Robust control of linear systems under small parameter variations

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Abstract

In this article, we investigate first-order parameter insensitization for linear controlled evolutions: given a family $A(\sigma)$ of generators on a Hilbert space H and a control operator B, does there exist a single control, independent of σ, that both steers the nominal system to a prescribed final state and cancels the first order sensitivity at the nominal system of the endpoint $\sigma \mapsto y_\sigma(T)$, where y_σ is the state of the system?

We start from an abstract framework that does not depend on the dimension or on the type of system (ODE or PDE), and we derive the associated sensitivity cascade system. Insensitization is then recast as a controllability problem for an extended state (y, z). We illustrate this approach with two significant examples: in finite dimension, we prove the equivalence between four natural formulations and provide a necessary and sufficient algebraic condition of Kalman-type rank. We also treat infinite-dimensional examples, namely: the heat equation with uncertain diffusion coefficient, and show that the natural null and mixed approximate/exact insensitization problems are solvable; the wave equation with uncertain potential under appropriate geometric conditions on the control region allowing to derive approximate insensitization properties.

This article is dedicated to our colleague and friend Jean-Michel Coron, on the occasion of his 70th birthday, and we thank him for his very influential works on control theory. Merci pour l'inspiration et les discussions toujours stimulantes !

1 Setting of the problem

1.1 Introduction

Most controlled systems are only known through mathematical models that involve coefficients depending on physical, geometric or environmental parameters. In practice, parameters such as diffusion coefficients in heat processes, damping terms in mechanical systems, or potentials in wave propagation cannot be measured precisely and are typically known only within a certain margin of uncertainty.

As a consequence, even if a system is controllable for a nominal parameter value, the control designed for this value may fail to steer the perturbed system to the desired target, or may produce an endpoint that is highly sensitive to small variations of the parameter. This raises a class of questions that go beyond classical controllability: can one design a single control input that remains effective for all small perturbations of the model, or at least one for which the impact of these perturbations on the terminal state is negligible?

A natural first step is to ask whether one can construct a control that

(i) steers the nominal system to a prescribed target;

(ii) makes the terminal state insensitive at first order to variations of the parameter. This is the so-called insensitizing or desensitizing control problem introduced by Lions [29], which transforms the study of robustness into the analysis of a coupled system linking the state and its sensitivity with respect to the parameter.

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Relationships between controllability of the nominal system and robustness of the perturbed one is delicate. Understanding when these robustness properties hold, and characterizing them through algebraic criteria or observability inequalities, is the core objective of this work.

More precisely, in this article we focus on the insensitization problem for linear controlled evolutions:

> *Given a parameterized family of generators $A(\sigma)$ and a control operator B, does there exist a control u (independent of the parameter σ) which both controls the nominal system at $\sigma = 0$ and makes the end-point map $\sigma \mapsto y_\sigma(T)$ insensitive to first order?*

We develop an abstract framework for first-order insensitization, establish necessary and sufficient conditions, and apply them to both finite-dimensional systems and classical PDE models of heat and wave type.

State of the art. Our study enters the broad question of *robustness* of control mechanisms, which is a wildly studied area, see for instance [12, Chap.9] or [42]. Usually, this question is rephrased as finding a robust feedback control operator.

Here, we are rather interested in open-loop controls, for which several types of robustness controls have been studied for parameter dependent control systems, in particular *simultaneous controllability*, *ensemble controllability* and *average controllability*, see for instance [27] for a review of these questions. Simultaneous controllability proposes to choose one control independent of the parameter which steers the initial datum to a target state whatever is the parameter. Ensemble controllability relaxes this notion, by asking that for all $\varepsilon > 0$, one can find a control independent of the parameter that steers the initial datum to the target one, up to an-ε neighborhood (this notion of course depends on the functional setting at hand). Finally, the concept of average controllability, introduced in [43] consists in controlling the average of the final state with respect to some parameter-dependent measures.

The notion that we are developing in this article seems to be new, and recalls the one of *insensitizing control* (sometimes known as *desensitizing control*), which was introduced by Jacques-Louis Lions and developed since the 90's (see, *e.g.*, [30]). In the PDE context one seeks a control that makes insensitive a prescribed functional of the state at first-order insensitive with respect to small perturbations of a system parameter. Most contributions concentrate on parabolic models. Early mathematical investigations established existence of insensiting controls for semilinear heat equations with respect to perturbations of the initial datum (see [14, 7]), and the general framework of null and approximate controllability for the linear heat equation, mainly based on Carleman estimates and spectral/observability approaches (see e.g. [20]), provides the analytic backbone for insensitization strategies. More recent works have broadened the class of admissible perturbations, for instance domain deformations or perturbations of coefficients, and clarified which geometric and regularity hypotheses are needed to obtain the required observability for the adjoint sensitivity cascade [18, 34, 35]. In this parabolic setting, the insensitizing problem is typically reduced, via linearization and duality, to proving observability inequalities for a coupled (block-upper triangular) adjoint system; unique continuation and refined Carleman estimates play a central role in those arguments.

Beyond the parabolic setting, insensitizing/desensitizing controls have also been established for hyperbolic models, in particular for the wave equation with locally distributed controls [13, 39]. We also mention the work [2], which provides exact insensitizing controls for the scalar wave equation and makes explicit the tight connection with the controllability of coupled cascade systems driven by a single control.

Robustness interpretation. The first-order insensitization requirement can be understood as a local open-loop robustness constraint on the end-point map. More precisely, whenever the map $\sigma \mapsto y_\sigma(T)$ is differentiable at $\sigma = 0$, the condition $\frac{d}{d\sigma} y_\sigma(T)\big|_{\sigma=0} = 0$ means that small parameter variations have no first-order impact on the terminal state; in particular, under a second-order expansion one expects $y_\sigma(T) = y_0(T) + \mathrm{O}(|\sigma|^2)$ as $\sigma \to 0$. This viewpoint complements the open-loop notions reviewed above (simultaneous/ensemble/average controllability) by targeting the sensitivity of the final state rather than uniform reachability properties across the parameter range.

Contributions and outline. We introduce the abstract framework and main assumptions in Section 1.2. Building on this, the paper makes four concrete contributions:

(i) We provide an abstract and unified formulation of "first-order parameter insensitization" for linear controlled evolutions. Under two alternative sets of semigroup hypotheses, we justify the sensitivity analysis, derive the coupled state–sensitivity cascade (1.6), and formalize several natural control objectives combining controllability of the nominal system and insensitization of the end-point map.

(ii) We develop an operator-theoretic duality viewpoint for these problems. In particular, we compute the adjoints of the end-point operators (Proposition 2.1) and derive sharp dual characterizations in terms of observability/unique-continuation properties for the adjoint cascade (2.1) (Proposition 2.2).

(iii) In the finite-dimensional setting, we obtain a complete characterization: all the control/insensitization formulations considered in this paper turn out to be equivalent, and they hold if and only if a Kalman-type rank condition restricted to a suitable subspace is satisfied (Theorem 1.4).

(iv) We illustrate the abstract results on PDE models. For the heat equation with uncertain diffusion, we prove the solvability of null/approximate insensitizing controls in any positive time (Theorem 1.6). For the wave equation with a uncertain potential, we establish approximate controllability and approximate insensitization under a geometric condition (Theorem 1.7). We also discuss higher-order extensions and open questions in Section 4.

The paper is organized as follows. Section 1.2 introduces the setting, the sensitivity system. Different control/insensitization problems are introduced in Section 1.3. Our main results are given on illustrative examples in Section 1.4 on several class of systems: finite-dimensional ones, heat and wave equations. Section 2 develops the abstract duality framework and the associated observability criteria. Section 3 contains the proofs of the main results. Section 4 presents extensions and open problems. Technical proofs of auxiliary results are provided in the appendix.

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1.2 An abstract framework for insensitization

1.2.1 Scope and standing assumptions

Let $\varepsilon > 0$ denote half the amplitude of the parameter variation that we consider. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real Hilbert space (the state space) and let U be a real Hilbert space (the control space)[1]. We consider a family of linear (possibly unbounded) operators $\{A(\sigma)\}_{\sigma\in(-\varepsilon,\varepsilon)}$ on H together with a control operator $B \in \mathcal{L}(U, Y)$ (where $\mathcal{L}$ stands for the linear continuous operators) for some other real Hilbert space Y, that will be specified later on, and will always be a subspace of $D((A(0)^*)'$. We introduce our perturbed control system as

$$\begin{cases} \dfrac{dy_\sigma}{dt}(t) = A(\sigma)\, y_\sigma(t) + B\, u(t), \quad t \in (0,T), \\ y_\sigma(0) = y_0 \in H, \end{cases} \qquad \text{with } u \in L^2((0,T),U). \tag{1.1}$$

For ease of presentation, we set $A_0 := A(0)$ the nominal operator and we assume:

(H0) **Common assumption.** For each $\sigma \in (-\varepsilon,\varepsilon)$, the domain $D(A(\sigma))$ of $A(\sigma)$, respectively $D(A(\sigma)^*)$ of $A(\sigma)^*$, coincide with the domain $D(A_0)$ of A_0, respectively $D(A_0^*)$ of A_0^*, that is

$$D\big(A(\sigma)\big) = D(A_0), \text{ and } D\big(A(\sigma)^*\big) = D(A_0^*) \qquad \text{for all } \sigma \in (-\varepsilon,\varepsilon).$$

In the sequel $D(A_0)$ is endowed with the graph norm $\|x\|_{D(A_0)} := \|x\|_H + \|A_0 x\|_H$.

[1] This is for the sake of simplicity; all our results are still valid in the complex case.

In what follows:

- $D(A_0^*)'$ denotes the dual space of $D(A_0^*)$ with respect to the pivot space H.
- $[H, D(A_0^*)]_{1/2}$ denotes the real interpolation space of order $1/2$ between H and $D(A_0^*)$ (for a definition of interpolation spaces, see [31, Chapter 1]). Its dual space (with respect to the pivot space H) is denoted by $([H, D(A_0^*)]_{1/2})'$.

We shall work under one of the two alternative sets of hypotheses presented below. Either set is sufficient for developing the sensitivity analysis and the extended state formulation.

Hypotheses set 1:

(A1a) **An analytic semigroup.** A_0 is the generator of an analytic semigroup $(e^{tA_0})_{t\geq 0}$ on H (see e.g. [17] for definitions and basic properties).

(A2a) **Regularity of the operators with respect to the parameter.** The map $\sigma \mapsto A(\sigma)$ is of class $C^1\big((-\varepsilon, \varepsilon), \mathcal{L}(D(A_0), H)\big)$. In particular, this implies that the derivative $A_1 := \frac{d}{d\sigma} A(\sigma)\big|_{\sigma=0}$ exists as an operator $\mathcal{L}(D(A_0), H)$.

(A3a) **A relatively bounded control operator.** B satisfies $B \in \mathcal{L}(U, ([H, D(A_0^*)]_{1/2})')$.

Hypotheses set 2:

(A1b) **A C_0-semigroup.** A_0 generates a C_0-semigroup $(e^{tA_0})_{t\geq 0}$ on H.

(A2b) **Regularity of the operators with respect to the parameter and bounded perturbations.** The perturbation map $\sigma \mapsto \widehat{A}(\sigma) := A(\sigma) - A_0$ is of class $C^1\big((-\varepsilon, \varepsilon), \mathcal{L}(H)\big)$. In particular, it follows that $A_1 := \frac{d}{d\sigma}(A(\sigma) - A_0)\big|_{\sigma=0} \in \mathcal{L}(H)$.

(A3b) **An admissibility condition for the control operator with respect to the semigroup $(e^{tA_0})_{t\geqslant 0}$.** The control operator $B \in \mathcal{L}(U, D(A_0^*)')$ is admissible for the semigroup $(e^{tA_0})_{t\geqslant 0}$ in the following sense: there exists $T_0 > 0$ such that the mapping

$$u \mapsto \int_0^{T_0} e^{(T_0 - s)A_0} B\, u(s) ds$$

belongs to $\mathcal{L}(L^2((0, T_0), U), H)$, *i.e.*, there exists $C_0 > 0$ such that for all $u \in L^2((0, T_0), U)$,

$$\left\| \int_0^{T_0} e^{(T_0 - s)A_0} B\, u(s)\, ds \right\|_H \leq C_0 \|u\|_{L^2((0,T_0),U)}. \tag{1.2}$$

We refer to [40, Chapters 4 and 5] for details and equivalent characterizations of admissibility. Note that in fact this definition does not depend on the choice of T_0.

Remark 1.1 (Comments on these assumptions). *Under either Hypotheses set 1 or set 2, and especially the assumptions (A3a) or (A3b), the nominal controlled system (1.1) with $\sigma = 0$ is well posed in the following sense: for every $u \in L^2((0, T), U)$ and every $y_0 \in H$, there exists a unique solution y of (1.1) with $\sigma = 0$ in $C^0([0, T], H)$, see [40]. Moreover, note that (A1a) is stronger than (A1b), which enables to have unbounded perturbations in (A2a) instead of bounded ones in (A2b).*

We also have the two following propositions, proved in the Appendix, which will allow us to justify next our arguments:

Proposition 1.2. *Under either Hypotheses set 1 or set 2, there exists a neighborhood I of 0 such that for each $\sigma \in I$, $A(\sigma)$ is the generator of a C_0-semigroup on H. Moreover, there exists $M, \omega > 0$ such that for any $\sigma \in I$, we have*

$$\|e^{tA(\sigma)}\|_{\mathcal{L}(H)} \leqslant M e^{\omega t}, \quad t \geqslant 0. \tag{1.3}$$

Moreover, for $\sigma \in I$, the control operator B is admissible for the semigroup $(e^{tA(\sigma)})_{t\geqslant 0}$.

We also introduce what we call the nominal system (which corresponds to (1.1) when $\sigma = 0$)

$$\begin{cases} y'(t) = A_0 y(t) + Bu(t), \quad t \in [0, T] \\ y(0) = y_0 \in H. \end{cases} \tag{1.4}$$

Then, we have the following property.

Proposition 1.3. *Under either Hypotheses set 1 or set 2, there exists a neighborhood 0 included in I (that we still call I, for the sake of simplicity) such that the parameter-to-state map $\sigma \in I \mapsto y_\sigma \in C^0([0,T], H)$, where y_σ is the solution of (1.1), is differentiable at $\sigma = 0$ for every fixed $y_0 \in H$ and $u \in L^2((0,T), U)$. The derivative $z(t) := \partial_\sigma y_\sigma(t)\big|_{\sigma=0}$ exists and satisfies the sensitivity equation*

$$\frac{dz}{dt}(t) = A_0 z(t) + A_1 y(t), \qquad z(0) = 0, \tag{1.5}$$

where $y(\cdot)$ stands for the state with $\sigma = 0$.

1.3 State and sensitivity equations: An abstract setting

According to Propositions 1.2 and 1.3, we are now interested in the coupled (sensitivity) system

$$\begin{cases} \dfrac{dy}{dt}(t) = A_0\, y(t) + B\, u(t), \\ \dfrac{dz}{dt}(t) = A_0\, z(t) + A_1\, y(t), \\ (y, z)(0) = (y_0, 0) \in H \times H. \end{cases} \tag{1.6}$$

In this article, we are concerned with the issue of controlling a perturbation of a controllable system. More precisely, we are interested in the existence of a control function u independent of σ such that

- u is an exact or null control of System (1.4) whenever $\sigma = 0$;
- u insensitizes the "end-point" mapping $\sigma \mapsto y_\sigma(T)$ at $\sigma = 0$.

From an abstract point of view, we will investigate the following control and insensibilization problems.

- **Problem EC+EI (Exact controllability + Exact insensitization).** For every $y_0, y_T \in H$, find $u \in L^2((0,T), U)$ so that the solution of (1.6) satisfies $(y, z)(T) = (y_T, 0)$.
- **Problem NC+EI (Null controllability + Exact insensitization).** For every $y_0 \in H$, find $u \in L^2((0,T), U)$ so that the solution of (1.6) satisfies $(y, z)(T) = (0, 0)$.
- **Problem AC+AI (Approximate controllability + Approximate insensitization).** For every $y_0, y_T \in H$, for every $\varepsilon > 0$, find $u \in L^2((0,T), U)$ so that the solution of (1.6) satisfies $\|y(T) - y_T\|_H \leqslant \varepsilon$ and $\|z(T)\|_H \leqslant \varepsilon$.
- **Problem NAC+AI (Null approximate controllability + Approximate insensitization).** For every $y_0 \in H$, for every $\varepsilon > 0$, find $u \in L^2((0,T), U)$ so that the solution of (1.6) satisfies $\|y(T)\|_H \leqslant \varepsilon$ and $\|z(T)\|_H \leqslant \varepsilon$.
- **Problem AC+EI (Approximate controllability + Exact insensitization).** For every $y_0, y_T \in H$ and any $\varepsilon > 0$, find a control function $u \in L^2((0,T), U)$ so that the solution of (1.6) satisfies $(y, z)(T) = (\tilde{y}_T, 0)$ with $\|\tilde{y}_T - y_T\|_H \leqslant \varepsilon$.

Note that we have the following implications:

$$\begin{array}{ccccc} \mathrm{EC+EI} & \Longrightarrow & \mathrm{AC+EI} & \Longrightarrow & \mathrm{AC+AI} \\ \Downarrow & & & & \Downarrow \\ \mathrm{NC+EI} & & \Longrightarrow & & \mathrm{NAC+AI} \end{array} \tag{1.7}$$

In the following, we address these issues in several contexts. Let us specify which of the two hypotheses sets introduced in Section 1.2 will be assumed for each worked example, together with a short justification.

Finite-dimensional systems. In the finite-dimensional setting ($H = \mathbb{R}^n$, $A(\sigma) \in \mathbb{M}_n(\mathbb{R})$, $B \in \mathbb{M}_{n,m}(\mathbb{R})$) all operators are bounded and the semigroup theory reduces to elementary linear ODE theory and is obviously analytic. We can therefore work either under *Hypotheses set 1* or *Hypotheses set 2*, as soon as $\sigma \mapsto A(\sigma)$ is C^1 locally around 0.

Heat equation, robustness with respect to the diffusion coefficient. We will also discuss in Section 1.4.2 a heat equation perturbed by a second order perturbation, namely $A_0 = \Delta$ with Dirichlet boundary conditions, $D(A_0) = H^2 \cap H_0^1(\Omega)$, $H = L^2(\Omega)$, and perturbations of the form $A(\sigma) = (1+\sigma)\Delta$. *Hypotheses set 1* is perfectly adapted to such cases as the heat semigroup is analytic on H, $D(A(\sigma)) = D(A_0)$ for all $\sigma \in (-1/2, 1/2)$, and $\sigma \mapsto A(\sigma)$ is $C^1((-1/2, 1/2), \mathcal{L}(D(A_0), H))$.

Wave equation, robustness with respect to variations with respect to the potential. We will also present an hyperbolic example, namely the wave equation with a small unknown potential of given shape. We will thus consider the natural energy space $H_0^1 \times L^2$ and the underlying generator A_0 is the standard skew-adjoint second-order wave operator which generates a C_0-group in H but not an analytic semigroup. The operators $A(\sigma)$ will then simply be given by

$$A(\sigma) = A_0 + \sigma \begin{pmatrix} 0 & 0 \\ -p & 0 \end{pmatrix},$$

in which p is a given potential in $L^\infty(\Omega)$. Therefore this corresponds to *Hypotheses set 2*.

1.4 Main results

1.4.1 In finite dimension

The first example we would like to investigate is the finite-dimensional case. Let n and m denote two nonzero integers. Let $\sigma \in [\sigma_-, \sigma_+]$, with $\sigma_- < 0$ and $\sigma_+ > 0$, and y_σ denote the solution of the parameter-dependent controlled autonomous linear system

$$\begin{cases} y_\sigma'(t) = A(\sigma) y_\sigma(t) + Bu(t), & t \in [0, T], \\ y_\sigma(0) = y_0, \end{cases} \tag{1.8}$$

where $A \in C^1([\sigma_-, \sigma_+], \mathbb{M}_n(\mathbb{R}))$ and $B \in \mathbb{M}_{n,m}(\mathbb{R})$, σ denotes a small parameter and $u(\cdot)$ is a control function in $L^2((0, T), \mathbb{R}^m)$. Such a setting has already been investigated in [41, Chapter 10] from the point of view of feedback and optimal control.

As a particular case of (1.6), we set

$$A_0 = A(0) \in \mathbb{M}_n(\mathbb{R}) \qquad \text{and} \qquad A_1 = \left.\frac{dA}{d\sigma}\right|_{\sigma=0} \in \mathbb{M}_n(\mathbb{R}),$$

and we introduce

$$\begin{cases} \dfrac{dy}{dt}(t) = A_0 y(t) + Bu(t), & t \in [0, T], \\ \dfrac{dz}{dt}(t) = A_0 z(t) + A_1 y(t), & t \in [0, T], \\ (y, z)(0) = (y_0, 0) \in \mathbb{R}^n \times \mathbb{R}^n. \end{cases} \tag{1.9}$$

Setting $X = (y, z)^\top$, the coupled control system (1.9) writes

$$X' = \mathcal{A}X + \mathcal{B}u, \tag{1.10}$$

with

$$\mathcal{A} = \begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}) \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix} \in \mathcal{M}_{2n,m}(\mathbb{R}).$$

Note that an easy induction shows that the Kalman Matrix associated to $\mathcal{A}$ and $\mathcal{B}$, which is by definition

$$[\mathcal{A}, \mathcal{B}] := \big(\mathcal{B}, \mathcal{A}\mathcal{B}, \dots \mathcal{A}^{2n-1}\mathcal{B}\big) \in \mathcal{M}_{2n,2nm}(\mathbb{R}),$$

is given by

$$[\mathcal{A},\mathcal{B}] = \begin{pmatrix} B & A_0B & \dots & \dots & A_0^{2n-1}B \\ 0 & A_1B & A_0A_1B + A_1A_0B & \dots & \sum_{i=1}^{2n-1} A_0^{i-1}A_1A_0^{2n-1-i}B \end{pmatrix}. \tag{1.11}$$

Note first that System (1.9) can be controlled whenever the pair $(\mathcal{A},\mathcal{B})$ satisfies the so-called Kalman rank condition (*i.e.* $(\mathcal{A},\mathcal{B})$ is of maximal rank $2n$). Nevertheless, it is likely that such an assumption can be significantly weakened since the control problem can be obviously solved in the case where $A_1 = 0$ and (A_0, B) satisfies the Kalman rank condition (this clearly appears in (1.9), and comes from the initial condition in the second equation of (1.9), which is 0, and thus the second component z of (1.9) vanishes identically). In this later case, due to expression (1.11), the pair $(\mathcal{A},\mathcal{B})$ never satisfies the Kalman rank condition.

Moreover, note that the fact that the pair (A_0, B) satisfies the usual Kalman rank condition is always necessary for solving the above control problem, since (1.4) for $\sigma = 0$ has to be controllable in the usual sense.

Before stating our results, we introduce the following objects.

- The extended Kalman matrix
$$[A_0, B] = [B, A_0B, A_0^2B, \dots A_0^{2n-1}B] \in \mathcal{M}_{n,2nm}(\mathbb{R}).$$
- We call $L \subset \mathbb{R}^{2nm}$ the nullspace of the matrix
$$[0, A_1B, \dots, \sum_{i=1}^{2n-1} A_0^{i-1}A_1A_0^{2n-1-i}B] \in \mathcal{M}_{n,2nm}(\mathbb{R}).$$

Our main result for finite-dimensional systems is the following one, which gives a complete answer to the questions.

Theorem 1.4. *In the above finite dimensional context, all problems **EC+EI, NC+EI, AC+EI, NAC+AI, AC+AI** are equivalent, and there are also all equivalent to the following "Kalman rank condition restricted on L":*

$$\operatorname{rank}([A_0, B]|_L) = n. \tag{1.12}$$

In particular, the solvability of any of these problems does not depend on the time horizon T.

Remark 1.5 (Explicit reformulation of (1.12))**.** *Let*

$$K_1 := \big[B,\ A_0B,\ \dots,\ A_0^{2n-1}B\big] \in \mathcal{M}_{n,2nm}(\mathbb{R}),$$
$$K_2 := \Big[0,\ A_1B,\ \dots,\ \sum_{i=1}^{2n-1} A_0^{i-1}A_1A_0^{2n-1-i}B\Big] \in \mathcal{M}_{n,2nm}(\mathbb{R}),$$

so that $L = \mathcal{N}(K_2)$, where $\mathcal{N}(K_2)$ denotes the kernel of the operator K_2. Then (1.12) is equivalent to the following constrained surjectivity property:

$$\forall\, y \in \mathbb{R}^n,\ \exists\, \xi \in \mathbb{R}^{2nm} \text{ such that } K_1\xi = y \quad \text{and} \quad K_2\xi = 0,$$

or, equivalently,

$$\forall\, y \in \mathbb{R}^n,\ \exists\, \xi \in \mathbb{R}^{2nm} \text{ such that } \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \xi = \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Writing $\xi = (v_0^\top, \dots, v_{2n-1}^\top)^\top$ with $v_k \in \mathbb{R}^m$, this means that one can solve

$$\sum_{k=0}^{2n-1} A_0^k B\, v_k = y \quad \text{under the constraint} \quad \sum_{k=1}^{2n-1} \Big(\sum_{i=0}^{k-1} A_0^i A_1 A_0^{k-1-i}\Big) B\, v_k = 0.$$

Note that, as expected, Condition (1.12) is stronger than the usual Kalman condition for A_0 and B, and weaker[2] than the Kalman condition for $\mathcal{A}$ and $\mathcal{B}$.

[2] We use the notations introduced in the remark above. Let us justify why (1.12) is weaker than the Kalman rank

1.4.2 Robustness with respect to the diffusion for the heat equation

Let Ω be a bounded, connected and open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) of class C^2, and ω a nonempty open subset of Ω.

Let $\sigma \in [-1/2, 1/2]$. Consider the family of heat equations

$$\begin{cases} \dfrac{\partial y_\sigma}{\partial t} - (1+\sigma)\Delta y_\sigma &= \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y_\sigma &= 0 & \text{on } (0,T)\times\partial\Omega, \\ y_\sigma(0,\cdot) &= y_0 & \text{in } \Omega, \end{cases} \tag{1.13}$$

for some $y_0 \in L^2(\Omega)$, where $u \in L^2((0,T)\times\Omega)$ is the control function, thus corresponding to our abstract setting with $H = U = L^2(\Omega)$. Remind that for any $\sigma \in (-1/2, 1/2)$, this equation is null-controllable in arbitrary small time (for any $T > 0$, for any y_0, there exists $u \in L^2((0,T)\times\Omega)$ such that $y(T) = 0$, see *e.g.* [20]), and that it is also approximately controllable in arbitrary small time (for any $T > 0$, for any $y_0, y_T \in L^2(\Omega)$, for any $\varepsilon > 0$ there exists $u \in L^2((0,T)\times\Omega)$ such that $\|y(T) - y_T\|_{L^2(\Omega)} \leqslant \varepsilon$ (this is a consequence of the previous property, as explained in [10, Theorem 2.45, Page 57]).

Corresponding to the abstract coupled system (1.6), we introduce the system:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y &= \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y &= 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) &= y_0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t} - \Delta z &= \Delta y & \text{in } (0,T)\times\Omega, \\ z &= 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,\cdot) &= 0 & \text{in } \Omega, \end{cases} \tag{1.14}$$

Remind that the problem **EC+EI** cannot be solved, since the heat equation is not exactly controllable due to the strong regularizing effects of the heat semigroup.

Our main result is then the following.

Theorem 1.6. *In any time $T > 0$, problems **NC+EI** and **AC+EI** are solvable.*

1.4.3 Robustness with respect to the potential for the wave equation

Another instance of interest is given by the wave equation with a potential term

$$\begin{cases} \dfrac{\partial^2 y_\sigma}{\partial t^2} - \Delta y_\sigma + \sigma p(x) y_\sigma = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y_\sigma = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(y_\sigma, \dfrac{\partial y_\sigma}{\partial t}\right)(0,\cdot) = (y_0, y_1) & \text{in } \Omega. \end{cases} \tag{1.15}$$

In this subsection, we only need $p \in L^\infty(\Omega)$, together with a non-degeneracy assumption in the control region:

$$\operatorname{int}(\operatorname{Supp} p) \cap \omega \neq \emptyset, \tag{1.16}$$

where we interpret $\operatorname{Supp}(p)$ as the essential support[3] of p.

condition for the extended pair $(\mathcal{A}, \mathcal{B})$. Recall that the Kalman matrix of $(\mathcal{A}, \mathcal{B})$ reads $[\mathcal{A}, \mathcal{B}] = \left(\mathcal{B}, \mathcal{A}\mathcal{B}, \dots, \mathcal{A}^{2n-1}\mathcal{B}\right) \in \mathcal{M}_{2n,2nm}(\mathbb{R})$, and that, using (1.11), it can be written in block form as

$$[\mathcal{A}, \mathcal{B}] = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}.$$

By definition, $L = \mathcal{N}(K_2) \subset \mathbb{R}^{2nm}$. If $(\mathcal{A}, \mathcal{B})$ satisfies Kalman's rank condition, i.e. $\operatorname{rank}([\mathcal{A}, \mathcal{B}]) = 2n$, then $[\mathcal{A}, \mathcal{B}]$ is onto from $\mathbb{R}^{2nm}$ to $\mathbb{R}^{2n}$. Hence, for every $y \in \mathbb{R}^n$, there exists $\xi \in \mathbb{R}^{2nm}$ such that

$$[\mathcal{A}, \mathcal{B}]\xi = \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Writing this equality in blocks yields $K_1\xi = y$ and $K_2\xi = 0$, so $\xi \in L$ and therefore $K_1|_L$ is onto $\mathbb{R}^n$. This is exactly (1.12), namely $\operatorname{rank}([A_0, B]|_L) = n$.

[3] In other words, $\operatorname{Supp}(p) := \Omega \setminus \bigcup\{U \subset \Omega \text{ open} : \ p = 0 \text{ a.e. in } U\}$.

The insensitizing problem then consists in considering the coupled wave system (corresponding to the abstract one in (1.6)):

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial^2 z}{\partial t^2} - \Delta z + p(x) y = 0 & \text{in } (0,T)\times\Omega, \\ y = z = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(y, \dfrac{\partial y}{\partial t}\right)(0,\cdot) = (y_0, y_1) & \text{in } \Omega, \\ \left(z, \dfrac{\partial z}{\partial t}\right)(0,\cdot) = (0,0) & \text{in } \Omega. \end{cases} \tag{1.17}$$

We then obtain the following result.

Theorem 1.7. *Assume $p \in L^\infty(\Omega)$ and (1.16). If*

$$T > 2 \sup_{x\in\Omega} \{ d_\Omega(x, (\mathrm{int}(\mathrm{Supp}\, p) \cap \omega)) \},$$

where d_Ω denotes the geodesic distance in Ω, the equation (1.15) is approximately controllable and approximately insensitizable at $\sigma = 0$, i.e., Problem **AC+AI** *is solvabl in time T.*

1.5 Related results

The sensitivity cascade (1.6) belongs to the broad class of cascade (or block-lower triangular) coupled systems, for which controllability with a reduced number of controls has been extensively studied in both parabolic and hyperbolic settings.

Null and approximate controllability of coupled parabolic systems with a single control (or with fewer controls than equations) has been addressed by Carleman and spectral methods; see for instance [24, 19]. Cascade structures are particularly relevant for robustness/insensitization questions, since the adjoint system naturally becomes triangular; unique continuation properties for parabolic cascade systems (and consequences for ε–insensitizing controls when the control/observation sets may be disjoint) are analyzed in [23]. For general cascade systems of m coupled parabolic equations controlled by a single force, we refer to [21].

In the hyperbolic framework, indirect controllability/observability phenomena for weakly or locally coupled systems are typically governed by geometric conditions. The geometric control condition (GCC) introduced in [5] plays a central role. For locally coupled wave-type systems, geometric criteria ensuring indirect controllability are developed in [3, 4]. Related controllability results for conservative cascade systems can be found in [37], and more general cascade-coupling controllability statements (with a reduced number of controls) are discussed in [1].

The insensitizing-control problem initiated by Lions has also been investigated for wave equations. In dimension one, insensitizing controls are studied in [13], and locally distributed desensitizing controls for the wave equation are addressed in [39]. Connections between exact insensitization for the scalar wave equation and exact controllability of two-equation cascade systems by a single control are established in [2].

In a related direction, insensitizing/desensitizing controls have been studied with respect to domain variations for the heat equation, where the objective is to make a given functional of the solution first-order insensitive to admissible deformations of the spatial domain, see [18, 34]. These contributions address partially unknown domains and boundary deformations, and provide positive results for approximate desensitization, together with more specific exact-type statements in linear settings, e.g. for finite-parameter deformation families or on finite-dimensional subspaces.

2 Abstract duality Theory

In this section, we go back to the abstract setting defined in Subsection 1.2, and give some dual versions of Problems **EC+EI, NC+EI, AC+AI** and **NAC+AI**, that will be useful for what follows and also for other studies.

As usual, let us introduce the following end-point maps:

$$S_T y_0 = \begin{pmatrix} y(T) \\ z(T) \end{pmatrix},$$

where (y, z) is the corresponding solution of (1.6) with initial condition $y_0 \in H$ and $u = 0$, whereas

$$F_T u = \begin{pmatrix} y(T) \\ z(T) \end{pmatrix},$$

where (y, z) is the corresponding solution of (1.6) with initial condition $y_0 = 0$ and control $u \in L^2((0,T), U)$. We have $S_T \in \mathcal{L}(H, H \times H)$ and $F_T \in \mathcal{L}(L^2((0,T), U), H \times H)$, using the admissibility condition (A3a) or (A3b).

Let us introduce the following adjoint system

$$\begin{cases} \psi'(t) = A_0^*\psi(t) + A_1^*\varphi(t), & t \in (0,T), \\ \varphi'(t) = A_0^*\varphi(t), & t \in (0,T), \\ (\psi(0), \varphi(0)) = (\psi_0, \varphi_0). \end{cases} \tag{2.1}$$

Setting $Z = (\psi, \varphi)^\top$, the coupled system (2.1) can be written as

$$Z' = \mathcal{A}^* Z, \quad t \in (0,T), \qquad \text{and } Z(0) = Z_0, \tag{2.2}$$

with

$$\mathcal{A}^* = \begin{pmatrix} A_0^* & A_1^* \\ 0 & A_0^* \end{pmatrix}.$$

Note that we also have

$$\mathcal{B}^* = \begin{pmatrix} B^* & 0 \end{pmatrix}.$$

Then, thanks to the help of this adjoint system, we can easily compute the adjoint of the operators S_T and F_T.

Proposition 2.1. *The adjoints of S_T and F_T are respectively given as follows:*

1. *$S_T^* : H \times H \to H$ is defined for $(\psi_0, \varphi_0) \in H \times H$ by $S_T^*(\psi_0, \varphi_0) = \psi(T)$, where (ψ, φ) is solution of (2.1) with $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.*

2. *$F_T^* : H \times H \to L^2((0,T), U)$ is defined for $(\psi_0, \varphi_0) \in H \times H$ by $F_T^*(\psi_0, \varphi_0) = B^*\psi(T - \cdot)$, where (ψ, φ) is solution of (2.1) with $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.*

Proof. By usual duality argument (see *e.g.* [10, Section 2.3.1], (y, z) is the solution of (1.9) with initial condition $X_0 = (y_0, 0)^\top$ if and only if for any $Z_0 = (\psi_0, \varphi_0)^\top \in H \times H$ and any $t \in [0, T]$, we have

$$\langle X(t), Z_0 \rangle_{H \times H} - \langle X_0, Z(t) \rangle_{H \times H} = \int_0^t \langle u(s), B^*\psi(t-s) \rangle_U ds, \tag{2.3}$$

where $X(t) = (y(t), z(t))^\top$ is the solution of (1.10) with initial datum X_0 and $Z(t) = (\psi(t), \varphi(t))^\top$ is the solution of (2.2) corresponding to the initial datum Z_0.

Adjoint of S_T. Taking $u = 0$ in (2.3) and $t = T$ gives that for any $Z_0 = (\psi_0, \varphi_0)^\top \in H \times H$, we have

$$\langle X(T), Z_0 \rangle_{H \times H} - \langle X_0, Z(T) \rangle_{H \times H} = 0,$$

i.e.,

$$\langle y_0, \psi(T) \rangle_H = \langle X(T), Z_0 \rangle_{H \times H} = \langle S_T y_0, Z_0 \rangle_{H \times H}.$$

By definition, the right-hand side is also equal to

$$\langle y_0, S_T^*(Z_0) \rangle_H,$$

whence our first point by identification.

Adjoint of F_T. Taking $y_0 = 0$ in (2.3) and $t = T$ gives that for any $Z_0 = (\psi_0, \varphi_0)^\top \in H \times H$, we have

$$\langle X(T), Z_0\rangle_{H\times H} = \int_0^T \langle u(s), B^*\psi(T-s)\rangle_U ds,$$

i.e.,

$$\langle F_T u, Z_0\rangle_H = \int_0^T \langle u(s), B^*\psi(T-s)\rangle_U ds.$$

By definition, the right-hand side is also equal to

$$\langle u, F_T^*(Z_0)\rangle_{L^2((0,T),H)},$$

whence our second point by identification. □

We are now ready to give a dual characterization of our controllability properties in terms of a dual "unique continuation" property. In what follows, $\mathcal{N}(T)$ represents the kernel of a linear operator T.

Proposition 2.2.

1. *Problem* ***EC+EI*** *is equivalent to the following property: there exists $C > 0$ such that for any $(\psi_0, \varphi_0) \in H \times H$,*

$$\|\psi_0\|_H^2 + \|\psi(T)\|_H^2 \leqslant C^2 \|B^*\psi\|^2_{L^2((0,T),U)}, \tag{2.4}$$

where (ψ, φ) is the solution of (2.1) with initial datum $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.

2. *Problem* ***NC+EI*** *is equivalent to the following property: there exists $C > 0$ such that for any $(\psi_0, \varphi_0) \in H \times H$, we have*

$$\|\psi(T)\|_H^2 \leqslant C^2 \|B^*\psi\|^2_{L^2((0,T),U)}, \tag{2.5}$$

where (ψ, φ) is the solution of (2.1) with initial datum $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.

3. *Problem* ***AC+AI*** *is equivalent to the following property: for any $(\psi_0, \varphi_0) \in H \times H$, we have*

$$B^*\psi(t) = 0 \text{ in } [0,T] \Rightarrow \psi_0 = \psi(T) = 0, \tag{2.6}$$

where (ψ, φ) is the solution of (2.1) with initial datum $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.

4. *Problem* ***NAC+AI*** *is equivalent to the following property: for any $(\psi_0, \varphi_0) \in H \times H$, we have*

$$B^*\psi(t) = 0 \text{ in } [0,T] \Rightarrow \psi(T) = 0, \tag{2.7}$$

where (ψ, φ) is the solution of (2.1) with initial datum $(\psi(0), \varphi(0)) = (\psi_0, \varphi_0)$.

5. *Problem* ***AC+EI*** *is equivalent to the two following properties:*
   - *there exists $C > 0$ such that for any $\varphi_0 \in H$, we have*

$$\|\psi(T)\|_H^2 \leqslant C^2 \|B^*\psi\|^2_{L^2((0,T),U)}, \tag{2.8}$$

   where (ψ, φ) is the solution of (2.1) with initial datum $(\psi(0), \varphi(0)) = (0, \varphi_0)$.
   - *If $\psi_0 \in H$ and $(\varphi_{0,n})_{n\in\mathbb{N}}$ is a sequence in H such that $(B^*\psi_n)_n$ goes to 0 in $L^2((0,T),U)$ as $n \to \infty$, where (ψ_n, φ_n) are the solution of (2.1) with initial datum $(\psi_n(0), \varphi_n(0)) = (\psi_0, \varphi_{0,n})$, then $\psi_0 = 0$ and $(\psi_n(T))_{n\in\mathbb{N}}$ goes to 0 in $L^2(\Omega)$.*

Proof. We develop the argument of each item separately.

Proof of item 1. Problem **EC+EI** can be reformulated as: for any $y_0, y_1 \in H$, find a control function $u \in L^2((0,T),U)$ such that

$$S_T y_0 + F_T u = \begin{pmatrix} y_1 \\ 0 \end{pmatrix},$$

i.e.,

$$F_T u = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} - S_T y_0.$$

Let us introduce the following linear continuous map from $H \times H$ into itself:

$$C_T(y_1, y_0) = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} - S_T y_0.$$

Clearly, Problem **EC+EI** is equivalent to saying that

$$\mathcal{R}(C_T) \subset \mathcal{R}(F_T),$$

where $\mathcal{R}$ represents the range of a linear operator. Hence, applying Douglas Theorem [15], we deduce that Problem **EC+EI** is equivalent to saying that there exists $C > 0$ such that for any $(\psi_0, \varphi_0) \in H \times H$, we have

$$\|C_T^*(\psi_0, \varphi_0)\|_{H\times H} \leqslant C\|F_T^*(\psi_0, \varphi_0)\|_{L^2((0,T),U)}. \tag{2.9}$$

Let us compute C_T^*. One readily sees that

$$C_T^*(\psi_0, \varphi_0) = \begin{pmatrix} \psi_0 \\ -S_T^*(\psi_0, \varphi_0) \end{pmatrix}. \tag{2.10}$$

Taking into account the expression of F_T^* and S_T^* given in Proposition 2.1, the estimate (2.9) is simply a reformulation of (2.4).

Proof of item 2. Problem **NC+EI** can be reformulated as: for any $y_0 \in H$, find a control function $u \in L^2((0,T), U)$ such that

$$S_T y_0 + F_T u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, it is equivalent to saying that

$$\mathcal{R}(S_T) \subset \mathcal{R}(F_T).$$

Hence, applying Douglas Theorem [15], we deduce that Problem **NC+EI** is equivalent to saying that there exists $C > 0$ such that for any $(\psi_0, \varphi_0) \in H \times H$, we have

$$\|S_T^*(\psi_0, \varphi_0)\|_{H\times H} \leqslant C\|F_T^*(\psi_0, \varphi_0)\|_{L^2((0,T),U)}.$$

Taking into account the expressions of F_T^* and S_T^* given in Proposition 2.1, we immediately deduce (2.5).

Proof of item 3. Problem **AC+AI** can be reformulated as: for any $y_0, y_1 \in H$, for any $\varepsilon > 0$, find a control function $u \in L^2((0,T), U)$ such that

$$\|C_T(y_1, y_0) - F_T u\|_{H\times H} \leqslant \varepsilon,$$

with C_T as defined above, which is equivalent to

$$\overline{\mathcal{R}(C_T)} \subset \overline{\mathcal{R}(F_T)}.$$

Hence, by passing to the orthogonal, Problem **AC+AI** is also equivalent to saying that

$$\mathcal{N}(F_T^*) \subset \mathcal{N}(C_T^*),$$

which is equivalent to (2.6), Taking into account and the expressions of F_T^* and S_T^* given in Proposition 2.1.

Proof of item 4. Problem **NAC+AI** can be reformulated as: for any $y_0 \in H$, for any $\varepsilon > 0$, find a control function $u \in L^2((0,T), U)$ such that

$$\|S_T(y_0) - F_T u\|_{H\times H} \leqslant \varepsilon,$$

which is equivalent to

$$\overline{\mathcal{R}(S_T)} \subset \overline{\mathcal{R}(F_T)}.$$

Hence, by passing to the orthogonal, Problem **NAC+AI** is also equivalent to saying that

$$\mathcal{N}(F_T^*) \subset \mathcal{N}(S_T^*),$$

which is equivalent to (2.7), taking into account the expressions of F_T^* and S_T^* given in Proposition 2.1.

Proof of item 5. This is again a duality statement, that is slightly more involved. The problem **AC+EI** can be reformulated as: for any $y_0 \in H$ and $y_1 \in H$, for any $\varepsilon > 0$, find a control function $u \in L^2((0,T),U)$ such that

$$\|P_y(C_T(y_1,y_0) - F_T u)\|_H \leqslant \varepsilon, \text{ and } P_z(S_T(y_0) - F_T u) = 0,$$

where P_y and P_z respectively stands for the projection on the y, respectively z, component, that is $P_y = (Id\ 0)$ and $P_z = (0\ Id)$. It is a mixed type condition. The duality result we invoke here is the one given in [32, Theorem 2.1, (ii)], which says that **AC+EI** is equivalent to the two following properties.

- There exists $C > 0$ such that for any $\varphi_0 \in H$, we have

$$\|C_T^*(P_y^*(\varphi_0))\|_H^2 \leqslant C^2 \|F_T^*(P_y^*(\varphi_0))\|_{L^2((0,T),U)}^2, \tag{2.11}$$

  which turns out to be exactly (2.8), by Proposition 2.1 and (2.10), since $P_y^*(\varphi_0) = (0,\varphi_0)^\top$ (which corresponds to considering (2.1) with $\psi_0 = 0$).

- If $\psi_0 \in H$ and $(\varphi_{0,n})_{n\in\mathbb{N}}$ is a sequence in H such that $(F_T^*(\psi_0,\varphi_{0,n})_{n\in\mathbb{N}})$ goes to 0 in $L^2((0,T),U)$ as $n \to \infty$, then, $(C_T^*(\psi_0,\varphi_{0,n})_{n\in\mathbb{N}})$ goes to 0 in $H \times H$ as $n \to \infty$. Taking into account Proposition 2.1 and (2.10), this can be easily reformulated as given in the second point of item 5.

□

Proposition 2.3. *If H is finite-dimensional, the problems **EC+EI**, **AC+EI** and **AC+AI** are equivalent, and the problems **NC+EI** and **NAC+AI** are also equivalent.*

Proof. The finite-dimensional case is easily deduced from the proofs above. Indeed, in the case where H is finite-dimensional, $\mathcal{R}(F_T)$, $\mathcal{R}(S_T)$ and $\mathcal{R}(C_T)$ are closed, as linear subspaces of the linear space of finite dimension H. Hence,

$$\mathcal{R}(C_T) \subset \mathcal{R}(F_T) \quad \Leftrightarrow \quad \overline{\mathcal{R}(C_T)} \subset \overline{\mathcal{R}(F_T)} \quad \text{and} \quad \mathcal{R}(S_T) \subset \mathcal{R}(F_T) \quad \Leftrightarrow \quad \overline{\mathcal{R}(S_T)} \subset \overline{\mathcal{R}(F_T)},$$

from which we deduce that **EC+EI** and **AC+AI** are equivalent, and **NC+EI** and **NAC+AI** are also equivalent. Concerning **AC+EI**, as a consequence of (1.7), it is also equivalent to **EC+EI** and **AC+AI**. □

3 Proof of the main results

3.1 Proof of Theorem 1.4

Let us prove the following result, that is quite surprising and not obvious from a conceptual point of view.

Proposition 3.1. *In the finite dimensional case, all the problems **EC+EI**, **AC+EI**, **AC+AI**, **NC+EI**, **NAC+AI** are equivalent. Besides, they are all solvable if and only if we have the following property: for any $(\psi_0,\varphi_0) \in H \times H$, we have*

$$B^*\psi = 0 \text{ in } [0,T] \Rightarrow \psi(t) = 0, \forall t \geqslant 0, \tag{3.1}$$

where (ψ,φ) is the solution of (2.1) with initial datum $(\psi(0),\varphi(0)) = (\psi_0,\varphi_0)$.

Remark 3.2. *In fact, since solutions of (2.1) are analytic in time, $B^*\psi = 0$ in $[0,T]$ is equivalent to $B^*\psi = 0$ in $\mathbb{R}_+$. It is thus clear that condition (3.1) is independent on the time horizon T. We deduce that if one of the above equivalent properties holds for some time $T_0 > 0$, then all properties hold for any $T > 0$.*

Proof. Since we are working in a finite-dimensional setting, from Proposition 2.3, Problems **EC+EI**, **AC+EI** and **AC+AI** are equivalent, and the problems **NC+EI** and **NAC+AI** are equivalent. Moreover, **EC+EI** implies **NC+EI** as remarked in (1.7). Hence, we are left to prove that **NAC+AI** implies the unique continuation property (3.1) and, for instance, that the unique continuation property (3.1) implies **NC+EI**.

Assume that **NAC+AI** holds true. From Proposition 2.2, we obtain the unique continuation property (2.7). Now, consider any $(\psi_0, \varphi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ such that the corresponding solution (ψ, φ) of (2.2) with initial condition (ψ_0, φ_0) verifies $B^*\psi = 0$ in $[0, T]$. By (2.7), we obtain $\psi(T) = 0$. Since the solution (ψ, φ) of (2.2) is analytic on $\mathbb{R}_+$, the solution in fact satisfies that $B^*\psi = 0$ in $\mathbb{R}_+$. Applying the unique continuation property (2.7) on $(\psi(\cdot + \delta), \varphi(\cdot + \delta))$ for $\delta \geqslant 0$, we deduce that for all $\delta \geqslant 0$, $\psi(T + \delta) = 0$. Since ψ is analytic in time, we necessarily have that ψ vanishes for all times.

The fact that the unique continuation property (3.1) implies **NC+EI** is obvious in view of (3.1) and the characterization (2.6) in Proposition 2.2. □

In fact, we can now prove Theorem 1.4, that gives a complete characterization in terms of a Kalman rank condition.

Proof of Theorem 1.4. From Proposition 3.1, we already know that the problems **EC+EI, AC+EI, NC+EI, AC+AI, NAC+AI** are equivalent. Assume that any of these problems is solvable, which, by Proposition 3.1, is equivalent to the unique continuation property (3.1). Our goal is thus simply to give necessary and sufficient conditions for the unique continuation property (3.1).

Let (ψ, φ) denote the solution of System (2.1) with initial condition (ψ_0, φ_0). The Duhamel formula shows that

$$\begin{cases} \varphi(t) = e^{tA_0^*}\varphi_0 \\ \psi(t) = e^{tA_0^*}\psi_0 + \displaystyle\int_0^t e^{(t-s)A_0^*} A_1^* e^{sA_0^*}\varphi_0 ds, \end{cases} \tag{3.2}$$

for all $t \in [0, T]$. Assume that $B^*\psi = 0$ in $(0, T)$. Recalling that $Z = (\psi, \varphi)^\top$ satisfies the finite dimensional equation (2.2), Z is obviously analytic in time, so that $B^*\psi = 0$ in $(0, T)$ is equivalent to $B^*\psi^{(k)}(0) = 0$, for all $k \in \mathbb{N}$, which turns out, after easy computations, to be equivalent to

$$\mathcal{B}^*(\mathcal{A}^*)^k \begin{pmatrix} \psi_0 \\ \varphi_0 \end{pmatrix} = 0, \ \forall k \in \mathbb{N}.$$

Since $\mathcal{A}^*$ is a matrix of size $2n \times 2n$, by Cayley-Hamilton theorem, we easily obtain that

$$B^*\psi = 0 \text{ in } (0, T) \quad \Leftrightarrow \quad \forall k \in \{0, \cdots, 2n-1\}, \ \mathcal{B}^*(\mathcal{A}^*)^k \begin{pmatrix} \psi_0 \\ \varphi_0 \end{pmatrix} = 0.$$

We thus compute the matrix $(\mathcal{A}^*)^k$ for $k \in \mathbb{N}$: by induction,

$$(\mathcal{A}^*)^k = \begin{pmatrix} (A_0^*)^k & \sum_{i=1}^k (A_0^*)^{k-i} A_1^* (A_0^*)^{i-1} \\ 0 & (A_0^*)^k \end{pmatrix}.$$

We then infer that $B^*\psi = 0$ in $(0, T)$ is equivalent to

$$K \begin{bmatrix} \psi_0 \\ \varphi_0 \end{bmatrix} = 0,$$

where

$$K = \begin{bmatrix} B^* & 0 \\ B^*A_0^* & B^*A_1^* \\ \vdots & \vdots \\ B^*A_0^{*2n-1} & \sum_{i=1}^{2n-1} B^*A_0^{*(2n-1-i)} A_1^* A_0^{*(i-1)} \end{bmatrix} \in \mathcal{M}_{2mn,2n}(\mathbb{C}).$$

Setting

$$\Pi = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{C}),$$

The unique continuation property (3.1) can then be rewritten as

$$K\begin{bmatrix}\psi_0\\ \varphi_0\end{bmatrix}=0\Rightarrow \Pi\begin{bmatrix}\psi_0\\ \varphi_0\end{bmatrix}=0,$$

i.e., $\mathcal{N}(K)\subset\mathcal{N}(\Pi)$, *i.e.*, by well-known properties of the annihilator of matrices, $\mathrm{Range}(\Pi^*)\subset\mathrm{Range}(K^*)$, where

$$K^*=\begin{bmatrix}B & A_0B & \dots & {A_0}^{2n-1}B\\ 0 & A_1B & \dots & \sum_{i=1}^{2n-1}A_0^{i-1}A_1A_0^{2n-1-i}B\end{bmatrix}\in\mathcal{M}_{2n,2nm}(\mathbb{C}).$$

In other words, the unique continuation property (3.1) is equivalent to the following property: for any $Y\in\mathbb{C}^n$, we want to find $(Z_1,\dots Z_{2n})\in(C^m)^{2n}$ such that

$$\sum_{k=1}^{2n}A_0^{k-1}BZ_k=Y\quad\text{ and }\quad\sum_{k=1}^{2n}\sum_{i=1}^{k-1}A_0^{i-1}A_1A_0^{2n-1-i}BZ_k=0,$$

which is equivalent to saying that $[(A_0,B)]_{|L}$ is onto and gives exactly the desired condition. □

3.2 Proof of Theorem 1.6

Let us introduce the adjoint system

$$\begin{cases}\dfrac{\partial\psi}{\partial t}-\Delta\psi & = & \Delta\varphi & \text{in } (0,T)\times\Omega,\\ \psi & = & 0 & \text{on } (0,T)\times\partial\Omega,\\ \psi(0,\cdot) & = & \psi_0 & \text{in } \Omega,\\ \dfrac{\partial\varphi}{\partial t}-\Delta\varphi & = & 0 & \text{in } (0,T)\times\Omega,\\ \varphi & = & 0 & \text{on } (0,T)\times\partial\Omega,\\ \varphi(0,\cdot) & = & \varphi_0 & \text{in } \Omega,\end{cases}\tag{3.3}$$

where $\psi_0,\varphi_0\in L^2(\Omega)$.

To analyze the unique continuation properties and their quantifications, we use [35, Theorem 4]. With the notations of this paper, the system (3.3) can be rewritten as

$$Z'=D\Delta Z,\qquad\text{where } D=\begin{pmatrix}1&1\\0&1\end{pmatrix},$$

and the observation operator $B\mathbb{1}_\omega$ is given by

$$B=\begin{pmatrix}1&0\end{pmatrix}.$$

Notably, D is elliptic as we have for all $x,y\in\mathbb{R}$,

$$\langle D\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\rangle_{\mathbb{R}^2}=x^2+xy+y^2\geqslant\frac{1}{2}(x^2+y^2).$$

Moreover, the following "Spectral Kalman rank condition" is verified: for any eigenvalue λ of the Dirichlet-Laplace operator, we have

$$\mathrm{Rank}\begin{pmatrix}B^* & -\lambda D^*B^*\end{pmatrix}=\mathrm{Rank}\begin{pmatrix}1&-\lambda\\0&-\lambda\end{pmatrix}=2,$$

since any such λ is positive. We deduce that [35, Theorem 4] applies, which in particular gives that for all $T>0$, there exists $C_T>0$ such that for any $(\psi_0,\varphi_0)\in(L^2(\Omega))^2$,

$$\int_\Omega\left(|\psi(T,x)|^2+|\varphi(T,x)|^2\right)\,dx\leqslant C_T\int_0^T\int_\omega\psi(t,x)^2\,dx\,dt,\tag{3.4}$$

where (ψ,φ) is the solution of (3.3) corresponding to the initial datum (ψ_0,φ_0).

In particular, (2.5) is satisfied. Hence, from Proposition 2.2, we immediately deduce that Problem **NC+EI** is solvable in any time.

To address Problem **AC+EI**, let us first note that the observability property (2.8) is satisfied since the estimate (3.4) holds. We should thus only check that if we have a sequence $(\varphi_{0,n})_{n\in\mathbb{N}}$ of elements of $L^2(\Omega)$ and $\psi_0 \in L^2(\Omega)$ such that the corresponding solution (ψ_n, φ_n) of (3.3) satisfies that $(\mathbb{1}_\omega \psi_n)_{n\in\mathbb{N}}$ goes to 0 in $L^2((0,T)\times\omega)$, then $\psi_0 = 0$ and $(\psi_n(T))_{n\in\mathbb{N}}$ goes to 0 in $L^2(\Omega)$.

The fact that $(\psi_n(T))_{n\in\mathbb{N}}$ (and also $(\varphi_n(T))_{n\in\mathbb{N}}$) goes to 0 in $L^2(\Omega)$ is an immediate consequence of (3.4).

Now, let $(e_k)_{k\geq 1}$ be an orthonormal basis of $L^2(\Omega)$ formed by eigenfunctions of the Dirichlet Laplacian:

$$-\Delta e_k = \lambda_k e_k, \qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \qquad (e_i, e_j)_{L^2(\Omega)} = \delta_{ij}.$$

We write the initial data and the solutions in this basis. For every n and k, define

$$\alpha_{n,k} := (\varphi_{0,n}, e_k)_{L^2(\Omega)}, \qquad \beta_k := (\psi_0, e_k)_{L^2(\Omega)}.$$

Since φ_n and ψ_n satisfy the Duhamel formula (3.2) (which is still valid in infinite dimension), the k-th Fourier coefficients satisfy the exact relations

$$(\varphi_n(T), e_k)_{L^2(\Omega)} = e^{-\lambda_k T}\,\alpha_{n,k}, \qquad (\psi_n(T), e_k)_{L^2(\Omega)} = e^{-\lambda_k T}\big(\beta_k - T\lambda_k \alpha_{n,k}\big).$$

Now, we use the previous convergences: one has

$$\varphi_n(T) \to 0 \quad \text{in } L^2(\Omega), \qquad \psi_n(T) \to 0 \quad \text{in } L^2(\Omega).$$

Multiplying by e_k, we obtain

$$e^{-\lambda_k T}\alpha_{n,k} = (\varphi_n(T), e_k)_{L^2(\Omega)} \xrightarrow[n\to\infty]{} 0,$$

hence $(\alpha_{n,k})_{n\in\mathbb{N}} \to 0$, for every $k \geqslant 1$. From the second relation,

$$e^{-\lambda_k T}\big(\beta_k - T\lambda_k \alpha_{n,k}\big) = (\psi_n(T), e_k)_{L^2(\Omega)} \xrightarrow[n\to\infty]{} 0,$$

so $(\beta_k - T\lambda_k\alpha_{n,k})_{n\in\mathbb{N}} \to 0$. Combining with $(\alpha_{n,k})_{n\in\mathbb{N}} \to 0$ yields $\beta_k = 0$ for every k. Therefore $\psi_0 = \sum_k \beta_k e_k = 0$ in $L^2(\Omega)$, which concludes the proof.

3.3 Proof of Theorem 1.7

Let us start by making precise the functional setting needed to deal with the wave equation.

The wave operator is given by

$$A_0 := \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix} \quad \text{on } X_0 = H_0^1(\Omega) \times L^2(\Omega), \quad \text{with domain } \mathscr{D}(A_0) = H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega),$$

and the operator A_1 is the bounded operator in X_0 given by

$$A_1 := \begin{pmatrix} 0 & 0 \\ -p & 0 \end{pmatrix}.$$

The control operator B is given by

$$B := \begin{pmatrix} 0 \\ \mathbb{1}_\omega \end{pmatrix},$$

and is defined as an operator from $U = L^2(\Omega)$ to X_0.

The operators $\mathcal{A}$ and $\mathcal{B}$ corresponding to (1.17) are then given by

$$\mathcal{A} := \begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix} \quad \text{on } X = X_0^2, \qquad \mathcal{B} := \begin{pmatrix} B \\ 0 \end{pmatrix} : U \to X.$$

Adjoint system and dual characterization Formally, the adjoint of $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\mathcal{A}^* = \begin{pmatrix} A_0^* & A_1^* \\ 0 & A_0^* \end{pmatrix}, \qquad \mathcal{B}^* = (B^*, 0).$$

This is true, but one needs to be careful with the choice of duality product that we take. For PDE, it is customary to choose to identify $L^2(\Omega)$ with its dual. In such case, $X_0' = L^2(\Omega) \times H^{-1}(\Omega)$. The usual choice is then to choose the duality product $\langle \cdot, \cdot \rangle_{X_0, X_0'}$ as follows:

$$\forall \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in X_0, \forall \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in X_0', \left\langle \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\rangle_{X_0, X_0'} = -\langle \varphi_1, \psi_2 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + \int_\Omega \varphi_2 \psi_1 \, dx$$
$$= -\int_\Omega \nabla \varphi_1 \cdot \nabla (-\Delta_D)^{-1} \psi_2 \, dx + \int_\Omega \varphi_2 \psi_1 \, dx,$$

where $-\Delta_D$ is the Dirichlet operator on $H^{-1}(\Omega)$ with domain $H_0^1(\Omega)$.

The adjoint of A_0 is then given by

$$A_0^* := -\begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix} \quad \text{on } X_0', \qquad \text{with domain } \mathscr{D}(A_0^*) = H_0^1(\Omega) \times L^2(\Omega),$$

and the adjoint of A_1 is

$$A_1^* := -\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}.$$

Finally, the adjoint of B is given by

$$B^* := (\mathbb{1}_\omega, 0).$$

The adjoint equation of (1.17) is thus given by

$$\begin{cases} \psi'(t) = A_0^* \psi(t) + A_1^* \varphi(t), \\ \varphi'(t) = A_0^* \varphi(t), \end{cases} \qquad (\psi, \varphi)(0) = (\psi_0, \varphi_0) \in X',$$

with $X' = X_0' \times X_0' = (L^2(\Omega) \times H^{-1}(\Omega))^2$.

Written in PDE form, the dual unknowns (ψ, φ) should thus satisfy:

$$\begin{cases} \partial_{tt} \psi - \Delta \psi + p \varphi = 0 & \text{in } (0,T) \times \Omega, \\ \partial_{tt} \varphi - \Delta \varphi = 0 & \text{in } (0,T) \times \Omega, \\ \psi = \varphi = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\psi, \partial_t \psi)(0, \cdot) = (\psi_0, \psi_1) & \text{in } \Omega, \\ (\varphi, \partial_t \varphi)(0, \cdot) = (\varphi_0, \varphi_1) & \text{in } \Omega. \end{cases} \tag{3.5}$$

Analysis of the problem AC+AI.

Proof of Theorem 1.7. Problem **AC+AI** is equivalent to showing that, if (ψ, φ) is a solution of (3.5) satisfying $\psi = 0$ in $(0,T) \times \omega$, then $\psi \equiv 0$ in $(0,T) \times \Omega$.

This is true provided ω satisfies suitable geometric conditions. Indeed, if $\psi = 0$ in $(0,T) \times \omega$, then $\partial_{tt} \psi = \Delta \psi = 0$ in $(0,T) \times \omega$ and (3.5) yields $p\,\varphi = 0$ in $(0,T) \times \omega$. By (1.16), this implies $\varphi = 0$ in $(0,T) \times (\mathrm{int}(\mathrm{Supp}\, p) \cap \omega)$. Accordingly, if $(T, \Omega, \mathrm{int}(\mathrm{Supp}\, p) \cap \omega)$ satisfies $T > 2 \sup_{x \in \Omega} \{d_\Omega(x, \mathrm{int}(\mathrm{Supp}\, p) \cap \omega)\}$, where d_Ω denotes the geodesic distance in Ω, by Holmgren's uniqueness theorem (see [22, Theorem 5.3.1]; for a more sophisticated version adapted to the wave equation and allowing to quantify this uniqueness property, we also refer to [26, Theorem 6.1]), $\varphi = 0$ in $(0,T) \times \Omega$.

Hence, ψ satisfies the free wave equation $\partial_{tt} \psi - \Delta \psi = 0$ in $(0,T) \times \Omega$, $\psi = 0$ on $(0,T) \times \partial\Omega$ and $\psi = 0$ in $(0,T) \times \omega$. Thus we immediately deduce, again using Holmgren's uniqueness theorem, that $\psi \equiv 0$ in $(0,T) \times \Omega$.

By the duality result of Proposition 2.2, this concludes the proof of Theorem 1.7. □

4 Some extensions and open problems

This section collects a few extensions of the framework developed in the previous sections and highlights open problems arising from the PDE examples. Our goal is not to be exhaustive, but rather to indicate natural directions where the abstract methodology remains applicable and to pinpoint the analytical difficulties that appear beyond the cases treated in this paper.

4.1 Multiple parameters

A natural extension is to replace the scalar parameter σ by a vector parameter $\theta = (\theta_1, \dots, \theta_q) \in \mathbb{R}^q$. Assume that $\theta \mapsto A(\theta)$ is differentiable at $\theta = 0$ (in the same sense as in Hypotheses set 1 or 2), and set

$$A_j := \partial_{\theta_j} A(\theta)\big|_{\theta=0} \qquad (j = 1, \dots, q).$$

Let y_θ be the state of the parameter-dependent system and set $y := y_{\theta=0}$. Define

$$z^{(j)}(t) := \partial_{\theta_j} y_\theta(t)\big|_{\theta=0}$$

as the first-order sensitivities along each coordinate direction. Differentiating the state equation yields the cascade

$$\begin{cases} y'(t) = A_0 y(t) + Bu(t), \\ (z^{(j)})'(t) = A_0 z^{(j)}(t) + A_j y(t), \quad j = 1, \dots, q, \\ y(0) = y_0, \qquad z^{(j)}(0) = 0, \quad j = 1, \dots, q. \end{cases}$$

Thus, first-order insensitization with respect to all components amounts to enforcing

$$z^{(j)}(T) = 0, \qquad j = 1, \dots, q,$$

together with the desired terminal requirement on $y(T)$.

Equivalently, with the extended state

$$X(t) := \big(y(t), z^{(1)}(t), \dots, z^{(q)}(t)\big) \in H^{q+1},$$

one is led to the controllability of the block-lower triangular system

$$X'(t) = \mathcal{A}\, X(t) + \mathcal{B}\, u(t), \qquad \mathcal{B} = \begin{pmatrix} B \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\mathcal{A} \in \mathcal{L}(H^{q+1})$ is given by

$$\mathcal{A} = \begin{pmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & 0 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_q & 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

In particular, the coupling operators A_j appear in the first column below the diagonal, namely $\mathcal{A}_{j+1,1} = A_j$ for $j = 1, \dots, q$.

For any fixed q, the multi-parameter extension is straightforward: one stacks the q first-order sensitivities into an extended state in H^{q+1}, and the well-posedness and duality arguments carry over verbatim. The only additional issue is quantitative: obtaining observability and hence control-cost estimates with explicit dependence on q, or uniform in q when q is large, since the adjoint coupling involves the operator $\mathcal{A}^*$ whose norm may grow with q.

4.2 Higher–order insensitization

We explain how to adapt the arguments above to obtain higher-order insensitization. Given an integer $k \geq 1$, we look for a control u (independent of σ) such that the end-point map has vanishing derivatives up to order k at $\sigma = 0$:

$$\frac{\partial^j y_\sigma(T)}{\partial \sigma^j}\Big|_{\sigma=0} = 0, \qquad j = 1, \dots, k.$$

The idea is to differentiate the state equation repeatedly with respect to σ and to reduce the problem to controllability of a larger block-lower triangular extended system. To perform k differentiations, one may assume that either Hypotheses set 1 or Hypotheses set 2 holds and strengthen the regularity in σ as follows:

(A1$_k$) The map $\sigma \mapsto A(\sigma)$ is of class $\mathcal{C}^k$ in the sense of Hypotheses set 1 (*i.e.*, in $\mathcal{L}(D(A_0), H)$) or Hypotheses set 2 (*i.e.*, in $\mathcal{L}(H)$), and we denote

$$A_j := \frac{\partial^j A}{\partial \sigma^j}\Big|_{\sigma=0} \qquad (j = 0, \dots, k).$$

(A3$_k$) The control operator B satisfies the same admissibility/boundedness assumption as in (A3a) or (A3b), so that the nominal system and all differentiated cascades are well posed in the corresponding state spaces.

For simplicity, and as in many PDE examples treated in this paper, one may additionally assume that each A_j extends to a bounded operator on H. More general, possibly unbounded, perturbations can be handled, at the cost of additional functional-analytic technicalities.

Cascade of sensitivity equations. For $\sigma \in I$, let y_σ solve

$$\partial_t y_\sigma = A(\sigma) y_\sigma + Bu, \qquad y_\sigma(0) = y_0.$$

Set for $r = 0, \dots, k$

$$y^{(r)}(t) := \frac{\partial^r y_\sigma(t)}{\partial \sigma^r}\Big|_{\sigma=0},$$

so that $y^{(0)} = y$ (the solution at $\sigma = 0$). Differentiating the evolution equation r times and using the product rule yields the triangular cascade

$$\begin{cases} \partial_t y^{(0)} = A_0\, y^{(0)} + B\, u, \\ \partial_t y^{(r)} = A_0\, y^{(r)} + \displaystyle\sum_{i=1}^{r} \binom{r}{i} A^{(i)}\, y^{(r-i)}, \qquad r = 1, \dots, k, \end{cases} \tag{4.1}$$

with initial conditions $y^{(0)}(0) = y_0$ and $y^{(r)}(0) = 0$ for $r \geq 1$.

Extended state formulation. Set the extended state $X := \big(y^{(0)}, y^{(1)}, \dots, y^{(k)}\big)^\top \in X_H := H^{\,k+1}$. Then (4.1) can be written as a single linear control system

$$\frac{d}{dt} X = \mathcal{A}^{(k)} X + \mathcal{B}^{(k)} u,$$

where $\mathcal{A}^{(k)}$ is block lower-triangular with A_0 on each diagonal block and linear combinations of powers of the A_j on the lower subdiagonals, while

$$\mathcal{B}^{(k)} = \begin{pmatrix} B \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

More precisely, for $r, s \in \{0, \ldots, k\}$ with $r \geq s$ the (r,s)–block of $\mathcal{A}^{(k)}$ is

$$\mathcal{A}^{(k)}_{r,s} = \begin{cases} A_0, & r = s, \\ \begin{pmatrix} r \\ r-s \end{pmatrix} A^{(r-s)}, & r > s, \\ 0, & r < s. \end{cases}$$

Higher-order insensitization problems can therefore be formulated as exact/null/approximate controllability questions for this extended cascade, and duality again leads to observability properties for the corresponding adjoint block-upper triangular system.

Higher–order control problems. For instance, the higher-order analogues of exact and null insensitization can be stated as follows.

Problem EC+EI$_k$ (Exact controllability + k-th order insensitization). For every $y_0, y_T \in H$, find $u \in L^2((0,T), U)$ such that the solution of (4.1) satisfies

$$y^{(0)}(T) = y_T, \qquad y^{(r)}(T) = 0 \quad (r = 1, \ldots, k).$$

Problem NC+EI$_k$ (Null controllability + k-th order insensitization). For every $y_0 \in H$, find $u \in L^2((0,T), U)$ such that

$$y^{(r)}(T) = 0 \quad (r = 0, \ldots, k).$$

4.3 Open problems

4.3.1 Heat equation with heterogeneous diffusion

A particularly interesting question is to understand insensitization for spatially heterogeneous diffusion perturbations. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2 and let $\omega \subset \Omega$ be nonempty and open. Let $a \in L^\infty(\Omega)$ satisfy

$$0 < a_- \leq a(x) \leq a_+ \qquad \text{for a.e. } x \in \Omega.$$

For $|\sigma| \leq 1/(2a_+)$, consider the family

$$\begin{cases} \dfrac{\partial y_\sigma}{\partial t} - \Delta y_\sigma - \sigma \operatorname{div}(a \nabla y_\sigma) = \mathbb{1}_\omega u & \text{in } (0,T) \times \Omega, \\ y_\sigma = 0 & \text{on } (0,T) \times \partial\Omega, \\ y_\sigma(0, \cdot) = y_0 & \text{in } \Omega. \end{cases}$$

Investigating the robustness with respect to σ at $\sigma = 0$ leads to the following coupled system:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial t} - \Delta z = \operatorname{div}(a \nabla y) & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

While the constant-coefficient case corresponding to $a \equiv 1$ fits into the framework treated above, the controllability of this heterogeneous cascade for general a remains unclear. Equivalently, one is led to the question of whether the associated adjoint cascade enjoys a suitable observability inequality from ω, or even a suitable unique continuation property, which remain unclear and also unknown in the abundant current literature of coupled parabolic systems (see *e.g.* [38, 8, 9] for recent results on coupled parabolic systems by lower-order terms, or [35, 16, 25], for recent results on the specific case where the coupling arises on the principal part of the operator). Establishing such an estimate would likely require new Carleman weights or refined microlocal arguments adapted to the variable diffusion structure.

4.3.2 Quantitative bounds, minimal time, and geometry

Even in the models for which insensitizing controls exist, several quantitative issues remain open. For instance, one may ask for sharp dependence of control costs on the time horizon T, on the size or geometry of ω, and on the amplitude of the perturbation, through suitable norms of A_1 or of PDE coefficients. In hyperbolic settings, it would also be natural to investigate whether insensitization can be achieved under the same minimal-time and geometric conditions as the controllability of the nominal equation, and how the coupling affects the relevant geometric optics mechanisms. For recent results on coupled wave systems, we refer to [33, 11, 36, 28].

A Appendix. Proofs of Proposition 1.2 and 1.3

Before going further, let us recall that if A is a bounded operator from $D(A)$ to H, densely defined and with a non-empty resolvent set, one can define its extension as a bounded operator from H to $\mathcal{D}(A^*)'$ by the extrapolation method, see for instance [40, Section 2.10]. Accordingly, we also have by interpolation that A is a bounded operator from $[H, D(A)]_{1/2}$ to $[D(A^*)', H]_{1/2} = [H, D(A^*)]'_{1/2}$. In view of Assumption (H0), we get that for any $\sigma \in (-\varepsilon, \varepsilon)$, $A(\sigma)$ is a bounded operator from $[H, D(A_0)]_{1/2}$ to $[H, D(A_0^*)]'_{1/2}$.

Proof of Proposition 1.2. We divide the proof depending on the set of Assumptions we consider, although both proofs are very similar.

Under the Hypotheses set 1. From hypothesis (A2a), using the mean value inequality, there exists $M > 0$ (depending on ε) such that for any $\sigma \in [-\varepsilon/2, \varepsilon/2]$, for every $x \in H$, we have

$$\|(A(\sigma) - A_0)(x)\|_H \leqslant M\sigma \|x\|_{D(A_0)}.$$

In particular, in view of the definition of the graph norm, for any $\alpha > 0$, there exists an open neighborhood I of 0 and a $C > 0$ such that for each $\sigma \in I$,

$$\|(A(\sigma) - A_0)(x)\|_H \leqslant \alpha(\|A_0 x\|_H + \|x\|_H).$$

Since A_0 is the generator of an analytic semigroup on H, a standard perturbation argument (see, *e.g.*, [17, Theorem III.2.10]) enables to deduce that for I corresponding to a small enough α, for any $\sigma \in I$, $A(\sigma)$ is also the generator of an analytic semigroup on H. Moreover, inspecting the proof of [17, Theorem III.2.10] easily leads to the fact that (1.3) holds independently on $\sigma \in I$.

Let $T_0 > 0$. Since the semigroup $(e^{tA_0})_{t \geqslant 0}$ is analytic on H (assumption (H1a)) and $B \in \mathcal{L}(U, ([H, D(A_0^*)]_{1/2})')$ is bounded (assumption (H3a)), by maximal regularity (*cf.* [6, Proposition 3.7], applied in the state space $([H, D(A_0^*)]'_{1/2})$), the solution y_0 of

$$y_0' = A_0 y_0 + Bu, \quad \text{in } (0, T_0) \quad \text{with } y_0(0) = 0 \tag{A.1}$$

satisfies $y_0 \in L^2((0, T_0), [H, D(A_0)]_{1/2}) \cap H^1((0, T_0), [H, D(A_0^*)]'_{1/2}) \hookrightarrow C^0([0, T_0], H)$, with

$$\|y_0\|_{L^2((0,T_0),[H,D(A_0)]_{1/2})} + \sup_{[0,T_0]} \|y_0(t)\|_H \leqslant C \|u\|_{L^2((0,T_0),U)}, \ \forall u \in L^2((0, T_0), U), \tag{A.2}$$

where C does not depend on u and might change from line to line.

Now, fix $u \in L^2((0, T_0), U)$ and y_0 the corresponding solution of (A.1). For $\sigma \in I$, we introduce z_σ the solution of

$$z_\sigma' = A(\sigma) z_\sigma + (A(\sigma) - A_0) y_0, \quad \text{in } (0, T_0) \quad \text{with } z_\sigma(0) = 0. \tag{A.3}$$

Using the previous regularity properties of y_0, we obtain

$$(A(\sigma) - A_0) y_0 \in L^2((0, T_0), [H, D(A_0^*)]'_{1/2}),$$

with the precise operator estimate

$$\|(A(\sigma) - A_0) y_0\|_{L^2((0,T_0),[H,D(A_0^*)]'_{1/2})} \leq \|A(\sigma) - A_0\|_{\mathcal{L}([H,D(A_0)]_{1/2},[H,D(A_0^*)]'_{1/2})} \cdot \|y_0\|_{L^2((0,T_0),[H,D(A_0)]_{1/2})}. \tag{A.4}$$

Moreover, by analyticity of the semigroup, $(e^{tA(\sigma)})_{t\geqslant 0}$, and the same maximal regularity argument as before, z_σ satisfies $z_\sigma \in L^2((0,T_0),[H,D(A_0)]_{1/2}) \cap H^1((0,T_0),[H,D(A_0^*)]'_{1/2}) \hookrightarrow C^0([0,T_0],H)$, with

$$\sup_{[0,T_0]} \|z_\sigma(t)\|_H \leqslant C\|(A(\sigma)-A_0)y_0\|_{L^2((0,T_0),[H,D(A_0^*)]'_{1/2})}. \tag{A.5}$$

From (A.2), (A.4) and (A.5), we deduce that

$$\sup_{[0,T_0]} \|z_\sigma(t)\|_H \leqslant C\,\|u\|_{L^2((0,T_0),U)}\,,\ \forall u \in L^2((0,T_0),U), \tag{A.6}$$

Finally, note that $y_\sigma = y_0 + z_\sigma$ satisfies

$$y'_\sigma = A(\sigma)y_\sigma + Bu, \quad \text{ in } (0,T_0) \quad \text{ with } y_\sigma(0)=0, \tag{A.7}$$

and belongs to $C^0([0,T_0],H)$, with norm bounded by $C\|u\|_{L^2((0,T_0),U)}$. In particular, we deduce that the end-point map $u \in L^2((0,T),U) \mapsto y(T) \in H$ is bounded, which exactly means that by the Duhamel formula, that

$$\left\| \int_0^{T_0} e^{(T_0-s)A(\sigma)} B\, u(s)\, ds \right\|_H \leq C_0 \|u\|_{L^2((0,T_0),U)}. \tag{A.8}$$

so that B is admissible for the semigroup $A(\sigma)$, for any $\sigma \in I$.

Under the Hypotheses set 2. By continuity of the perturbation in $\mathcal{L}(H)$, the bounded perturbation theorem [17, Theorem III.1.3] ensures that for $I=(-\varepsilon/2,\varepsilon/2)$ (for instance) and any $\sigma \in [-\varepsilon/2,\varepsilon/2]$, $A(\sigma)$ generates a C_0-semigroup verifying (1.3), independently on $\sigma \in I$. Note that by the Duhamel formula, the admissibility assumption (H3b) is equivalent to saying that for any function $u \in L^2((0,T_0),U)$, the solution y_0 of (A.1) satisfies $y_0(T_0) \in H$. Using [40, Proposition 4.2.4], in fact we even have that $y_0 \in C^0([0,T_0],H)$, and

$$\sup_{[0,T_0]} \|y_0(t)\|_H \leqslant C\,\|u\|_{L^2((0,T_0),U)}\,.$$

Also note that the arguments in [40] allows us to show that, for any $u \in L^2((0,T_0),U)$, the solution y_σ to the equation

$$y'_\sigma = A(\sigma)y_\sigma + Bu, \quad \text{ in } (0,T_0) \quad \text{ with } y_\sigma(0)=0$$

is well-defined and unique in $C^0([0,T_0],D(A(\sigma)^*)')$. Now, fix $u \in L^2((0,T_0),U)$ and y_0 the corresponding solution of (A.1). For $\sigma \in (-\varepsilon,\varepsilon)$, we introduce z_σ as the solution of (A.3).

Note that assumption (H2b) implies that the operator $A(\sigma)-A_0$ is bounded on H. Accordingly, the source term $(A(\sigma)-A_0)y_0 \in C^0([0,T_0],H)$. The fact that $(e^{tA(\sigma)})_{t\geqslant 0}$ is a C_0-semigroup on H implies that $z_\sigma \in C^0([0,T_0],H)$.

We then set $y_\sigma = y_0 + z_\sigma$. By construction, it solves the equation (A.7). Since

$$\|y_\sigma(T_0)\|_H \leqslant \|y_0(T_0)\|_H + \|z_\sigma(T_0)\|_H \leqslant C(1 + C\|A(\sigma)-A_0\|_{\mathcal{L}(H)})\,\|u\|_{L^2((0,T_0),U)}\,,$$

we have proved that (A.8) holds, so that B is admissible for the semigroup $A(\sigma)$, for any $\sigma \in I$. ☐

Proof of Proposition 1.3. Here again, we divide the proof depending on the set of Assumptions we consider, although both proofs are very similar. We will also build on the proof of Proposition 1.2 to show the result.

To start with, let us note that under both sets of assumptions, the solution z of (1.5) belongs to $C^0([0,T],H)$.

Let us remark that

$$\frac{1}{\sigma}(A(\sigma)-A_0) = A_1 + R(\sigma), \tag{A.9}$$

where $R(\sigma)$ goes to 0 as $\sigma \to 0$ in $\mathcal{L}(D(A_0),H)$ when working with the Hypotheses set 1, or in $\mathcal{L}(H)$ when working with the Hypotheses set 2.

Now, we set

$$r_\sigma = \frac{y_\sigma - y}{\sigma} - z, \tag{A.10}$$

where z is the solution of (1.5). By definition,

$$r_\sigma' = \frac{y_\sigma' - y'}{\sigma} - z'.$$

Using (1.1) and (1.4), we obtain

$$\frac{y_\sigma' - y'}{\sigma} = \frac{A(\sigma)y_\sigma - A_0 y}{\sigma},$$

and therefore, by (1.5), we have

$$r_\sigma' = \frac{A(\sigma)y_\sigma - A_0 y}{\sigma} - \big(A_0 z + A_1 y\big).$$

Now, we write

$$A(\sigma)y_\sigma - A_0 y = A(\sigma)(y_\sigma - y) + (A(\sigma) - A_0)y.$$

(A.10) and (A.9), we infer that

$$r_\sigma' = A(\sigma)(r_\sigma + z) + (A_1 + R(\sigma))y - A_0 z - A_1 y.$$

We deduce that r_σ solves the equation

$$\begin{cases} \dfrac{dr_\sigma}{dt}(t) = A(\sigma)r_\sigma + (A(\sigma) - A_0)z + R(\sigma)y, \quad t \in (0,T), \\ r_\sigma(0) = 0, \end{cases}$$

The conclusion that $\|r_\sigma\|_{C^0([0,T],H)} \to 0$ as $\sigma \to 0$ will follow, which is equivalent to saying that the mapping $\sigma \in I \mapsto y_\sigma \in C^0([0,T],H)$ is differentiable at $\sigma = 0$ with derivative z.

Indeed, under Hypotheses set 1, $y \in L^2((0,T),[H,D(A_0)]_{1/2})$, and $A_1 \in \mathcal{L}([H,D(A_0)]_{1/2}),([H,D(A_0^*)]_{1/2})')$, so $z \in L^2((0,T),[H,D(A_0)]_{1/2})$. Thus, $(A(\sigma) - A_0)z \in L^2((0,T),([H,D(A_0^*)]_{1/2})')$ with norm going to 0 as $\sigma \to 0$. Similarly, $R(\sigma)y \in L^2((0,T),([H,D(A_0^*)]_{1/2})')$ with norm going to 0. Using then the analyticity of the semigroup $(e^{tA(\sigma)})_{t\geqslant 0}$ for $\sigma \in I$, we deduce by maximal regularity, the Duhamel formula, and (1.3), that r_σ goes to 0 in $C^0([0,T],H)$ as $\sigma \to 0$.

Under Hypotheses set 2, y and z both belong to $C^0([0,T],H)$, and the assumptions give that $(A(\sigma) - A_0)z + R(\sigma)y$ belongs to $C^0([0,T],H)$, with norm going to 0 as $\sigma \to 0$. Accordingly, by the Duhamel formula and (1.3), r_σ goes to 0 in $C^0([0,T],H)$ as $\sigma \to 0$. □

References

[1] F. Alabau-Boussouira. Controllability of cascade coupled systems of multi-dimensional evolution pdes by a reduced number of controls. *Comptes Rendus. Mathématique*, 350(11-12):577–582, 2012.

[2] F. Alabau-Boussouira. Insensitizing exact controls for the scalar wave equation and exact controllability of-coupled cascade systems of pde's by a single control. *Mathematics of Control, Signals, and Systems*, 26(1):1–46, 2014.

[3] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled systems under geometric conditions. *Comptes Rendus. Mathématique*, 349(7-8):395–400, 2011.

[4] F. Alabau-Boussouira and M. Léautaud. Indirect controllability of locally coupled wave-type systems and applications. *Journal de Mathématiques Pures et Appliquées*, 99(5):544–576, 2013.

[5] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM journal on control and optimization*, 30(5):1024–1065, 1992.

[6] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and Control of Infinite Dimensional Systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Boston, MA, 2nd edition, 2007.

[7] O. Bodart, M. González-Burgos, and R. Pérez-Garcia. Insensitizing controls for a semilinear heat equation with a superlinear nonlinearity. *Comptes Rendus Mathematique*, 335(8):677–682, 2002.

[8] F. Boyer and M. Morancey. Analysis of non scalar control problems for parabolic systems by the block moment method. *C. R. Math. Acad. Sci. Paris*, 361:1191–1248, 2023.

[9] F. Boyer and G. Olive. Boundary null controllability of some multi-dimensional linear parabolic systems by the moment method. *Ann. Inst. Fourier (Grenoble)*, 74(5):1943–2012, 2024.

[10] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[11] Y. Cui, C. Laurent, and Z. Wang. On the observability inequality of coupled wave equations: the case without boundary. *ESAIM Control Optim. Calc. Var.*, 26:Paper No. 14, 37, 2020.

[12] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.

[13] R. Dáger. Insensitizing controls for the 1-d wave equation. *SIAM Journal on Control and Optimization*, 45(5):1758–1768, 2006.

[14] L. de Teresa. Controls insensitizing the norm of the solution of a semilinear heat equation in unbounded domains. *ESAIM: Control, Optimisation and Calculus of Variations*, 2:125–149, 1997.

[15] R. G. Douglas. On majorization, factorization, and range inclusion of operators on hilbert space. *Proceedings of the American Mathematical Society*, 17(2):413–415, 1966.

[16] M. Duprez, M. González-Burgos, and D. A. Souza. Remarks on the controllability of parabolic systems with non-diagonalizable diffusion matrix. *Discrete Contin. Dyn. Syst. Ser. S*, 16(6):1346–1382, 2023.

[17] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer, New York, 2000.

[18] S. Ervedoza, P. Lissy, and Y. Privat. Desensitizing control for the heat equation with respect to domain variations. *Journal de l'École polytechnique — Mathématiques*, 9:1397–1429, 2022.

[19] E. Fernández-Cara, M. González-Burgos, and L. de Teresa. Boundary controllability of parabolic coupled equations. *Journal of Functional Analysis*, 259(7):1720–1758, 2010.

[20] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[21] M. González-Burgos and L. de Teresa. Controllability results for cascade systems of m coupled parabolic pdes by one control force. *Portugaliae Mathematica*, 67(1):91–113, 2010.

[22] L. Hörmander. *Linear partial differential operators*. Die Grundlehren der mathematischen Wissenschaften, Bd. 116. Academic Press Inc., Publishers, New York, 1963.

[23] O. Kavian and L. De Teresa. Unique continuation principle for systemsof parabolic equations. *ESAIM: Control, Optimisation and Calculus of Variations*, 16(2):247–274, 2010.

[24] F. A. Khodja, A. Benabdallah, C. Dupaix, and I. Kostin. Null-controllability of some systems of parabolic typeby one control force. *ESAIM: Control, Optimisation and Calculus of Variations*, 11(3):426–448, 2005.

[25] A. Koenig and P. Lissy. Null-controllability of underactuated linear parabolic-transport systems with constant coefficients, 2023.

[26] C. Laurent and M. Léautaud. Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves. *J. Eur. Math. Soc. (JEMS)*, 21(4):957–1069, 2019.

[27] M. Lazar and J. Lohéac. Control of parameter dependent systems. In *Handbook of Numerical Analysis*, volume 23, pages 265–306. Elsevier, 2022.

[28] T. Li and B. Rao. Exact mixed synchronization by groups for a coupled system of wave equations. *Eur. J. Control*, 85:Paper No. 101278, 8, 2025.

[29] J.-L. Lions. Quelques notions dans l'analyse et le contrôle de systemes à données incompletes. In *Proceedings of the XIth Congress on Differential Equations and Applications/First Congress on Applied Mathematics (Spanish)(Málaga, 1989)*, pages 43–54. Univ. Málaga, Málaga, 1990.

[30] J.-L. Lions. *Sentinelles pour les systèmes distribués à données incomplètes*, volume 21 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1992.

[31] J.-L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications. Vol. I*, volume 181 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth.

[32] P. Lissy. Mixing Douglas's and weak majorization and factorization theorems. *Preprint submitted*, https://arxiv.org/abs/2411.09305, 2024.

[33] P. Lissy and J. Niu. Controllability of a coupled wave system with a single control and different speeds. *J. Math. Pures Appl. (9)*, 205:Paper No. 103808, 55, 2026.

[34] P. Lissy, Y. Privat, and Y. Simporé. Insensitizing control for linear and semi-linear heat equations with partially unknown domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 25:50, 2019.

[35] P. Lissy and E. Zuazua. Internal observability for coupled systems of linear partial differential equations. *SIAM J. Control Optim.*, 57(2):832–853, 2019.

[36] J. Niu. Simultaneous control of wave systems. *SIAM J. Control Optim.*, 59(3):2381–2409, 2021.

[37] L. Rosier and L. de Teresa. Exact controllability of a cascade system of conservative equations. *Comptes Rendus Mathematique*, 349(5-6):291–296, 2011.

[38] E. H. Samb. Boundary null-controllability of two coupled parabolic equations: simultaneous condensation of eigenvalues and eigenfunctions. *ESAIM Control Optim. Calc. Var.*, 27:Paper No. S29, 43, 2021.

[39] L. Tebou. Locally distributed desensitizing controls for the wave equation. *Comptes Rendus. Mathématique*, 346(7-8):407–412, 2008.

[40] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[41] A. Weinmann. *Uncertain Models and Robust Control*. Springer Vienna, 1 edition, 1991.

[42] R. K. Yedavalli. *Robust control of uncertain dynamic systems*, volume 10. Springer, 2014.

[43] E. Zuazua. Averaged control. *Automatica*, 50(12):3077–3087, 2014.