W} \to (\alpha+\mathscr{A})^{-1}\mathscr{A}\mathscr{W} \text{ as } \varepsilon \to 0^+ \text{in } \mathscr{L}^3_0 \times \mathscr{L}^3_{-1}. \tag{3.52}$$

Remind that

$$D(\mathscr{A}) = \{U \in \mathscr{L}^3_0 \times \mathscr{L}^3_{-1} | \frac{d}{dt}(e^{-t\mathscr{A}})_{t=0^+} \text{ converges}\}. \tag{3.53}$$

Since $||(\alpha+\mathscr{A})^{-1}\cdot||_{\mathscr{L}^3_0\times\mathscr{L}^3_{-1}}$ is a norm, (3.52) means that $(Id - e^{-\epsilon\mathscr{A}})_{\varepsilon>0}$ is convergent for this norm. Since all norms are equivalent on the finite-dimensional linear subspace $\mathscr{N}(T)$, we notably deduce that $(Id - e^{-\epsilon\mathscr{A}})\mathscr{W}$ converges in $\mathscr{L}^3_0 \times \mathscr{L}^3_{-1}$, so that $\mathscr{W} \in D(\mathscr{A})$ by (3.53). We deduce that $N(T-\delta) \subset D(\mathscr{A})$. Since this equality is true for any $\delta \in (0,T)$, we deduce that $N(T) \subset D(\mathscr{A})$. Hence, for $\mathscr{W} \in N(T)$, we have

$$\frac{d}{dt}(e^{-t\mathscr{A}}(\mathscr{W}))_{t=0^+} = -\mathscr{A}\mathscr{W}.$$

Since $\mathcal{N}(T)$ is clearly stable by differentiation with respect to t, we deduce that $\mathscr{A}\mathscr{W} \in N(T)$. This implies that $\mathscr{A}\mathscr{N}(T) \subset \mathscr{N}(T) \subset \mathscr{L}^3_0 \times \mathscr{L}^3_{-1}$. Since $\mathscr{N}(T)$ is a finite dimensional closed subspace of $D(\mathscr{A})$, and stable by the action of the operator $\mathscr{A}$, it contains an eigenfunction of $\mathscr{A}$. Let us consider such an eigenfunction $(\phi_1^0, \phi_2^0, \phi_3^0, \phi_1^1, \phi_2^1, \phi_3^1) \in \mathscr{N}(T)$, associated to an eigenvalue $\nu \in \mathbb{C}$, so that we have

(3.54)
$$\begin{cases} \phi_1^1 & = \nu\phi_1^0, \\ \phi_2^1 & = \nu\phi_2^0, \\ \phi_3^1 & = \nu\phi_3^0, \\ -d_1\Delta_D\phi_1^0 & = \nu\phi_1^1, \\ -d_2\Delta_D\phi_2^0 - \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}\phi_1^1 - \frac{a_1(-\Delta_D)^{-1}}{d_2}\phi_3^1 & = \nu\phi_2^1, \\ -d_2\Delta_D\phi_3^0 - a_2\phi_3^0 + \frac{a_1(-\Delta_D)^{-1}}{d_2}\phi_3^0 + \frac{\alpha_1 a_1 d_1^2(-\Delta_D)^{-1}}{d_2(d_2-d_1)^2}\phi_1^0 - \frac{\alpha_1 a_2 d_1^2}{(d_2-d_1)^2}\phi_1^0 + \phi_2^1 & = \nu\phi_3^1, \\ \left(\frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0\right)|_\omega & = 0. \end{cases}$$

Let us define a change of variables:

$$\begin{cases} \varphi_1 = d_1^2\Delta_D^2\phi_1^0, \\ \varphi_2 = \nu\phi_2^0 + \frac{\alpha_1 d_1^2}{d_2-d_1}\Delta_D\phi_1^0 + \frac{a_1(-\Delta_D)^{-1}}{d_2}\left(\frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0\right), \\ \varphi_3 = \frac{\alpha_1 d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_3^0. \end{cases} \tag{3.55}$$

Remark 3.13. *We could make a link between the transform $\mathcal{S}'$ and* (3.55). *Formally, we are able to write*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \mathcal{S}'(\nu, \Delta_D) \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \\ \phi_3^0 \end{pmatrix}. \tag{3.56}$$

Here we use the notation $\mathcal{S}'(\nu, \Delta_D)$ to denote the transform replacing formally D_t by the eigenvalue ν (remind that $\mathcal{S}'$ involves only D_t and powers of Δ_D).

Then, we obtain a new system

$$\begin{cases} -d_1\Delta_D\varphi_1 & = \nu^2\varphi_1, \\ -d_2\Delta_D\varphi_2 + \alpha_1\varphi_1 - a_1\varphi_3 & = \nu^2\varphi_2, \\ -d_2\Delta_D\varphi_3 - a_2\varphi_3 + \varphi_2 & = \nu^2\varphi_3, \\ \varphi_3|_\omega & = 0. \end{cases} \tag{3.57}$$

Using the last equation of (3.57), we have

$$\varphi_2|_\omega = \left(\nu^2\varphi_3 + d_2\Delta_D\varphi_3 + a_2\varphi_3\right)|_\omega = 0.$$

Similarly, using the second equation of (3.57), we obtain $\varphi_1|_\omega = 0$. Since $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is the solution of the elliptic System (3.57) verifying $\varphi|_\omega = 0$, by usual unique continuation for elliptic systems, we obtain that $\varphi \equiv 0$ on Ω.

Let us now go back to the eigenvector $(\phi_1^0, \phi_2^0, \phi_3^0, \phi_1^1, \phi_2^1, \phi_3^1)$. The first line of (3.57) gives that $\alpha_1 d_1^2\Delta_D^2\phi_1^0 = 0$ on Ω. Since $\alpha_1 \neq 0$ by (3.3) and $\phi_1^0 = \Delta\phi_1^0 = 0$ on $\partial\Omega$, we deduce that $\phi_1^0 = 0$ on Ω. The first line of (3.54) also provides that $\phi_1^1 = 0$ on Ω. Working on the second line of (3.57) and then on the last line of (3.57), we obtain similarly that $\phi_2^0 = \phi_2^1 = \phi_3^0 = \phi_3^1 = 0$ on Ω, which concludes the proof.

□

3.3. **The case $\tilde{\alpha}_2 \neq 0$.** According to Lemma 2.2, given the initial condition

$$(\tilde{u}_1^{1,0}, \tilde{u}_1^{1,1}) \in H^3_\Omega(\Delta_D) \times H^2_\Omega(\Delta_D),$$

the solution $\tilde{u}_1^1$ to the first line of (3.5) satisfies

$$\tilde{u}_1^1 \in C^0([0,T], H^3_\Omega(\Delta_D)) \cap C^1([0,T], H^2_\Omega(\Delta_D)).$$

For technical reasons, we would like to work in symmetric spaces. We introduce a change of variables

$$\begin{cases} v_1^1 = D_t^2 \tilde{u}_1^1 + \dfrac{\tilde{\alpha}_2 d_2}{d_1 - d_2} \tilde{u}_2^2, \\ v_1^2 = D_t \tilde{u}_1^2, \\ v_2^2 = \tilde{u}_2^2. \end{cases}$$

with the inverse transform defined by

$$\begin{cases} \tilde{u}_1^1 = \frac{(-\Delta_D)^{-1}}{d_1} v_1^1 - \frac{\alpha_1 (-\Delta_D)^{-2}}{d_1 d_2} v_1^2 + \frac{(-\Delta_D)^{-1}}{d_1} (\alpha_1 - \frac{\alpha_2 d_1}{d_1 - d_2}) v_2^2, \\ \tilde{u}_1^2 = \frac{(-\Delta_D)^{-1}}{d_2} (D_t v_1^2 - v_2^2), \\ \tilde{u}_2^2 = v_2^2. \end{cases}$$

The exact controllability of System (3.5) is equivalent to the exact controllability in the state space $\mathscr{L}_1^3 \times \mathscr{L}_0^3$ of the system:

(3.58)

$$\begin{cases} \Box_1 v_1^1 + (\alpha_1 - \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) D_t v_1^2 - (\frac{a_2 \alpha_2 d_1}{d_1 - d_2} + \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) v_2^2 & = \frac{\alpha_2 d_1}{d_1 - d_2} f, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0, \\ \Box_2 v_2^2 - \frac{a_1 (-\Delta_D)^{-1}}{d_2} D_t v_1^2 + (\frac{a_1 (-\Delta_D)^{-1}}{d_2} - a_2) v_2^2 & = f, \\ v_1^1|_{\partial\Omega} = 0,\ v_j^2|_{\partial\Omega} & = 0,\ j = 1, 2, \\ (v_1^1, v_1^2, v_2^2, \partial_t v_1^1, \partial_t v_1^2, \partial_t v_2^2)|_{t=0} & \in \mathscr{L}_1^3 \times \mathscr{L}_0^3. \end{cases}$$

It is equivalent to proving the following observability inequality: $\exists C > 0$ such that for any solutions of the adjoint system

(3.59)

$$\begin{cases} \Box_1 w_1^1 & = 0, \\ \Box_2 w_1^2 + (\alpha_1 - \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) D_t w_1^1 - \frac{a_1 (-\Delta_D)^{-1}}{d_2} D_t w_2^2 & = 0, \\ \Box_2 w_2^2 + D_t w_1^2 + (\frac{a_1 (-\Delta_D)^{-1}}{d_2} - a_2) w_2^2 - (\frac{a_2 \alpha_2 d_1}{d_1 - d_2} + \frac{\alpha_2 a_1 d_1 (-\Delta_D)^{-1}}{d_2 (d_1 - d_2)}) w_1^1 & = 0, \\ w_1^1|_{\partial\Omega} = 0,\ w_j^2|_{\partial\Omega} & = 0\, j = 1, 2, \\ (w_1^1, w_1^2, w_2^2)|_{t=0} = (w_1^{1,0}, w_1^{2,0}, w_2^{2,0}) & \in \mathscr{L}_0^3, \\ (\partial_t w_1^1, \partial_t w_1^2, \partial_t w_2^2)|_{t=0} = (w_1^{1,1}, w_1^{2,1}, w_2^{2,1}) & \in \mathscr{L}_{-1}^3. \end{cases}$$

we have the following observability inequality

$$C \int_0^T \int_\omega \left| \frac{\alpha_2 d_1}{d_1 - d_2} w_1^1 + w_2^2 \right|^2 dx dt \geq ||W(0)||^2_{\mathscr{L}_0^3 \times \mathscr{L}_{-1}^3}. \tag{3.60}$$

We follow the same procedure to prove the inequality (3.60) as we presented in Subsection 3.2. The proof is totally similar for the high frequency part. For the low frequency part, the same computations lead to consider a unique continuation property of the form

$$\begin{cases} -d_1 \Delta_D \varphi_1 & = \nu^2 \varphi_1, \\ -d_2 \Delta_D \varphi_2 + \alpha_1 \varphi_1 - a_1 \varphi_3 & = \nu^2 \varphi_2, \\ -d_2 \Delta_D \varphi_3 + \alpha_2 \varphi_1 + \varphi_2 - a_2 \varphi_3 & = \nu^2 \varphi_3, \\ \varphi_3|_\omega & = 0. \end{cases} \tag{3.61}$$

This system is very similar to (3.57). The main difference is that from the two last lines of (3.61), we only obtain for the moment that

$$\alpha_2 \varphi_1 + \varphi_2 = 0 \text{ on } \omega. \tag{3.62}$$

Using (3.62) with the first line of (3.61), we deduce that

$$d_1\Delta_D\varphi_2 = -d_1\alpha_2\Delta_D\varphi_1 = \nu^2\alpha_2\varphi_1 \text{ on } \omega. \tag{3.63}$$

From (3.63) and the second line of (3.61), we deduce that

$$\left(d_1\alpha_1 - \alpha_2 d_2\nu^2\right)\varphi_1 - \nu^2 d_1\varphi_2 = 0 \text{ on } \omega. \tag{3.64}$$

The unique solution of (3.62) and (3.64) is $\varphi_1 = \varphi_2 = 0$ on ω if

$$(\alpha_2)\left(-\nu^2 d_1\right) - 1\left(d_1\alpha_1 - \alpha_2 d_2\nu^2\right) \neq 0,$$

i.e.

$$\alpha_2\nu^2\left(d_1 - d_2\right) + d_1\alpha_1 \neq 0.$$

The first line of (3.61) implies that there exists $\lambda \in \sigma(-\Delta_D)$ such that $\nu^2 = d_1\lambda$. Hence, $\varphi_1 = \varphi_2 = 0$ on ω if

$$\alpha_2\lambda\left(d_1 - d_2\right) + \alpha_1 \neq 0,$$

which is the case thanks to (3.3). Hence, we have $\varphi_1 = \varphi_2 = \varphi_3 = 0$ on ω, and we can then conclude exactly as in the previous case $\tilde{\alpha}_2 = 0$.

4. Proof of the sufficient part of Theorem 1.16

We organize this section a little bit differently from the previous section. We start by a modal problem to introduce the compatibility condition in this setting. We follow by a reformulation procedure of System (1.2). At last, we finish the proof of our main Theorem 1.16.

4.1. **The modal case.** Let $f \in L^2((0,T), L^2(\Omega))$. For a fixed $1 \leq s \leq n_2$, we consider the following system as a modal problem

$$\begin{cases} \Box_1 u_1^1 + \sum_{j=1}^{s}\alpha_j u_j^2 & = 0 & \text{in } (0,T)\times\Omega,\\ \Box_2 u_1^2 + u_2^2 & = 0 & \text{in } (0,T)\times\Omega,\\ \vdots & & \\ \Box_2 u_{n_2-1}^2 + u_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega,\\ \Box_2 u_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j}u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\ u_1^1 = 0, u_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1\leq j\leq n_2,\\ (u_1^1, u_1^2, \cdots, u_{n_2}^2)|_{t=0} & = (u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) & \text{in } \Omega,\\ (\partial_t u_1^1, \partial_t u_1^2, \cdots, \partial_t u_{n_2}^2)|_{t=0} & = (u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) & \text{in } \Omega. \end{cases} \tag{4.1}$$

In this section, we aim to prove the exact controllability of System (4.1) with the help of proper compatibility conditions. For this modal System (4.1), we have the following well-posedness property:

Proposition 4.1. *Assume that the initial conditions verify*

$$(u_1^{1,0}, u_1^{2,0}, \cdots, u_{n_2}^{2,0}) \in H_\Omega^{n_2+3-s}(\Delta_D)\times H_\Omega^{n_2}(\Delta_D)\times\cdots\times H_\Omega^{1}(\Delta_D),$$
$$(u_1^{1,1}, u_1^{2,1}, \cdots, u_{n_2}^{2,1}) \in H_\Omega^{n_2+2-s}(\Delta_D)\times H_\Omega^{n_2-1}(\Delta_D)\times\cdots\times H_\Omega^{0}(\Delta_D).$$

Additionally, let us define $\tilde{U}^0$ and $\tilde{U}^1$ by

(4.2)
$$\tilde{U}^0 = (-d_1\Delta)^{n_2-s+1}u_1^{1,0} + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^{2,0}$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^{2,0},$$

and

(4.3)
$$\tilde{U}^1 = (-d_1\Delta)^{n_2-s+1}u_1^{1,1} + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^{2,1}$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^{2,1}.$$

Assume that $\tilde{U}^0 \in H^1_\Omega(\Delta_D)$ and $\tilde{U}^1 \in H^0_\Omega(\Delta_D)$. Then, the solution $(u_1^1, u_1^2, \cdots, u_{n_2}^2)$ satisfies

$$\begin{aligned} &u_1^1 \in C^0([0,T], H_\Omega^{n_2+3-s}(\Delta)) \cap C^1([0,T], H_\Omega^{n_2+2-s}(\Delta)),\\ &u_j^2 \in C^0([0,T], H_\Omega^{n_2+1-j}(\Delta)) \cap C^1([0,T], H_\Omega^{n_2-j}(\Delta)), 1 \le j \le n_2. \end{aligned} \tag{4.4}$$

Furthermore, we have

(4.5)
$$\Bigg((-d_1\Delta)^{n_2-s+1}u_1^{1} + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k} \alpha_j \binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^{2}$$
$$+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^{2}\Bigg)$$
$$\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

Remark 4.2. *Let $n_2 = 2, s = 1, \alpha_1 = 1$, then (4.5) becomes the following condition:*

$$\Bigg((-d_1\Delta)^2u_1^1 + \sum_{k=0}^{1}\sum_{l=0}^{1-k}\binom{1-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{1-k-l}u_{1+l}^2$$
$$+\sum_{k=0}^{1}\sum_{l=0}^{1-k}\frac{d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{1-k}{l}(-d_2\Delta)^{1-k-l}u_{1+k+l}^2\Bigg)$$
$$\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

Simplifying the formula, we obtain that

$$\left((-d_1\Delta)^2u_1^1 + \frac{d_1}{d_1-d_2}(-d_1\Delta)u_1^2 + \frac{d_1^2}{(d_1-d_2)^2}u_2^2\right)$$
$$\in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D)).$$

This is just the compatibility condition in the previous section.

Proof. As we have shown in the proof of Proposition 3.2, it is classical to obtain the regularity of the solutions given in (4.4), following Lemma 2.2. Now, we focus on the proof of the compatibility conditions (4.5), so we restrict to the case $s < n_2$ according to Remark 1.10. We perform the similar reformulation for the solutions of System (4.1):

$$\begin{cases} v_1^1 = D_t^{n_2+2-s}u_1^1, \\ v_1^2 = D_t^{n_2-1}u_1^2, \\ \vdots \\ v_{n_2}^2 = u_{n_2}^2. \end{cases} \tag{4.6}$$

The transform above is "invertible", and there are four different cases for the form of the inverse, that is, n_2 and $n_2 - s$ are both even or odd, n_2 is even while $n_2 - s$ is odd and the converse, that we do not detail here. We perform the same strategy as we have already shown in the proof of the Proposition 3.2. Thus, we obtain a system for $v_1^1, v_1^2, \cdots, v_{n_2}^2$ given by

$$\begin{cases} \Box_1 v_1^1 + \sum_{j=1}^{s} \alpha_j D_t^{n_2+2-s} u_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_1^2 + D_t v_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 v_{n_2-1}^2 + D_t v_{n_2}^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2 v_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j} u_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ v_1^1 = 0, v_j^2 & = 0 & \text{on } (0,T)\times\partial\Omega, 1\le j\le n_2, \end{cases} \tag{4.7}$$

with initial conditions

$$\begin{aligned} (v_1^1, v_1^2, \cdots, v_{n_2}^2)|_{t=0} &= (v_1^{1,0}, v_1^{2,0}, \cdots, v_{n_2}^{2,0}), \\ (\partial_t v_1^1, \partial_t v_1^2, \cdots, \partial_t v_{n_2}^2)|_{t=0} &= (v_1^{1,1}, v_1^{2,1}, \cdots, v_{n_2}^{2,1}). \end{aligned} \tag{4.8}$$

We focus on the first equation. Let $y_0^1 = v_1^1 + \frac{\alpha_s d_2}{d_1 - d_2} v_s^2$. Then, we obtain

$$\begin{aligned} \Box_1 y_0^1 &= \Box_1 v_1^1 + \frac{\alpha_s d_2}{d_1 - d_2}\Box_1 v_s^2 \\ &= -\sum_{j=1}^{s} \alpha_j D_t^{n_2-s+2} u_j^2 + \frac{\alpha_s d_2}{d_1 - d_2}\Box_2 v_s^2 + \frac{\alpha_s d_2}{d_1 - d_2}(d_2 - d_1)\Delta v_s^2 \\ &= -\sum_{j=1}^{s} \alpha_j D_t^{n_2-s+2} u_j^2 - \frac{\alpha_s d_2}{d_1 - d_2} D_t v_{s+1}^2 - \alpha_s d_2 \Delta v_s^2. \end{aligned}$$

Since v_s^2 satisfies the equation $\Box_2 v_s^2 + D_t v_{s+1}^2 = 0$ by (4.1), we obtain that

$$\begin{aligned} -\alpha_s D_t^{n_2-s+2} u_s^2 - \alpha_s d_2 \Delta v_s^2 &= -\alpha_s (D_t^2 v_s^2 + d_2\Delta) v_s^2 \\ &= \alpha_s \Box_2 v_s^2 \\ &= -\alpha_s D_t v_{s+1}^2. \end{aligned}$$

This implies that

$$\begin{aligned}\Box_1 y_0^1 &= -\sum_{j=1}^{s-1}\alpha_j D_t^{n_2-s+2}u_j^2 - \frac{\alpha_s d_2}{d_1-d_2}D_t v_{s+1}^2 - \alpha_s D_t v_{s+1}^2\\ &= -\sum_{j=1}^{s-1}\alpha_j D_t^{n_2-s+2}u_j^2 - \alpha_s(\frac{d_2}{d_1-d_2}+1)D_t v_{s+1}^2\\ &= -\sum_{j=1}^{s-1}\alpha_j D_t^{n_2-s+2}u_j^2 - \frac{\alpha_s d_1}{d_1-d_2}D_t v_{s+1}^2.\end{aligned}$$

As a consequence, using the definition $v_{s-1}^2 = D_t^{n_2-s+1}u_{s-1}^2$, we know that y_0^1 satisfies the equation

$$\Box_1 y_0^1 + \sum_{j=1}^{s-2}\alpha_j D_t^{n_2-s+2}u_j^2 + \frac{\alpha_s d_1}{d_1-d_2}D_t v_{s+1}^2 + \alpha_{s-1}D_t v_{s-1}^2 = 0. \tag{4.9}$$

Define by induction

$$y_j^1 = D_t y_{j-1}^1 + \sum_{k=0}^{j}\frac{\alpha_{s-k}d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}}v_{s+j-2k}^2, 1\le j\le n_2-s-1. \tag{4.10}$$

Let $\alpha_j = 0$ for $j\in\mathbb{Z}\backslash\{1,2,\cdots,s\}$. We have the following lemmas, which are proved afterwards.

Lemma 4.3. *y_j^1 $(1\le j\le n_2-s-1)$ satisfies the equation*

$$\Box_1 y_j^1 + \sum_{k=-\infty}^{s-2-j}\alpha_k D_t^{n_2-s+j+2}u_k^2 + \sum_{k=0}^{j+1}\frac{\alpha_{s-k}d_1^{j+1-k}}{(d_1-d_2)^{j+1-k}}D_t v_{s+j+1-2k}^2 = 0. \tag{4.11}$$

Remark 4.4. *$\sum_{k=-\infty}^{s-2-l}\alpha_k D_t^{n_2-s+2}u_k^2$ is a sum of finite terms, since for $k\le 0$, $\alpha_k\equiv 0$.*

Let $y_{comp} = D_t y_{n_2-s-1}^1 + \sum_{k=0}^{n_2-s}\frac{\alpha_{s-k}d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}}v_{n_2-2k}^2$.

Lemma 4.5. *y_{comp} satisfies the equation*

$$\begin{aligned}\Box_1 y_{comp} = &-\sum_{k=1}^{n_2-s+1}\frac{\alpha_{s-k}d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}}D_t v_{n_2+1-2k}^2 - \sum_{k=-\infty}^{2s-2-n_2}\alpha_k D_t^{2n_2-2s+2}u_k^2\\ &+\sum_{k=1}^{n_2}\frac{a_{n_2+1-k}\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}f.\end{aligned} \tag{4.12}$$

Lemma 4.6. *For y_{comp}, we have*

$$\begin{aligned}y_{comp} &= (-d_1\Delta)^{n_2-s+1}u_1^1\\ &+\sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2\\ &+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_{s-k}d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^2.\end{aligned} \tag{4.13}$$

Assume for the moment that these Lemmas are true and let us complete the proof of Proposition 4.1. Define

$$F = -\sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}} D_t v^2_{s+j+1-2k} - \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 + \sum_{k=1}^{n_2} \frac{a_{n_2+1-k}\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f. \tag{4.14}$$

Since

$$u_k^2 \in C^0([0,T], H_\Omega^{n_2+1-k}(\Delta_D)) \cap C^1([0,T], H_\Omega^{n_2-k}(\Delta_D)),$$

we know that

$$D_t^{2n_2-2s+2} u_k^2 \in L^1((0,T), H^0_\Omega(\Delta_D)),\ k \le 2s-2-n_2,$$

which implies that $F \in L^1((0,T), H^0_\Omega(\Delta_D))$. Now, we remark that by (4.4) and (4.5), y_{comp} satisfies

$$y_{comp}|_{t=0} = \tilde{U}^0 \in H^1_\Omega(\Delta_D),$$
$$\partial_t y_{comp}|_{t=0} = \tilde{U}^1 \in H^0_\Omega(\Delta_D).$$

Consequently, from (4.12), (4.14) and the fact that $F \in L^1((0,T), H^0_\Omega(\Delta_D))$,we conclude that $y_{comp} \in C^0([0,T], H^1_\Omega(\Delta_D)) \cap C^1([0,T], H^0_\Omega(\Delta_D))$. □

It only remains to prove Lemma 4.3, Lemma 4.5 and Lemma 4.6.

Proof of Lemma 4.3 *and Lemma* 4.5. We prove these lemmas by induction. For y_0^1, according to (4.9), we know that y_0^1 satisfies (4.11) for $j=1$. Assume that for $l<j$, y_l^1 satisfies (4.11). Thus, using the definition of y_j^1 and the equation for y_{j-1}^1, we know that y_j^1 satisfies the following equation

$$\begin{aligned}\Box_1 y_j^1 &= D_t \Box_1 y_{j-1}^1 + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \Box_1 v^2_{s+j-2k} \\ &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 - \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} D_t^2 v^2_{s+j-2k} \\ &+ \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} \Box_2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} (d_2-d_1)\Delta v^2_{s+j-2k}.\end{aligned}$$

By simple observation, we know that

$$\begin{aligned}&-\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} D_t^2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} (d_2-d_1)\Delta v^2_{s+j-2k} \\ =&\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} \partial_t^2 v^2_{s+j-2k} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} (-d_2\Delta) v^2_{s+j-2k} \\ =&\sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} \Box_2 v^2_{s+j-2k}.\end{aligned}$$

Therefore, we simplify the equation for y_j^1,

$$\begin{aligned}\Box_1 y_j^1 &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k}}{(d_1-d_2)^{j-k}} (\frac{d_2}{d_1-d_2}+1) \Box_2 v_{s+j-2k}^2 \\ &= -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} \Box_2 v_{s+j-2k}^2.\end{aligned}$$

Using the equation $\Box_2 v_{s+j-2k}^2 = -D_t v_{s+1+j-2k}^2$ coming from (4.7), we obtain

$$\Box_1 y_j^1 = -\sum_{k=-\infty}^{s-1-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 - \sum_{k=0}^{j} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} D_t v_{s+j-2k+1}^2.$$

Now we look at the term $\alpha_{s-1-j} D_t^{n_2-s+j+2} u_{s-1-j}^2$. If $j \leq s-1$, we obtain

$$\alpha_{s-1-j} D_t^{n_2-s+j+2} u_{s-1-j}^2 = \alpha_{s-1-j} D_t v_{s-1-j}^2;$$

if $j > s-1$, $\alpha_{s-1-j} = 0$. Hence, we have

$$\Box_1 y_j^1 = -\sum_{k=-\infty}^{s-2-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 - \sum_{k=0}^{j+1} \frac{\alpha_{s-k} d_1^{j-k+1}}{(d_1-d_2)^{j-k+1}} D_t v_{s+j-2k+1}^2.$$

By induction, this implies that $y_j^1 (1 \leq j \leq n_2-s-1)$ satisfies the equation

$$\Box_1 y_j^1 + \sum_{k=-\infty}^{s-2-j} \alpha_k D_t^{n_2-s+j+2} u_k^2 + \sum_{k=0}^{j+1} \frac{\alpha_{s-k} d_1^{j+1-k}}{(d_1-d_2)^{j+1-k}} D_t v_{s+j+1-2k}^2 = 0. \tag{4.15}$$

Using the definition of y_{comp}, we obtain

$$\Box_1 y_{comp} = D_t \Box_1 y_{n_2-s-1}^1 + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}} \Box_1 v_{n_2-2k}^2.$$

Following the same procedure, we have the following equation

$$\Box_1 y_{comp} = -\sum_{k=-\infty}^{2s-1-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_1^{n_2-s-k+1}}{(d_1-d_2)^{n_2-s-k+1}} \Box_2 v_{n_2-2k}^2.$$

Using the equation $\Box_2 v_{n_2}^2 = \sum_{k=1}^{n_2} a_{n_2+1-k} u_k^2 + f$ coming from (4.7), we obtain

$$\begin{aligned}\Box_1 y_{comp} = &-\sum_{k=-\infty}^{2s-1-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 + \sum_{k=1}^{n_2-s} \frac{\alpha_{s-k} d_1^{n_2-s-k+1}}{(d_1-d_2)^{n_2-s-k+1}} D_t v_{n_2-2k+1}^2 \\ &+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2-s+1}}{(d_1-d_2)^{n_2-s+1}} u_k^2 + \frac{\alpha_s d_1^{n_2-s+1}}{(d_1-d_2)^{n_2-s+1}} f.\end{aligned}$$

Now look at the term $\alpha_{2s-1-n_2} D_t^{2n_2-2s+2} u^2_{2s-1-n_2}$. If $2s-1-n_2 \le 0$, we know that $\alpha_{2s-1-n_2} \equiv 0$. Otherwise, we know that $D_t^{2n_2-2s+2} u^2_{2s-1-n_2} = D_t v^2_{2s-1-n_2}$. Consequently, we obtain the equation for y_{comp}:

$$\begin{aligned}\Box_1 y_{comp} = &-\sum_{k=1}^{n_2-s+1} \frac{\alpha_{s-k} d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}} D_t v^2_{n_2+1-2k} - \sum_{k=-\infty}^{2s-2-n_2} \alpha_k D_t^{2n_2-2s+2} u_k^2 \\ &+ \sum_{k=1}^{n_2} \frac{a_{n_2+1-k} \alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} u_k^2 + \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f,\end{aligned}$$

which is exactly the equation (4.12). □

Proof of Lemma 4.6. Recall the definition of y_{comp},

$$y_{comp} = D_t y^1_{n_2-s-1} + \sum_{k=0}^{n_2-s} \frac{\alpha_{s-k} d_2 d_1^{n_2-s-k}}{(d_1-d_2)^{n_2-s+1-k}} v^2_{n_2-2k},$$

and the definition of $y^1_j (1 \le j \le n_2-s-1)$,

$$y^1_j = D_t y^1_{j-1} + \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} v^2_{s+j-2k}.$$

Therefore, by iteration, we have the following expression for y_{comp}

$$y_{comp} = D_t^{n_2-s} y^1_0 + \sum_{j=1}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{n_2-s-j} v^2_{s+j-2k}. \tag{4.16}$$

Using the definitions of $y^1_0 = v^1_1 + \frac{\alpha_s d_2}{d_1-d_2} v^2_s$ and $v^2_j = D_t^{n_2+1-j} u^2_j, 1 \le j \le n_2$ given in (4.6), we simplify the formula above:

$$y_{comp} = D_t^{2n_2-2s+2} u^1_1 + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k}.$$

According to the equation $D_t^2 u^1_1 = -d_1 \Delta u^1_1 + \sum_{j=1}^{s} \alpha_j u^2_j$ coming from (4.1), we obtain

$$\begin{aligned} y_{comp} &= D_t^{2n_2-2s}(-d_1 \Delta u^1_1 \\ &\quad + \sum_{j=1}^{s} \alpha_j u^2_j) + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k} \\ &= (-d_1 \Delta) D_t^{n_2-s} u^1_1 + \sum_{j=1}^{s} \alpha_j D_t^{n_2-s} u^2_j \\ &\quad + \sum_{j=0}^{n_2-s} \sum_{k=0}^{j} \frac{\alpha_{s-k} d_2 d_1^{j-k}}{(d_1-d_2)^{j+1-k}} D_t^{2n_2-2s-2j+2k} u^2_{s+j-2k}. \end{aligned}$$

By iteration, we are able to obtain that

$$y_{comp} = (-d_1\Delta)^{n_2-s+1}u_1^1 + \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\alpha_j(-d_1\Delta)^k D_t^{2n_2-2s-2k}u_j^2$$
$$+\sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2.$$

Now we introduce the following lemma to describe the term $D_t^{2k}u_j^2$.

Lemma 4.7. *Let u_j^2 be solutions to the system (4.1). If $k+j\leq n_2$, we have*

$$D_t^{2k}u_j^2 = \sum_{l=0}^{k}\binom{k}{l}(-d_2\Delta)^l u_{j+k-l}^2. \tag{4.17}$$

We shall prove this lemma in Appendix B. Now, we use this lemma to simplify the formula of y_{comp}. In the term $\sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\alpha_j(-d_1\Delta)^k D_t^{2n_2-2s-2k}u_j^2$, since $j\leq s$ and $k\geq 0$, we know that $n_2-s-k+j\leq n_2-k\leq n_2$. Thus, according to Lemma 4.7, we obtain

$$D_t^{2n_2-2s-2k}u_j^2 = \sum_{l=0}^{n_2-s-k}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2. \tag{4.18}$$

On the other hand, in the term $\sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2$, since $k\geq 0$, we know that $(s+j-2k)+(n_2-s-j+k)=n_2-k\leq n_2$. Therefore, according to Lemma 4.7, we obtain

$$D_t^{2n_2-2s-2j+2k}u_{s+j-2k}^2 = \sum_{l=0}^{n_2-s-j+k}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2. \tag{4.19}$$

As a consequence, we obtain that

$$\begin{aligned} y_{comp} &= (-d_1\Delta)^{n_2-s+1}u_1^1 \\ &+ \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2 \\ &+ \sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\sum_{l=0}^{n_2-s-j+k}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2. \end{aligned} \tag{4.20}$$

For the last term in the formula above, since $\alpha_{s-k}=0$ for $k\geq s$, we know that

$$\sum_{j=0}^{n_2-s}\sum_{k=0}^{j}\sum_{l=0}^{n_2-s-j+k}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2$$
$$=\sum_{k=0}^{s-1}\sum_{j=k}^{n_2-s}\sum_{l=0}^{n_2-s-j+k}\frac{\alpha_{s-k}d_2d_1^{j-k}}{(d_1-d_2)^{j+1-k}}\binom{n_2-s-j+k}{l}(-d_2\Delta)^{n_2-s-j+k-l}u_{s+j-2k+l}^2$$
$$=\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2d_1^{k}}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^2.$$

The last equality holds after a change of the sum index. Therefore, we obtain the form for y_{comp}

$$\begin{aligned} y_{comp} &= (-d_1\Delta)^{n_2-s+1}u_1^1 \\ &+ \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2 \\ &+\sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^2. \end{aligned} \tag{4.21}$$

□

We also have the similar theorem as we proved in the previous section:

Theorem 4.8. *Given $T > 0$, suppose that:*

(1) (ω, T, p_{d_i}) satisfies GCC, $i = 1, 2$.
(2) Ω has no infinite order of tangential contact with the boundary.

Then System (4.1) *is exactly controllable in $\mathcal{H}_1^s \times \mathcal{H}_0^s$.*

As before, proving Theorem 4.8 is equivalent to proving the exact controllability of the following system:

$$\begin{cases} \Box_1 v_1^1 + R(v_1^2,\cdots,v_{n_2}^2) &= \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \Box_2 v_1^2 + D_t v_2^2 &= 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2 v_{n_2}^2 - \sum_{k=1}^{n_2} a_{n_2+1-k}\mathcal{S}_k^{-1}(v_k^2,\cdots,v_{n_2}^2) &= f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ v_1^1 = 0, v_1^2 = \cdots = v_{n_2}^2 &= 0 & \text{on } (0,T)\times\partial\Omega,, \\ (v_1^1, v_1^2,\cdots,v_{n_2}^2)|_{t=0} \in \mathscr{L}_1^{n_2+1} & & \\ (\partial_t v_1^1, \partial_t v_1^2,\cdots,\partial_t v_{n_2}^2)|_{t=0} \in \mathscr{L}_0^{n_2+1}, & & \end{cases} \tag{4.22}$$

with

$$\begin{aligned} R(v_1^2,\cdots,v_{n_2}^2) = &\sum_{k=1}^{n_2-s+1}\frac{\alpha_{s-k}d_1^{n_2-s+1-k}}{(d_1-d_2)^{n_2-s+1-k}}D_t v_{s+j+1-2k}^2 \\ &+\sum_{k=-\infty}^{2s-2-n_2}\alpha_k D_t^{2n_2-2s+2}\mathcal{S}_k^{-1}(v_k^2,\cdots,v_{n_2}^2) \\ &+\sum_{k=1}^{n_2}\frac{a_{n_2+1-k}\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}\mathcal{S}_k^{-1}(v_k^2,\cdots,v_{n_2}^2). \end{aligned}$$

Here we use the transform $\mathcal{S}$ given by

$$\mathcal{S}\begin{pmatrix} u_1^1 \\ u_1^2 \\ \vdots \\ u_{n_2}^2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ \vdots \\ v_{n_2}^2 \end{pmatrix},$$

where

$$\left\{\begin{array}{l} v_1^1 = y_{comp} \\ v_1^2 = D_t^{n_2-1}u_1^2, \\ \vdots \\ v_{n_2}^2 = u_{n_2}^2, \end{array}\right. \tag{4.23}$$

with

$$\begin{aligned} y_{comp} &= (-d_1\Delta)^{n_2-s+1}u_1^1 \\ &+ \sum_{k=0}^{n_2-s}\sum_{j=1}^{s}\sum_{l=0}^{n_2-s-k}\alpha_j\binom{n_2-s-k}{l}(-d_1\Delta)^k(-d_2\Delta)^{n_2-s-k-l}u_{j+l}^2 \\ &+ \sum_{j=1}^{s}\sum_{k=0}^{n_2-2s+j}\sum_{l=0}^{n_2-s-k}\frac{\alpha_j d_2 d_1^k}{(d_1-d_2)^{k+1}}\binom{n_2-s-k}{l}(-d_2\Delta)^{n_2-s-k-l}u_{j+k+l}^2. \end{aligned}$$

Remark that Proposition 3.2 together with (4.23) ensures that

$$(v_1^1, v_1^2, \dots, v_{n_2}^2) \in C^0([0,T], \mathscr{L}_1^{n_2+1}) \cap C^1([0,T], \mathscr{L}_0^{n_2+1}).$$

We use $\mathcal{S}^{-1}$ to denote the inverse transform given by

$$\left\{\begin{array}{l} u_1^1 = \mathcal{S}_0^{-1}(v_1^1, v_1^2, \cdots, v_{n_2}^2), \\ u_1^2 = \mathcal{S}_1^{-1}(v_1^2, \cdots, v_{n_2}^2), \\ \vdots \\ u_{n_2-j}^2 = \mathcal{S}_j^{-1}(v_{n_2-j}^2, \cdots, v_{n_2}^2), 0 \le j \le n_2-1, \\ \vdots \\ u_{n_2}^2 = \mathcal{S}_{n_2}^{-1}(v_{n_2}^2) = v_{n_2}^2. \end{array}\right. \tag{4.24}$$

Then, we treat exactly the same way as we did in the proof of Proposition 3.2 to obtain the form of the inverse transform of $\mathcal{S}$. There are two different cases. For $n_2 = 2k+1$, which is an odd integer, we are able to obtain that

$$\left\{\begin{array}{ll} u_{2k+1}^2 & = v_{2k+1}^2, \\ u_{2k}^2 & = (-d_2\Delta_D)^{-1}D_t v_{2k}^2 + T(2k, 2k+1)(-d_2\Delta_D)^{-1}v_{2k+1}^2, \\ \vdots & \\ u_1^2 & = (-d_2\Delta_D)^{-k}v_1^2 + T(1,2)(-d_2\Delta_D)^{-k-1}D_t v_2^2 \cdots \\ & \quad +T(1,2k+1)(-d_2\Delta_D)^{-2k}v_{2k+1}^2. \end{array}\right. \tag{4.25}$$

It is similar for the even integer $n_2 = 2k$:

(4.26)

$$\left\{\begin{array}{ll} u_{2k}^2 & = v_{2k}^2, \\ u_{2k-1}^2 & = (-d_2\Delta_D)^{-1}D_t v_{2k-1}^2 + T(2k-1, 2k)(-d_2\Delta_D)^{-1}v_{2k}^2, \\ \vdots & \\ u_1^2 & = (-d_2\Delta_D)^{-k}D_t v_1^2 + T(1,2)(-d_2\Delta_D)^{-k}v_2^2 \cdots + T(1,2k)(-d_2\Delta_D)^{1-2k}v_{2k}^2. \end{array}\right.$$

Here the coefficients $\{T(i,j)\}_{1\le i<j\le n}$ are uniquely determined by System (4.1), but their exact value is not really important.

Remark 4.9. *As explained in Remark* 3.3*, we are able to rewrite the system* (4.22) *as follows:*

$$(\partial_t^2 - \mathcal{S}D\mathcal{S}^{-1}\Delta + \mathcal{S}A\mathcal{S}^{-1})V = \mathcal{S}\hat{b}f,$$

and we have

$$\mathcal{S}\hat{b}f = \begin{pmatrix} \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} f, \\ 0, \\ \vdots \\ 0, \\ f \end{pmatrix}.$$

Moreover, we could notice that both $\mathcal{S}$ *and* $\mathcal{S}^{-1}$ *only involve* D_t *and* $(-\Delta_D)^k, k \in \mathbb{Z}$.

According to the Hilbert Uniqueness Method, we only need to prove the observability inequality

$$C\int_0^T \int_\omega \left| \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^1 + w_{n_2}^2 \right|^2 dxdt \geq ||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}}. \tag{4.27}$$

for any solution of the adjoint system:

$$\begin{cases} \Box_1 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \Box_2 w_1^2 + \Lambda_1 w_{n_2}^2 + \tilde{\Lambda}_1 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \Box_2 w_2^2 + D_t w_1^2 + \Lambda_2 w_{n_2}^2 + \tilde{\Lambda}_2 w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ \vdots & \\ \Box_2 w_{n_2}^2 + D_t w_{n_2-1}^2 + \Lambda_{n_2} w_{n_2}^2 + \tilde{\Lambda}_{n_2} w_1^1 & = 0 \quad \text{in } (0,T)\times\Omega, \\ w_1^1 = 0, w_1^2 = \cdots = w_{n_2}^2 & = 0 \quad \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{4.28}$$

with initial conditions

$$(w_1^1, w_1^2, \cdots, w_{n_2}^2)|_{t=0} \in (L^2(\Omega))^{n_2+1} = \mathscr{L}_0^{n_2+1}$$
$$(\partial_t w_1^1, \partial_t w_1^2, \cdots, \partial_t w_{n_2}^2)|_{t=0} \in (H_\Omega^{-1}(\Delta_D))^{n_2+1} = \mathscr{L}_{-1}^{n_2+1},$$

where the operators $(\Lambda_j)_{1\leq j\leq n_2}$ and $(\tilde{\Lambda}_j)_{1\leq j\leq n_2}$ are uniquely determined by the transform (4.23) and additionally are bounded operators in $L^2(\Omega)$. As usual, we divide the proof of the observability inequality (4.27) into two steps.

Remark 4.10. *We are able to rewrite the adjoint system* (4.28) *as follows*

$$(\partial_t^2 - (\mathcal{S}')^{-1} D\mathcal{S}'\Delta + (\mathcal{S}')^{-1} A^*\mathcal{S}')W = 0.$$

Here the transform $\mathcal{S}'$ *denotes the invertible transform between the adjoint systems. Moreover, we could notice that both* $\mathcal{S}'$ *and* $(\mathcal{S}')^{-1}$ *only involve* D_t *and* $(-\Delta_D)^k, k \in \mathbb{Z}$.

4.1.1. *Step 1: establish a relaxed observability inequality.* First, we can establish a relaxed observability inequality for the adjoint System (4.28).

Proposition 4.11. *For solutions of System* (4.28)*, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}} \\ &\leq C\left(\int_0^T \int_\omega \left| \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}} w_1^1 + w_{n_2}^2 \right|^2 dxdt + ||W(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}}\right). \end{aligned} \tag{4.29}$$

Proof. We argue by contradiction. Suppose that the observability inequality (4.29) is not satisfied. Thus, there exists a sequence $(W^k)_{k\in\mathbb{N}}$ the solutions of System (4.28) such that

$$||W^k(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}} = 1, \tag{4.30}$$

$$\int_0^T\int_\omega \left|\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}w_1^{1,k} + w_{n_2}^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty, \tag{4.31}$$

$$||W^k(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}} \to 0 \text{ as } k\to\infty. \tag{4.32}$$

By the continuity of the solution with respect to the initial data of System (3.27), we know that the sequence $(W^k)_{k\in\mathbb{N}}$ is bounded in $(L^2((0,T)\times\Omega))^{n_2+1}$ and moreover, $W^k \rightharpoonup 0$ in $(L^2((0,T)\times\Omega))^{n_2+1}$. We have W^k satisfying the following system

$$\begin{cases} \Box w_1^{1,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \Box w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \Box w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box w_{n_2}^{2,k} + D_t w_{n_2-1}^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} & \text{in } (0,T)\times\Omega. \end{cases} \tag{4.33}$$

Hence, we obtain two microlocal defect measures $\underline{\mu}_1 \in \underline{\mathcal{M}}^+$ and $\underline{\mu}_2 \in \underline{\mathcal{M}}^+$ associated with $(w_1^{1,k})_{k\in\mathbb{N}}$ and $(W^{2,k})_{k\in\mathbb{N}}$ respectively. From the definition in Proposition 2.6, we know that

$$\forall A \in \underline{\mathcal{A}}, \quad \langle \underline{\mu}_1, \sigma(A)\rangle = \lim_{k\to\infty}(A\underline{w}_1^{1,k}, \underline{w}_1^{1,k})_{L^2},$$
$$\langle \underline{\mu}_2(i,j), \sigma(A)\rangle = \lim_{k\to\infty}(A\underline{w}_i^{2,k}, \underline{w}_j^{2,k})_{L^2}, 1\le i,j\le 2.$$

Here $\underline{\mu}_2 = (\underline{\mu}_2(i,j))_{1\le i,j\le n_2}$ is the matrix measure associated with the sequence $(W^{2,k})_{k\in\mathbb{N}} = (w_1^{2,k},\cdots,w_{n_2}^{2,k})_{k\in\mathbb{N}}$ and moreover, $\underline{w}_1^{1,k}$ and $\underline{w}_i^{2,k}$ is the extension by 0 across the boundary of $\Omega(1\le i\le n_2)$. As we already presented in the Subsection 3.2, the two measures are mutually singular in $(0,T)\times\Omega$. Then provided with

$$\int_0^T\int_\omega \left|\frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}w_1^{1,k} + w_{n_2}^{2,k}\right|^2 dxdt \to 0 \text{ as } k\to\infty,$$

we obtain that for $\chi \in C_0^\infty((0,T)\times\omega)$

$$\langle \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}\chi w_1^{1,k}, \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}\chi w_1^{1,k}\rangle \to 0,$$
$$\langle \chi w_{n_2}^{2,k}, \chi w_{n_2}^{2,k}\rangle \to 0, \quad \text{as } k\to\infty.$$

Thus, we know that

$$\underline{\mu}_1|_{(0,T)\times\omega} = 0, \text{ and } \underline{\mu}_2(n_2,n_2)|_{(0,T)\times\omega} = 0. \tag{4.34}$$

For $\underline{\mu}_1$, since $\underline{\mu}_1$ is invariant along the along the general bicharacteristics of p_{d_1}, combining with GCC, we know that $\underline{\mu}_1 \equiv 0$. For $\underline{\mu}_2$, we consider the other definition of the microlocal defect measure. From Proposition 2.8, we know that there exists a measure $\mu_2 \in \mathcal{M}^+$ such that

$$\forall A \in \mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A))\rangle = \lim_{k\to\infty}(AW^{2,k}, W^{2,k})_{L^2}. \tag{4.35}$$

Here $\mu_2 = (\mu_2(i,j))_{1\le i,j\le n_2}$ is a matrix measure. Since $\underline{\mu}_2|_{Char(p_{d_2})} = \mu_2$ μ_2-almost surely, we obtain that $\mu_2(n_2,n_2)|_{(0,T)\times\omega} = 0$. As we already presented in the Subsection 3.2, we would like to use Lemma 2.10. So we adapt this lemma under our setting here.

Lemma 4.12. *Assume that μ_2 is the corresponding microlocal defect measure defined by*

$$\forall A \in \mathcal{A}, \quad \langle \mu_2, \kappa(\sigma(A))\rangle = \lim_{k\to\infty} (AW^{2,k}, W^{2,k})_{L^2}. \tag{4.36}$$

for the sequence $W^{2,k} = (w_1^{2,k},\cdots,w_{n_2}^{2,k})_{k\in\mathbb{N}}$ which satisfies the following system:

$$\begin{cases} \Box w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} \quad in\ (0,T)\times\Omega, \\ \Box w_2^{2,k} + D_t w_1^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} \quad in\ (0,T)\times\Omega, \\ \vdots & \\ \Box w_{n_2}^{2,k} + D_t w_{n_2-1}^{2,k} & = o(1)_{H_\Omega^{-1}(\Delta_D)} \quad in\ (0,T)\times\Omega. \end{cases} \tag{4.37}$$

If we denote the general bicharacteristic by $s\mapsto\gamma(s)$, then along $\gamma(s)$ there exists a continuous function $s\mapsto M(s)$ such that M satisfies the differential equation:

$$\frac{d}{ds}(M(s)) = iE(\tau)M(s), M(0) = Id,$$

and μ_2 is invariant along the flow associated with M, which means that

$$\frac{d}{ds}(M^*\mu_2 M) = 0.$$

Here we denote by $E(\tau)$ the matrix $\begin{pmatrix} 0 & \tau & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \tau \\ 0 & \cdots & 0 & 0 \end{pmatrix}$.

Remark 4.13. *For the differential equation satisfied by M and the form of the matrix E, one can refer to* [10, Section 3.2] *for more details.*

Here, M has the form of $\begin{pmatrix} 1 & i\tau s & \cdots & \frac{(i\tau s)^{n_2-1}}{(n_2-1)!} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i\tau s \\ 0 & \cdots & 0 & 1 \end{pmatrix}$, where τ is a nonzero constant along the generalized bicharacteristic.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_{n_2} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ be the canonical basis for $\mathbb{R}^{n_2}$. For any point $\rho_0 \in \operatorname{supp}(\mu_2)$, by the geometric control condition (GCC), we know that there exists a unique general bicharacteristic $s\mapsto\gamma(s)$ such that $\gamma(0) = \rho_0$. Moreover, there exists $\epsilon > 0$, sufficiently small, such that $\gamma((-2\epsilon, 2\epsilon)) \subset \pi^{-1}((0,T)\times\omega)$. Since μ_2 is invariant along the flow associated with M, *i.e.* $\frac{d}{ds}(M^*\mu_2 M) = 0$, we obtain that for any $t_0 \in (0, 2\varepsilon)$, we have

$$\mu_2(0) = M(t_0)^*\mu_2(t_0)M(t_0).$$

Noticing that $\mathrm{supp}(\mu_2)(n_2,n_2)\cap\pi^{-1}((0,T)\times\omega)=\emptyset$ (which also implies that $\mu_2(t_0)e_{n_2} = 0$ by an already developed argument), we obtain that

$$M(-t_0)^*\mu_2(0)M(-t_0)e_{n_2}=\mu_2(t_0)e_{n_2}=0.$$

Hence, $\mu_2(0)M(-t_0)e_{n_2}=0$. Moreover, considering $n-1$ times $t_1,\dots t_{n-1}$ such that $t_0<t_1<\ldots<t_{n-1}<\varepsilon$, the same argument leads to

$$\begin{cases} \mu_2(0)M(-t_0)e_{n_2} & =0,\\ \mu_2(0)M(-t_1)e_{n_2} & =0,\\ \mu_2(0)M(-t_2)e_{n_2} & =0,\\ \vdots & \\ \mu_2(0)M(-t_{n-1})e_{n_2} & =0. \end{cases} \tag{4.38}$$

From the expression of M, we obtain that $\{M(-t_i)e_{n_2}\}_{i\in[|0,n-1|]}$ is a basis of $\mathbb{R}^n$ (its determinant is proportional to the Vandermonde determinant $\prod_{i<j}(-t_i+t_j)$). Hence, (4.38) implies that $\mu_2(0)=0$. According to the arbitrary choice of $\rho_0\in\mathrm{supp}(\mu_2)$, we are able to conclude that $\mathrm{supp}(\mu_2)=\emptyset$, *i.e.* $\mu_2\equiv 0$. Then, we conclude that the relaxed observability inequality (4.29) holds for all the solutions of System (4.28). □

4.1.2. *Step 2: analysis on the invisible solutions.* We first define for any $T>0$ the set of invisible solutions from $]0,T[\times\omega$

$$\begin{aligned}\mathscr{N}_{n_2}(T)=\{\mathscr{W}=(w_1^{1,0},w_1^{2,0},\cdots,w_{n_2}^{2,0},w_1^{1,1},w_1^{2,1},\cdots,w_{n_2}^{2,1})^t\in\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}\\ \text{such that the associated solution of System (4.28)}\\ \text{satisfies } \frac{\alpha_s d_1^{n_2+1-s}}{(d_1-d_2)^{n_2+1-s}}w_1^1(x,t)+w_{n_2}^2(x,t)=0,\forall(x,t)\in(0,T)\times\omega\}.\end{aligned}$$

With the relaxed observability inequality of (4.29), we only need to prove the following key lemma:

Lemma 4.14. $\mathscr{N}_{n_2}(T)=\{0\}$.

Proof of Lemma 4.14. According to the relaxed observability inequality (4.29), for $\mathscr{W}\in\mathscr{N}_{n_2}(T)$, we obtain that

$$||W(0)||^2_{\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}}\le C||W(0)||^2_{\mathscr{L}_{-1}^{n_2+1}\times\mathscr{L}_{-2}^{n_2+1}}. \tag{4.39}$$

We know that $\mathcal{N}_{n_2}(T)$ is a closed subspace of $\mathscr{L}_0^{n_2+1}\times\mathscr{L}_{-1}^{n_2+1}$. By the compact embedding $L^2(\Omega)\times H^{-1}(\Omega)\hookrightarrow H^{-1}(\Omega)\times H^{-2}(\Omega)$, we know that $\mathcal{N}_{n_2}(T)$ has a finite dimension. Then, we define the operator $\mathscr{A}_{n_2}$ to be the generator associated with System (4.28). We know that the solution $(w_1^1,w_1^2,\cdots,w_{n_2}^2,D_tw_1^1,D_tw_1^2,\cdots,D_tw_2^{n_2})^t$ can be written as

$$\begin{pmatrix} w_1^1\\ w_1^2\\ \vdots\\ w_{n_2}^2\\ D_tw_1^1\\ D_tw_1^2\\ \vdots\\ D_tw_{n_2}^2\end{pmatrix}=e^{-t\mathscr{A}_{n_2}}\mathscr{W}.$$

It suffices to prove a unique continuation property for eigenfunctions of the operator $\mathscr{A}_{n_2}$. Let us take $\Phi = (\Phi^0, \Phi^1) = (\phi_1^0, \cdots, \phi_{n_2+1}^0, \phi_1^1, \cdots, \phi_{n_2+1}^1) \in \mathscr{N}_{n_2}(T)$, satisfying

$$(4.40) \qquad \begin{cases} \mathscr{A}_{n_2}\Phi & = \lambda\Phi, \\ \left(\frac{d_1^2}{(d_1-d_2)^2}\phi_1^0 + \phi_{n_2+1}^0\right)|_\omega & = 0. \end{cases}$$

Then, it is equivalent to a the system

$$(4.41) \qquad \begin{cases} (-D\Delta_D + A^*)\varphi & = \lambda^2\varphi, \\ \hat{b}^*\varphi|_\omega & = 0. \end{cases}$$

Indeed, as explained in Remark 3.13, Φ and φ verify the relation $\varphi = \mathcal{S}'(\lambda, \Delta)\Phi$ (where we replace formally D_t by λ). The study of (4.41) is totally similar to the one of (3.57): using the analyticity, we know that $\hat{b}^*\varphi \equiv 0$. Then, we obtain that $\hat{b}^*(-D\Delta_D + A^*)^k\varphi = 0$, for any $k \in \mathbb{N}$, *i.e.*$\varphi \in Ker(\mathcal{K}^*) = \{0\}$, so that $\varphi \equiv 0$, which concludes our proof. ☐

4.2. **Reformulation of the system in the general case.** According to Proposition 1.8, we already know that the operator Kalman rank condition is necessary for the exact controllability of System (1.1). In this section, provided with the operator Kalman rank condition $Ker(\mathcal{K}^*) = \{0\}$, we plan to give a reformulation of System (1.1).

As a consequence of Proposition 1.6, we know that (A_2, B) satisfies Kalman rank condition. Therefore, applying Theorem 3.1, there exists an invertible matrix P such that we reformulate System (1.1) into the following system

(4.42)
$$\begin{cases} \Box_1\tilde{u}_1^1 + \sum_{j=1}^{n_2}\tilde{\alpha}_j\tilde{u}_j^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \Box_2\tilde{u}_1^2 + \tilde{u}_2^2 & = 0 & \text{in } (0,T)\times\Omega, \\ \vdots & & \\ \Box_2\tilde{u}_{n_2-1}^2 + \tilde{u}_{n_2}^2 = 0 \text{ in } (0,T)\times\Omega, & & \\ \Box_2\tilde{u}_{n_2}^2 - \sum_{j=1}^{n_2} a_{n_2+1-j}\tilde{u}_j^2 & = f\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ \tilde{u}_1^1 = 0, \tilde{u}_1^2 = \cdots \tilde{u}_{n_2}^2 & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\tilde{u}_1^1, \tilde{u}_1^2, \cdots, \tilde{u}_{n_2}^2)|_{t=0} & = (\tilde{u}_1^{1,0}, \tilde{u}_1^{2,0}, \cdots, \tilde{u}_{n_2}^{2,0}) & \text{in } \Omega, \\ (\partial_t\tilde{u}_1^1, \partial_t\tilde{u}_1^2, \cdots, \partial_t\tilde{u}_{n_2}^2)|_{t=0} & = (\tilde{u}_1^{1,1}, \tilde{u}_1^{2,1}, \cdots, \tilde{u}_{n_2}^{2,1}) & \text{in } \Omega, \end{cases}$$

where $\tilde{u}_1^1 = u_1^1$, $\tilde{U}_2 = PU_2$ and $(\tilde{\alpha}_1, \cdots, \tilde{\alpha}_n) = (\alpha_1, \cdots, \alpha_n)P^{-1}$. Define $s = \max\{1 \leqslant j \leqslant n_2; \tilde{\alpha}_j \neq 0\}$. From Proposition 4.1, the appropriate state space for (4.42) is $\mathcal{H}_1^s \times \mathcal{H}_0^s$. Moreover, by Theorem 4.8, under our hypotheses, we have exact controllability of System (4.42) in the state space $\mathcal{H}_1^s \times \mathcal{H}_0^s$. This immediately leads to the conclusion of Theorem 1.16.

5. Some comments

As we can see, the system (1.2) is only an example of a more general system as follows:

$$(5.1) \qquad \begin{cases} (\partial_t^2 - D\Delta_D)U + AU & = \hat{b}f\mathbf{1}_{(0,T)}(t)\mathbf{1}_\omega(x) & \text{in } (0,T)\times\Omega, \\ U & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (U, \partial_t U)|_{t=0} & = (U^0, U^1) & \text{in } \Omega, \end{cases}$$

with here

$$D = \begin{pmatrix} d_1 Id_{n_1} & 0 \\ 0 & d_2 Id_{n_2} \end{pmatrix}_{n\times n}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n},$$
$$\hat{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{n\times m}, \text{ and } f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}_{m\times 1} \tag{5.2}$$

where $n = n_1 + n_2$ and $f_j \in L^2((0,T)\times\omega), j = 1, 2, \cdots, m$. In this very general system (5.1), there are three different kinds of effective parts acting on the controllability problem, that is, control functions and two different types of coupling.

The first part is obviously the control functions. The more control functions we have, the more sophisticated structure we demand for the coupled matrix to obtain the controllability. It is very related to the Brunovský Normal Form and when we consider more than one control function, the standard Brunovský Normal Form has more than one block in the coupling matrix, which increases the complicity of the calculation to obtain an explicit formula of the compatibility conditions (as we have seen, for instance, in (1.8)). However, when we deal with the case with more than one control functions, we usually rely on the Brunovský Normal Form to put the coupling matrix into the standard form and then, deal with the problem block by block. This means that we first need to establish the result with only one block, *i.e.* with only one control function. In the system (1.2), we choose that $\tilde{b}$ only acts on the second part of the system. The reason is that if we give both parts the effective control function, we cannot observe the influence of the coupling term because of the regularity.

The second part we considered is the coupling with the same speed, which corresponds to A_{11} and A_{22}, and on the other hand, the third part is the coupling effects of the different speeds, which corresponds to A_{12} and A_{21}. As we can see in the proof of the Theorem 1.16, coupling with same speed, we are able to observe a phenomena of regularity increase by one with successive solutions. While we can prove that the regularity gap between two coupled solutions with different speeds is two (one can see in Subsection 2.2). This difference gives us the motivation to consider that the simplest example of coupled wave system containing the two different coupling effects, *i.e.* the system (1.2). We try to use this example to analyse the different influence of these two types of coupling terms. When one introduces the fully coupling matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{n\times n}$, it is complicated to analyse the two different types of coupling. Because they are combined too closely, it is difficult to separate them. From a technical point of view, it seems very hard to derive an appropriate normal form similar to Brunovský form to obtain the compatibility conditions and the appropriate state space.

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Appendix A. On the operator Kalman rank condition

Proof of Proposition 1.6. Let $\lambda \in \sigma(-\Delta_D)$ and $\mathcal{K}(\lambda) = [(\lambda D + A)|\hat{b}] \in \mathcal{M}_n(\mathbb{R})$ (remind that $\hat{b} = {}^t(0, b) \in \mathbb{R}^n$). Firstly, we compute the form of the matrix $\mathcal{K}(\lambda)$ by induction.

$$\mathcal{K}(\lambda) = \begin{pmatrix} S_{n-1}(\lambda) & \cdots & S_j(\lambda) & \cdots & A_1 b & 0 \\ (d_2\lambda + A_2)^{n-1} b & \cdots & (d_2\lambda + A_2)^j b & \cdots & (d_2\lambda + A_2) b & b \end{pmatrix}. \tag{A.1}$$

The general term $S_j(\lambda)$, $1 \le j \le n-1$ is defined by

$$S_j(\lambda) = A_1 \left(\sum_{k=0}^{j-1} d_1^k \lambda^k (d_2\lambda + A_2)^{j-1-k} \right) b. \tag{A.2}$$

Since the rank of a matrix is invariant under elementary operations on the columns (that we will shorten in column transformation in what follows), it is easy to see that $rank(\mathcal{K}(\lambda)) = rank(\tilde{\mathcal{K}}(\lambda))$, where

$$\tilde{\mathcal{K}}(\lambda) = \begin{pmatrix} \tilde{S}_{n-1}(\lambda) & \cdots & \tilde{S}_j(\lambda) & \cdots & A_1 b & 0 \\ A_2^{n-1} b & \cdots & A_2^j b & \cdots & A_2 b & b \end{pmatrix}, \tag{A.3}$$

with

$$\tilde{S}_j(\lambda) = A_1 \left(\sum_{k=0}^{j-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b. \tag{A.4}$$

Let us first prove the necessity of the conditions. Suppose that $n_1 > 1$ and let us prove that the Kalman matrix $K(\lambda)$ is not of full rank. We take the n_1-th column of the matrix $\tilde{\mathcal{K}}(\lambda)$, *i.e.*

$$\begin{pmatrix} \tilde{S}_{n_2}(\lambda) \\ A_2^{n_2} b \end{pmatrix}.$$

Let $\chi(X) = X^{n_2} + \sum_{j=0}^{n_2-1} a_j X^j$ be the characteristic polynomial of the matrix A_2. By the Cayley-Hamilton Theorem, $A_2^{n_2} = -\sum_{j=0}^{n_2-1} a_j A_2^j$.

By using an adequate column transformation, we can put the n_1-th column into the form

$$\begin{pmatrix} T_{n_2}(\lambda) \\ 0 \end{pmatrix}, \tag{A.5}$$

where $T_{n_2}(\lambda) = \tilde{S}_{n_2}(\lambda) + \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda)$. By (A.4),

$$\begin{aligned} \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda) &= \sum_{j=1}^{n_2-1} a_j A_1 \left(\sum_{k=0}^{j-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b \\ &= A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_j (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b. \end{aligned}$$

Using the expression of $\tilde{S}_{n_2}(\lambda)$ given in (A.4), we obtain that

$$\begin{aligned} T_{n_2}(\lambda) &= \tilde{S}_{n_2}(\lambda) + \sum_{j=1}^{n_2-1} a_j \tilde{S}_j(\lambda) \\ &= A_1 \left(\sum_{k=0}^{n_2-1} (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b + A_1 \left(\sum_{k=0}^{n_2-2} \sum_{j=k+1}^{n_2-1} a_j (d_1 - d_2)^k \lambda^k A_2^{j-1-k} \right) b \\ &= A_1 \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \left(A_2^{n_2-1-k} + \sum_{j=k+1}^{n_2-1} a_j A_2^{j-1-k} \right) + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} \right) b, \end{aligned}$$

i.e.

$$T_{n_2}(\lambda) = A_1 \left(\sum_{k=0}^{n_2-2} (d_1 - d_2)^k \lambda^k \sum_{j=k+1}^{n_2} a_j A_2^{j-1-k} + (d_1 - d_2)^{n_2-1} \lambda^{n_2-1} \right) b. \tag{A.6}$$

Here and hereafter, we use the notation $a_{n_2} = 1$ in order to obtain a clean from. Now, we take the $(n_1 - 1)-$th column of the matrix $\tilde{\mathcal{K}}(\lambda)$, *i.e.*

$$\begin{pmatrix} \tilde{S}_{n_2+1}(\lambda) \\ A_2^{n_2+1} b \end{pmatrix}$$

Again using the characteristic polynomial of the matrix A_2, we obtain that

$$\begin{aligned} A_2^{n_2+1} &= -A_2 \sum_{j=0}^{n_2-1} a_j A_2^j \\ &= -\sum_{j=0}^{n_2-2} a_j A_2^{j+1} - a_{n_2-1} A_2^{n_2} \\ &= -\sum_{j=0}^{n_2-2} a_j A_2^{j+1} + a_{n_2-1} \sum_{j=0}^{n_2-1} a_j A_2^j \\ &= \sum_{j=1}^{n_2-1} (a_j a_{n_2-1} - a_{j-1}) A_2^j + a_{n_2-1} a_0. \end{aligned}$$

By applying an adequate column transformation, we can put the $(n_1 - 1)$-th column into the form:

$$\begin{pmatrix} T_{n_2+1}(\lambda) \\ 0 \end{pmatrix},$$

where $T_{n_2+1}(\lambda)$ satisfies

$$
\begin{aligned}
T_{n_2+1}(\lambda) &= \tilde{S}_{n_2+1}(\lambda) - \sum_{j=1}^{n_2-1}(a_j a_{n_2-1} - a_{j-1})\tilde{S}_j(\lambda) \\
&= A_1\left(\sum_{k=0}^{n_2}(d_1-d_2)^k\lambda^k A_2^{n_2-k}\right) b \\
&- \sum_{j=1}^{n_2-1}(a_j a_{n_2-1} - a_{j-1})A_1\left(\sum_{k=0}^{j-1}(d_1-d_2)^k\lambda^k A_2^{j-1-k}\right) b \\
&= A_1\left(\sum_{k=0}^{n_2}(d_1-d_2)^k\lambda^k A_2^{n_2-k}\right) b \\
&- A_1\left(\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1}(a_j a_{n_2-1} - a_{j-1})(d_1-d_2)^k\lambda^k A_2^{j-1-k}\right) b \\
&= A_1\left(\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^k A_2^{n_2-k} + (d_1-d_2)^{n_2}\lambda^{n_2} + (d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_2\right) b \\
&+ A_1\left(\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1} a_{j-1}(d_1-d_2)^k\lambda^k A_2^{j-1-k}\right. \\
&\left. -a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1} a_j(d_1-d_2)^k\lambda^k A_2^{j-1-k}\right) b.
\end{aligned}
$$

Now consider the sum

$$
\begin{aligned}
&\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^k A_2^{n_2-k} + \sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1} a_{j-1}(d_1-d_2)^k\lambda^k A_2^{j-1-k} \\
&= \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(A_2^{n_2-k} + \sum_{j=k+1}^{n_2-1} a_{j-1}A_2^{j-1-k}\right) + A_2^{n_2} + \sum_{j=1}^{n_2-1} a_{j-1}A_2^{j-1} \\
&= \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1} a_{j-1}A_2^{j-1-k}\right) - \sum_{j=0}^{n_2-1} a_j A_2^j \\
&+ \sum_{j=1}^{n_2-1} a_{j-1}A_2^{j-1} - \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k a_{n_2-1}A_2^{n_2-1-k}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right) - \sum_{j=0}^{n_2-1}a_jA_2^j \\
&+ \sum_{j=1}^{n_2-1}a_{j-1}A_2^{j-1} - \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k a_{n_2-1}A_2^{n_2-1-k} \\
&= \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right) - a_{n_2-1}A_2^{n_2-1} \\
&-a_{n_2-1}\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^kA_2^{n_2-1-k} \\
&= \sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\left(\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right) - a_{n_2-1}\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^kA_2^{n_2-1-k}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
T_{n_2+1}(\lambda) &= A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k}\right)b \\
&+ A_1\Bigg(-a_{n_2-1}\sum_{k=0}^{n_2-2}(d_1-d_2)^k\lambda^kA_2^{n_2-1-k} + (d_1-d_2)^{n_2}\lambda^{n_2} \\
&+(d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_2\Bigg)b + A_1\left(-a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2-1}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)b \\
&= A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k+1}^{n_2+1}a_{j-1}A_2^{j-1-k} + (d_1-d_2)^{n_2}\lambda^{n_2}\right)b \\
&+A_1\left((d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_2 - a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)b.
\end{aligned}
$$

Then, we aim to find a connection between the terms $T_{n_2+1}(\lambda)$ and $T_{n_2}(\lambda)$. By calculation, we obtain

$$
\begin{aligned}
(d_1-d_2)\lambda T_{n_2}(\lambda) &= A_1\left(\sum_{k=0}^{n_2-2}(d_1-d_2)^{k+1}\lambda^{k+1}\sum_{j=k+1}^{n_2}a_jA_2^{j-1-k} + (d_1-d_2)^{n_2}\lambda^{n_2}\right)B \\
&= A_1\left(\sum_{k=1}^{n_2-2}(d_1-d_2)^k\lambda^k\sum_{j=k}^{n_2}a_jA_2^{j-k} + (d_1-d_2)^{n_2}\lambda^{n_2}\right)B \\
&+ (d_1-d_2)^{n_2-1}\lambda^{n_2-1}A_1A_2B \\
&= T_{n_2+1}(\lambda) + A_1\left(a_{n_2-1}\sum_{k=0}^{n_2-2}\sum_{j=k+1}^{n_2}a_j(d_1-d_2)^k\lambda^kA_2^{j-1-k}\right)B \\
&= T_{n_2+1}(\lambda) + a_{n_2-1}T_{n_2}(\lambda).
\end{aligned}
$$

Hence, we know that $T_{n_2+1}(\lambda) = ((d_1 - d_2)\lambda - a_{n_2-1})\, T_{n_2}(\lambda)$, which means that the two columns

$$\begin{pmatrix} T_{n_2}(\lambda) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T_{n_2+1}(\lambda) \\ 0 \end{pmatrix}$$

are linearly dependent. This means that

$$\begin{pmatrix} \tilde{S}_{n_2}(\lambda) \\ A_2^{n_2} b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{S}_{n_2+1}(\lambda) \\ A_2^{n_2+1} b \end{pmatrix}$$

are linearly dependent. By the expression of $\tilde{\mathcal{K}}(\lambda)$ given in (A.3) and the definition of $\tilde{S}_j$ given in (A.4), we deduce that all the j-th columns of $\tilde{\mathcal{K}}(\lambda)$, for $j \leqslant n_1$, are linearly dependent. We deduce that $\tilde{K}(\lambda)$ is of rank less that $n - n_1 + 1 = n_2 + 1$. This is in contradiction with the fact that $\tilde{\mathcal{K}}(\lambda) \in \mathcal{M}_n(\mathbb{R})$ is of full rank $n = n_1 + n_2 > n_2 + 1$ since we assumed that $n_1 > 1$. So we deduce that $n_1 = 1$.

Concerning the two other conditions, remark that the first column of $\tilde{K}(\lambda)$ can be changed by a previously introduced column transformation into (A.5), where $T_{n_2}(\lambda)$ verifies (A.6). We deduce that the rank of $K(\lambda)$ is equal to the rank of the matrix

$$\begin{pmatrix} T_{n_2}(\lambda) & \tilde{S}_{n_2-1}(\lambda) & \cdots & \tilde{S}_j(\lambda) & \cdots & A_1 b & 0 \\ 0 & A_2^{n_2-1} & \cdots & A_2^{j-1} b & \cdots & A_2 b & b \end{pmatrix}.$$

This matrix is of full rank $n = n_2 + 1$ (if and) only if $T_{n_2}(\lambda) \neq 0$ (which gives (1.4) thanks to (A.6)) and

$$\begin{pmatrix} A_2^{n_2-1} & \cdots & A_2^{j-1} b & \cdots & A_2 b & b \end{pmatrix} \in \mathcal{M}_{n_2,n_2}(\mathbb{R})$$

is of full rank n_2, which is exactly meaning that (A_2, b) verifies the usual Kalman rank condition.

The sufficiency of the three conditions given in Proposition 1.6 is also straightforward, by the same arguments.

■

Appendix B. Proof of Lemma 4.7

We first look at $u_{n_2}^2$. Since $j + k \leq n_2$, we know for $j = n_2$, the conclusion is trivial. For $1 \leq j \leq n_2 - 1$, we argue by induction. When $k = 0$, the conclusion holds for sure. Assume that

$$D_t^{2k-2} u_j^2 = \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2 \Delta)^l u_{j+k-1-l}^2. \tag{B.1}$$

Then for $D_t^{2k} u_j^2$, we know that

$$\begin{aligned} D_t^{2k} u_j^2 &= D_t^2 D_t^{2k-2} u_j^2 \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2 \Delta)^l D_t^2 u_{j+k-1-l}^2. \end{aligned} \tag{B.2}$$

Using the equation $D_t^2 u^2_{j+k-1-l} = -d_2\Delta u^2_{j+k-1-l} + u^2_{j+k-l}$, we obtain that

$$
\begin{aligned}
D_t^{2k} u_j^2 &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^{l+1} u^2_{j+k-1-l} + \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^{l} u^2_{j+k-l} \\
&= \sum_{l=1}^{k} \binom{k-1}{l-1} (-d_2\Delta)^{l} u^2_{j+k-l} + \sum_{l=0}^{k-1} \binom{k-1}{l} (-d_2\Delta)^{l} u^2_{j+k-l} \qquad \text{(B.3)} \\
&= \sum_{l=1}^{k-1} \left(\binom{k-1}{l-1} + \binom{k-1}{l}\right)(-d_2\Delta)^{l} u^2_{j+k-l} + (-d_2\Delta)^k u_j^2 + u^2_{j+k}.
\end{aligned}
$$

Since $(\binom{k-1}{l-1} + \binom{k-1}{l}) = \binom{k}{l}$, we obtain the conclusion.

■

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