A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation

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Abstract

In this article, we revisit a result by A. Koenig concerning the non-controllability of the half-heat equation posed on $\mathbb{R}$, with a control domain that is a measurable set whose exterior contains an interval. The main novelty of the present article is to disprove the corresponding observability inequality by using as an initial condition a family of prolate spheroidal wave function (PSWF) translated in the Fourier space, associated to a parameter c that goes to ∞. The proof is essentially based on the dual nature of the PSWF together with direct computations, showing that the solution "does not spread out" too much during time, with respect to the parameter c. As a consequence, we obtain a new non-controllability result on the Grushin equation posed on $\mathbb{R} \times \mathbb{R}$.

Keyworlds: Controllability, observability, fractional parabolic equations, prolate spheroidal wave functions.

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1 Introduction

Let $T > 0$, and let ω be a mesurable subset of $\mathbb{R}$, such that $\overline{\omega} \neq \mathbb{R}$ (which means that $\mathbb{R} \setminus \omega$ contains at least a nonempty interval). We are interested in the following control problem, that we will call half-heat equation with distributed control:

$$\begin{cases} \partial_t y\,(t,x) + |\nabla|\, y\,(t,x) = 1_\omega v(t,x) \text{ in } (0,T)\times\mathbb{R}, \\ y\,(0,x) = y^0\,(x) \text{ in } L^2(\mathbb{R}). \end{cases} \tag{1}$$

The operator $|\nabla|$ is defined as a Fourier multiplier: for any $h \in H^1(\mathbb{R})$ and any $\xi \in \mathbb{R}$, we have

$$\widehat{|\nabla|\, h}(\xi) = |\xi|\widehat{h}(\xi).$$

Here and in what follows, for $h \in L^2(\mathbb{R})$, $\widehat{h}$ is the Fourier transform of h given by

$$\widehat{h}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

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Notice that $|\nabla|$ can also be defined equivalently (and also in many other ways, see *e.g.* [30]) as a singular integral operator by the following formula: for any $h \in H^1(\mathbb{R})$ and almost any $x \in \mathbb{R}$,

$$|\nabla|\, h(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(x)-h(y)}{|x-y|^2} dy.$$

It is well-known that as soon as $v \in L^2((0,T)\times\mathbb{R})$ and $y^0 \in L^2(\mathbb{R})$, there exists a unique solution y to (1) verifying moreover $y \in C^0([0,T], L^2(\mathbb{R}))$, and there exists $C(T) > 0$ such that for any $v \in L^2((0,T)\times\mathbb{R})$ and any $y^0 \in L^2(\mathbb{R})$, we have

$$||y||_{C^0([0,T],L^2(\mathbb{R}))} \leqslant C(T) \left(||y_0||_{L^2(\mathbb{R})} + ||v||_{L^2((0,T)\times\mathbb{R})}\right)$$

(see *e.g.* [19, Theorem 2.37]). Our main result is the following.

Theorem 1.1 *System* (1) *is not null-controllable for no time $T > 0$, in the following sense: for any $T > 0$, there exists at least one initial condition $y^0 \in L^2(\mathbb{R})$ such that there exists no $v \in L^2((0,T)\times\mathbb{R})$ for which the solution y of* (1) *verifies $y(T) = 0$.*

Our second result concerns the following Grushin equation on $\mathbb{R}^2$. We consider $\Omega = \mathbb{R}\times\omega$. We are interested in the following Grushin equation, controlled on Ω:

$$\begin{cases} \partial_t f\,(t,x,y) - \partial^2_{xx} f(t,x,y) - x^2 \partial^2_{yy} f(t,x,y) = 1_\Omega(t,x,y) g(t,x,y) \text{ in } (0,T)\times\mathbb{R}^2, \\ f\,(0,x,y) = f^0\,(x,y) \text{ in } \mathbb{R}^2. \end{cases} \tag{2}$$

As for the previous equation, it is easy to prove that for $g \in L^2((0,T)\times\mathbb{R}^2)$ and $f^0 \in L^2(\mathbb{R}^2)$, there exists a unique solution f to (2) verifying moreover $f \in C^0([0,T], L^2(\mathbb{R}^2))$, and there exists $C(T) > 0$ such that for any $g \in L^2((0,T), L^2(\mathbb{R}^2))$ and any $f^0 \in L^2(\mathbb{R}^2)$, we have

$$||f||_{C^0([0,T],L^2(\mathbb{R}))} \leqslant C(T) \left(||f_0||_{L^2(\mathbb{R}^2)} + ||g||_{L^2((0,T)\times\mathbb{R}^2)}\right).$$

Our second main result is the following.

Theorem 1.2 *System* (2) *is not null-controllable for no time $T > 0$.*

1.1 Comparison with the existing literature

Controllability properties of fractional heat equations have been an increasing subject of interest these last years. The most well-understood case is the fractional heat equation on a bounded interval of $\mathbb{R}$, with a scalar control, for the so-called spectral Laplace operator (*i.e.* obtained by functional calculus from the decomposition of the Dirichlet-Laplace operator in its basis of eigenfunctions). The framework of a scalar control encompasses most of the $1D$ situations where we have either a boundary control, or a distributed control that is imposed to be with separated variables, *i.e.* under the form $u(t,x) = v(t)f(x)$, where $f(x)$ is a fixed profile (with some conditions on its support and the behavior of its Fourier coefficients), and $v(t)$ is a one-dimensional control belonging for example to a L^2-space. In this context, null-controllability turns out to be essentially equivalent to the construction of appropriate bi-orthogonal functions to a family of exponential and strongly relies on the applications of the generalized Müntz theorem given in [52] (see also [4] and [48, Appendix]). To be more precise, the first positive result for $1D$ fractional heat equations (of exponent $\alpha > 1/2$) on a bounded interval with scalar control was provided in [23] thanks to the celebrated moment method (see also [15] for a close result with a definition of the fractional Laplace operator similar to the one used in the present article). Precise estimates on the cost of controllability in this case have been obtained in [48, 49, 40, 41]. Negative results in the case of exponents $\alpha \leqslant 1/2$ have been obtained in [22] and made more precise in [45]. We also mention [48] for Neumann boundary conditions instead of Dirichlet boundary conditions. See also [48, Appendix] for some extensions in the case where the control belongs to a finite-dimensional space.

The multi-dimensional setting has been less explored for fractional heat equations, even if the case of the usual heat equation on a bounded domain of $\mathbb{R}^n$ or on a smooth compact Riemannian manifold with or without boundary has been known since the seminal works [37] and [27]. Positive results for distributed control on compact manifolds for exponents $\alpha > 1/2$ are given in [48], with precise estimates on the cost of controllability.

The case of unbounded domains is also not well-understood for the moment. In [48], a positive result is given for exponents $\alpha > 1/2$ when the control set is the exterior of a compact set. For the usual heat equation, related works are also [46] and [47], for some counterexamples on the half-line and the half-space, with boundary controls. Let us mention that for the heat equation on $\mathbb{R}^n$, the optimal controllability results have been only recently obtained in [58] and [21], the main tool being the Logvinenko-Sereda uncertainty principle proved in [43] (see also [28]) and the use of the Lebeau-Robbiano strategy (see [37] and [38]). The only known negative results for $\alpha < 1/2$ on $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) are given in [33]. An extension to the half-heat equation (1) is provided in [32, Chapter 2] (notice that in the two previous references, a little bit more general model is studied here, the "rotated fractional heat equations"). The proof of [32, Chapter 2] is based on the study of some inequalities for precise classes of holomorphic functions, but turns out to give a less precise result than ours. Notably, it cannot be applied to the Grushin operator.

Notice that proving negative controllability results for fractional heat equations and distributed control is much more difficult than the case of scalar input controls. For instance, for equation (1), at almost each time $t \in [0, T]$, the control lives in the infinite-dimensional space $L^2(\omega)$, hence, there is no reason that we can restrict to a scalar (or finite-dimensional) input control as one could think for positive controllability results. Notably, the application of the generalized Müntz Theorem is not enough to deduce negative controllability results.

On the other hand, controllability properties of evolution equations involving hypoelliptic operators have recently become an active field of study (see [2, 3, 10, 11, 12, 13, 14, 36] for hypoelliptic diffusions different that the Grushin ones, which will be discussed into details later on), due to the specific difficulties arising in this context, namely geometric conditions on the control region together with the appearance of minimal times of controllability. One simple example (but of interest) of such equations is the Grushin equation, that has been widely studied in the literature but is still not completely understood. The example of the Grushin plane can be seen as a first step to study the controllability properties of more general heat equations on singular Riemannian manifolds, which explains the interest of understanding deeply this case. Since the seminal work [6], some improvements have been obtained in [8, 9, 31, 20]. Notably, for the Grushin equation posed on $(-1, 1) \times (0, 1)$, the situation is the following:

- If the control set is a vertical strip that does not touch the singularity $\{x = 0\}$, controllability holds only in large time. The minimal control time can be exactly characterized and depends on the distance between the vertical strip and the singularity. Notably, if the vertical strip touches the singularity, we have controllability in arbitrary small time ([6, 8, 9, 1]).
- On the contrary, if the control set is the exterior of a horizontal strip, we never have null controllability ([31]).
- To finish, if the control region is the neighbourhood of a curve coming from the "bottom" of $(-1, 1) \times (0, 1)$ and going to the top of $(-1, 1) \times (0, 1)$, we also have a result of null-controllability in sufficiently large time ([20]).

For some generalizations in higher dimension, see [7]. See also [50, 17] for a study of the Grushin operator with a singular potential, and [42] for generalizations in the case of measurable sets of controls. Remark that most of the above results concern controllability regions that have a rectangular structure. This is due to the fact that one of the main ingredients of the results described above is the use of Fourier series in one direction in order to reduce the problem to the study of a family of one-dimensional PDEs with a parameter (the Fourier coefficient). Trying to generalize the above results to other geometrical situations (general control domain, global geometry that is not cartesian) seems to be a very difficult challenge. Notice that some of the above results can be extended to the case $(-1-1)\times\mathbb{T}$, $\mathbb{R}\times\mathbb{T}$ or $\mathbb{R}\times(0,1)$ (see notably [32, Chapter 2]). However, to our knowledge, the case where the variable y belongs to the whole space $\mathbb{R}$ has not been investigated so far. This is one of the motivations of the present work.

We also mention [18, 24, 25, 44] for other examples of control problems of parabolic type with interior degeneracy in one space dimension.

Concerning the PSWF, since the seminal works [54, 34, 35, 55], they have been widely studied from a theoretical and numerical point of view. They also turned out to have many applications, notably in sampling and signal or image processing. For more explanations on the PSWF and their applications, we refer to the surveys [56, 57], the books [51, 29] and the references therein. However, as far as the author knows, this is the first time that they are used in the context of controllability properties of PDEs.

1.2 Heuristic and outline of the paper

Here, our goal is to give a different proof of the results given in [32, Chapter 2], relying on an explicit construction of a family of counter-examples given by the PSWF. Another advantage of our technique is that it enables to give a corresponding result on the Grushin equation on $\mathbb{R}\times\mathbb{R}$, and it is more likely to be extendable in the multi-dimensional case or on more general non-compact manifolds (see our concluding Section 5).

More precisely, we use the first PSWF as initial condition in order to disprove some observability inequality for the free half-heat equation (15). Our approach seems quite natural: as we will see, the first PSWF are bandlimited functions that saturate the Logvinenko-Sereda uncertainty principle outside of the interval $(-1,1)$. Notably, they strongly concentrate in $(-1,1)$, and they become more and more concentrated (at an exponential rate) as soon as the band-limit increases. Hence, they are very natural candidates in order to disprove some observability inequality when the observation is some set at least not contains $(-1,1)$. Of course, because of the dissipativity of the half-heat equation, one cannot expect that the associated solution to the free half-heat equation keeps the same concentration properties. However, it can be proved (and this is the main point of the present study) that during time, the associated solution still remains "quite concentrated". This will enable us to conclude that the observation on $(0,T)\times\omega$ will be "small" by comparison to the whole L^2-norm of the associated solution at time T. The extension to the Grushin plane is quite straightforward, using the particular form of our control set Ω and an appropriate Fourier decomposition.

The paper is organized as follows. In Section 2.1, we recall some known properties of the PSWF that we will need. In Section 3.1, we reduce the problem to the disproof of an adequate observability on a dual problem. In Section 3.2, we give some properties of the particular family of solutions of the adjoint problem we will consider. In Section 3.3, we give some useful estimates on the complex PSWF. Section 3.4 is the core of the paper: we give two main Lemmas 3.4 and 3.5 that express the fact that our family of solutions, which is very well concentrated at initial time, does not "spread out" too much during time. We can then conclude to the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2 (which will mainly be a consequence of the already proved estimates in Section 2). In Section 5, we give some concluding remarks.

2 Proof of Theorem 1.1

2.1 Preliminaries on the PSWF

In this section, we regroup some basic facts on the first PSWF. For the sake of clarity, we follow closely the presentation given in [51, Section 2.4] and we refer as often as possible to [51], even if many results have been in general known for a longer time. Let $c>0$, destined to tend to $+\infty$. We introduce the following operator

$$F_c : \varphi \in L^2(-1,1) \mapsto \left(x \mapsto \int_{-1}^{1} e^{icx\xi} \varphi(\xi)\, d\xi \right) \in L^2(-1,1).$$

It is easy to prove that F_c is a compact and normal operator on $L^2(-1,1)$, with distinct eigenvalues (see [51, Theorem 2.3]). We call λ_c the largest (in modulus) eigenvalue of F_c. We will call ψ_c "the" (up to a normalization and orientation that will be detailed afterwards) first eigenvector of F_c. ψ_c is called the first (the one of index 0) PSWF with parameter c. By definition, ψ_c verifies, for almost every $x \in (-1,1)$,

$$\lambda_c \psi_c(x) = \int_{-1}^{1} e^{icx\xi} \psi_c(\xi)\, d\xi.$$

Hence, ψ_c is a bandlimited function with bandlimit c, and its Fourier transform is in $L^2(-c,c)$ by the Plancherel Theorem and an easy change of variable. From the Paley-Wiener Theorem (see e.g. [53, Theorem 19.3, Page 370]), we deduce that ψ_c can be extended on the entire plane $\mathbb{C}$, and ψ_c is an entire function of exponential type c which lies in $L^2(\mathbb{R})$, Moreover, the above expression can be extended in the complex plane as follows: for any $z = x + it \in \mathbb{C}$,

$$\psi_c(x+it) = \frac{1}{\lambda_c} \int_{-1}^{1} e^{ic(x+it)\xi} \psi_c(\xi)\, d\xi. \tag{3}$$

We have the following properties on ψ_c (see [51, Theorem 2.3]).

Proposition 2.1 *ψ_c is real and even on $\mathbb{R}$ (and hence ψ_c is even also on $\mathbb{C}$), and has no roots on $(-1,1)$.*

From (3) and Proposition 2.1, a straightforward computation gives the following property for ψ_c on the complex plane $\mathbb{C}$.

Proposition 2.2 *For any $z = x + it \in \mathbb{C}$,*

$$\psi_c(-x+it) = \psi_c(x-it) = \overline{\psi_c(x+it)}. \tag{4}$$

Now, we explain how we normalize ψ_c. We choose ψ_c such that ψ_c is normalized in $L^2(\mathbb{R})$ $-$norm and $\psi_c > 0$ in $(-1,1)$ (it has no roots on $(-1,1)$ by Proposition 2.1).

In what follows, we will also need to consider the operator

$$Q_c : \varphi \in L^2(-1,1) \mapsto \left(x \mapsto \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c(x-\xi))}{x-\xi} \varphi(\xi)\, d\xi \right) \in L^2(-1,1).$$

Remark that $Q_c = P_c \circ E$, where E is the extension operator (by 0) from $L^2(-1,1)$ to $L^2(\mathbb{R})$, and

$$P_c : \varphi \in L^2(\mathbb{R}) \mapsto \left(x \mapsto \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c(x-\xi))}{x-\xi} \varphi(\xi)\, d\xi \right) \in L^2(\mathbb{R}).$$

P_c turns out to be exactly the celebrated projection in $L^2(\mathbb{R})$ on its closed subspace of bandlimited functions with bandlimit c. By [51, Corollary 2.1], we have

$$F_c F_c^* = F_c^* F_c = \frac{2\pi}{c} Q_c.$$

As in [51, (3.48)], we introduce

$$\mu_c = \frac{c}{2\pi} \lambda_c^2. \tag{5}$$

Then, ψ_c is clearly an eigenvector for Q_c associated with the eigenvalue μ_c:

$$\mu_c \psi_c(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sin(c(x-\xi))}{x-\xi} \psi_c(\xi)\, d\xi. \tag{6}$$

From [51, Theorem 7.6] and our choice of normalization (as explained in [51, Page 236], the authors chose a normalization in $L^2(-1,1)$-norm, which explains the difference of expression with [51, Theorem 7.6]), we have the following equality.

Lemma 2.1 *We have*

$$\int_{-1}^{1} \psi_c(x)^2\, dx = \mu_c \left(= \mu_c \int_{\mathbb{R}} \psi_c(x)^2\, dx \right). \tag{7}$$

An important feature of the first PSWF is the following concentration property (see [51, Theorem 3.53]): any other function f which is c-bandlimited and has $L^2(\mathbb{R})$-norm equal to 1 is such that

$$\int_{\mathbb{R}\setminus[-1,1]} f^2(x)dx \geqslant 1 - \mu_c. \tag{8}$$

It means that ψ_c is the c-bandlimited function which concentrates the most on $[-1,1]$. Notice that (8) is a version of the Logvinenko-Sereda uncertainty principle for c-bandlimited functions given in [43], in the very particular case where the observation is made on the thick set $\mathbb{R} \setminus [-1,1]$. Hence, $(1-\mu_c)^{-1}$ is the "best constant" in the Logvinenko-Sereda uncertainty principle for c-bandlimited functions for an observation on the thick set $\mathbb{R}\setminus[-1,1]$, reached only for the linear subspace generated by the first PSWF ψ_c.

From [26, Theorem 1] applied with $a = \frac{2C_1 c}{\log(c)}$ and $n = 0$ (see the introduction of [26]), we also deduce the following estimate.

Lemma 2.2 *We have*

$$1 - \mu_c \sim 4\sqrt{\pi} c^{\frac{1}{2}} e^{-2c} \text{ as } c \to \infty. \tag{9}$$

Let us now explain the dual nature of the PSWF, that are both eigenfunctions for an integral and a differential operator. Let us call L_c the differential operator

$$L_c : \varphi \in C^2(-1,1) \mapsto \left(x \mapsto -\left(1-x^2\right)\varphi''(x) + 2x\varphi'(x) + c^2x^2\varphi(x)\right) \in C^0(-1,1).$$

Then, it can be proved (see [51, Theorem 2.8]) that F_c and L_c commute. Hence, ψ_c is also an eigenvector of L_c, with associated eigenvalue χ_c. Moreover, by [51, (3.4)], we have

$$0 < \chi_c < c^2. \tag{10}$$

We also have the following asymptotic formula (see [51, Theorem 2.6]):

$$\chi_c \sim c \text{ as } c \to \infty. \tag{11}$$

By definition, ψ_c verifies: for any $x \in (-1,1)$,

$$-\left(1-x^2\right)\psi_c''(x) + 2x\psi_c'(x) + c^2x^2\psi_c(x) = \chi_c\psi_c(x).$$

By analyticity, this expression can be extended on the whole complex plane, so that ψ_c verifies: for any $z \in \mathbb{C}$,

$$-\left(1-z^2\right)\psi_c''(z) + 2z\psi_c'(z) + c^2z^2\psi_c(z) = \chi_c\psi_c(z). \tag{12}$$

To conclude, let us now give an equivalent on $|\psi_c(1)|^2$ as $c \to \infty$.

Lemma 2.3

$$|\psi_c(1)| \sim 2\pi^{\frac{1}{4}} c^{\frac{3}{4}} e^{-c} \text{ as } c \to \infty. \tag{13}$$

Proof of Lemma 2.3. We remark that by (6), ψ_c verifies, for any $x \in (-1,1)$,

$$\mu_c \psi_c(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sin(c(x-\xi))}{x-\xi}\psi_c(\xi)\,d\xi.$$

Hence, $\psi_c\left(\frac{\cdot}{\sqrt{c}}\right)$ verifies, for any $x \in (-\sqrt{c}, \sqrt{c})$,

$$\mu_c \psi_c\left(\frac{x}{\sqrt{c}}\right) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sin(\sqrt{c}x - c\xi)}{x/\sqrt{c}-\xi}\psi_c(\xi)\,d\xi.$$

Using the change of variable $\xi' = \sqrt{c}\xi$, we obtain

$$\mu_c \psi_c\left(\frac{x}{\sqrt{c}}\right) = \frac{1}{\sqrt{c}\pi}\int_{-\sqrt{c}}^{\sqrt{c}} \frac{\sin(\sqrt{c}(x-\xi'))}{x/\sqrt{c}-\xi'/\sqrt{c}}\psi_c\left(\frac{\xi'}{\sqrt{c}}\right)d\xi' = \frac{1}{\pi}\int_{-\sqrt{c}}^{\sqrt{c}} \frac{\sin(\sqrt{c}(x-\xi'))}{x-\xi'}\psi_c\left(\frac{\xi'}{\sqrt{c}}\right)d\xi'.$$

Hence, $\psi_c\left(\frac{\cdot}{\sqrt{c}}\right)$ is exactly the first eigenfunction of the operator

$$\varphi \in L^2\left(-\sqrt{c},\sqrt{c}\right) \mapsto \left(x \mapsto \int_{-\sqrt{c}}^{\sqrt{c}} \frac{\sin(\sqrt{c}(x-\xi'))}{x-\xi'}\varphi(\xi')\,d\xi'\right) \in L^2(-\sqrt{c},\sqrt{c}),$$

which is exactly the setting of [26] with $\sqrt{c} = a$ and $n = 0$. Let us set

$$f(x) = \frac{1}{c^{\frac{1}{4}}\sqrt{\mu_c}}\psi_c\left(\frac{x}{\sqrt{c}}\right). \tag{14}$$

Then, we have by the definition of μ_c given in (7) that

$$\int_{-\sqrt{c}}^{\sqrt{c}} f(x)^2\,dx = \frac{1}{\sqrt{c}\mu_c}\int_{-\sqrt{c}}^{\sqrt{c}} \psi_c\left(\frac{x}{\sqrt{c}}\right)^2 dx = \frac{1}{\mu_c}\int_{-1}^{1}\psi_c(y)^2\,dy = 1.$$

We can now apply [26, III, Page 329] since we are in the correct normalization setting, and we obtain that

$$|f\left(\sqrt{c}\right)| \sim 2\pi^{\frac{1}{4}}c^{\frac{1}{2}}e^{-c}\mu_c \text{ as } c \to \infty.$$

Going back to ψ_c thanks to (14) and taking into account that $\mu_c \to 1$ as $c \to \infty$ by (9), we deduce that (13) holds. ■

3 Proof of Theorem 1.1

3.1 Reduction to the disproof of an observability inequality

Our proof mainly relies on a classical duality argument: the null-controllability of (1) at time $T > 0$ is equivalent to the following property: there exists $C(T) > 0$ such that for any $u^0 \in L^2(\mathbb{R})$, the solution u of

$$\begin{cases} \partial_t u(t,x) + |\nabla|\, u(t,x) = 0 \text{ in } (0,T)\times\mathbb{R}, \\ u(0,x) = u^0(x) \end{cases} \tag{15}$$

verifies the following inequality, called observability inequality:

$$\int_{\mathbb{R}} |u(T,x)|^2 dx \leqslant C(T)\int_0^T\int_{\omega}|u(t,x)|^2 dxdt. \tag{16}$$

For more explanations, see *e.g.* [19, Theorem 2.44, Page 56]. Hence, in order to prove Theorem 1.1, it is sufficient to exhibit a family of initial conditions u_c^0 depending on some parameter $c>0$ such that the corresponding family of solutions u_c to (15) make the quotient

$$\frac{\int_0^T \int_\omega |u_c(t,x)|^2 dxdt}{\int_{\mathbb{R}} |u_c(T,x)|^2 dx}$$

go to 0 as $c \to \infty$.

Let us explain some important reduction for what follows. Since ω is some measurable subset of $\mathbb{R}$ such that $\overline{\omega} \neq \mathbb{R}$, $\mathbb{R} \setminus \overline{\omega}$ is an open subset of $\mathbb{R}$ which is not empty, so, there exists some ball $B(y_0,\varepsilon)$ in $\mathbb{R}\setminus\overline{\omega}$ with $\varepsilon>0$ and $y_0 \in \mathbb{R}$. Without loss of generality, we can assume that $y_0 = 0$ (this is just a question of translating in space the initial condition of (15)). Moreover, we assume without loss of generality that $\omega = \mathbb{R}\setminus[-\varepsilon,\varepsilon]$. Indeed, if (15) were controllable on ω, it would be also controllable on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$ since $\omega \subset \mathbb{R}\setminus[-\varepsilon,\varepsilon]$. Hence, we will disprove the observability on $\mathbb{R}\setminus[-\varepsilon,\varepsilon]$.

3.2 A particular family of solutions

First of all, let us give the following crucial lemma.

Lemma 3.1 *If $u^0(x) = \psi_c(x)e^{icx}$, then the solution u of (15) is given for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}$ by*

$$u(t,x) = e^{ic(x+it)}\psi_c(x+it). \tag{17}$$

Moreover, its Fourier transform in space is given by

$$\widehat{u}(t,\xi) = \frac{2\pi}{c\lambda_c} 1_{[-c,c]}(\xi - c)\, e^{-t|\xi|}\psi_c\left(\frac{\xi-c}{c}\right). \tag{18}$$

Proof of Lemma 3.1. First of all, we have

$$\widehat{u}(t,\xi) = e^{-t|\xi|}\widehat{\psi_c(\cdot)\,e^{ic\cdot}}(\xi) = e^{-t|\xi|}\widehat{\psi_c}(\xi - c).$$

Due to the expression of the PSWF given in (3) for $t=0$, an easy change of variable and the fact that ψ_c is even on $\mathbb{R}$ by Proposition 2.1, we deduce that

$$\psi_c(x) = \frac{1}{c\lambda_c}\int_{-c}^{c} e^{ix\xi'}\psi_c\left(\frac{\xi'}{c}\right)d\xi' = \frac{1}{c\lambda_c}\int_{\mathbb{R}} 1_{[-c,c]}(\xi')\,e^{ix\xi'}\psi_c\left(\frac{\xi'}{c}\right)d\xi'.$$

Hence, the Fourier inversion formula of the Fourier transform gives

$$\widehat{\psi_c}(\xi) = \frac{2\pi}{c\lambda_c}1_{[-c,c]}(\xi)\,\psi_c\left(\frac{\xi}{c}\right).$$

We deduce that (18) holds.

Using one more time the Fourier inversion formula and the change of variable $\xi' = \frac{\xi-c}{c}$, we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{c\lambda_c}\int_{\mathbb{R}} e^{-t|\xi|+ix\xi}\psi_c\left(\frac{\xi-c}{c}\right)1_{[-c,c]}(\xi-c)\,d\xi \\ &= \frac{1}{\lambda_c}\int_{-1}^{1} e^{-ct|\xi'+1|+icx(\xi'+1)}\psi_c(\xi')\,d\xi' \\ &= \frac{1}{\lambda_c}\int_{-1}^{1} e^{-ct(\xi'+1)+icx(\xi'+1)}\psi_c(\xi')\,d\xi' \\ &= \frac{1}{\lambda_c}e^{-ct+icx}\int_{-1}^{1} e^{-ct\xi'+icx\xi'}\psi_c(\xi')\,d\xi'. \end{aligned}$$

We conclude by remarking that $-ct+icx = ic(x+it)$ and by applying identity (3). ∎

Remark that if u is a solution of (15) associated with the initial condition u_0, then, for any $\beta > 0$, $u(\beta\cdot, \beta\cdot)$ is still a solution of (15) associated with the initial condition $u_0(\beta\cdot)$. Hence, we deduce the following result.

Corollary 3.2 *Let $\alpha > 0$. If $u^0_{c,\varepsilon,\alpha}(x) = e^{i\frac{\alpha c^2}{\log(c)\varepsilon}x}\psi_c\left(\frac{\alpha c}{\log(c)\varepsilon}x\right)$, then the corresponding solution of (15) is given for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}$ by*

$$u_{c,\varepsilon,\alpha}(t,x) = e^{i\frac{\alpha c^2}{\log(c)\varepsilon}(x+it)}\psi_c\left(\frac{\alpha c}{\log(c)\varepsilon}(x+it)\right). \tag{19}$$

3.3 The behaviour of ψ_c on the upper half plane for large x

Our proof requires to obtain some fine estimates on ψ_c in the complex plane by using growth properties of second order ordinary differential equations with complex coefficients. Let us remind the following general theorem (see [51, Theorem 2.21] and [39]).

Proposition 3.1 *Let $a < b$ be real numbers. Consider $w, u, \beta, \gamma : [a,b] \to \mathbb{C}$ that are of class C^1, verifying the following differential system : for any $t \in [a,b]$, we have*

$$\begin{pmatrix} w'(t) \\ u'(t) \end{pmatrix} = \begin{pmatrix} 0 & \beta(t) \\ \gamma(t) & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ u(t) \end{pmatrix}. \tag{20}$$

Assume moreover that β and γ do not vanish on $[a,b]$. Introduce

$$R(t) = \frac{|\beta(t)|}{|\gamma(t)|} \tag{21}$$

and

$$Q(t) = |w(t)|^2 + R(t)|w'(t)|^2. \tag{22}$$

Then, for any $a \leqslant t_0 \leqslant t_1 \leqslant b$, we have

$$\frac{\sqrt{Q(t_0)}}{R(t_0)^{\frac14}} \leqslant \frac{\sqrt{Q(t_1)}}{R(t_1)^{\frac14}} \exp\left(\int_{t_0}^{t_1} \sqrt{\left(\frac{R'(s)}{4R(s)}\right)^2 + \frac{|\beta(s)\gamma(s)| + Re\left(\beta(s)\gamma(s)\right)}{2}}ds\right). \tag{23}$$

We can deduce from Proposition 3.1 the following estimate of ψ_c on the upper half plane, for large values of x, that might be of independent interest.

Proposition 3.2 *There exists two constants $C, C_1 > 0$ such that for any $x \geqslant 2$, $t \geqslant 0$ and $c > 0$ large enough (independently of x or t), we have*

$$|\psi_c(x+it)| \leqslant Ce^{ct}\frac{\psi_c(1)}{c\lambda_c\sqrt{x^2+t^2}}\exp\left(\frac{C_1 ct}{x^2+t^2}\right). \tag{24}$$

Proof of Proposition 3.2. In the following proof, $C > 0$ denotes a numerical constant, that does not depend on x, t or c, and might change from inequality to inequality. For $x + it \in \mathbb{C}$, with $x \geqslant 2$ and $t \geqslant 0$, we introduce

$$\phi_c(x+it) = \psi_c(x+it)\sqrt{(x+it)^2 - 1}. \tag{25}$$

It is well-known that this change of unknown enables to remove the first-order term in (12), and that ϕ_c verifies (see *e.g.* [51, Section 4.2.3])

$$\phi_c''(x+it) + \left(\frac{c^2(x+it)^2 - \chi_c}{(x+it)^2 - 1} + \frac{1}{\left((x+it)^2 - 1\right)^2}\right)\phi_c(x+it) = 0. \tag{26}$$

We introduce the following notations, according to (26) and Proposition 3.1 applied with $w(x) = \phi_c(x+it)$ and $u(x) = \phi_c'(x+it)$:

$$\beta(x) = 1,\ \gamma(x) = -\left(\frac{c^2(x+it)^2 - \chi_c}{(x+it)^2-1} + \frac{1}{\left((x+it)^2-1\right)^2}\right),\ R(x) = \frac{1}{\left|\frac{c^2(x+it)^2-\chi_c}{(x+it)^2-1} + \frac{1}{((x+it)^2-1)^2}\right|}. \tag{27}$$

Remark that R verifies that for any $t \geqslant 0$, any $x \geqslant 2$ and any $c \geqslant 1$ (thanks to (10)),

$$\frac{1}{Cc^2} \leqslant |R(x)| \leqslant \frac{C}{c^2}. \tag{28}$$

We also introduce

$$Q(x) = |\phi_c(x+it)|^2 + R(x)\left|\phi_c'(x+it)\right|^2. \tag{29}$$

From now on, we call $y_0 = \sqrt{\frac{\chi_c}{c^2}} \in (0,1)$ by (10). An explicit computation gives

$$\mathrm{Re}\left(\gamma(x)\right) = -\frac{c^2\left(t^4 + t^2\left(2x^2+y_0^2+1\right) + \left(x^2-1\right)(x-y_0)(x+y_0)\right)}{2\left(t^4+2t^2\left(x^2+1\right)+\left(x^2-1\right)^2\right)} - \frac{t^4+t^2\left(2-6x^2\right)+\left(x^2-1\right)^2}{\left(t^4+2t^2\left(x^2+1\right)+\left(x^2-1\right)^2\right)^2} \tag{30}$$

and

$$\mathrm{Im}\left(\gamma(x)\right) = -\frac{2c^2tx\left(y_0^2-1\right)}{t^4+2t^2\left(x^2+1\right)+\left(x^2-1\right)^2} - \frac{4tx\left(t^2-x^2+1\right)}{\left(t^4+2t^2\left(x^2+1\right)+\left(x^2-1\right)^2\right)^2} \tag{31}$$

From (30), since for any $x \geqslant 2$ and $t \geqslant 0$,

$$\left|\frac{t^4+t^2\left(2-6x^2\right)+\left(x^2-1\right)^2}{\left(t^4+2t^2\left(x^2+1\right)+\left(x^2-1\right)^2\right)^2}\right| \leqslant C,$$

and $y_0 \in (0,1)$, it is clear that for any $x \geqslant 2, t \geqslant 0$ and $c > 0$ large enough (not depending on x or t, which will be assumed implicitly from now on), we have

$$|\mathrm{Re}(\gamma(t))| \geqslant Cc^2 \text{ and } \mathrm{Re}(\gamma(t)) \leqslant 0. \tag{32}$$

From (31), since for any $x \geqslant 2$ and $t \geqslant 0$, we have

$$t^4 + 2t^2(x^2+1) + (x^2-1) \geqslant (t^2+x^2-1)^2$$

and

$$|t^2 - x^2 + 1| \leqslant |t^2+x^2-1|,$$

combined with the fact that $y_0 \in (0,1)$, we also deduce that for $c \geqslant 2$,

$$|\mathrm{Im}(\gamma(x))| \leqslant \frac{2c^2tx}{(t^2+x^2-1)^2}. \tag{33}$$

Hence, using the inequality $\sqrt{a+b} - \sqrt{a} \leqslant b/(2\sqrt{a})$, true for any $a, b \geqslant 0$, since $\mathrm{Re}(\gamma(t)) \leqslant 0$ by (32), we obtain that

$$\frac{|\beta(s)\gamma(s)| + \mathrm{Re}\left(\beta(s)\gamma(s)\right)}{2} = \frac{\sqrt{\mathrm{Re}(\gamma(s))^2 + \mathrm{Im}(\gamma(s))^2} - |\mathrm{Re}\left(\gamma(s)\right)|}{2} \leqslant \frac{\mathrm{Im}(\gamma(s))^2}{4|\mathrm{Re}(\gamma(s))|}. \tag{34}$$

From (32), (33) and (34), we deduce that

$$\frac{|\beta(s)\gamma(s)| + \mathrm{Re}\,(\beta(s)\gamma(s))}{2} \leqslant C \frac{c^2 t^2 x^2}{(t^2+x^2-1)^4}. \tag{35}$$

Now, we apply Proposition 3.1. We obtain that for any $2 \leqslant x \leqslant x_1$, any $t \geqslant 0$ and any $c > 0$, we have

$$\frac{\sqrt{Q(x)}}{R(x)^{\frac14}} \leqslant \frac{\sqrt{Q(x_1)}}{R(x_1)^{\frac14}} \exp\left(\int_x^{x_1} \sqrt{\left(\frac{R'(s)}{4R(s)}\right)^2 + \frac{|\beta(s)\gamma(s)| + \mathrm{Re}\,(\beta(s)\gamma(s))}{2}} ds \right).$$

Using (28) and the inequality $\sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}$, true for any $a, b \geqslant 0$, we deduce that for $c > 0$ large enough,

$$\sqrt{Q(x)} \leqslant C\sqrt{Q(x_1)} \exp\left(\int_x^{x_1} \left| \frac{R'(s)}{4R(s)} \right| ds \right) \exp\left(\int_x^{x_1} \sqrt{\frac{|\beta(s)\gamma(s)| + \mathrm{Re}\,(\beta(s)\gamma(s))}{2}} ds \right). \tag{36}$$

Since R is a rational function of x, R'/R can change sign only a finite number of times on $[2, +\infty)$. Let us decompose $[x, x_1]$ as

$$[x, x_1] = \bigsqcup_{i=1}^{p} [a_{i-1}, a_i],\ a_{i-1} < a_i,$$

where R'/R is of strict constant sign on (a_i, a_{i+1}). Assume without loss of generality that R'/R is positive on (a_0, a_1) and that p is even (if not, it is very easy to adapt the following reasoning). Then,

$$\begin{aligned} \exp\left(\int_x^{x_1} \left| \frac{R'(s)}{4R(s)} \right| ds \right) &= \exp\left(\sum_{i=0}^{(p-2)/2} \int_{a_{2i}}^{a_{2i+1}} \frac{R'(s)}{4R(s)} ds - \sum_{i=0}^{(p-2)/2} \int_{a_{2i+1}}^{a_{2i+2}} \frac{R'(s)}{4R(s)} ds \right) \\ &= \left(\prod_{i=0}^{(p-2)/2} \frac{R(a_{2i+1})}{R(a_{2i})} \prod_{i=0}^{(p-2)/2} \frac{R(a_{2i+1})}{R(a_{2i+2})} \right)^{\frac14}. \end{aligned}$$

Using (28) and the fact that p is independent of c on t, x and x_1, (because the degree in x of the numerator of R does not depend on c or t, so p remains bounded when c , t, x and x_1 vary), the above expression provides

$$\exp\left(\int_x^{x_1} \left| \frac{R'(s)}{4R(s)} \right| ds \right) \leqslant C. \tag{37}$$

Now, let us remind the following expansions of ψ_c and ψ_c', coming from the expression provided in [51, Theorems 6.21 and 6.22] and extended in the complex plane:

$$\psi_c(x+it) = \frac{2\psi_c(1)\sin(c(x+it))}{c(x+it)\lambda_c} + o\left(\frac{1}{x}\right) \quad \text{as } x \to \infty$$

and

$$\psi_c'(x+it) = \frac{2\psi_c(1)\cos(c(x+it))}{(x+it)\lambda_c} + o\left(\frac{1}{x}\right) \quad \text{as } x \to \infty.$$

From (25), (28), (29) and the above expansions, we deduce that for any $t \geqslant 0$ and any $c > 0$ large enough,

$$\limsup_{x_1 \to +\infty} Q(x_1) \leqslant C e^{2ct} \frac{\psi_c(1)^2}{c^2 \lambda_c^2}.$$

Combining this estimate with (36), (37) and (35), we deduce that for any $x \geqslant 2$, any $t \geqslant 0$ and any $c > 0$ large enough,

$$\sqrt{Q(x)} \leqslant Ce^{ct}\frac{\psi_c(1)}{c\lambda_c}\exp\left(\int_x^{+\infty}\frac{Ccts}{(s^2+t^2-1)^2}ds\right) \leqslant Ce^{ct}\frac{\psi_c(1)}{c\lambda_c}\exp\left(\frac{Cct}{(x^2+t^2-1)}\right). \tag{38}$$

Let us now investigate $|\sqrt{(x+it)^2-1}|$. We have $|\sqrt{(x+it)^2-1}|^2 = \sqrt{(x^2-t^2-1)^2+4x^2t^2}$. Now, we remark that

$$\begin{aligned}(x^2-t^2-1)^2+4x^2t^2 &= (x^2-1)^2+t^4-2t^2(x^2-1)+4x^2t^2\\ &\geqslant (x^2-1)^2+t^4-2t^2(x^2-1)+4t^2(x^2-1)\\ &\geqslant (x^2-1)^2+t^4+2t^2(x^2-1)\\ &\geqslant ((x^2-1)+t^2)^2.\end{aligned}$$

We deduce that for any $x \geqslant 2$ and any $t \geqslant 0$, we have

$$|\sqrt{(x+it)^2-1}| \geqslant \frac{1}{4}\sqrt{x^2+t^2}. \tag{39}$$

Hence, combining (38), (29), (25) and (39) easily leads to (24). ∎

From now on, we consider

$$\alpha = 2C_1, \tag{40}$$

where C_1 is the constant appearing in (24). An important consequence is the following.

Corollary 3.3 *There exists $C > 0$ such that for any $c > 0$ large enough and any $t \geqslant 0$, we have*

$$\left(\int_{\mathbb{R}\setminus[-\frac{2C_1c}{\log(c)},\frac{2C_1c}{\log(c)}]}|\psi_c(x+it)|^2dx\right)e^{-2ct} \leqslant C\sqrt{\log(c)}\frac{1-\mu_c}{\frac{c}{\log(c)}+t}. \tag{41}$$

Remark 1 — *Estimate (41) roughly says that $\psi_c(x+it)e^{ic(x+it)}$ remains "well-concentrated" in space at least on $[-\frac{2C_1c}{\log(c)},\frac{2C_1c}{\log(c)}]$. Remark that it is important here to consider large enough x. Indeed, if x is too small, the term $\exp\left(\frac{Cct}{(x^2+t^2)}\right)$ in (24) will give an extra exponential term that enters in competition with the exponential decreasing given in (9).*

— *Let us emphasize that if we look at the proof, we would directly obtain that*

$$\left(\int_{\mathbb{R}\setminus[-\frac{2C_1c}{\log(c)},\frac{2C_1c}{\log(c)}]}|\psi_c(x+it)|^2dx\right)e^{-2ct} \leqslant C\sqrt{\log(c)}\frac{|\psi_c(1)|^2}{c(\frac{c}{\log(c)}+t)}.$$

This means that introducing μ_c and using estimation (9) is stricto sensu not needed in our proof. However, we think that expression (41) is more eloquent, due to the interpretation in terms of concentration of solutions we gave in the previous point of this remark.

Proof of Corollary 3.3. In the following proof, $C > 0$ denotes a numerical constant, that does not depend on x, t or c, and might change from inequality to inequality. Using (24) and (40), for any $c > 0$ large enough, $x \geqslant \frac{2C_1c}{\log(c)}$ and $t \geqslant 0$, we have

$$|\psi_c(x+it)|^2 \leqslant Ce^{2ct}\frac{\psi_c(1)^2}{c^2\lambda_c^2(x^2+t^2)}\exp\left(\frac{2C_1ct}{\frac{4C_1^2c^2}{\log(c)^2}+t^2}\right).$$

Since the maximum of the function

$$t \geqslant 0 \mapsto \frac{2C_1ct}{\frac{4C_1^2c^2}{\log(c)^2}+t^2}$$

is $\log(c)/2$, we deduce that

$$|\psi_c(x+it)|^2e^{-2ct} \leqslant C\sqrt{\log(c)}\frac{\psi_c(1)^2}{c^2\lambda_c^2(x^2+t^2)}.$$

Using (5), (9) and (13), we deduce that for any $c>0$ large enough, $x \geqslant \frac{2C_1c}{\log(c)}$ and $t \geqslant 0$, we have

$$|\psi_c(x+it)|^2e^{-2ct} \leqslant C\sqrt{\log(c)}\frac{1-\mu_c}{x^2+t^2}.$$

Integrating this expression between $\frac{2C_1c}{\log(c)}$ and $+\infty$ and using (4) leads to

$$\left(\int_{\mathbb{R}\setminus[-\frac{2C_1c}{\log(c)},\frac{2C_1c}{\log(c)}]}|\psi_c(x+it)|^2dx\right)e^{-2ct} \leqslant C\sqrt{\log(c)}(1-\mu_c)\frac{\mathrm{Arctan}(\frac{t\log(c)}{2C_1c})}{t}.$$

The easy estimate

$$\frac{\mathrm{Arctan}(y)}{y} \leqslant \frac{2}{1+y},\ y \geqslant 0,$$

applied to $y=\frac{t\log(c)}{2C_1c}$ enables to conclude. ■

3.4 Proof of the non-controllability result

Let us remind that $T>0$ is fixed. We still consider α chosen in (40), and we consider the solution $u_{c,\varepsilon,\alpha}$ given in (19) and the quotient

$$Q(c,\varepsilon)=\frac{\int_0^T||u_{c,\varepsilon,\alpha}(t,\cdot)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}dt}{||u_{c,\varepsilon,\alpha}(T,\cdot)||^2_{L^2(\mathbb{R})}}. \tag{42}$$

As already explained in Section 3.1, our goal is to prove that at fixed T, $Q(c,\varepsilon)\to 0$ as $c\to\infty$, which will exactly mean that the observability inequality (16) cannot be true.

We first estimate from above the numerator. This is the purpose of the following Lemma.

Lemma 3.4 *There exists $C(\varepsilon)>0$, depending only on ε, such that for any $t\geqslant 0$ and any c large enough (independently on t, but possibly depending on ε),*

$$||u_{c,\varepsilon,\alpha}(t,\cdot)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])} \leqslant C(\varepsilon)\left(\log(c)\right)^{\frac{5}{2}}\frac{(1-\mu_c)}{c^2(1+t)}. \tag{43}$$

Remark 2 *The term "$(1+t)$" will be particularly useful in the proof of Theorem 1.2.*

Proof of Lemma 3.4. From (19) with α as in (40) and an easy change of variables, we have

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}|u_{c,\varepsilon,\alpha}(t,x)|^2dx &= e^{-\frac{4C_1c^2}{\log(c)\varepsilon}t}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\left|\psi_c\left(\frac{2C_1c}{\log(c)\varepsilon}(x+it)\right)\right|^2\\ &=\frac{\varepsilon\log(c)}{2C_1c}e^{-\frac{4C_1c^2}{\log(c)\varepsilon}t}\int_{\mathbb{R}\setminus[-\frac{2C_1c}{\log(c)},\frac{2C_1c}{\log(c)}]}\left|\psi_c\left(x+i\frac{2C_1ct}{\log(c)\varepsilon}\right)\right|^2dx.\end{aligned}$$

Using (41), we deduce that

$$\int_{\mathbb{R}\backslash[-\varepsilon,\varepsilon]} |u_{c,\varepsilon,\alpha}(t,x)|^2 dx \leqslant C\sqrt{\log(c)}\frac{\varepsilon\log(c)}{2C_1 c}\frac{(1-\mu_c)}{\frac{c}{\log(c)}+\frac{2C_1 ct}{\log(c)\varepsilon}}.$$

Our result immediately follows. ■

Now, we estimate the denominator.

Lemma 3.5 *There exist $K(T,\varepsilon)>0$ and $C(T,\varepsilon)>0$ (depending only on T and ε) such that for any $c\geqslant K(T,\varepsilon)$, we have*

$$\int_{\mathbb{R}} |u_{c,\varepsilon,\alpha}(T,x)|^2 dx \geqslant C(T,\varepsilon)\frac{|\psi_c(1)|^2\log(c)^3}{c^3}. \tag{44}$$

Proof of Lemma 3.5. From (19) with α given in (40) and an easy change of variables, we have

$$\int_{\mathbb{R}} |u_{c,\varepsilon,\alpha}(T,x)|^2 dx = e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T}\frac{\log(c)\varepsilon}{2C_1 c}\int_{\mathbb{R}}\left|\psi_c\left(x+i\frac{2C_1c}{\log(c)\varepsilon}T\right)\right|^2 dx.$$

Using (18) and the Plancherel Theorem, we have

$$\int_{\mathbb{R}} |u_{c,\varepsilon,\alpha}(T,x)|^2 dx = \frac{\log(c)\varepsilon}{2C_1 c}\frac{2\pi e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T}}{c\lambda_c^2}\int_{-1}^{1}|\psi_c(\xi)|^2 e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi}d\xi. \tag{45}$$

We introduce the auxiliary function f defined on $(-1,0)$ given by

$$f(\xi)=\psi_c(\xi)e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi}.$$

We remark that

$$f'(\xi)=\left(\psi_c'(\xi)-\frac{4C_1c^2}{\log(c)\varepsilon}T\psi_c(\xi)\right)e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi}.$$

Using (12) at point $x=-1$ and $t=0$, we know that $\psi_c'(-1)=\frac{c^2-\chi_c}{2}\psi_c(-1)$. Hence, we have

$$f'(-1)=\left(\frac{c^2-\chi_c}{2}-\frac{4C_1c^2}{\log(c)\varepsilon}T\right)\psi_c(-1)e^{\frac{4C_1c^2}{\log(c)\varepsilon}T}.$$

Moreover, using (11) and the fact that for any $c>0$, $\psi_c(-1)=\psi_c(1)>0$ by [51, (7.62)], we deduce that for c larger than a constant depending only on T and ε, $f'(-1)>0$, so that $f'>0$ on a neighbourhood of -1 (depending on c). If $f'>0$ on $(-1,0)$, we call $\xi_0=0$. Otherwise, let us call ξ_0 the smallest point of $(-1,0)$ where $f'(\xi)=0$, *i.e.* the first point ξ_0 verifying

$$\psi_c'(\xi_0)=\frac{4C_1c^2}{\log(c)\varepsilon}T\psi_c(\xi_0). \tag{46}$$

ξ_0 is well-defined since f' is analytic on $\mathbb{C}$, so it has a finite number of zeros on $(-1,0)$. It is clear that at this point, one must have $f''(\xi_0)\leqslant 0$. Using (46), if we compute $f''(\xi_0)$, we obtain

$$\begin{aligned} f''(\xi_0) &= \left(\psi_c''(\xi_0)-\frac{8C_1c^2}{\log(c)\varepsilon}T\psi_c'(\xi_0)+\frac{16C_1^2c^4}{\log(c)^2\varepsilon}T^2\psi_c(\xi_0)\right)e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi_0} \\ &= \left(\psi_c''(\xi_0)-\frac{16C_1^2c^4}{\log(c)^2\varepsilon}T^2\psi_c(\xi_0)\right)e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi_0}. \end{aligned}$$

Using (12) together with (46) in the above expression, we obtain that

$$\begin{aligned}\left(1-\xi_0^2\right) f''\left(\xi_0\right) e^{cT\xi_0} &= \left(c^2\xi_0^2 - \chi_c - \left(1-\xi_0^2\right)\frac{16C_1^2c^4}{\log(c)^2\varepsilon}T^2\right)\psi_c\left(\xi_0\right) + 2\xi_0\psi_c'\left(\xi_0\right)\\ &= \left(c^2\xi_0^2 - \chi_c - \left(1-\xi_0^2\right)\frac{16C_1^2c^4}{\log(c)^2\varepsilon}T^2 + \frac{8C_1c^2}{\log(c)\varepsilon}T\xi_0\right)\psi_c\left(\xi_0\right).\end{aligned}$$

The smallest root of the polynomial (in the variable ξ_0) is given by

$$\xi_1(T,\varepsilon) = \frac{-\frac{8c^2C_1T}{\epsilon\log(c)} - \sqrt{\frac{64c^4C_1^2T^2}{\epsilon^2\log^2(c)} - 4\left(\frac{16c^4C_1^2T^2}{\epsilon^2\log^2(c)} + c^2\right)\left(-\frac{16c^4C_1^2T^2}{\epsilon^2\log^2(c)} - \chi_c\right)}}{2\left(\frac{16c^4C_1^2T^2}{\epsilon^2\log^2(c)} + c^2\right)}.$$

An explicit calculation gives that

$$\xi_1(T,\varepsilon) \sim \frac{\epsilon^2\log(c)^2}{32c^2C_1^2T^2} - 1 \text{ as } c \to \infty. \tag{47}$$

We deduce that for c larger than a constant $K(T,\varepsilon)$ depending only on T and ε, we have $\xi_1 \in (-1,0)$. Hence, for c larger than some constant depending on T and ε, since we must have $f''\left(\xi_0\right) \leqslant 0$, we are sure that $\xi_0 \geqslant \xi_1(T,\varepsilon)$. We have finally proved that for any $c \geqslant K(T,\varepsilon)$, f is increasing on $(-1,\xi_1(T,\varepsilon))$. Going back to (45), we deduce that for $c \geqslant K(T,\varepsilon)$,

$$\begin{aligned}\int_{\mathbb{R}} |u_{c,\varepsilon,\alpha}\left(T,x\right)|^2 dx &\geqslant \frac{\log(c)\varepsilon}{2C_1c}\frac{2\pi e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T}}{c\lambda_c^2}\int_{-1}^{\xi_1(T,\varepsilon)} |\psi_c\left(\xi\right)|^2 e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T\xi}d\xi\\ &\geqslant \frac{2\pi\log(c)\varepsilon e^{-\frac{4C_1c^2}{\log(c)\varepsilon}T}}{2C_1c^2\lambda_c^2}\left(1-\xi_1(T,\varepsilon)\right)|\psi_c\left(-1\right)|^2 e^{\frac{4C_1c^2}{\log(c)\varepsilon}T}\\ &\geqslant \frac{2\pi\log(c)\varepsilon}{2C_1c^2\lambda_c^2}\left(1-\xi_1(T,\varepsilon)\right)|\psi_c\left(-1\right)|^2.\end{aligned}$$

By using (5), (9), (47) and the parity of ψ_c, the conclusion easily follows. ■

Proof of Theorem 1.1.

We are now able to conclude. Remind that $T>0$ and $\varepsilon>0$ are fixed. From (42), (43) and (44), we deduce that for any $c \geqslant K(T,\varepsilon)$, we have

$$Q(c,\varepsilon) \leqslant C'\left(T,\varepsilon\right)\frac{c\left(1-\mu_c\right)}{\sqrt{log(c)}|\psi_c\left(1\right)|^2},$$

for $C'(T,\varepsilon)$ depending only on T and ε. Hence, using (9) together with (13), we deduce that for $c \geqslant K(T,\varepsilon)$ and some constant $C''(T,\varepsilon)$ depending only on T and ε,

$$Q(c,\varepsilon) \leqslant C'\left(T,\varepsilon\right) \leqslant \frac{C''\left(T,\varepsilon\right)}{\sqrt{\log(c)}}.$$

This ends the proof of the non-observability result by making c go to ∞.

Remark 3 *The family of initial conditions (and then the family of solutions) we use are complex-valued. However, it is quite natural to try to find counter-examples in the case of real initial conditions and real-valued solutions. However, since the half-Laplace operator is a real Fourier* *multiplier, it can be proved that either Re(u) or Im(u) gives a real counterexample to inequality* (16)*. Indeed, assume that we decompose the initial condition* u^0 *in real and imaginary part as*

$$u^0\left(x\right)=f^0(x)+ig^0(x).$$

Then, since $(-\Delta)^{\frac{1}{2}}$ *is a real Fourier multiplier, one easily see that in any time* $t>0$,

$$u(t,x)=f(t,x)+ig(t,x),$$

where f *is the solution to* (15) *with initial condition* f^0 *and* g *is the solution to* (15) *with initial condition* g^0*. Hence, the quotient* (42) *can be rewritten as*

$$Q(c,\varepsilon)=\frac{\int_0^T\left(||f\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}+||g\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\right)dt}{||f\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}+||g\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}}. \tag{48}$$

Our goal is to prove that either

$$\frac{\int_0^T\left(||f\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\right)dt}{||f\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}}$$

or

$$\frac{\int_0^T\left(||g\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\right)dt}{||g\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}}$$

has a subsequence that goes to 0 *as* $c\to\infty$*. By positivity and by passing to the inverse, it is enough to prove that either*

$$InvQ_r(c)=\frac{||f\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}}{\int_0^T\left(||f\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}+||g\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\right)dt}$$

or

$$InvQ_i(c)=\frac{||g\left(T,\cdot\right)||^2_{L^2(\mathbb{R})}}{\int_0^T\left(||f\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}+||g\left(t,\cdot\right)||^2_{L^2(\mathbb{R}\setminus[-\varepsilon,\varepsilon])}\right)dt}$$

has a subsequence that goes to ∞ *as* $c\to\infty$*. If it was not the case, it would mean that both sequences are bounded and so is* $InvQ_r(c)+InvQ_i(c)$*. This is impossible since this quantity is supposed to go to* $+\infty$*:* $InvQ_r(c)+InvQ_i(c)=\frac{1}{Q(c,\varepsilon)}$ *where* $Q(c,\varepsilon)$ *is given in* (48) *and goes to* 0*.*

4 Proof of Theorem 1.2

We only give the main ingredients, since the proof is very close to the one of Theorem 1.1. Let $T>0$. As for the half-heat equation (1), we can restrict our study to disprove the null-controllability of (2) on the control set $\omega=\mathbb{R}\times(\mathbb{R}\setminus[-\varepsilon,\varepsilon])$. As for the half-heat equation, the null-controllability of (2) at time $T>0$ is equivalent to the following property: there exists $C(T)>0$ such that for any $v_0\in L^2(\mathbb{R}^2)$, the solution v of

$$\begin{cases}\partial_t v\left(t,x,y\right)-\partial^2_{xx}v(t,x,y)-x^2\partial^2_{yy}v(t,x,y)=0 \text{ in } (0,T)\times\mathbb{R}^2,\\ \qquad v\left(0,x,y\right)=v^0\left(x,y\right) \text{ in } \mathbb{R}^2,\end{cases} \tag{49}$$

verifies

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v(T,\cdot)|^2 dydx \leqslant C(T)\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} |v(t,x,y)|^2 dydxdt. \tag{50}$$

If we consider the Fourier transform in the second space variable y that we still call $\widehat{\ }$, we obtain

$$\begin{cases} \partial_t \widehat{v}(t,x,\xi) - \partial_{xx}^2 \widehat{v}(t,x,\xi) + x^2\xi^2 v(t,x,y) = 0 \text{ in } (0,T)\times\mathbb{R}^2, \\ \widehat{v}(0,x,\xi) = \widehat{v^0}(x,\xi) \text{ in } \mathbb{R}^2. \end{cases}$$

The elliptic operator $-\partial_{xx}^2 + x^2\xi^2$ is exactly the harmonic oscillator. Notably, the first eigenfunction is given by the first Hermite function, correctly rescaled, namely $x \mapsto e^{-\frac{|\xi|x^2}{2}}$, associated to the eigenvalue $|\xi|$. Hence, if we consider the initial condition given by

$$v^0_{c,\varepsilon,\alpha}(x,y) = e^{i\frac{2C_1c^2}{\log(c)\varepsilon}\left(y+i\frac{x^2}{2}\right)}\psi_c\left(\frac{2C_1c}{\log(c)\varepsilon}y\right),$$

straightforward computations show that the corresponding solution of (49) is given by

$$v_{c,\varepsilon,\alpha}(t,x,y) = e^{i\frac{2C_1c^2}{\log(c)\varepsilon}\left(y+i\left(t+\frac{x^2}{2}\right)\right)}\psi_c\left(\frac{2C_1c}{\log(c)\varepsilon}\left(y+i\left(t+\frac{x^2}{2}\right)\right)\right). \tag{51}$$

Hence $v_{c,\varepsilon,\alpha}(t,x,y) = u_{c,\varepsilon,\alpha}\left(t+\frac{x^2}{2},y\right)$, where $u_{c,\varepsilon,\alpha}$ is the solution of (15) used in Section 2. It is now quite straightforward to conclude. Indeed, according to the inequality (50) that we want to disprove, we introduce the quotient

$$Q'(c,\varepsilon) = \frac{\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} |v_{c,\varepsilon,\alpha}(t,x,y)|^2 dydxdt}{\int_{\mathbb{R}}\int_{\mathbb{R}} |v_{c,\varepsilon,\alpha}(T,x,y)|^2 dydx}. \tag{52}$$

Let us look at the numerator. Using (51), (19) with α given in (40), and (43), we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} |v_{c,\varepsilon,\alpha}(t,x,y)|^2 dydxdt &\leqslant C(\varepsilon)T(\log(c))^{\frac{5}{2}}\frac{(1-\mu_c)}{c^2}\int_{\mathbb{R}}\frac{1}{(1+\frac{x^2}{2})}dx \\ &\leqslant C(\varepsilon)T\sqrt{2}\pi(\log(c))^{\frac{5}{2}}\frac{(1-\mu_c)}{c^2}. \end{aligned}$$

Concerning the denominator, let us remark that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v_{c,\varepsilon,\alpha}(T,x,y)|^2 dydx \geqslant \int_0^1\int_{\mathbb{R}}\left|u_{c,\varepsilon,\alpha}\left(T+\frac{x^2}{2},y\right)\right|^2 dydx.$$

Using the dissipativity of the solutions of (15),

$$x\in(0,1)\int_{\mathbb{R}}\left|u_{c,\varepsilon,\alpha}\left(T+\frac{x^2}{2},y\right)\right|^2 dy$$

is decreasing, so

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v_{c,\varepsilon,\alpha}(T,x,y)|^2 dydx \geqslant \int_{\mathbb{R}}\left|u_{c,\varepsilon,\alpha}\left(T+\frac{1}{2},y\right)\right|^2 dy.$$

Hence, using (44) where T is replaced by $T+1/2$, we deduce that there exist $K(T,\varepsilon)>0$ and $C(T,\varepsilon)>0$ depending only on T such that for any $c\geqslant K(T,\varepsilon)$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v_{c,\varepsilon,\alpha}(T,x,y)|^2 dydx \geqslant C(T,\varepsilon)\frac{|\psi_c(1)|^2\log(c)^3}{c^3}.$$

Hence, going back to expression (52) and using the same computations as for $u_{c,\varepsilon,\alpha}$ in Section 2, we deduce as before that $Q'(c,\varepsilon)\to 0$ as $c\to\infty$, which gives the desired non-controllability result. ■

5 Conclusion

In this article, we explained how we can use the PSWF in order to obtain negative controllability results for the half-heat equation on the whole line $\mathbb{R}$ and the Grushin plane on the whole plane $\mathbb{R}^2$. Since PSWF are functions that are naturally bandlimited and that concentrate very strongly on $(-1, 1)$, they are quite natural candidates as initial conditions in order to disprove observability inequalities. Our hope is that this natural idea can be exploited in order to derive other non-controllability results for weakly diffusive equations on general geometries. Using the results given in [55], we hope to extend our results in the case of the whole space $\mathbb{R}^n$, $n \geqslant 2$. A more ambitious goal would be the following. If we consider some smooth Riemannian manifold M (that can be unbounded), can we use some generalized PSWF that saturate some Logvinenko-Sereda uncertainty principle in order to obtain non-null controllability results for the half-heat equation on M? Now, if we consider a non-compact complete singular manifold for instance with an interior singularity that is an hypersurface, is it possible to use arguments similar to the one developed in Section 1.2 to derive some non-controllability results for the corresponding hypoelliptic diffusion?

However, one major limitation is that our study relies strongly on what Slepian calls in [56] "the lucky accident": the integral operator F_c commutes with the differential operator L_c. Note that this property is crucial in our proof, since it is used both in the proof of Lemmas 3.4 and 3.5. In some particular cases, such a lucky accident still happens (notably in $\mathbb{R}^n$ for concentration on a ball, see [55]). It seems that in more complex geometries, one cannot expect to exhibit some adequate differential operator L_c, making impossible to reproduce the proof given here.

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