Internal observability for coupled systems of linear partial differential equations

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Abstract

We deal with the internal observability for some coupled systems of partial differential equations with constant or time-dependent coupling terms by means of a reduced number of observed components. We prove new general observability inequalities under some Kalman-like or Silverman-Meadows-like condition. Our proofs combine the observability properties of the underlying scalar equation with algebraic manipulations.

In the more specific case of systems of heat equations with constant coefficients and non-diagonalizable diffusion matrices, we also give a new necessary and sufficient condition for observability in the natural L^2-setting. The proof relies on the use of the Lebeau-Robbiano strategy together with a precise study of the cost of controllability for linear ordinary differential equations, and allows to treat the case where each component of the system is observed in a different subdomain.

Keyworlds: partial differential equations; systems; observability inequalities; rank conditions.

MSC: 35E99; 93B07.

1 Introduction

1.1 General presentation of the problem

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ $(N \in \mathbb{N}^*)$ or a smooth compact connected Riemannian manifold of dimension N $(N \in \mathbb{N}^*)$, with or without boundary.

We consider the following "scalar" evolution equation

$$\left\{ \begin{array}{rcl} \partial_t z &=& Pz \text{ in } (0,T)\times\Omega, \\ z(0) &=& z^0, \end{array} \right. \tag{1.1}$$

where P is a linear partial differential operator with domain $D(P) \subset \mathcal{H} = L^2(\Omega, \mathbb{K})$ of arbitrary order with time-independent and (possibly) space-dependent coefficients, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The initial datum z^0 is in $\mathcal{H}$.

To ensure the well-posedness of the system, we assume that P is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$. Moreover we assume, without loss of generality, that $0 \notin \rho(P)$

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(this property can be ensured by translating P to $P - \beta Id$ for some $\beta \in \mathbb{K}$ if necessary), so that for every $k \in \mathbb{N}$, one endows $D(P^k)$ with the norm

$$||z||_{D(P^k)} = ||P^k z||_{L^2(\Omega)}.$$

Moreover, we assume from now on that the following backward uniqueness property is verified for a solution z of equation (1.1):

$$z(T) = 0 \Rightarrow z = 0 \text{ on } [0, T] \times \Omega. \tag{1.2}$$

Given an open subset ω of Ω and a positive time $T > 0$, this paper is concerned with observability inequalities of the form

$$||z(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||z(t,x)||^2 dx dt. \tag{1.3}$$

More precisely, assuming that this inequality holds for the scalar equation (1.1), i.e. that it holds with a uniform constant $C > 0$ for all initial data z_0, our goal is to prove new observability inequalities of the same kind for systems of evolution equations coupling scalar equations of the form above, observations being made on a limited number of components of the state.

In the context of finite-dimensional PDEs, the classical Kalman rank condition provides a complete answer to the problem. Our goal here is to combine existing observability inequalities for partial differential equations (PDE) and algebraic techniques to achieve similar results for PDE systems. We emphasize that we do not present here new results on the observability of scalar equations. Rather, our goal is to develop a systematic method, inspired in finite-dimensional theory and employing algebraic manipulations, allowing to transfer the existing observability results on scalar PDE to systems of PDE, observing a limited number of components of the state.

We shall mainly consider two situations: a) The system is constituted by a finite number of copies of the scalar equation (1.1) coupled through a lower order term; b) The system couples, through the principal part, various scalar equations of the form (1.1). Special attention will be devoted to the case of parabolic systems with constant coefficients, where sharp results will be presented.

1.2 State of the art

There is a large literature on the controllability and observability of systems of partial differential equations. Several techniques have been applied to derive observability inequalities by means of observations done on a reduced number of components of the system in various situations. This paper is mainly devoted to coupled systems and distributed controls.

Although these questions arise naturally for all kind of systems, the problem has been investigated specially in the case of parabolic systems. Here we list some of the existing results and references:

- By means of Carleman estimates [7] addresses the internal control of coupled systems of heat equations with the same diffusion coefficients and constant or time-dependent coupling terms of zero order and [8] gives a generalization to the case of different diffusion coefficients on each equation.
- In [41] the switching control of a scalar heat equation is analyzed. Although the model under consideration is scalar the kind of ideas employed to deal with two alternating (in time) controllers inspires our method to deal with parabolic systems.
- More specific results of internal or boundary controllability in the case of variable coefficients in the one-dimensional case are proved in [13], [14], [11], [25] and [19]. These one-dimensional results also allow to deal with some simple geometries like cylinders (see [18]).
- Article [49] analyzes coupled systems of two heat equations, or the coupling of different dynamics, e.g. heat and wave equations, and it inspires the first part of the present article. We also refer to [17], where similar ideas are developed for the obtention of decay estimates for partially dissipative hyperbolic systems.

- Articles [30], [32] and [29] (by means of Carleman inequalities), [31], [26] and [27] (using the fictitious control method) deal with the internal control of parabolic systems in higher space dimension, with variable coefficients and lower order coupling terms of order 0 and 1.

- We also refer to [9], [10], [6], [31], [21] , [29] and [22] for results on systems of non-linear (or semi-linear) heat equations.

- To conclude, we also would like to mention [39], where the internal controllability of a system of heat equations with analytic non-local coupling terms is investigated.

For further informations on this specific topic, we refer to the survey article [12].

Hyperbolic and dispersive systems have been less studied and the results obtained are of different nature. The following ones are worth mentioning:

- Article [3] deals with the controllability of second order in time cascade or bidiagonal systems under suitable coercivity conditions on the coupling terms, and [5] with a system of two wave equations with one control and asymmetric coupling matrices satisfying some additional technical properties, using a multi-energy method that has been introduced in [1] and [2] in the case of two equations.

- Article [24] is devoted to analyzing a cascade system of two wave equations with one control on a compact manifold without boundary, where a necessary and sufficient condition for controllability, in terms of the geometry of the control domain and the coupling region, is proved using microlocal techniques.

- Article [40] deals with he specific study of the Schrödinger equation without using transmutation techniques, in the case of a cascade system of two equations with one control force, using Carleman estimates.

- Article [4] treats the case of some linear systems of two periodic and one-dimensional non-conservative transport equations with the same speed of propagation, space-time varying coupling matrix and one control are also analyzed, together with some nonlinear variants, thanks to the fictitious control method.

Remind that results on abstract wave equations can be combined with the *transmutation method* (see [45], [42] or [28]) to address abstract heat and Schrödinger equations, under strong (and probably not sharp in general) geometric restrictions on the coupling and control regions.

Some results in a more abstract setting were also obtained, see notably:

- In [47] abstract periodic groups of isometries with bounded self-adjoint control were considered, with an application to the Schrödinger equation in arbitrary dimension on the torus, under the assumption that the observation time has to be large enough (which is probably of purely technical nature).

- In [38] a Kalman rank condition of controllability was also proved using the fictitious control method, in the abstract setting of groups of operators with bounded control in the case of constant coupling coefficients, with applications to some systems of wave and Schrödinger equations.

To finish, we also refer to the book [23] for the control of networks of $1-d$ wave-like equations, which is closely linked to the topic of control of coupled systems.

In the present article, our goal is twofold:

- In Section 2 we present a simple method that can be applied to any first order in time PDE with internal control, giving results of weak observability (in the sense that the observation is made in higher order Sobolev norms) for systems of equations with zero order coupling terms and constant or time-dependent coefficients with a reduced number of observations. Our study is a complement of the results of [38] to the case of all scalar (notably non-conservative) PDE whose solutions verify (1.1), in a different functional setting though. Our results are then applied to systems of Schrödinger equations in Section 2.3 and to systems of wave equations (under some minor modifications) in Section 2.4.

- In Section 3, we give a necessary and sufficient condition for the internal observability of systems of heat equations with constant coefficients and zero-order coupling terms. The main difficulties are that the diffusion matrix is not necessarily diagonalizable and the open subsets of observation can be different on each equation. This generalizes the result of [8] where only the case of diagonalizable diffusion matrices was investigated with the same observation subset on all components, and also the result of [44], where the author considered open subsets (or boundary observations) that may be different for each component of the control, but under the restrictive condition that D was the identity matrix.

2 Weak observability results

2.1 The case of constant matrices

In what follows, we consider different kind of systems involving the scalar equation (1.1), with constant coupling matrices. Let $n \in \mathbb{N}^*$ be the number of equations and $m \in \mathbb{N}^*$ be the number of observed components, with possibly $m < n$.

We will focus on two different situations.

1. The diagonal case with the same operator P on each line

$$\left\{\begin{array}{rcll} \partial_t Z & = & I_n P Z + AZ & \text{in} \quad (0,T)\times\Omega, \\ Z(0) & = & Z^0, & \end{array}\right. \tag{2.1}$$

with $Z^0 \in \mathcal{H}^n$ and $A \in \mathcal{M}_n(\mathbb{K})$ some coupling matrix with constant coefficients and I_n is the identity matrix of size n. The observation is given by $\mathbb{1}_\omega BZ$, where $B \in \mathcal{M}_{m,n}(\mathbb{K})$.

2. The case where the coupling arises in the principal part:

$$\left\{\begin{array}{rcll} \partial_t Z & = & DPZ & \text{in} \quad (0,\tilde{T})\times L^2(\Omega)^n, \\ Z(0) & = & Z^0, & \end{array}\right. \tag{2.2}$$

with $Z^0 \in \mathcal{H}^n$ and a diffusion matrix with constant coefficients $D \in \mathcal{M}_n(\mathbb{K})$, where D is assumed to be **diagonalizable** with positive eigenvalues (note however that, when P is the generator of a group of operators, the positivity of the eigenvalues of P is not necessary, one may only assume non-zeros eigenvalues). From now on, without loss of generality we assume that $D = diag(d_1,\ldots,d_n) \in \mathcal{M}_n(\mathbb{K})$, where $d_i > 0$ ($i \in [|1,n|]$), and we introduce the following observation time

$$\tilde{T} := \max_i \frac{T}{d_i}. \tag{2.3}$$

Of course, when the observation time T can be taken to be arbitrarily small, $\tilde{T}$ can be taken to be arbitrarily small as well.

The observation is given by $\mathbb{1}_\omega BZ$, where $B \in \mathcal{M}_{m,n}(\mathbb{K})$.

We introduce the following notations:

$$K_A := [B^*|A^*B^*|\dots|A^{*(n-1)}B^*] \in \mathcal{M}_{n,nm}(\mathbb{K}), \tag{2.4}$$

and

$$K_D := [B^*|D^*B^*|\dots|D^{*(n-1)}B^*] \in \mathcal{M}_{n,nm}(\mathbb{K}). \tag{2.5}$$

Remark 1. In systems (2.1) and (2.2), we denote by A, D the coupling matrices and B the matrix observation for the sake of clarity. This leads to the usual Kalman matrices (2.4) and (2.5), often introduced in the context of the dual control problem. The controlled dynamics would be associated to the matrices A^*, B^*, D^* while the system under consideration would correspond to the dual or adjoint one for which the question of observability under discussion in the present paper arises naturally.

For $\varphi \in \cap_{k=0}^{n-1} H^k((0,T), D(P)^{n-1-k})^m$, we introduce

$$||\varphi||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)} := \sum_{k=0}^{n-1}\int_0^T\int_\omega ||(\partial_t - I_m P)^k\varphi(t,x)||^2 dxdt, \tag{2.6}$$

For $\varphi \in \cap_{k=0}^{n-1} H^k((0,\tilde{T}), D(P)^{n-1-k})^m$, we also introduce

$$||\varphi||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)} := \sum_{k=0}^{n-1}\int_0^{\tilde{T}}\int_\omega ||\partial_t^{(k)} I_m P^{n-1-k}\varphi(t,x)||^2 dxdt. \tag{2.7}$$

One has the following result:

Theorem 1. *Assume that the scalar equation (1.1) verifies the observability inequality (1.3). Then,*

- *System (2.1) is observable in time T in norm $\mathcal{H}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0 \in D(I_nP^{n-1})$, the solution Z of (2.1) verifies*

$$||Z(T)||^2_{L^2(\Omega)} \leqslant C||BZ||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)} \tag{2.8}$$

  if and only if

$$\text{rank } K_A = n. \tag{2.9}$$

- *System (2.2) is observable in time $\tilde{T}$ in norm $\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0 \in D(I_nP^{n-1})$, the solution Z of (2.2) verifies*

$$||Z(\tilde{T})||^2_{D(I_nP^{n-1})} \leqslant C||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)} \tag{2.10}$$

  if and only if

$$\text{rank } K_D = n. \tag{2.11}$$

Remark 2. Several remarks are in order:

- Inequality (2.10) implies, in particular,

$$||Z(\tilde{T})||^2_{L^2(\Omega)} \leqslant C||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)}.$$

- Theorem 1 provides observability inequalities in weaker norms than the usual L^2-norm. These inequalities may be improved in some cases. This issue will be addressed in Section 3 for systems of heat equations.

- One remarks that the definitions of the two norms $||\cdot||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)}$ and $||\cdot||^2_{\mathcal{I}_{n,m}((0,T)\times\omega)}$ differ. This may be explained by the fact that each of these norms is in some sense adapted to the specific algebraic structure of the equations. However, remarking that for any φ regular enough and any $(t,x)\in Q_T$, one has

$$\sum_{k=1}^{n-1}||(\partial_t - I_mP)^k\varphi(t,x)||^2 = \sum_{k=0}^{n-1}\left|\left|\sum_{l=0}^{k}\begin{pmatrix} k \\ l \end{pmatrix}(-1)^l\partial_t^{(k-l)}I_mP^l\varphi(t,x)\right|\right|^2 \leqslant C\sum_{k=0}^{n-1}\left|\left|\partial_t^{(k)}I_mP^{n-1-k}\varphi(t,x)\right|\right|^2 dxdt,$$

  we see that the norm $\mathcal{I}_{n,m}((0,T)\times\omega)$ is stronger that the norm $\mathcal{H}_{n,m}((0,T)\times\omega)$. Moreover, both norms contain the same type of derivatives.

- The hypothesis that D is diagonalizable is crucial in our proof of the second point Theorem 1. Indeed, the first step of our proof is devoted to proving the following observability inequality on the system (2.2):

$$||Z(\tilde{T})||^2_{L^2(\Omega)} \leqslant C\int_0^{\tilde{T}}\int_\omega ||Z(t,x)||^2dxdt,$$

  the observation being done on all of the components of the state.

  However, unless D is diagonalizable, one cannot simply deduce this inequality from (1.3) and this kind of inequality may even be hard to obtain. For example, in [29], systems of heat equations with (time and space-varying) zero-order coupling terms are treated by means of Carleman estimates, but the proof only works under the condition that the Jordan blocks of D are of size less that 4 (which seems to be a purely technical condition that until now has not been removed).

  In Section 3 we describe how to deal with a general constant diffusion matrix D, constant coupling matrix A and constant observation matrix B, using the Lebeau-Robbiano strategy (see [37] and [33]). This argument may hardly be adapted to treat more general cases (different diffusion operators in the various equations entering in the system, different lower order potentials, etc.) due to the specific arguments of the proof.

- The time of observation $\tilde{T}$ comes from an easy rescaling argument in time allowing to use the observability inequality (1.3) for the initial problem (1.1).

- In the case where D is diagonal (and not only diagonalizable), the Kalman condition (2.11) may be rewritten in a more explicit way. Assume that $D = diag(d_1,\ldots,d_n)$ and $d_i \neq 0$ for every $i\in[|1,n|]$. Let us decompose $B := (b_{ji})_{j\in[|1,n|],i\in[|1,m|]}$. For $i\in[|1,m|]$, set

$$B_i^* := \begin{pmatrix} b_{1i} & 0 & \ldots & 0 \\ 0 & b_{2i} & \ldots & 0 \\ \vdots & & & \\ 0 & \ldots & 0 & b_{ni} \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{K}),$$

  and $\widehat{B}^* = (B_1^*|\ldots|B_m^*) \in \mathcal{M}_{n,nm}(\mathbb{K})$. Then, it is easy to check that there exists a permutation matrix $\sigma\in GL_n(\mathbb{R})$ such that

$$K_D = \sigma\text{Vander}(d_1,\ldots,d_n)\widehat{B}^*,$$

  where $\text{Vander}(d_1,\ldots,d_n)$ is the Vandermonde matrix of size n associated to $d_1,\ldots,d_n$. Hence, for K_D to be of maximal rank n, all the d_i need to be distinct and $\widehat{B}^*$ has to be of maximal rank, which means that for every $i\in[1,n|]$, there exists $j\in[|1,m|]$ such that $b_{ij\neq 0}$.

■

Proof of the first point of Theorem 1. Let us first consider system (2.1) and prove the inverse part of the equivalence. We assume that the Kalman rank condition (2.9) is verified. First of all, let us establish the following observability inequality on the solutions of (2.1):

$$||Z(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||Z(t,x)||^2 dxdt. \tag{2.12}$$

Indeed, if Z verifies (2.1), then $\tilde{Z} := \exp(-tA)Z$ verifies

$$\begin{cases} \partial_t \tilde{Z} = I_n P \tilde{Z} & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ \tilde{Z}(0) = Z^0. \end{cases} \tag{2.13}$$

Hence, applying inequality (1.3) on each line of system (2.13), we obtain that

$$||\tilde{Z}(T)||^2_{L^2(\Omega)} \leqslant C \int_0^T \int_\omega ||\tilde{Z}(t,x)||^2 dxdt.$$

Inequality (2.12) is then easily deduced by remarking that

$$C_1 ||\tilde{Z}(t,x)||^2 \leqslant ||Z(t,x)||^2 \leqslant C_2 ||\tilde{Z}(t,x)||^2$$

for some constants C_1 and C_2 independent of t and x (but depending on T).

Now, let us consider $||BZ||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)}$. Using (2.6), the fact that B has constant coefficients and equation (2.1), we deduce that

$$||BZ||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)} = \sum_{k=0}^{n-1} \int_0^T \int_\omega ||BA^k Z(t,x)||^2 dxdt. \tag{2.14}$$

Since (2.9) is verified, the following map

$$z = (z_1, \dots z_n) \in \mathbb{R}^n \mapsto \sum_{k=0}^{n-1} ||BA^k z||^2$$

defines a norm on $\mathbb{R}^n$, equivalent to the euclidean norm $z \mapsto ||z||^2$. Hence, we obtain, using (2.14), that

$$||BZ||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)} \geqslant C \int_0^T \int_\omega ||Z(t,x)||^2 dxdt,$$

which enables us to deduce (2.8) thanks to (2.12).

The fact that (2.8) implies (2.9) is classical and can be handled for example by using the strategy of [7, Sections 3.1 & 3.2]: we first prove the result for $m = 1$ by transforming (2.1) in the Brunovsky canonical form and we treat the general case $m > 1$ by transforming (2.1) into a block triangular system where each diagonal block is in the Brunovsky canonical form. Note that system (2.1) also verifies a backward uniqueness property, which is required for the above strategy to hold. This can be easily proved by using the uncoupled equation (2.13) on $\tilde{Z}$ together with (1.2).

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Proof of the second point of Theorem 1. Let us now consider system (2.2) and prove the inverse part of the equivalence. We assume that the Kalman rank condition (2.11) is verified.

First of all, let us prove the following observability inequality on the solutions of (2.2):

$$||Z(\tilde{T})||^2_{L^2(\Omega)} \leqslant C \int_0^{\tilde{T}} \int_\omega ||Z(t,x)||^2 dxdt. \tag{2.15}$$

Let us consider the $i-th$ line of system (2.2) on the time interval $(0,\tilde{T})$, i.e.

$$\left\{\begin{array}{rcll} \partial_t z_i &=& d_i P Z_i & \text{in} \quad (0,\tilde{T})\times L^2(\Omega),\\ z_i(0) &=& z_i^0. \end{array}\right.$$

We perform the change of unknowns $\tilde{z}_i(t,x) := z_i(t/d_i,x)$. $\tilde{z}$ is now defined on $(0,T_i)$, where $T_i = d_i\tilde{T}$, and $\tilde{z}$ verifies (1.1).

By definition (2.3) of $\tilde{T}$, one has $T_i \geqslant T$, hence using (1.3) (on the time interval $(0,T)$) together with the well-posedness of equation (1.1), we obtain

$$||\tilde{z}_i(T_i)||^2_{L^2(\Omega)} \leqslant C\int_0^{T_i}\int_\omega ||\tilde{z}_i(t,x)||^2 dxdt,$$

i.e.

$$||z_i(\tilde{T})||^2_{L^2(\Omega)} \leqslant C\int_0^{\tilde{T}}\int_\omega ||z_i(t,x)||^2 dxdt.$$

Adding on i we obtain (2.15).

Now, we consider $||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)}$. Using (2.7), the fact that B has constant coefficients and equation (2.1), we deduce that

$$||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)} = \sum_{k=0}^{n-1}\int_0^{\tilde{T}}\int_\omega ||BD^kP^{n-1}Z(t,x)||^2 dxdt. \tag{2.16}$$

Since (2.11) is verified, the following map

$$z=(z_1,\dots z_n)\in\mathbb{R}^n \mapsto \sum_{k=0}^{n-1}||BD^kz||^2$$

is a norm on $\mathbb{R}^n$, equivalent to the euclidian one $z\mapsto ||z||^2$. Hence, we obtain, using (2.16), that

$$||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)} \geqslant C\int_0^{T}\int_\omega ||I_nP^{n-1}Z(t,x)||^2 dxdt.$$

Applying (2.15) to $I_nP^{n-1}Z$, instead of Z, we know that

$$||Z(\tilde{T})||^2_{D(I_nP^{n-1})} \leqslant C||BZ||^2_{\mathcal{I}_{n,m}((0,\tilde{T})\times\omega)},$$

from which we deduce (2.10).

The fact that (2.10) implies (2.11) is classical and will be omitted here (see [7, Sections 3.1 & 3.2] for example).

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2.2 The case of time-dependent matrices

In this section we assume, for the sake of simplicity, that system (1.1) is observable **in arbitrary small time**. We will explain in Remark 3 how to treat the case where the minimal time of observation is positive. We consider the following systems involving the scalar equation (1.1):

1. The diagonal case with the same operator P on each line and time-dependent coupling terms and observation operators:
$$\left\{\begin{array}{ll} \partial_t Z &= I_n P Z + A(t) Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z(0) &= Z^0, \end{array}\right. \tag{2.17}$$
with $Z^0 \in \mathcal{H}^N$ and $A \in C^\infty([0,T], \mathcal{M}_n(\mathbb{K}))$. The observation is given by $\mathbb{1}_\omega B Z$, where $B \in C^\infty([0,T], \mathcal{M}_{m,n}(\mathbb{K}))$.

2. The case where the coupling arises in the principal part:
$$\left\{\begin{array}{lll} \partial_t Z &=& D(t) I_n P Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z(0) &=& Z^0, \end{array}\right. \tag{2.18}$$
with $Z^0 \in \mathcal{H}^N$ and $D \in C^\infty([0,T], \mathcal{M}_n(\mathbb{K}))$ assumed to be **diagonal** with positive eigenvalues at all time. We write D as $D = diag(d_1, \ldots, d_n)$, where $d_i(t) > 0$ for all $t \in [0,T]$. The observation is given by $\mathbb{1}_\omega B Z$, where $B \in C^\infty([0,T], \mathcal{M}_{m,n}(\mathbb{K}))$.

We introduce the notations
$\overline{B}_0^* = \tilde{B}_0^* = B^*$, $\overline{B}_i^* = A^* \overline{B}_{i-1}^* - \overline{B}_{i-1}^{*'}$ and $\tilde{B}_i^* = D^* \tilde{B}_{(i-1)}^* - \tilde{B}_{i-1}^{*'}$ for $i \in \mathbb{N}^*$.
One has the following result:

Theorem 2. *Assume that (1.1) is observable* ***in arbitrary small time.***

- *System (2.17) is observable in norm $\mathcal{H}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C > 0$ such that for every $Z^0 \in D(I_n P^{n-1})$, the solution Z of (2.17) verifies (2.8) if there exists $\bar{t} \in [0,T]$ such that*
$$Span\ \{\overline{B}_k^*(\bar{t})(\mathbb{R}^m) | k \in \mathbb{N}\} = \mathbb{R}^n. \tag{2.19}$$
- *System (2.18) is observable in norm $\mathcal{I}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C > 0$ such that for every $Z^0 \in D(I_n P^{n-1})$, the solution Z of (2.18) verifies (2.10) if there exists $\tilde{t} \in [0,T]$ such that*
$$Span\ \{\tilde{B}_k^*(\tilde{t})(\mathbb{R}^m) | k \in \mathbb{N}\} = \mathbb{R}^n. \tag{2.20}$$

Remark 3.
- The regularity conditions on A, B and D can be weakened. Notably, Theorem 2 still holds provided that we assume that there exists $p \in \mathbb{N}$ such that $A \in C^p([0,T], \mathcal{M}_n(\mathbb{K}))$, $D \in C^p([0,T], \mathcal{M}_n(\mathbb{K}))$ and $B \in C^{p+1}([0,T], \mathcal{M}_{m,n}(\mathbb{K}))$, and that (2.19) and (2.20) are replaced by the stronger conditions
$$\text{Span } \{\overline{B}_k^*(\bar{t})(\mathbb{R}^m) | k \in [|0, p+1|]\} = \mathbb{R}^n$$
and
$$\text{Span } \{\tilde{B}_k^*(\tilde{t})(\mathbb{R}^m) | k \in [|0, p+1|]\} = \mathbb{R}^n.$$
- The condition that the minimal time of observation needs to be arbitrarily small comes from conditions (2.19) and (2.20), which require that one has to observe on a small interval of time near $\bar{t}$ or $\tilde{t}$ (this will be made more precise during the proof of Theorem 2). One can readily see from the proof of Theorem 2 that it is possible to get rid of the condition of observability in arbitrary small time by assuming respectively the following conditions: there exists $K \in \mathbb{N}$ such that for every $t \in [0,T]$,
$$\text{Span } \{\overline{B}_k^*(t)(\mathbb{R}^m) | k \in [0, K]\} = \mathbb{R}^n,$$
and there exists $K \in \mathbb{N}$ such that for every $t \in [0,T]$,
$$\text{Span } \{\tilde{B}_k^*(t)(\mathbb{R}^m) | k \in [0, K]\} = \mathbb{R}^n.$$
We would then obtain a result of observability in time T for (2.17) and $\tilde{T} := \max_i \int_0^T (d_i(s))^{-1} ds$ for (2.18).

Proof of the first point of Theorem 2. Let us first consider system (2.1). We assume that condition (2.19) is verified.

We will need the following Lemma, that essentially says that in condition (2.19), we can restrict to a finite number of $\overline{B}_i^*$ on an interval close to $\bar{t}$.

Lemma 2.1. *Assume that (2.19) is verified. Then there exists $\varepsilon > 0$ such that for every $t \in [|\bar{t} - \varepsilon, \bar{t} + \varepsilon|] \setminus \{\bar{t}\}$,*

$$Span\ \{\overline{B}_k(t)^*(\mathbb{R}^m) | k \in [|0, n-1|]\} = \mathbb{R}^n. \tag{2.21}$$

This Lemma is proved in [20, Proposition 1.19, Page 11].

Hence, we now consider some interval $[t_0, t_1] \subset [0, T]$ such that for every $t \in [t_0, t_1]$, one has $\{\overline{B}_k(t)^*(\mathbb{R}^m) | k \in [|0, n-1|]\} = \mathbb{R}^n$.

We proceed as in the proof of Theorem 1. We want to prove the following observability inequality on the solutions of (2.17):

$$||Z(t_1)||^2_{L^2(\Omega)} \leqslant C \int_{t_0}^{t_1} \int_\omega ||Z(t,x)||^2 dx dt. \tag{2.22}$$

Since we assumed that (1.1) holds in arbitrary small time, we know that there exists $C > 0$ such that for any solution z of (1.1), we have

$$||z(t_1)||^2_{L^2(\Omega)} \leqslant C \int_{t_0}^{t_1} \int_\omega ||z(t,x)||^2 dx dt. \tag{2.23}$$

We now use the change of variables $\tilde{Z} := R_{-A}(t, t_0) Z$, where R_{-A} is the resolvent operator associated to the ordinary differential equation $y' = -A(t) y$. Then $\tilde{Z}$ is solution of

$$\begin{cases} \partial_t \tilde{Z} = I_n P \tilde{Z} & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ \tilde{Z}(0) = Z^0. \end{cases} \tag{2.24}$$

Using inequality (2.23) on each equation of (2.24), we deduce that

$$||\tilde{Z}(t_1)||^2_{L^2(\Omega)} \leqslant C \int_{t_0}^{t_1} \int_\omega ||\tilde{Z}(t,x)||^2 dx dt.$$

Inequality (2.22) is then easily deduced by remarking that, thanks to the regularity of A, one has

$$C_1 ||\tilde{Z}(t,x)||^2 \leqslant ||Z(t,x)||^2 \leqslant C_2 ||\tilde{Z}(t,x)||^2$$

for some constants C_1 and C_2 independent of t and x (but depending on t_0, t_1).

Thanks to (2.17), one can prove by an easy induction that for any $i \in [|0, n-1|]$, one has

$$(\partial_t - I_m P)^i B Z(t,x) = \overline{B}_i Z(t,x),$$

where $Z \in \cap_{k=0}^{n-1} H^k((t_0, t_1), D(P)^{n-1-k})^n$ is the solution of (2.17).

Using (2.6), we deduce that

$$||BZ||^2_{\mathcal{H}_{n,m}((t_0,t_1)\times\omega)} = \sum_{i=0}^{n-1} \int_{t_0}^{t_1} \int_\omega ||\overline{B}_i(t) Z(t,x)||^2 dx dt. \tag{2.25}$$

Since (2.21) is verified, for every $t \in (t_0, t_1)$, the following map

$$x = (x_1, \dots x_n) \in \mathbb{R}^n \mapsto \sum_{i=0}^{n-1} ||\overline{B}_i(t) x||^2$$

is a norm on $\mathbb{R}^n$, equivalent to the euclidean one $x \mapsto ||x||^2$, which means that there exists some constant $C(t) > 0$ such that

$$\sum_{i=0}^{n-1} ||\overline{B}_i(t)x||^2 \geqslant C(t)||x||^2.$$

Moreover, by restricting (t_0, t_1) if necessary, we may always assume that $C(t) > C$ for $t \in [t_0, t_1]$ since $C(t)$ can be chosen as the smallest singular value of $[|\overline{B}_0^*(t)|\overline{B}_1^*(t) \dots |\overline{B}_n^*(t)|]^*$, which is continuous with respect to t. We deduce that for any $t \in [t_0, t_1]$, one has

$$\sum_{i=0}^{n-1} ||\overline{B}_i(t)x||^2 \geqslant C||x||^2. \tag{2.26}$$

Hence, we obtain, using (2.14) and (2.26), that

$$||BZ||^2_{\mathcal{H}_{n,m}((t_0,t_1)\times\omega)} \geqslant C \int_{t_0}^{t_1} \int_{\omega} ||Z(t,x)||^2 dxdt,$$

which enable us to deduce (2.8) thanks to (2.22) and the well-posedness of (2.17). ■

Proof of the second point of Theorem 2 The proof is similar and then omitted.

■

2.3 An example: coupled systems of Schrödinger equations

In this Section, we explain how we can apply our previous results to different families of systems of Schrödinger equations. We set $P = i\Delta$ with domain $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$. Remark that the backward uniqueness property (1.2) is verified by conservation of the energy in time. With the same notations as in the previous subsections, we consider the following systems:

$$\left\{ \begin{array}{rcl} \partial_t Z &=& i\Delta Z + AZ \quad \text{in} \quad (0,T)\times\Omega, \\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega, \\ Z(0) &=& Z^0 \in L^2(\Omega), \end{array} \right. \tag{2.27}$$

$$\left\{ \begin{array}{rcl} \partial_t Z &=& iD\Delta Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega, \\ Z(0) &=& Z^0 \in L^2(\Omega), \end{array} \right. \tag{2.28}$$

$$\left\{ \begin{array}{rcl} \partial_t Z &=& i\Delta Z + A(t)Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega, \\ Z(0) &=& Z^0 \in L^2(\Omega), \end{array} \right. \tag{2.29}$$

$$\left\{ \begin{array}{rcl} \partial_t Z &=& iD(t)\Delta Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega, \\ Z(0) &=& Z^0 \in L^2(\Omega). \end{array} \right. \tag{2.30}$$

being implicit that $\partial\Omega = \emptyset$ if Ω is a manifold without boundary.

We assume from now on that ω verifies the geometric control condition (see [46] and [16]), so that the scalar Schrödinger equation is observable in arbitrary small time (see [36]). Remark that this sufficient condition might be weakened in some particular geometries. We introduce the following norms:

$$||\varphi||^2_{\mathcal{H}_{n,m}((0,T)\times\omega)} := \sum_{k=0}^{n-1} \int_0^T \int_{\omega} ||(\partial_t - i\Delta)^k \varphi(t,x)||^2 dxdt,$$

and

$$||\varphi||^2_{\mathcal{I}_{n,m}((0,T)\times\omega)} := \sum_{k=0}^{n-1}\int_0^T\int_\omega ||\partial_t^{(k)}(i\Delta)^{n-1-k}\varphi(t,x)||^2 dxdt.$$

Applying directly Theorems 1 and 2, we obtain the following result.

Corollary 2.1. *Assume that ω verifies the geometric control condition.*

- *System (2.27) is observable in any time T in norm $\mathcal{H}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0\in D(\Delta^{n-1})$, the solution Z of (2.27) verifies (2.8) if and only if (2.9) is verified.*
- *System (2.28) is observable in any time T in norm $\mathcal{I}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0\in D(\Delta^{n-1})$, the solution Z of (2.28) verifies (2.10) if and only if (2.11) is verified.*
- *System (2.29) is observable in norm $\mathcal{H}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0\in D(\Delta^{n-1})$, the solution Z of (2.29) verifies (2.8) if there exists $\bar{t}\in[0,T]$ such that (2.19) holds.*
- *System (2.30) is observable in norm $\mathcal{I}_{n,m}((0,T)\times\omega)$ in the sense that there exists $C>0$ such that for every $Z^0\in D(\Delta^{n-1})$, the solution Z of (2.30) verifies (2.10) if there exists $\tilde{t}\in[0,T]$ such that (2.20) holds.*

2.4 Another example: coupled systems of wave equations

In this Section, we explain how we can apply (under some minor modifications) our previous results to different families of systems of wave equations. We set $P=\Delta$ with domain $D(P)=H^2(\Omega)\cap H^1_0(\Omega)$. We consider then the following "scalar" wave equations

$$\left\{\begin{array}{rcl}\partial^2_{tt}z &=& \Delta z \text{ in } (0,T)\times\Omega,\\ z &=& 0 \text{ on } (0,T)\times\partial\Omega,\\ (z(0),\partial_t z(0)) &=& (z^0,z^1)\in H^1_0(\Omega)\times L^2(\Omega),\end{array}\right. \tag{2.31}$$

being implicit that $\partial\Omega=\emptyset$ if Ω is a manifold without boundary. Remark that the backward uniqueness property (1.2) is verified by conservation of the energy in time. Here, we cannot enter directly the framework of this article, because of the second derivative in time. However, the same result holds, under minor modifications in the proof that will be explained afterwards.

We assume from now on that ω verifies the geometric control condition (see [46] and [16]), so that (2.31) is observable in sufficiently large time $T_0>0$. With the same notations as in the previous subsections, we consider the following systems:

$$\left\{\begin{array}{rcl}\partial^2_{tt}Z &=& \Delta Z + AZ \quad\text{in}\quad (0,T)\times\Omega,\\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega,\\ (Z(0),\partial_t Z(0) &=& (Z^0,Z^1)\in H^1_0(\Omega)^n\times L^2(\Omega)^n,\end{array}\right. \tag{2.32}$$

$$\left\{\begin{array}{rl}\partial^2_{tt}Z & = D\Delta Z \quad\text{in}\quad (0,T)\times\Omega,\\ Z & = 0 \text{ on } (0,T)\times\partial\Omega,\\ (Z(0),\partial_t Z(0) & = (Z^0,Z^1)\in H^1_0(\Omega)^n\times L^2(\Omega)^n,\end{array}\right. \tag{2.33}$$

$$\left\{\begin{array}{rcl}\partial^2_{tt}Z &=& D(t)\Delta Z \quad\text{in}\quad (0,T)\times\Omega,\\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega,\\ (Z(0),\partial_t Z(0)) &=& (Z^0,Z^1)\in H^1_0(\Omega)^n\times L^2(\Omega)^n.\end{array}\right. \tag{2.34}$$

We introduce the following observation time for (2.33) and (2.34): $\tilde{T}_0 := \max_i \int_0^{T_0}(d_i(s))^{-1}ds$.

We also introduce the following norms:

$$||\varphi||^2_{\mathcal{H}_{n,m}((0,T_0)\times\omega)} := \sum_{k=0}^{n-1}\int_0^{T_0}\int_\omega ||(\partial^2_{tt}-\Delta)^k\varphi(t,x)||^2 dxdt,$$

and

$$||\varphi||^2_{\mathcal{I}_{n,m}((0,\tilde{T}_0)\times\omega)} := \sum_{k=0}^{n-1}\int_0^{\tilde{T}_0}\int_\omega ||\partial_t^{(2k)}\Delta^{n-1-k}\varphi(t,x)||^2 dxdt.$$

We have the following results:

Theorem 3. *Assume that ω verifies the geometric control condition.*

- *System (2.32) is observable at time T_0 in norm $\mathcal{H}_{n,m}((0,T_0)\times\omega)$ in the sense that there exists $C>0$ such that for every $(Z^0,Z^1)\in D(\Delta^{n-1/2})\times D(\Delta^{n-1}))$, the solution Z of (2.32) verifies (2.8) if and only if (2.9) is verified.*
- *System (2.33) is observable at time $\tilde{T}_0$ in norm $\mathcal{I}_{n,m}((0,\tilde{T}_0)\times\omega)$ in the sense that there exists $C>0$ such that for every $(Z^0,Z^1)\in D(\Delta^{n-1/2})\times D(\Delta^{n-1}))$, the solution Z of (2.33) verifies (2.10) if and only if (2.11) is verified.*
- *System (2.34) is observable at time T_0 in norm $\mathcal{I}_{n,m}((0,\tilde{T}_0)\times\omega)$ in the sense that there exists $C>0$ such that for every $(Z^0,Z^1)\in D(\Delta^{n-1/2})\times D(\Delta^{n-1})$, the solution Z of (2.34) verifies (2.10) if there exists $K\in\mathbb{N}$ such that for every $t\in[0,\tilde{T}_0]$,*

$$Span\ \{\tilde{B}_k(t)^*(\mathbb{R}^m)|k\in[0,K]\} = \mathbb{R}^n.$$

Remark 4. We do not know how to obtain a similar result in the case

$$\left\{\begin{array}{rcl} \partial^2_{tt}Z &=& \Delta Z + A(t)Z \quad \text{in} \quad (0,T)\times\Omega, \\ Z &=& 0 \text{ on } (0,T)\times\partial\Omega, \\ (Z(0),\partial_t Z(0)) &=& (Z^0,Z^1)\in H^1_0(\Omega)^n\times L^2(\Omega)^n, \end{array}\right. \tag{2.35}$$

Indeed, the first step of Theorem 2 (i.e. observing each component on $(0,T)\times\omega$) cannot be performed using the same reasoning, and also cannot be performed by a compactness-uniqueness argument (see what follows).

Sketch of the proof of Theorem 3. The first step is to prove that there exists $C>0$ such that for any solution Z of (2.32), we have

$$||Z(T_0)||^2_{L^2(\Omega)} \leqslant C\int_0^{T_0}\int_\omega ||Z(t,x)||^2 dxdt, \tag{2.36}$$

and that there exists $C>0$ such that for any solution Z of (2.33), and (2.34), we have

$$||Z(\tilde{T}_0)||^2_{L^2(\Omega)} \leqslant C\int_0^{\tilde{T}_0}\int_\omega ||Z(t,x)||^2 dxdt. \tag{2.37}$$

For (2.33) and (2.34), it is obvious that (2.37) can be proved by using the same reasoning as in the proof of Theorems 1 and 2. Concerning (2.32), the same reasoning as in Theorem 1 cannot be used. However, (2.36) can be easily recovered by using a standard compactness-uniqueness argument, for example the one given in [38, Proof of Proposition 3.1].

As soon as (2.36) is proved, the reasoning is totally similar as in the case of first-order systems in time. ■

3 Sharp results for non-diagonalizable systems of parabolic equations with constant coefficients

In this Section, we are interested in the question of observing the following system of heat equations

$$\left\{\begin{array}{rcll} \partial_t Z & = & D\Delta Z + AZ & \text{in} \quad (0,T)\times\Omega, \\ Z(0) & = & Z^0, & \end{array}\right. \tag{3.1}$$

with $Z^0 \in L^2(\Omega)^n$, $A \in \mathcal{M}_n(\mathbb{R})$ and $D \in \mathcal{M}_n(\mathbb{R})$ verifying an ellipticity condition given by

$$\langle DX, X\rangle \geqslant \alpha ||X||^2, \forall X \in \mathbb{R}^n, \tag{3.2}$$

for some $\alpha > 0$. This condition is sufficient to ensure the well-posedness of (3.1), since the system is strongly parabolic in the sense of [34, Chapter 7, Definition 7].

The observation is done on

$$\sum_{i=1}^{m} B_i Z 1_{\omega_i},$$

where B_i is the i-th line of $B \in \mathcal{M}_{n,m}(\mathbb{K})$, and ω_i $(i \in [|1,m|])$ are some open subsets of Ω. These observation subsets can be chosen arbitrarily. In particular all the subdomains ω_i may be disjoint.

Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions and $e_k \in H_0^1(\Omega)$ be the corresponding eigenfunctions, constituting an orthonormal basis of $L^2(\Omega)$.

We also introduce the one-parameter $(\lambda > 0)$ family of matrices

$$K(\lambda) := [B^*|(-\lambda D^* + A^*)B^*| \dots \,|(-\lambda D^* + A^*)^{n-1}B^*]. \tag{3.3}$$

The main result of this section is as follows:

Theorem 4. *System (3.1) is observable on $(0,T)$ in the sense that for every $Z^0 \in L^2(\Omega)^n$, the solution Z of (3.1) verifies*

$$||Z(T)||^2_{L^2(\Omega)} \leqslant C\int_0^T \int_\Omega \sum_{i=1}^{m} \left(B_i Z(t,x) 1_{\omega_i}(x)\right)^2 dxdt \tag{3.4}$$

if and only if

$$rank\ K(\lambda_p) = n, \forall p \geqslant 1. \tag{3.5}$$

Remark 5.

- Contrary to the observability results presented in Section 2, it is here very easy to give a related controllability result on the adjoint system, since we are working in the usual L^2 setting. Notably, (3.1) verifies (3.4) (i.e. (3.5) is verified) if only if for any $Y^0 \in L^2(\Omega)^n$, there exists $U \in L^2(\Omega)^m$ such that the solution Y of

$$\left\{\begin{array}{rcll} \partial_t Y & = & D^*\Delta Y + A^*Y + \sum_{i=1}^m B_i^* Y 1_{\omega_i} & \text{in} \quad (0,T)\times\Omega, \\ Y(0) & = & Y^0, & \end{array}\right.$$

verifies $Y(T,\cdot) = 0$.

- Here we do not need to make any extra assumption on the Jordan blocks of D contrary to [29]. This comes from the fact that we restrict to constant coupling terms here, which enables us to use a different strategy.

- If all ω_i coincide, the same condition was obtained in [7] under the restriction that D had to be a diagonalizable matrix and in [29] under a restriction on the Jordan blocks of D.

- Some results for systems of parabolic equations with internal or boundary controls with different control domains have already been proved in [44], under the restrictive condition that D is the identity matrix. The author derived Kalman rank conditions similar to the ones of [7].
- The strategy developed in the proof relies heavily on the spectral observability estimate proved by Lebeau and Robbiano (see (3.7) below). The method of proof we develop here can be applied as soon as an appropriate spectral inequality similar to (3.7) for the scalar equation (1.1) is available. For instance, similar results could be obtained for a system of Kolmogorov equations on the whole space (see [35]), or for any heat equation where $-\Delta$ is replaced by $(-\Delta)^s$ (the spectral fractional Dirichlet-Laplace operator of order s) for $s > 1/2$.

■

Proof of Theorem 4. We first prove the inverse part of Theorem 4. We assume that (3.5) holds. We use the Lebeau-Robbiano strategy.

First of all, we decompose the initial condition Z^0 as

$$Z^0(x) = \sum_{k=1}^{\infty} Z_k^0 e_k(x), \;\; Z_k^0 \in \mathbb{R}^n.$$

Then the solution Z of (3.1) can be written as

$$Z(t,x) = \sum_{k=1}^{\infty} Z_k(t) e_k(x),$$

where Z_k is the unique solution of the ordinary differential system

$$\left\{ \begin{array}{rcl} Z_k' & = & (-\lambda_k D + A) Z_k, \\ Z_k(0) & = & Z_k^0. \end{array} \right. \tag{3.6}$$

Let us recall the Lebeau-Robbiano inequality for eigenfunctions of the Dirichlet-Laplace operator as obtained in [33]: for any non-empty subset ω_i of Ω, there exists $C_i > 0$ such that for any $J > 0$ and any finite linear combination of the e_k ($k \leqslant J$) given by $e(x) := \sum_{k \leqslant J} a_k e_k(x)$, we have

$$\sum_{k \leqslant J} |a_k|^2 = \int_\Omega \left(\sum_{k \leqslant J} a_k e_k(x) \right)^2 dx \leqslant C_i e^{C_i \sqrt{\lambda_J}} \int_{\omega_i} \left(\sum_{k \leqslant J} a_k e_k(x) \right)^2 dx. \tag{3.7}$$

Writing (3.7) for each component of BZ_k and adding on n we obtain that there exists $C > 0$ such that

$$\sum_{k \leqslant J} ||BZ_k(t)||^2 \leqslant C e^{C\sqrt{\lambda_J}} \sum_{i=1}^{n} \int_{\omega_i} \left(\sum_{k \leqslant J} B_i Z_k(t) e_k(x) \right)^2 dx. \tag{3.8}$$

Integrating (3.8) between 0 and T, we obtain

$$\int_0^T \sum_{k \leqslant J} ||BZ_k(t)||^2 dt \leqslant C e^{C\sqrt{\lambda_J}} \sum_{i=1}^{n} \int_0^T \int_{\omega_i} \left(\sum_{k \leqslant J} B_i Z_k(t) e_k(x) \right)^2 dx dt. \tag{3.9}$$

Now, we consider equation (3.6). Thanks to the assumption (3.5), we deduce that for every $k \in [|1, J|]$, system (3.6) is observable and we have the existence of some constant $C(T,k,n) > 0$ (depending on k, T and n) such that

$$||Z_k(T)||^2 \leqslant C(T,k,n) \int_0^T ||BZ_k(t)||^2 dt. \tag{3.10}$$

Moreover, one can prove the following Lemma:

Lemma 3.1. *There exists some $C > 0$ (independent on T,k, depending only on n) such that (3.10) holds with*

$$C(T,n,k) \leqslant C\left(\sqrt{T}+\frac{1}{T^{n-\frac{1}{2}}}\right)F(\lambda_k), \tag{3.11}$$

where F is a rational function of λ_k.

This lemma is proved in Appendix A.

Using (3.9), (3.10) and (3.11) we deduce that

$$\begin{aligned}\sum_{k\leqslant J}||Z_k(T)||^2 &\leqslant C\max_{k\leqslant J}\{\sqrt{C(T,k,n)}\}e^{C\sqrt{\lambda_J}}\int_0^T\int_\omega\left(\sum_{k\leqslant J}B_iZ_k(t)e_k(x)\right)^2 dx\\ &\leqslant \tilde{C}\left(1+\frac{1}{T^{p_1}}\right)e^{\tilde{C}\sqrt{\lambda_J}}\int_0^T\int_\omega\left(\sum_{k\leqslant J}B_iZ_k(t)e_k(x)\right)^2 dx\end{aligned} \tag{3.12}$$

for some new constant $\tilde{C} \geqslant C$. Once we have (3.12), it is very classical that one can deduce the observability inequality (3.4) by coupling (3.12) with a dissipation estimate (one can for example apply directly [43, Theorem 2.2]).

The proof of the inverse part of the equivalence is finished.

Let us now prove the direct part of Theorem 4. We argue by contradiction as in [8]. We assume that (3.5) does not hold, i.e. there exists $p_0 \in \mathbb{N}^*$ such that

$$\text{rank } K(\lambda_{p_0}) < n.$$

Then, by the usual Kalman rank condition for ODEs, for $k = p_0$, (3.6) is not observable and there exists a solution Z_{p_0} to (3.6) verifying $BZ_{p_0}(t) = 0$ for every $t \in (0,T)$. It is clear that $Z(t,x) = Z_{p_0}(t)e_{p_0}(x)$ is a solution of (3.1) verifying moreover $BZ(t,x) = 0$ for every $(t,x) \in (0,T)\times\Omega$, which means that (3.1) is not observable and concludes the proof.

■

4 Conclusion and open problems

In this article, we presented a simple method for finding algebraic sufficient (and sometimes necessary) conditions for weak observability of coupled systems of partial differential equations with constant or time-dependent coefficients and a reduced number of observations, and we applied the Lebeau-Robbiano strategy on a non-diagonalizable system of heat equations in order to derive a spectral necessary and sufficient condition of observability on distinct subsets. We address to following natural open questions arising after our study:

- In the case of systems (2.1), (2.2), (2.17) and (2.18), can we obtain observability inequalities in the natural L^2-norm? If not, what are the optimal Sobolev norms that one can estimate in the left-hand side?
- In the case of systems (2.2) and (2.18), can we obtain the same Kalman or Silverman-Meadows condition for non-diagonalizable coupling matrices D^*?
- In the case of systems (2.17) (resp. (2.18)), is the Silverman-Meadows condition necessary in the case where A (resp. D) and B are analytic? If A (resp. D) or B is not analytic, is it possible to find a counter-example showing that the Silverman-Meadows condition may be not verified whereas null-controllability is still verified?

- In the case where P is self-adjoint but no spectral inequality similar to (3.7) is known for the eigenfunctions of P, or in the case of parabolic systems of order two with time and space-dependent coefficients, can we obtain by other means necessary and sufficient conditions similar to (3.4) for general systems of the form $\partial_t Z = D^* I_n P Z + A^* Z$?
- One may also ask the question of finding necessary and sufficient conditions for general systems of the form $\partial_t Z = D^* I_n P Z + A^* Z$ for unitary groups of operators like Schrödinger or wave equations.
- In the case of systems (2.1), (2.2), (2.17) and (2.18), can we obtain observability inequalities with different (and possibly disjoint) observation subsets ω_i as in the case of (3.1)?
- Can we obtain the same characterizations if we couple different dynamics (for example systems of mixed heat and wave equations), as in [49]?
- Can obtain the same results for an infinite number of coupled equations (i.e. $n = \infty$)?

A Proof of Lemma 3.1

We consider the corresponding control problem naturally associated to the adjoint problem (3.6) given by

$$\left\{ \begin{array}{ll} Y_k' & = (-\lambda_k D^* + A^*) Y_k + B^* U_k, \\ Y_k(0) & = Y_k^0, \end{array} \right. \tag{A.1}$$

where $U_k \in L^2((0,T), \mathbb{R}^m)$.

In order to prove Lemma 3.1, we will use a construction of (non-necessarily optimal) controls given in [48] (see also [15]). For $k \in \mathbb{N}^*$, we call $C(\lambda_k) := -\lambda_k D^* + A^*$. We introduce the following linear continuous operator

$$Q(\lambda_k) : (u_0, u_1, \ldots u_{n-1}) \in (\mathbb{R}^m)^n \mapsto B^* u_0 + C(\lambda_k) B^* u_1 + \ldots + C(\lambda_k)^{n-1} B^* u_{n-1} \in \mathbb{R}^n.$$

Thanks to (3.5), we know that for any $k \in \mathbb{N}^*$, $Q(\lambda_k)$ is onto, so that there exists a right inverse $E(\lambda_k) : \mathbb{R}^n \to (\mathbb{R}^m)^n$ to $Q(\lambda_k)$. If we identify $E(\lambda_k)$ with its matrix in the canonical basis, It is clear that all the elements of $E(\lambda_k)$ are fractional functions of λ_k, since $E(\lambda_k)$ can be constructed using Gauss elimination on the matrix that represents $Q(\lambda_k)$ in the canonical basis. Writing $E(\lambda_k)$ as $E(\lambda_k) = (E_0(\lambda_k), \ldots E_{n-1}(\lambda_k))$, by definition of $E(\lambda_k)$ we have

$$\sum_{i=0}^{n-1} C_k^i B^* E_i(\lambda_k) = Id_{\mathbb{R}^n}.$$

Let us consider the function $\varphi \in C^n([0,T], \mathbb{R})$ defined as the normalization in L^1-norm of the function $t \mapsto t^n (T-t)^n$. This function verifies

$$\varphi^{(i)}(0) = \varphi^{(i)}(T) = 0, i \in [|0, n-1|], \text{ and } \int_0^T \varphi = 1.$$

Moreover, we know by [15, Page 891] that for any $i \in [|0, n-1|]$, there exists a constant $C_i > 0$ such that

$$||\varphi^{(i)}||_{\infty,[0,T]} \leqslant C_i \max\left(\frac{1}{T}, \frac{1}{T^{i+1}}\right) \leqslant C_i \max\left(\frac{1}{T}, \frac{1}{T^n}\right) \leqslant C_i \left(1 + \frac{1}{T^n}\right),$$

so that taking the maximum of all these C_i, there exists some constant $C > 0$ such that for any $i \in [|0, n-1|]$, we have

$$||\varphi^{(i)}||_{\infty,[0,T]} \leqslant C \left(1 + \frac{1}{T^n}\right). \tag{A.2}$$

Then, following [48] and [15], we may give the following expression of a control U_k steering Y_k^0 to 0 at time T:

$$U_k(t) := \sum_{i=0}^{n-1} E_i(\lambda_k)\psi^{(i)}(t), \tag{A.3}$$

where $\psi(t) = -e^{tC(\lambda_k)}Y_k^0\varphi(t)$. Now, using the Leibniz rule, we have that

$$\psi^{(i)}(t) = -\sum_{l=0}^{i} \binom{l}{i} C(\lambda_k)^{(i-l)} e^{tC(\lambda_k)} Y_k^0 \varphi^{(l)}(t). \tag{A.4}$$

We remark that for $X \in \mathbb{R}^n$, using (3.2) and denoting by $|||.|||$ the matrix norm associated to the usual euclidian norm, we have

$$\langle C(\lambda_k)X, X\rangle = -\lambda_k\langle DX, X\rangle + \langle AX, X\rangle \leqslant (-\lambda_k\alpha + |||A|||)||X||^2,$$

so that for λ_k large enough (which is sufficient for our purpose), we have existence of some $\beta > 0$ (independent of k) such that

$$\langle C(\lambda_k)X, X\rangle \leqslant -\beta||X||^2.$$

Then, it is standard to deduce that we have, for any $t \geqslant 0$,

$$||e^{tC(\lambda_k)}Y_k^0|| \leqslant e^{-\beta t}||Y_k^0|| \leqslant ||Y_k^0||. \tag{A.5}$$

Combining (A.4) together with (A.2) and (A.5), we deduce that for any $i \in [|, 0, n-1|]$, we have

$$||\psi^{(i)}||_{\infty,[0,T]} \leqslant \left(C\left(1 + \frac{1}{T^n}\right) \sum_{l=0}^{i} |||C(\lambda_k)|||^{i-l} \right) ||Y_k^0||.$$

Since $C(\lambda_k)$ has coefficients that depend linearly on $\lambda_k > 0$, one can find some $C > 1$ such that $|||C(\lambda_k)||| \leqslant C(1 + \lambda_k)$. We deduce that

$$||\psi^{(i)}||_{\infty,[0,T]} \leqslant \left(C\left(1 + \frac{1}{T^n}\right) C^i(1+\lambda_k)^i \right) ||Y_k^0|| \leqslant \left(C\left(1 + \frac{1}{T^n}\right) C^n(1+\lambda_k)^n \right) ||Y_k^0||.$$

Using this last inequality together with (A.3), we deduce that there exists come $C > 0$ (depending only on n) such that

$$||U_k(t)||_{\infty,[0,T]} \leqslant \left(C\left(1 + \frac{1}{T^n}\right) (1+\lambda_k)^n \right) \max_{i\in[|0,n-1|]} ||E_i(\lambda_k)||,$$

from which we deduce

$$||U_k(t)||_{L^2(0,T} \leqslant \sqrt{T}||U_k(t)||_{\infty,[0,T]} \leqslant \left(C\left(\sqrt{T} + \frac{1}{T^{n-\frac{1}{2}}}\right) (1+\lambda_k)^n \right) \max_{i\in[|0,n-1|]} ||E_i(\lambda_k)||.$$

Since the coefficients of $E_i(\lambda_k)$ depend only on λ_k and are fractional in λ_k, the desired result follows. ∎

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Conflict of Interest

The authors declare that they have no conflict of interest.

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