Optimal filtration for the approximation of boundary controls for the one-dimensional wave equation using finite-difference method

Pierre Lissy *　　Ionel Rovenţa†

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Abstract

We consider a finite-difference semi-discrete scheme for the approximation of boundary controls for the one-dimensional wave equation. The high frequency numerical spurious oscillations lead to a loss of the uniform (with respect to the mesh-size) controllability property of the semi-discrete model in the natural setting. We prove that, by filtering the high frequencies of the initial data in an optimal range, we restore the uniform controllability property. Moreover, we obtain a relation between the range of filtration and the minimal time of control needed to ensure the uniform controllability. The proof is based on the moment method.

Keywords: wave equation, control approximation, moment problem, biorthogonal families.

Mathematical subject codes: 93B05, 30E05, 65M06.

1 Introduction

This paper is concerned with the problem of the null boundary controllability (which is equivalent to the exact boundary controllability) for a finite-difference semi-discrete scheme of the one-dimensional wave equation on the space interval $(0,1)$. It is well-known that for the wave equation, given $T > 2$ and $(u^0, u^1) \in H_0^1((0,1),\mathbb{C}) \times L^2((0,1),\mathbb{C})$, there exists a control function $v \in H^1((0,T),\mathbb{C}) \subset C^0([0,T],\mathbb{C})$ such that the solution of the wave equation

$$\begin{cases} u''(t,x) - u_{xx}(t,x) = 0 & t \in (0,T),\ x \in (0,1), \\ u(t,0) = 0 & t \in (0,T), \\ u(t,1) = v(t) & t \in (0,T), \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1.1}$$

satisfies

$$u(T,x) = u'(T,x) = 0 \qquad (x \in (0,1)).$$

*CEREMADE, Université Paris-Dauphine & CNRS UMR 7534, PSL Research University, 75016 Paris, France. E-mail: `lissy@ceremade.dauphine.fr`.

†Department of Mathematics, University of Craiova, 200585, Romania. E-mail: `ionelroventa@yahoo.com`.

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. For $T > 0$, we consider the following semi-discrete space approximation of the wave equation by the explicit finite-differences method:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} = 0 & 1 \leq j \leq N,\ t > 0, \\ u_0(t) = 0 & t \in (0,T), \\ u_{N+1}(t) = v_h(t) & t \in (0,T), \\ u_j(0) = u_j^0, \quad u_j'(0) = u_j^1 & 1 \leq j \leq N. \end{cases} \tag{1.2}$$

Given $T \geq 2$, $h > 0$ and $((u_j^0, u_j^1))_{1\leq j\leq N} \in \mathbb{C}^{2N}$, we study the existence of a control function $v_h \in C^0([0,T])$ such that the solution of the equation (1.2) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

More precisely, our aim is to study the existence of a uniformly bounded sequence of controls $(v_h)_{h>0}$ with respect to the mesh size h, by using the moment method. Remind that the discretization of the wave equation (with finite-differences schemes but also finite-element schemes) is known for a long time (see notably [10] and [11]) to lead to high-frequency spurious solutions generated by the discretization process that make the discrete controls diverge when the mesh-size goes to zero. Basically, this difficulty can be overcame by at least four strategies:

- A Tychonoff regularization of the HUM cost functional, which consist in constructing a fictitious control in the whole domain vanishing in the limit $h \to 0$, that we would not discuss here (see notably [11] and the survey [22]).
- An appropriate filtering technique, which consists in relaxing the control requirement by controlling only the low-frequency (of order $|n| << N$) part of the solution in order to eliminate the short wave length components of the solutions of the discrete system of the discretized problem.
- A change of the numerical scheme, notably using mixed finite elements (see e.g. [2]), a vanishing viscosity parameter depending on h (see e.g. [16]), or other type of finite-difference schemes (see e.g. [18]).
- An approximation of discrete controls (as in [3]), which does not lead exactly the discrete solution to zero for fixed discretization parameter $h > 0$, but converges to an exact control of the continuous problem as $h \to 0$.

Let us explain into more details different filtering techniques used in the literature. In [13] (see also [5, Section 2.4]), the author proved a uniform observability result by eliminating short wave length components of the *whole solution* for large enough times (depending on the filtration).

Here the approach we chose is the one of [15], where we directly cut frequencies of the initial condition. This approach is very different because even if the initial condition is filtered, the control will *excite all frequencies*, and then the study of the uniform controllability is much more intricate from a theoretical point of view. However, the authors believe that this approach is more natural and convenient from a practical point of view. In [15] it was proved that if the initial data are given by

$$\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1\leq j\leq N} = \sum_{1\leq |n|\leq M} a_{hn}^0 \Phi_h^n,$$

where Φ_h^n is defined in (2.4) and with $M = \sqrt{N}$, then there exists a sequence of bounded controls $(v_h)_{h>0}$ for (1.2) provided that the initial condition verifies some conditions on its Fourier coefficients and that the time is large enough (but no quantitative estimate of this minimal time is given).

In the present paper, we drastically improve the results of [15] by filtering in an optimal way the initial condition: here M can be chosen to be any N^α with $\alpha \in (0,1)$, or even $M = \delta N$ with $\delta \in (0,1)$. Moreover, we obtain a precise estimate on the minimal time needed that turns out to be optimal as soon as we filter enough frequencies, and contrary to the result of [15], we can choose any initial condition in $H_0^1((0,1),\mathbb{C}) \times L^2((0,1),\mathbb{C})$. Moreover, the controls can be chosen to be continuous (and not only $L^2(0,T)$) in the present work. Note that the optimal space of controlled initial data with controls in $L^2(0,T)$ is $L^2((0,1),\mathbb{C}) \times H^{-1}((0,1),\mathbb{C})$, however our proof requires to have a little bit more regularity on the initial data.

We emphasize that beyond the theoretical interest of our result, it is likely that it is of interest to try to allow filtrations which contain as many modes as possible, in order to improve the precision of the approximation.

For more general uniform controllability results, by using filtered spaces, with different schemes or non-uniform meshes and covering some higher-dimensional situations, the interested reader is referred to [4, 6, 7, 8, 17, 22]. Let us also mention that our approach is based on a *discrete approach* but there also exists a *continuous approach*, see notably [3, 5] and the references therein.

Before stating our main theorem, we introduce some notations. In all what follows, an initial condition $(u^0, u^1) \in H_0^1((0,1),\mathbb{C}) \times L^2((0,1),\mathbb{C})$ will be decomposed as

$$(u^0, u^1) = \sum_{n \in \mathbb{Z}^*} a_n \Phi^n(x), \tag{1.3}$$

with $(a_n)_{n\in\mathbb{Z}^*} \in l^2(\mathbb{Z}^*, \mathbb{C})$, where by definition

$$\Phi^n(x) := \begin{pmatrix} \frac{1}{in\pi} \sin(n\pi x) \\ -\sin(n\pi x) \end{pmatrix},$$

which correspond to the eigenfunctions of the elliptic problem associated to the homogeneous version of (1.1). We will also consider the following filtered version of the initial condition (u^0, u^1) given by

$$(u_M^0, u_M^1) = \sum_{|n| \leqslant M, n \neq 0} a_n \Phi^n(x), \tag{1.4}$$

where $M \in \mathbb{N}^*$ (M will depend on the mesh size h in what follows).

In general, for the semi-discrete equation (1.2), we will consider the following discretization of the initial condition (u^0, u^1) given by

$$U_h^0 = \sum_{|n| \leqslant M, n \neq 0} a_n \tilde{\Phi}_h^n, \tag{1.5}$$

with $M \leqslant N$, where

$$\tilde{\Phi}_h^n = \begin{pmatrix} \frac{1}{in\pi}\varphi_h^n \\ -\varphi_h^n \end{pmatrix} \qquad (1 \le |n| \le N),$$

and

$$(\varphi_h^n)_{1\le|n|\le N} = \begin{pmatrix} \sin(n\pi h) \\ \sin(2n\pi h) \\ \dots \\ \sin(Nn\pi h) \end{pmatrix} \in \mathbb{C}^N.$$

It is clear that if $M \to \infty$ as $N \to \infty$, then $U_h^0 \to (u^0, u^1)$. Let us emphasize that the $\tilde{\Phi}_h^n$ *does not correspond* to the eigenvalues of the elliptic problem associated to the homogeneous version of (1.2) (for detailed explanations, see [15, Page 758] and Section 2.1).

Our main result is the following.

Theorem 1.1. *Let $(u^0, u^1) \in H_0^1((0,1),\mathbb{C}) \times L^2((0,1),\mathbb{C})$ some initial condition that we decompose as in* (1.3)*. Let $f : \mathbb{N}^* \to \mathbb{N}^*$ be an increasing function (which corresponds to the filtering function of the modes) verifying $f(N) \leqslant N$ for every $N \in \mathbb{N}^*$ and $f(N) \to \infty$ as $N \to \infty$. We denote by*

$$\Gamma(f) := \limsup_{N\to\infty} \frac{f(N)}{N} \in [0,1]. \tag{1.6}$$

In addition, we assume that $\Gamma(f) < 1$ and we consider the filtered initial condition given by (1.4)*, with $M = f(N)$, and its discretized version given by* (1.5)*. Then, for any $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$, there exists a control $v_h \in C^0([0,T],\mathbb{C})$ bringing the solution of* (1.2) *(with initial condition U_h^0) to (0,0) such that the sequence $(v_h)_{h>0}$ is bounded in $C^0([0,T],\mathbb{C})$.*

For example, if $f(N) = \delta N$ with $\delta \in (0,1)$ then $\Gamma(f) = \delta$ and the minimal time that ensures the uniform controllability is $\frac{4}{1-\sin\left(\frac{\pi\delta}{2}\right)}$. Moreover, if $f(N) = o(N)$ (this is notably the case if $f(N) = N^\alpha$ with $\alpha \in (0,1)$), then $\Gamma(f) = 0$, and in this case, we obtain the uniform controllability for any time $T > 4$, which is twice the minimal time needed to control the corresponding continuous system (1.1). Note that this result is sharp, because it is well-known that one cannot choose $\Gamma(f) = 1$, in this case the controls might explode exponentially when h goes to 0 (see [15]).

On the other hand, our main result establish the area where the minimal time of uniform controllability worsens. This precise area is located in the range where the gap between the eigenvalues of the discrete problem becomes smaller than the gap between the corresponding eigenvalues of the continuous problem. In this range it appears spurious high frequency oscillations which gives bad approximations of the controls that can be observed numerically (see [22]).

Let us remind that as soon as the sequence $(v_h)_{h>0}$ is bounded, one can extract a subsequence converging weakly to some $v \in L^\infty(0,T)$ that will be a control for the continuous problem (1.1) (it can be easily deduced by using the same computations as in [15, Theorem 4.3]). Another consequence is that the HUM controls (i.e. the control of minimal L^2-norm) associated to the control problem (1.2) are also bounded in $L^2(0,T)$ as soon as the initial condition is filtrated as in Theorem 1.1.

The paper is organized as follows. In Section 1, we present some known results concerning the question of the numerical control of the $1-D$ wave equations and we present the precise scope of the paper. In Section 2, we give the proof of Theorem 1.1. More precisely, In Section 2.1, we begin with recalling some spectral properties of the semi-discretized problem and we set our moment problem. In Section 2.2, we give some crucial estimations on a product involving the eigenvalues associated to the discretized problem. In Section 2.3, we exhibit an adequate multiplier for the moment method and give some estimations on it. Finally, in Section 2.4, we end our reasoning by constructing our controls thanks to the Paley-Wiener Theorem and proving the properties of uniform controllability as stated in Theorem 1.1. In Section 3 we mention some final remarks and open results.

2 Proof of Theorem 1.1

2.1 Spectral properties and the moment problem

In this section we recall some well known facts about the spectral properties of our problem.

Let us consider the corresponding homogeneous adjoint problem:

$$\begin{cases} w_j''(t) - \frac{w_{j+1}(t)-2w_j(t)+w_{j-1}(t)}{h^2} = 0 & 1 \le j \le N,\ t > 0, \\ w_0(t) = 0 & t \in (0,T), \\ w_{N+1}(t) = 0 & t \in (0,T), \\ w_j'(0) = w_j^0, \quad w_j' = w_j^1 & 1 \le j \le N. \end{cases} \tag{2.1}$$

We define the matrix $A_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ as follows:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The adjoint problem (2.1) can be rewritten in a matricial form as follows:

$$\begin{cases} W''(t) + A_h W(t) = 0 & t > 0, \\ W(0) = W^0, \quad W'(0) = W^1, \end{cases} \tag{2.2}$$

where $W(t) = (w_1(t), ..., w_N(t))^T \in \mathbb{C}^N$ and the initial data is $\begin{pmatrix} W^0 \\ W^1 \end{pmatrix} = \begin{pmatrix} (w_j^0)_{1\le j\le N} \\ (w_j^1)_{1\le j\le N} \end{pmatrix} \in \mathbb{C}^{2N}$.

Now, if we set $Z(t) = \begin{pmatrix} W(t) \\ W'(t) \end{pmatrix}$ and $Z^0 = \begin{pmatrix} W^0 \\ W^1 \end{pmatrix}$, then (2.2) has the following equivalent vectorial form

$$\begin{cases} Z'(t) + \mathcal{A}_h Z(t) = 0 \\ Z(0) = Z^0, \end{cases}$$

where the operator $\mathcal{A}_h$ is given by $\mathcal{A}_h = \begin{pmatrix} 0 & -I_N \\ A_h & 0 \end{pmatrix}$ and I_N is the identity matrix of size N.

The eigenvalues of $\mathcal{A}_h$ are given by the family $(i\,\lambda_n)_{1\leq|n|\leq N}$, where

$$\lambda_n = \frac{2}{h}\sin\left(\frac{n\pi h}{2}\right), \qquad 1\leq |n|\leq N, \tag{2.3}$$

and the corresponding eigenvectors are

$$\Phi_h^n = \begin{pmatrix} \frac{1}{i\lambda_n}\varphi_h^n \\ -\varphi_h^n \end{pmatrix} \qquad (1\leq |n|\leq N), \tag{2.4}$$

where

$$(\varphi_h^n)_{1\leq|n|\leq N} = \begin{pmatrix} \sin(n\pi h) \\ \sin(2n\pi h) \\ \dots \\ \sin(n\pi h N) \end{pmatrix} \in \mathbb{C}^N$$

are the eigenvectors of A_h.

Note that $(\Phi_h^n)_{1\leq|n|\leq N}$ forms an orthonormal basis in $\mathbb{C}^{2N}$.

The next result gives a sufficient and necessary condition for the null-controllability of (1.2). Its proof follows immediately multiplying (1.2) by the solution of (2.2) and integrating by parts in time (for more details, see [15, Proposition 3.4]).

Lemma 2.1. *Given $T>0$, system* (1.2) *is null-controllable at time T if, and only if, for any initial data $U^0 = (u_j^0, u_j^1)_{1\leq j\leq N}\in\mathbb{C}^{2N}$, there exists $v_h\in C^0([0,T],\mathbb{C})$ which verifies*

$$\int_0^T v_h(t)\,\frac{\overline{w_N(t)}}{h}\,dt = h\sum_{1\leq j\leq N}(u_j^0\overline{w}_j^1 - u_j^1\overline{w}_j^0) \qquad \left(\forall\ (w_j^0, w_j^1)_{1\leq j\leq N}\in\mathbb{C}^{2N}\right),$$

where W is the solution of (2.2).

We are now able to transform our controllability problem into a moment problem (see [15, Proposition 3.5]).

Proposition 2.1. *Given $T>0$, system* (1.2) *is null-controllable at time T if, and only if, for any initial data $(u_j^0, u_j^1)_{1\leq j\leq N} = \sum_{1\leq|n|\leq N} a_n\Phi_h^n$ there exists $v_h\in C^0([0,T],\mathbb{C})$ which verifies*

$$\int_0^T v_h(t)\,e^{-i\lambda_n t}\,dt = \frac{(-1)^n h}{\sin(n\pi h)}a_n, \tag{2.5}$$

for every $1\leq|n|\leq N$.

Let us now give a precise description of the next steps.

Our aim is to construct and evaluate an explicit biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq|n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$. We shall do that in several steps (which will be done in the next two sections):

1. We construct an entire function P_m, with the property that $P_m(\lambda_n)=\delta_{mn}$.

2. We give an estimate of the product P_m on the real axis.

3. We construct a smart multiplier M_m with rapid decay on the real axis such that $P_m M_m$ is bounded on the real axis and $M_m(\lambda_m) = 1$.

4. The Fourier transform of the entire function $\psi_m(z) := P_m(z)M_m(z)$ gives the element θ_m of a biorthogonal sequence to the family $(e^{i\lambda_n t})_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$. Moreover, an estimate for the L^∞-norm of θ_m is also obtained thanks to an estimate on the L^1-norm of ψ_m.

Once we have a biorthogonal sequence $(\theta_m)_{1\leq |m|\leq N}$ to the family $(e^{i\lambda_n t})_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ we can construct a control thanks to the following formula (see [15, Page 759]):

$$v_h(t) = \sum_{1\leq |n|\leq N} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_n \frac{T}{2}} a_n(h)\theta_n \left(t - \frac{T}{2}\right),$$

where $a_n(h)$ is related to a_n by the following relations (see [15, Page 758]):

$$a_n(h) := \begin{cases} 0, & |n| > f(N), \\ \frac{1}{2}\left(\frac{\lambda_n}{n\pi} + 1\right) a_n + \frac{1}{2}\left(\frac{\lambda_n}{n\pi} - 1\right) a_{-n}, & |n| \leqslant f(N). \end{cases} \tag{2.6}$$

The method considered for the construction of a biorthogonal family was first introduced in [20] and in the context of boundary controllability problems in [9]. The main difficulties in our analysis is to obtain optimal estimates for the behavior of the Weierstrass product on the real axis and also to construct an adequate multiplier. These two difficulties will be treated in the next two subsections.

Remark 2.1. *If one approximates (u^0, u^1) by $U_h^0 = \sum_{1\leq |n|\leq N} a_n \Phi_h^n$ instead of (1.5), then the same result can be obtained by taking a control function v_h as above with a_n instead of $a_n(h)$ (see [15, Pages 756-758].*

2.2 The product estimates

In this section we define a Weierstrass product P_m, with the property that $P_m(\lambda_n) = \delta_{mn}$ and we obtain an optimal estimate (in the sense that the asymptotic behavior both when $x \to \infty$ and $h \to 0$ of the function φ introduced in (2.19) cannot be improved, otherwise we would be able to obtain results of uniform controllability for $T < 2$, which is of course impossible) of the product P_m on the real axis. In the sequel, $C > 0$ denotes an absolute constant (independent on n, N and h) which may vary from line to line. Let us emphasize that λ_n as defined in (2.3) depends on h.

For every $1 \leq |m| \leq N$, we define the function

$$P_m(z) = \prod_{\substack{1\leq |n|\leq N \\ n\neq m}} \left(\frac{z}{\lambda_n} - 1\right) \prod_{\substack{1\leq |n|\leq N \\ n\neq m}} \frac{\lambda_n}{\lambda_m - \lambda_n} := P_m^1(z) S_m \qquad (z \in \mathbb{C}), \tag{2.7}$$

where

$$P_m^1(z) = \prod_{\substack{1\leq |n|\leq N \\ n\neq m}} \left(\frac{z}{\lambda_n} - 1\right),$$

$$S_m = \prod_{\substack{1\le|n|\le N\\ n\neq m}} \frac{\lambda_n}{\lambda_m-\lambda_n}.$$

We start our estimates with the following technical result concerning the second term of the product P_m.

Lemma 2.2. *For every $1\le|m|\le N$, we have that*

$$|S_m| = \cos^2\frac{m\pi h}{2}. \tag{2.8}$$

Proof of Lemma 2.2. From the symmetry of the sequence $(\lambda_n)_{1\le|n|\le N}$ (*i.e.* $\lambda_{-n}=-\lambda_n$), it is sufficient to consider only the case $1\le m\le N$. We remark that

$$S_m = \prod_{\substack{1\le|n|\le N\\ n\neq m}} \left|\frac{\lambda_n}{\lambda_n-\lambda_m}\right| = \frac{1}{2}\prod_{\substack{1\le n\le N\\ n\neq m}} \left|\frac{\lambda_n^2}{\lambda_n^2-\lambda_m^2}\right| = \frac{1}{2}\prod_{\substack{1\le n\le N\\ n\neq m}} \frac{\sin^2\left(\frac{n\pi h}{2}\right)}{\left|\sin\left(\frac{(m-n)\pi h}{2}\right)\sin\left(\frac{(m+n)\pi h}{2}\right)\right|}. \tag{2.9}$$

We study this last product by splitting it into two parts. We have that

$$\prod_{\substack{1\le n\le N\\ n\neq m}} \frac{\sin\left(\frac{n\pi h}{2}\right)}{\left|\sin\left(\frac{(n-m)\pi h}{2}\right)\right|} = \frac{\prod\limits_{1\le n\le m-1}\sin\left(\frac{n\pi h}{2}\right)\prod\limits_{m+1\le n\le N}\sin\left(\frac{n\pi h}{2}\right)}{\prod\limits_{1\le n\le m-1}\sin\left(\frac{n\pi h}{2}\right)\prod\limits_{1\le n\le N-m}\sin\left(\frac{n\pi h}{2}\right)}$$

$$= \frac{\prod\limits_{k=m+1}^{N}\sin\left(\frac{k\pi h}{2}\right)}{\prod\limits_{k=1}^{N-m}\sin\left(\frac{k\pi h}{2}\right)} = \frac{\prod\limits_{k=m+1}^{N}\sin\left(\frac{k\pi h}{2}\right)}{\prod\limits_{k=m+1}^{N}\cos\left(\frac{k\pi h}{2}\right)},$$

and also

$$\prod_{\substack{1\le n\le N\\ n\neq m}} \frac{\sin\left(\frac{n\pi h}{2}\right)}{\sin\left(\frac{(n+m)\pi h}{2}\right)} = \frac{\sin\left(\frac{\pi h}{2}\right)\ldots\sin\left(\frac{N\pi h}{2}\right)}{\sin\left(\frac{m\pi h}{2}\right)}\;\frac{\sin\left(\frac{2m\pi h}{2}\right)}{\sin\left(\frac{(m+1)\pi h}{2}\right)\ldots\sin\left(\frac{(m+N)\pi h}{2}\right)}$$

$$= 2\cos\left(\frac{m\pi h}{2}\right)\frac{\prod\limits_{k=1}^{m}\sin\left(\frac{k\pi h}{2}\right)}{\prod\limits_{k=N+1}^{m+N}\sin\left(\frac{k\pi h}{2}\right)} = 2\cos^2\left(\frac{m\pi h}{2}\right)\frac{\prod\limits_{k=1}^{m}\sin\left(\frac{k\pi h}{2}\right)}{\prod\limits_{k=1}^{m}\cos\left(\frac{k\pi h}{2}\right)},$$

where for the last estimate we have used the fact that $\sin\left(\frac{k\pi h}{2}\right) = \cos\left(\frac{(N+1-k)\pi h}{2}\right)$.

From the last two relations together with (2.9) we obtain that

$$S_m = \cos^2\left(\frac{m\pi h}{2}\right)\frac{\prod\limits_{k=1}^{N}\sin\left(\frac{k\pi h}{2}\right)}{\prod\limits_{k=1}^{N}\cos\left(\frac{k\pi h}{2}\right)} = \cos^2\left(\frac{m\pi h}{2}\right),$$

and the proof is complete. ∎

In the next Lemma, we give an intermediate estimate that will be useful to estimate P_m^1. In what follows, for any $0 \le x < \frac{2}{h}$, we consider

$$n_x = \operatorname{argmin}_{1 \le n \le N-1} \{|x - \lambda_n|\}. \tag{2.10}$$

Lemma 2.3. *Let $x \in \mathbb{R}$. We have that*

$$P(x) := \prod_{\substack{1 \le n \le N \\ n \neq n_x}} \frac{\left|x^2 - \lambda_n^2\right|}{\lambda_n^2} \le \frac{C}{\cos^2 \frac{v\pi h}{2}} \qquad \left(|x| < \frac{2}{h}\right), \tag{2.11}$$

$$\prod_{n=1}^{N} \frac{x^2 - \lambda_n^2}{\lambda_n^2} = 1 \qquad \left(|x| = \frac{2}{h}\right), \tag{2.12}$$

$$\prod_{n=1}^{N} \frac{x^2 - \lambda_n^2}{\lambda_n^2} \le \exp\left(\frac{2}{h} \ln\left(\frac{xh}{2} + \sqrt{\frac{x^2h^2}{4} - 1}\right)\right) \qquad \left(|x| > \frac{2}{h}\right), \tag{2.13}$$

where $v \in \left[0, \frac{1}{h}\right)$ is defined such that $x = \frac{2}{h} \sin \frac{v\pi h}{2}$.

Proof of Lemma 2.3. We remark that

$$\left|P_m^1(x)\right| = \frac{|\lambda_m|}{|x - \lambda_m|} \prod_{1 \le n \le N} \frac{\left|x^2 - \lambda_n^2\right|}{\lambda_n^2}. \tag{2.14}$$

If $|x| = \frac{2}{h}$ then

$$\prod_{n=1}^{N} \frac{x^2 - \lambda_n^2}{\lambda_n^2} = \prod_{n=1}^{N} \frac{\cos^2\left(\frac{n\pi h}{2}\right)}{\sin^2\left(\frac{n\pi h}{2}\right)} = 1,$$

and (2.12) holds.

If $|x| > \frac{2}{h}$, let $r^2 = \frac{x^2h^2}{4} - 1$. We have that

$$\prod_{1 \le n \le N} \frac{x^2 - \lambda_n^2}{\lambda_n^2} = \prod_{n=1}^{N} \left(\frac{\cos^2\left(\frac{n\pi h}{2}\right) + r^2}{\sin^2\left(\frac{n\pi h}{2}\right)}\right) = \prod_{n=1}^{N} \frac{\cos^2\left(\frac{n\pi h}{2}\right)}{\sin^2\left(\frac{n\pi h}{2}\right)} \prod_{n=1}^{N} \left(1 + \frac{r^2}{\cos^2\left(\frac{n\pi h}{2}\right)}\right)$$
$$= \prod_{n=1}^{N} \left(1 + \frac{r^2}{\cos^2\left(\frac{n\pi h}{2}\right)}\right) \le \exp(I),$$

where

$$I := \int_0^{N+1} \ln\left(1 + \frac{r^2}{\cos^2\left(\frac{t\pi h}{2}\right)}\right) dt.$$

In the following, we are going to compute exactly the value of the integral I. If we consider the change of variable given by $u = \tan\left(\frac{t\pi h}{2}\right)$, we infer that

$$I = \frac{2}{\pi h} \int_0^{\infty} \frac{\ln\left(1 + r^2(1 + u^2)\right)}{1 + u^2} \, du = \frac{2}{\pi h} F(1),$$

where

$$F(\alpha) = \int_0^\infty \frac{\ln\left(1 + r^2(1+\alpha u^2)\right)}{1+u^2}\, du.$$

It is easy to prove that F is of class C^1 and its derivative F' is given by

$$F'(\alpha) = \int_0^\infty \frac{u^2 r^2}{(1+u^2)(1+r^2(1+\alpha u^2))}\, du = \frac{r^2}{\alpha r^2 - r^2 - 1}\int_0^\infty \frac{1}{1+u^2} - \frac{1+r^2}{1+r^2+\alpha u^2 r^2}\, du$$

$$= \frac{r^2}{\alpha r^2 - r^2 - 1}\left(\arctan u - \frac{1+r^2}{\alpha r^2}\frac{1}{\sqrt{\frac{1+r^2}{\alpha r^2}}}\arctan\left(\frac{u}{\sqrt{\frac{1+r^2}{\alpha r^2}}}\right)\right)\Bigg|_0^\infty$$

$$= \frac{r^2}{\alpha r^2 - r^2 - 1}\frac{\pi}{2}\left(1 - \sqrt{\frac{1+r^2}{\alpha r^2}}\right).$$

Hence, we obtain that there exists some constant $\mathcal{C} > 0$ such that

$$F(\alpha) = \pi \ln(\sqrt{\alpha r^2} + \sqrt{1+r^2}) + \mathcal{C}.$$

Since $F(0) = \frac{\pi}{2}\ln(1+r^2)$ we deduce that $\mathcal{C} = 0$. Thus, we have that

$$I = \frac{2}{\pi h}F(1) = \frac{2}{h}\ln(r + \sqrt{1+r^2}) = \frac{2}{h}\ln\left(\frac{xh}{2} + \sqrt{\frac{x^2h^2}{4} - 1}\right),$$

and inequality (2.13) holds.

Our next aim is to prove (2.11). Since $x \mapsto P(x)$ is an even function we study only the case $x \geq 0$. If $0 \leq x < \frac{2}{h}$, let $x = \frac{2}{h}\sin\frac{v\pi h}{2}$, where $v \in \left[0, \frac{1}{h}\right)$. We also consider

$$[v] = \operatorname{argmin}_{1\leq n\leq N-1}\{|v-n|\} \quad \text{and} \quad \{v\} = v - [v].$$

We remark that $|\{v\}| \leq \frac{1}{2}$. From (2.10) we deduce that $[v] = n_x$. We have that

$$P(x) = \prod_{n=1}^{n_x-1}\left(\frac{\sin^2\frac{v\pi h}{2}}{\sin^2\frac{n\pi h}{2}} - 1\right)\prod_{n=n_x+1}^{N}\left(1 - \frac{\sin^2\frac{v\pi h}{2}}{\sin^2\frac{n\pi h}{2}}\right)$$

$$= \prod_{n=1}^{n_x-1}\frac{\cos(n\pi h) - \cos(v\pi h)}{2\sin^2\frac{n\pi h}{2}}\prod_{n=n_x+1}^{N}\frac{\cos(v\pi h) - \cos(n\pi h)}{2\sin^2\frac{n\pi h}{2}}$$

$$= \prod_{n=1}^{n_x-1}\frac{\sin\frac{([v]+\{v\}-n)\pi h}{2}\sin\frac{([v]+\{v\}+n)\pi h}{2}}{\sin^2\frac{n\pi h}{2}}\prod_{n=n_x+1}^{N}\frac{\sin\frac{(n-[v]-\{v\})\pi h}{2}\sin\frac{([v]+\{v\}+n)\pi h}{2}}{\sin^2\frac{n\pi h}{2}}.$$

We deduce that

$$P(x) = \prod_{k=1}^{[v]-1} \sin\frac{(k+\{v\})\pi h}{2} \prod_{k=[v]+1}^{2[v]-1} \sin\frac{(k+\{v\})\pi h}{2}$$
$$\times \prod_{k=1}^{N-[v]} \sin\frac{(k-\{v\})\pi h}{2} \prod_{k=2[v]+1}^{N+[v]} \sin\frac{(k+\{v\})\pi h}{2} \prod_{\substack{1\le k\le N\\ k\neq [v]}} \frac{1}{\sin^2\frac{k\pi h}{2}}.$$

$$= \prod_{\substack{k=1\\ k\neq [v],\,2[v]}}^{N} \sin\frac{(k+\{v\})\pi h}{2} \prod_{\substack{k=N+1\\ k\neq 2[v]}}^{N+[v]} \sin\frac{(2N+2-k-\{v\})\pi h}{2} \prod_{k=1}^{N-[v]} \sin\frac{(k-\{v\})\pi h}{2} \prod_{\substack{1\le k\le N\\ k\neq [v]}} \frac{1}{\sin^2\frac{k\pi h}{2}}.$$

Hence, we have that

$$P(x) = \prod_{\substack{k=1\\ k\neq [v],\,2[v]}}^{N} \sin\frac{(k+\{v\})\pi h}{2} \prod_{\substack{k=1\\ k\neq N-[v]+1,\,2N+2-2[v]}}^{N+1} \sin\frac{(k-\{v\})\pi h}{2} \prod_{\substack{1\le k\le N\\ k\neq [v]}} \frac{1}{\sin^2\frac{k\pi h}{2}}$$
$$= \frac{\sin^2\frac{[v]\pi h}{2}\cos\frac{\{v\}\pi h}{2}}{\sin\frac{v\pi h}{2}\sin\frac{(v+[v])\pi h}{2}\cos\frac{v\pi h}{2}} \prod_{k=1}^{N} \frac{\sin\frac{(k+\{v\})\pi h}{2}\sin\frac{(k-\{v\})\pi h}{2}}{\sin^2\frac{k\pi h}{2}}. \tag{2.15}$$

From the concavity of the functions $f_1(x)=\ln(\sin x)$, $x\in(0,\pi)$, the following inequality holds:

$$\sin(x+y)\sin(x-y)\le \sin^2(x) \quad \left(0\le y<x<\frac{\pi}{2}\right). \tag{2.16}$$

By using (2.15) and (2.16) we deduce that there exists $C>0$ such that

$$P(x) \le \frac{\sin\frac{[v]\pi h}{2}}{2\sin\frac{(v+[v])\pi h}{4}\cos\frac{(v+[v])\pi h}{4}\cos\frac{v\pi h}{2}} \le \frac{C}{\cos^2\frac{v\pi h}{2}}. \tag{2.17}$$

If $[v]=N$, in a similar way we have that

$$P(x) \le \frac{C}{\cos^2\frac{v\pi h}{2}}. \tag{2.18}$$

From (2.17) and (2.18) we deduce that (2.11) holds and the proof is complete.

■

Now we have all the ingredients needed to estimate the behavior of P_m on the real axis.

Proposition 2.2. *Let $x\in\mathbb{R}$. There exists a constant $C>0$ such that for any $1\le |m|\le N$ we have that*

$$|P_m(x)| \le \begin{cases} C & \left(|x|<\frac{2}{h}\right) \\ C\exp(\varphi(x)) & \left(|x|\ge\frac{2}{h}\right), \end{cases} \tag{2.19}$$

where

$$\varphi(x)=\frac{2}{h}\ln\left(\frac{xh}{2}+\sqrt{\frac{x^2h^2}{4}-1}\right).$$

Proof of Proposition 2.2. If $|x| \geq \frac{2}{h}$, from (2.8) and (2.14) we obtain that

$$|S_m P_m^1(x)| \leq \cos^2 \frac{m\pi h}{2} \frac{\lambda_{|m|}}{|x| - \lambda_{|m|}} \prod_{1 \leq n \leq N} \frac{x^2 - \lambda_n^2}{\lambda_n^2}$$

$$\leq C \frac{\frac{2}{h} \sin \frac{|m|\pi h}{2} \cos^2 \frac{m\pi h}{2}}{x - \frac{2}{h} \sin \frac{|m|\pi h}{2}} \prod_{1 \leq n \leq N} \frac{x^2 - \lambda_n^2}{\lambda_n^2} \leq \frac{\sin \frac{|m|\pi h}{2} \cos^2 \frac{m\pi h}{2}}{1 - \sin \frac{|m|\pi h}{2}} \prod_{1 \leq n \leq N} \frac{x^2 - \lambda_n^2}{\lambda_n^2}.$$

From the above inequality, (2.12) and (2.13) we obtain that

$$|P_m(x)| \leq C \exp\left(\varphi(x)\right) \qquad \left(|x| \geq \frac{2}{h}\right).$$

If $|x| < \frac{2}{h}$, from (2.14) and the fact that $\frac{x + \lambda_{n_x}}{\lambda_{n_x}} \leq \frac{\lambda_{n_x+1} + \lambda_{n_x}}{\lambda_{n_x}} \leq C$ we have that

$$\left|P_m^1(x)\right| = \frac{|\lambda_m|}{|x - \lambda_m|} \frac{\left|x^2 - \lambda_{n_x}^2\right|}{\lambda_{n_x}^2} \prod_{\substack{1 \leq n \leq N \\ n \neq n_x}} \frac{\left|x^2 - \lambda_n^2\right|}{\lambda_n^2} \leq C \frac{\lambda_{|m|}}{\left||x| - \lambda_{|m|}\right|} \frac{|x - \lambda_{n_x}|}{\lambda_{n_x}} P(x). \tag{2.20}$$

Let $v \in \left[0, \frac{1}{h}\right)$ such that $x = \frac{2}{h} \sin \frac{v\pi h}{2}$. From (2.10) we deduce that $[v] = n_x$. From (2.11) and (2.20) we have that

$$\left|P_m^1(x)\right| \leq \frac{C}{\cos^2 \frac{v\pi h}{2}} \frac{\lambda_{|m|}}{\lambda_{n_x}} \frac{|x - \lambda_{n_x}|}{\left||x| - \lambda_{|m|}\right|}. \tag{2.21}$$

If $|m| = n_x$ from (2.8) and (2.21) we obtain that

$$|P_m(x)| \leq C \qquad (|m| = n_x).$$

If $|m| \neq n_x$ from (2.8) , (2.21) and taking to account that $|x - \lambda_{n_x}| \leq \lambda_{n_x+1} - \lambda_{n_x}$ we have that

$$|P_m(x)| \leq C \frac{\cos^2 \frac{m\pi h}{2}}{\cos^2 \frac{v\pi h}{2}} \frac{\lambda_{|m|}}{\lambda_{n_x}} \frac{|x - \lambda_{n_x}|}{\left||x| - \lambda_{|m|}\right|} \leq C \frac{(N+1-m)^2}{(N+1-v)^2} \frac{|m|}{n_x} \frac{\sin \frac{\pi h}{4} \cos \frac{(2n_x+1)\pi h}{4}}{\sin \frac{|v-m|\pi h}{4} \cos \frac{(v+m)\pi h}{4}}$$

$$\leq C \frac{(N+1-m)^2}{(N+1-v)^2} \frac{|m|}{n_x} \frac{2N + 1 - 2n_x}{|v - m|(2N + 2 - v - m)}.$$

By taking into account the different cases for m and $n_x = [v]$ we deduce that

$$|P_m(x)| \leq C \qquad (|m| \neq n_x),$$

and the proof is complete. ■

Remark 2.2. *We remark that (2.19) is optimal. More precisely, in the range $|x| \geqslant \frac{2}{h}$ we are able to prove a similar lower bound for $|P_m|$, with a different constant C.*

The last result of this section gives a rough estimate on the whole complex plane for P_m^1 that will be useful in our Paley-Wiener strategy.

Lemma 2.4. *For $z \in \mathbb{C}$, one has for some constant $C(h)$ depending on h,*

$$|P_m^1(z)| \leqslant (1 + C(h)|z|)^{\frac{2}{h}}. \tag{2.22}$$

Proof of Lemma 2.4. We have

$$|P_m^1(z)| \leqslant \prod_{k\neq 0, k=-N}^{N} \left(1 + \frac{|z|}{|\lambda_n|}\right) \leqslant \prod_{k\neq 0, k=-N}^{N} \left(1 + \frac{|z|}{\lambda_1}\right) \leqslant \left(1 + \frac{|z|}{\lambda_1}\right)^{2N}.$$

■

2.3 Construction of the multiplier

In this section we introduce a smart multiplier M_m with rapid decay on the real axis such that the product $P_m M_m$ is bounded on the real axis and $M_m(\lambda_m) = 1$. The Fourier transform of such a product gives the element θ_m of a biorthogonal sequence. We are able to obtain an estimate for the L^∞-norm of θ_m.

To get our multiplier, we consider the following construction. Let us remind that $2/h = 2(N+1) \in \mathbb{N}$. We follow step by step the construction given in [12, Pages 19-20] (see also [14]) in order to obtain a multiplier with compact support, and we use the estimates given there that we extend a little bit (be careful that here we work on the interval $[-b, b]$). For every $b > 0$ we set

$$H_b := \frac{1_{[-b,b]}}{2b}.$$

We remark that

$$\int_{\mathbb{R}} H_b = \int_{-b}^{b} H_b = 1.$$

We introduce some parameter $\eta \in (0, 1)$, which will be chosen sufficiently small. We denote by

$$a := \frac{h(1-\eta)}{2+h},$$

so that

$$\sum_{n=1}^{2/h+1} a + \eta = 1.$$

We then introduce $b_0 = b_1 = \frac{\eta}{2}$ and $b_2 = \ldots = b_{2/h+2} = a$ and we consider the product

$$u := H_{b_0} * \ldots * H_{b_{\frac{2}{h}+2}},$$

where $*$ represents the convolution product. Let us emphasize that the sequence $(b_k)_{k\in[|0,2/h+2|]}$ is decreasing as soon as h is small enough. One has the following result:

Lemma 2.5. *u is of class $C^{2/h+1}$ and is compactly supported in $[-1, 1]$. Moreover, one has $u^{(2/h+1)} \in W^{1,\infty}(\mathbb{R})$, and the following estimates hold:*

$$\int_{-1}^{1} u = ||u||_1 = 1, \tag{2.23}$$

$$||u^{(2)}||_1 \leqslant \frac{4}{\eta^2} \tag{2.24}$$

and

$$||u^{(2/h+2)}||_1 \leqslant \frac{4}{\eta^2}\left(\frac{(2+h)}{h(1-\eta)}\right)^{\frac{2}{h}}. \tag{2.25}$$

The proof of Lemma 2.5 is straightforward using the formulas given in [12, Page 20] that one can easily generalize until the $u^{(2/h+1)}$-th derivative by remarking that the generalized derivative $u^{(2/h+2)}$ exists and is piecewise continuous.

From now on, we denote by $T^- := T(1-\delta)$, where $\delta \in (0,1)$ is some sufficiently small constant. We now denote by

$$M_m(z) := \int_{-1}^{1} u(t) e^{-i\frac{T^-}{2}(z-\lambda_m)t} dt. \tag{2.26}$$

One has the following properties on M_m.

Lemma 2.6. *One has*

$$M_m(\lambda_m) = 1 \tag{2.27}$$

and for every $x \in \mathbb{R}$,

$$|M_m(x)| \leqslant 1. \tag{2.28}$$

Moreover, for every $x \in \mathbb{R}$,

$$|M_m(x)| \leqslant \frac{16}{(\eta T^-|x-\lambda_m|)^2}, \tag{2.29}$$

$$|M_m(x)| \leqslant \frac{16}{(\eta T^-|x-\lambda_m|)^2}\left(\frac{2(2+h)}{h(1-\eta)|x-\lambda_m|T^-}\right)^{\frac{2}{h}}. \tag{2.30}$$

Relation (2.27) is clear from the definition of M_m given in (2.23). Estimate (2.29) can be easily deduced by using (2.24) and performing 2 integrations by parts, whereas (2.30) is easily deduced by using (2.25) and performing $2/h+2$ integrations by parts (which is possible thanks to the regularity of u).

Now, we consider the function

$$\psi_m(z) := P_m(z) M_m(z) \qquad (z \in \mathbb{C}). \tag{2.31}$$

The main proposition of this section is the following.

Proposition 2.3. *The function ψ_m is of exponential type $T/2$. Moreover, if $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$ (where $\Gamma(f)$ is defined in (1.6)), then $\psi_m \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and, for any $1 \leq m \leq f(N)$ we have*

$$||\psi_m||_{L^1(\mathbb{R})} \leqslant C(T, \Gamma(f)), \tag{2.32}$$

for some constant $C(T,\Gamma(f))$ depending on T and $\Gamma(f)$.

Proof of Proposition 2.3.

It is easy to verify that ψ_m is of exponential type $T/2$. Indeed, using (2.31), (2.26), (2.23) and (2.22) we obtain that for any $z \in \mathbb{C}$, one has

$$|\psi_m(x)| \leqslant (1 + C(h)|z|)^{\frac{2}{h}}\, e^{(1-\delta)|z|\frac{T}{2}}.$$

Since the application $z \in \mathbb{C} \mapsto (1 + C(h)|z|)^{\frac{2}{h}}\, e^{-\delta\frac{T|z|}{2}}$ is bounded, there exists some constant $C(h, T, \delta) > 0$ (that might depend on h, T and δ) such that for any $z \in \mathbb{C}$, one has

$$(1 + C(h)|z|)^{\frac{2}{h}}\, e^{-\delta|z|\frac{T}{2}} \leqslant C(h, T, \delta).$$

Hence, using this inequality we deduce that

$$|\psi_m(x)| \leqslant C(h, T, \delta) e^{\frac{T|z|}{2}},$$

from which we deduce that ψ_m is of exponential type $T/2$.

Now, in order to obtain (2.32), we need to perform careful estimations.

1. If $|x| \leqslant \lambda_m - 1$, then using (2.31), (2.19) and (2.29) one has for $T > 4$ and δ small enough (independently on h)

$$|\psi_m(x)| \leqslant \frac{C}{\eta^2 |x - \lambda_m|^2}. \tag{2.33}$$

2. If $|x| \in [\lambda_m - 1, \lambda_m + 1]$, then using (2.31), (2.19) and (2.28) one has

$$|\psi_m(x)| \leqslant C. \tag{2.34}$$

3. If $|x| \in [\lambda_m + 1, \frac{2}{h}]$, then using (2.31),(2.19) and (2.29) one has for $T > 4$ and δ small enough (independently on h)

$$|\psi_m(x)| \leqslant \frac{C}{\eta^2 |x - \lambda_m|^2}. \tag{2.35}$$

4. If $|x| > \frac{2}{h}$,then using (2.31), (2.19) (which implies that $|P_m(x)| \leqslant C(|x|h)^{\frac{2}{h}}$) and (2.30), one has for $T > 4$ and δ small enough (independently on h)

$$\begin{aligned} |\psi_m(x)| &\leqslant \tfrac{C}{\eta^2} \tfrac{1}{(|x-\lambda_m|)^2} (|x|h)^{\frac{2}{h}} \left(\tfrac{2(2+h)}{h(1-\eta)|x-\lambda_m|T^-}\right)^{\frac{2}{h}} \\ &\leqslant \tfrac{C}{\eta^2} \tfrac{1}{(|x-\lambda_m|)^2} \left(\tfrac{2(2+h)}{(1-\eta)T^-}\right)^{\frac{2}{h}} \left(\tfrac{|x|}{|x-\lambda_m|}\right)^{\frac{2}{h}}. \end{aligned} \tag{2.36}$$

We recall that $2/h > \lambda_m$. Hence, we deduce that if $|x| > 2/h$ we have

$$\frac{|x|}{|x - \lambda_n|} \leqslant \frac{2/h}{2/h - \lambda_m} \leqslant \frac{1}{1 - h\lambda_m/2}. \tag{2.37}$$

Let $\varepsilon > 0$. We observe that

$$\frac{h\lambda_m}{2} = \sin\left(\frac{m\pi h}{2}\right) = \sin\left(\frac{m\pi}{2(N+1)}\right) \leqslant \sin\left(\frac{f(N)\pi}{2N}\right).$$

Hence, for N large enough (i.e. h small enough) we obtain thanks to the previous estimate together with the definition of $\Gamma(f)$ given in (1.6) and (2.37) that

$$\frac{|x|}{|x-\lambda_n|} \leqslant \frac{1}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon}.$$

Using the previous estimate together with (2.36) and choosing N large enough so that we also have $h \leqslant \varepsilon$, we deduce that

$$|\psi_m(x)| \leqslant \frac{C}{\eta^2}\left(\frac{2(2+\varepsilon)}{(1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon)T^-}\right)^{\frac{2}{h}}\left(\frac{1}{|x-\lambda_m|}\right)^2. \tag{2.38}$$

Combining (2.33), (2.34), (2.35) and (2.38), we deduce easily that $\psi_m \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\begin{aligned}
||\psi_m||_{L^1(\mathbb{R})} &\leqslant C(1-\tfrac{1}{\lambda_m}) + \tfrac{C}{\eta^2} + C\int_{\lambda_m+1}^{\frac{2}{h}}\left(\tfrac{1}{|x-\lambda_m|}\right)^2 dx \\
&+\tfrac{C}{\eta^2}\left(\tfrac{2(2+\varepsilon)}{(1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon)T^-}\right)^{\frac{2}{h}}\int_{2/h}^{\infty}\left(\tfrac{1}{|x-\lambda_m|}\right)^2 dx \\
&\leqslant \tfrac{C}{\eta^2} + C\int_{\lambda_m+1}^{\infty}\left(\tfrac{1}{|x-\lambda_m|}\right)^2 dx + \tfrac{C}{\eta^2}\left(\tfrac{2(2+\varepsilon)}{(1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon)T^-}\right)^{\frac{2}{h}}\tfrac{1}{2/h-\lambda_m} \\
&\leqslant \tfrac{C}{\eta^2} + \tfrac{C}{\eta^2}\tfrac{h}{2(1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon)}\left(\tfrac{2(2+\varepsilon)}{(1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)-\varepsilon)T^-}\right)^{\frac{2}{h}}.
\end{aligned}$$

Hence, for every $T > \frac{4}{1-\sin\left(\frac{\pi\Gamma(f)}{2}\right)}$, by choosing $\delta, \varepsilon, \eta$ small enough (depending on $\Gamma(f)$ and T but independent of h), we deduce that $||\psi_m||_{L^1(\mathbb{R})}$ is bounded independently of h and λ_m and we deduce (2.32). ■

2.4 Construction and estimation of the discretized control

Proof of Theorem 1.1. We denote by θ_n the inverse Fourier transform of ψ_n. By Proposition 2.3, $\psi_n \in L^2(\mathbb{R})$ and ψ_n is of exponential type $T/2$, hence, applying the version of the Paley-Wiener Theorem given in [19, Theorem 19.3, Page 370], θ_n is of compact support $[-T/2, T/2]$ and $\theta_n \in L^2(-T/2, T/2)$. Moreover, since $\psi_n \in L^1(\mathbb{R})$ we know that $\theta_n \in C^0([0,T], \mathbb{R})$ and using (2.32) one has

$$||\theta_n||_\infty \leqslant C. \tag{2.39}$$

Now, we define our control as follows:

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_n\frac{T}{2}} a_n(h)\theta_n\left(t-\frac{T}{2}\right),$$

where $a_n(h)$ was defined in (2.6). It is clear that v_h is supported in $[0,T]$ since the θ_n are supported on $[-T/2, T/2]$. Thanks to (2.6), (2.7), (2.27) and (2.31), one easily verifies that (2.5) is verified and then the control v_h is such that the corresponding solution of (1.2) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (j = 1, 2, ..., N).$$

Let us emphasize that our control is continuous as a finite sum of continuous functions.

We remark that

- If $n \in [0, N/2]$, then
$$\frac{h}{\sin(n\pi h)} \leqslant \frac{1}{2n}.$$
- If $n \in [N/2, f(N)]$, then, since there exists $C < 1$ such that $f(N) \leqslant CN$, we have
$$\frac{h}{\sin(n\pi h)} = \frac{h}{\sin(\pi(1-nh))} \leqslant \frac{h}{2(1-nh)} \leqslant \frac{h}{2(1-C)} \leqslant \frac{1}{2N(1-C)}.$$

Hence, in both cases we have
$$\frac{h}{\sin(n\pi h)} \leqslant \frac{C}{n}.$$
Moreover, thanks to (2.6), one has

$$|a_n(h)| \leqslant 2(|a_n| + |a_{-n}|),$$

hence, using the two previous estimates together with (2.39) and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} ||v_h||_\infty \leqslant C(T, \Gamma(f)) \left(\sum_{1 \leq |n| \leq N} \frac{1}{n^2} \right)^{\frac{1}{2}} \left(\sum_{1 \leq |n| \leq N} a_n^2 \right)^{\frac{1}{2}} &\leqslant C(T, \Gamma(f)) ||a_n||_2 \\ &\leqslant C(T, \Gamma(f)) ||(u^0, u^1)||_{H_0^1(0,1) \times L^2(0,1)}, \end{aligned}$$

which ends the proof of Theorem 1.1. ∎

3 Conclusions and open problems

In this section we briefly present some conclusions, remarks and open problems.

Our result says that, if the initial data are given by

$$\begin{pmatrix} u_j^0 \\ u_j^1 \end{pmatrix}_{1 \leq j \leq N} = \sum_{1 \leq |n| \leq M} a_{hn}^0 \Phi_h^n,$$

with M less than δN with $\delta \in (0,1)$, then there exists a sequence of bounded controls $(v_h)_{h>0}$ for (1.2). By filtering in an optimal way the initial condition we are able to compute the minimal time needed to control. As expected, the optimal range of filtration is localized in the area where the gap between the eigenvalues of the discrete model becomes very small.

We also mention that our results are mainly based on some optimal estimates of a Weierstrass product P_m on the real axis and on finding a smart family of multipliers. Having explicit discrete eigenvalues, we are able to obtain very precise and tricky estimates of some intricate products. This is the reason why we can prove that the product $P_m(x)$ is bounded until the range $2/h$, and beyond this range the behavior is of polynomial type in x. Such tricky estimates are of course very specific to the case of the wave equation discretized using the finite-difference method (the discrete eigenvalues depend on the numerical scheme).

An interesting (and maybe difficult) open question that remains unclear after this work is the following: for a given range of filtration given by $n \leqslant f(N)$, can we determine precisely the minimal time that ensures the uniform controllability? If we compare with the results given in [5, Section 2.4], it is likely that one cannot do better than $T > 2/\cos(\Gamma(f)\pi/2)$, which is sharp in the context where we filter the whole solution, but we have no intuition if it would be possible to recover this minimal time in our context.

Another question arising is to prove that v_h converges strongly to v, and if it is the case, to give an estimation of the speed of convergence.

Of course, one can ask if it is possible to apply the same strategy for other one-dimensional problems, notably the beam equations or equations involving fractional Laplace operators.

The last proposed open problem is given by the study of a similar strategy as in [1], where a finite difference semi-discrete scheme for the approximation of the boundary controls of a 1-D equation modeling the transversal vibrations of a hinged beam has been considered. Due to the high frequency numerical spurious oscillations, the uniform (with respect to the mesh-size) controllability property of the semi-discrete model fails in the natural setting. Hence, the convergence of the approximate boundary controls corresponding to initial data in the finite energy space cannot be guaranteed. In [1] it was proved that, by adding a vanishing numerical viscosity, the uniform controllability property and the convergence of the scheme is ensured. The discrete wave equation perturbed by a vanishing viscosity term has been studied in [16], and by choosing the parameter of viscosity ϵ to be exactly equal to h, the uniform controllability properties was restored. It remains as an open problem if it is enough to choose a smaller parameter of the form $\epsilon = h^\alpha$, with $\alpha \in (1, 2)$ in order to restore the uniform controllability properties (it has been proved in [21] that the controls explode if $\varepsilon = h^2$). The numerical experiments from [16] confirm this fact, and based on our present result, a more difficult theoretical study might be successfully done.

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