Non-localization of eigenfunctions for Sturm-Liouville operators and applications

Thibault Liard* Pierre Lissy† Yannick Privat‡§

Abstract

In this article, we investigate a non-localization property of the eigenfunctions of Sturm-Liouville operators $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ runs over the bounded nonnegative potential functions on the interval $(0, L)$ with $L > 0$. More precisely, we address the extremal spectral problem of minimizing the L^2-norm of a function $e(\cdot)$ on a measurable subset ω of $(0, L)$, where $e(\cdot)$ runs over all eigenfunctions of A_a, at the same time with respect to all subsets ω having a prescribed measure and all L^∞ potential functions $a(\cdot)$ having a prescribed essentially upper bound. We provide some existence and qualitative properties of the minimizers, as well as precise lower and upper estimates on the optimal value. Several consequences in control and stabilization theory are then highlighted.

Keywords: Sturm-Liouville operators, eigenfunctions, extremal problems, calculus of variations, control theory, wave equation.

AMS classification: 34B24, 49K15, 47A75, 49J20, 93B07, 93B05

1 Introduction

1.1 Localization/Non-localization of Sturm-Liouville eigenfunctions

In a recent survey article concerning the Laplace operator ([10]), D. Grebenkov and B.T. Nguyen introduce, recall and gather many possible definitions of the notion of *localization of eigenfunctions*. In particular, in section 7.7 of their article, they consider the Dirichlet-Laplace operator Δ on a given bounded open set Ω of $\mathbb{R}^n$, a Hilbert basis of eigenfunctions $(e_j)_{j\in\mathbb{N}^*}$ in $L^2(\Omega)$ and use as a measure of localization of the eigenfunctions on a measurable subset $\omega \subset \Omega$ the following criterion

$$C_p(\omega) = \inf_{j\in\mathbb{N}^*} \frac{\|e_j\|^p_{L^p(\omega)}}{\|e_j\|^p_{L^p(\Omega)}},$$

where $p \geqslant 1$. For instance, evaluating this quantity for different choices of subdomains ω if Ω is a ball or an ellipse allows to illustrate the so-called *whispering galleries* or *bouncing ball* phenomena. At the opposite, when Ω denotes the d-dimensional box $(0, \ell_1) \times \cdots \times (0, \ell_d)$ (with $\ell_1, \ldots, \ell_d > 0$),

*Université Pierre et Marie Curie (Univ. Paris 6), CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (thibault.liard@upmc.fr).

†Ceremade, Université Paris-Dauphine, CNRS UMR 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France (pierre.lissy@ceremade.dauphine.fr).

‡CNRS, Université Pierre et Marie Curie (Univ. Paris 6), UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (yannick.privat@upmc.fr).

§The third author is supported by the ANR projects AVENTURES - ANR-12-BLAN-BS01-0001-01 and OPTIFORM - ANR-12-BS01-0007

it is recalled that $C_p(\omega) > 0$ for any $p \geqslant 1$ and any measurable subset $\omega \subset \Omega$ whenever the ratios $(\ell_i/\ell_j)^2$ are not rational numbers for every $i \neq j$.

Many other notions of localization have been introduced in the literature. Regarding the Dirichlet/Neumann/Robin Laplacian eigenfunctions on a bounded open domain Ω of $\mathbb{R}^n$ and using a semi-classical analysis point of view, the notions of *quantum limit* or *entropy* have been widely investigated (see e.g. [1, 3, 4, 9, 9, 13, 20]) and provide an account for possible strong concentrations of eigenfunctions. Notice that the properties of $C_p(\omega)$ are intimately related to the behavior of high-frequency eigenfunctions and especially to the set of quantum limits of the sequence of eigenfunctions considered. Identifying such limits is a great challenge in quantum physics ([4, 9, 40]) and constitute a key ingredient to highlight non-localization/localization properties of the sequence of eigenfunctions considered.

Given a nonzero integer p, the *non-localization property* of a sequence $(e_j)_{j\in\mathbb{N}^*}$ of eigenfunctions means that the real number $C_p(\omega)$ is positive for every measurable subset $\omega \subset \Omega$. Concerning the one-dimensional Dirichlet-Laplace operator on $\Omega = (0, \pi)$, it has been highlighted in the case where $p = 2$ (for instance in [12, 24, 36]) that

$$\inf_{|\omega|=r\pi} C_2(\omega) = \inf_{|\omega|=r\pi} \inf_{j\in\mathbb{N}^*} \frac{2}{\pi} \int_\omega \sin(jx)^2 \, dx > 0,$$

for every $r \in (0, 1)$.

Motivated by these considerations, the present work is devoted to studying similar issues in the case $p = 2$, for a general family of one-dimensional Sturm-Liouville operators of the kind $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ is a nonnegative essentially bounded potential defined on the interval $(0, L)$. More precisely, we aim at providing lower quantitative estimates of the quantity $C_2(\omega)$, where $(e_{a,j})_{j\in\mathbb{N}^*}$ denotes now a sequence of eigenfunctions of A_a, in terms of the measure of ω and the essential supremum of $a(\cdot)$ by minimizing this criterion at the same time with respect to ω and $a(\cdot)$, over the class of subsets ω having a prescribed measure and over a well-chosen class of potentials $a(\cdot)$, relevant from the point of view of applications. Independently of its intrinsic interest, the choice "$p = 2$" is justified by the fact that the quantity $C_2(\omega)$ plays a crucial role in many mathematical fields, notably the control or stabilization of the linear wave equation (see for example [25] and [12]) and the randomised observation of linear wave, Schrödinger or heat equations (see for example [33], [37] or [38]).

Explicit lower bounds of $C_2(\omega)$ have already been obtained in [31] in the case $a(\cdot) = 0$. In the case where the potential $a(\cdot)$ differs from 0, some partial estimates of $C_2(\omega)$ are gathered in [12] holding in a restricted class of potentials with very small variations around constants. Up to our knowledge, there are no other articles investigating this precise problem.

The article is organized as follows: in Section 1.2, the extremal problem we will investigate is introduced. The main results of this article are stated in Section 2: a comprehensive analysis of the extremal problem is performed, reducing in some sense (that will be made precise in the sequel) this infinite-dimensional problem to a finite one. Moreover, we provide very simple lower and upper estimates of the optimal value. The whole section 3 is devoted to the proofs of the main and intermediate results. Finally, consequences and applications of our main results for observation and control theory and several numerical illustrations and investigations are gathered in Section 4.

1.2 The extremal problem

Let L be a positive real number and $a(\cdot)$ be an essentially nonnegative function belonging to $L^\infty(0, L)$. We consider the operator

$$A_a := -\partial_{xx} + a(\cdot), \tag{1}$$

defined on $\mathcal{D}(A_a) = H^1_0(0,L) \cap H^2(0,L)$. As a self-adjoint operator, A_a admits a Hilbert basis of $L^2(0,L)$ made of eigenfunctions denoted $e_{a,j} \in \mathcal{D}(A_a)$ and there exists a sequence of increasing positive real numbers $(\lambda_{a,j})_{j\in\mathbf{N}^*}$ such that $e_{a,j}$ solves the eigenvalue problem

$$\begin{cases} -e''_{a,j}(x) + a(x)e_{a,j}(x) = \lambda^2_{a,j} e_{a,j}(x), \ x \in (0,L), \\ \qquad e_{a,j}(0) = e_{a,j}(L) = 0. \end{cases} \tag{2}$$

By definition, the normalization condition

$$\int_0^L e^2_{a,j}(x)\, dx = 1 \tag{3}$$

is satisfied and we also impose that $e'_{a,j}(0) > 0$, so that $e_{a,j}$ is uniquely defined.

With regards to the explanations of Section 1.1, we are interested in the non-localization property of the sequence of eigenfunctions $(e_{a,j})_{j\in\mathbf{N}^*}$. The quantity of interest, denoted $J(a,\omega)$, is defined by

$$J(a,\omega) = \inf_{j\in\mathbf{N}^*} \frac{\int_\omega e^2_{a,j}(x)\, dx}{\int_0^L e^2_{a,j}(x)\, dx} = \inf_{j\in\mathbf{N}^*} \int_\omega e^2_{a,j}(x)\, dx, \tag{4}$$

where ω denotes a measurable subset of $(0,L)$ of positive measure.

The real number $J(a,\omega)$ is the equivalent for the Sturm-Liouville operators A_a of the quantity $C_2(\omega)$ introduced in Section 1.1 for the one-dimensional Dirichlet-Laplace operator.

It is natural to assume the knowledge of *a priori* informations about the subset ω and the potential function $a(\cdot)$. Indeed, we will choose them in some classes that are small enough to make the minimization problems we will deal with non-trivial, but also large enough to provide "explicit" (at least numerically) values of the criterion for a large family of potential.

Hence, in the sequel, we assume that:

- the measure (or at least a lower bound of the measure, which leads to the same solution of the optimal design problem we consider) of the subset ω is given;
- the potential function $a(\cdot)$ is nonnegative and essentially bounded.

Such conditions are relevant and commonly used in the context of control or inverse problems.

Fix $M > 0$, $r \in (0,1)$, α and β two real numbers such that $\alpha < \beta$. Let us introduce the class of admissible observation subsets

$$\Omega_r(\alpha,\beta) = \{\text{Lebesgue measurable subset } \omega \text{ of } (\alpha,\beta) \text{ such that } |\omega| = r(\beta-\alpha)\}, \tag{5}$$

as well as the class of admissible potentials

$$\mathcal{A}_M(\alpha,\beta) = \left\{a \in L^\infty(\alpha,\beta) \text{ such that } 0 \leqslant a(x) \leqslant M \text{ a.e. on } (\alpha,\beta)\right\}, \tag{6}$$

Let us now introduce the optimal design problem we will investigate.

Extremal spectral problem. *Let $M > 0$, $r \in (0,1)$ and $L > 0$ be fixed. We consider*

$$m(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a,\omega), \tag{$\mathcal{P}_{L,r,M}$}$$

where the functional J is defined by (4), $\Omega_r(0,L)$ and $\mathcal{A}_M(0,L)$ are respectively defined by (5) and (6).

In the sequel, we will call *minimizer of the problem* ($\mathcal{P}_{L,r,M}$) a triple $(a^*, \omega^*, j_0) \in \mathcal{A}_M(0, L) \times \Omega_r(0, L) \times \mathbb{N}^*$ (whenever it exists) such that

$$\inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J(a, \omega) = \int_{\omega^*} e_{a^*,j_0}(x)^2 \, dx.$$

Remark 1. Noting that for a given real number c, the operator $A_a + c\,\mathrm{Id}$ has the same eigenfunctions as A_a, we claim that all the results and conclusions of this article will also hold if we replace the class of potentials $\mathcal{A}_M(\alpha, \beta)$ by the larger class

$$\widetilde{\mathcal{A}}_M(\alpha, \beta) := \left\{ \rho \in L^\infty(\alpha, \beta) \text{ such that } \sup_{(\alpha,\beta)} \rho(\cdot) - \inf_{(\alpha,\beta)} \rho(\cdot) \leqslant M \right\}.$$

Indeed, for any $\rho \in \widetilde{\mathcal{A}}_M(0, L)$, set $a = \rho - \inf_{(0,L)} \rho$. Then, $a \in \mathcal{A}_M(0, L)$, and every eigenfunction $e_{\rho,j}$ solving the system

$$\begin{cases} -e''_{\rho,j}(x) + \rho(x)e_{\rho,j}(x) = \mu_{\rho,j}e_{\rho,j}(x), \; x \in (0, L), \\ \qquad e_{\rho,j}(0) = e_{\rho,j}(L) = 0, \end{cases}$$

is also a solution of the system

$$\begin{cases} -e''_{\rho,j}(x) + a(x)e_{\rho,j}(x) = \lambda^2_{a,j}e_{\rho,j}(x), \; x \in (0, L), \\ \qquad e_{\rho,j}(0) = e_{\rho,j}(L) = 0 \end{cases}$$

with $\lambda^2_{a,j} = \mu_{\rho,j} + \inf_{(0,L)} \rho$, whence the claim.

2 Main results and comments

Let us state the main results of this article. The next theorems are devoted to the analysis of the optimal design problems ($\mathcal{P}_{L,r,M}$).

We also stress on the fact that the following estimates of $J(a, \omega)$ are valid for every measurable subset ω of prescribed measure and that we do not need to add any topological assumption on it.

Theorem 1. *Let $r \in (0, 1)$ and $M \in \mathbb{R}^*_+$.*

i Problem ($\mathcal{P}_{L,r,M}$) has a solution (a^*, ω^*, j_0). In particular, there holds

$$m(L, M, r) \quad = \quad \min_{a\in\mathcal{A}_M(0,L)} \min_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j_0}(x)^2 \, dx,$$

and the solution a^* of Problem ($\mathcal{P}_{L,r,M}$) is bang-bang, (i.e. equal to 0 or M a.e. in $(0, L)$).

ii Assume that $M \in (0, \pi^2/L^2]$. Then, ω^* is the union of $j_0 + 1$ intervals, and a^* has at most $3j_0 - 1$ and at least j_0 switching points[1]*.*

Moreover, one has the estimate

$$\gamma \min(r, \underline{r}_2)^3 \leqslant m(L, M, r) \leqslant r - \frac{\sin(\pi r)}{\pi}, \tag{7}$$

with $\gamma = \frac{7\sqrt{3}}{8}(3 - 2\sqrt{2}) \simeq 0.2600$ and $\underline{r}_2 = \frac{\sqrt{5}}{5} + \frac{\sqrt{10}}{10} \simeq 0.7634$.

[1] Recall that a switching point of a *bang-bang* function is a point at which this function is not continuous.

In the following result, we highlight the necessity of imposing a pointwise upper bound on the potentials functions $a(\cdot)$ to get the existence of a minimizer.

Theorem 2. *Let $r \in (0,1)$ and $j \in \mathbb{N}^*$. Then, the optimal design problem of finding a minimizer to*

$$\inf_{a \in \mathcal{A}_\infty(0,L)} \inf_{\omega \in \Omega_r(0,L)} J(a,\omega), \tag{$\mathcal{P}_{L,r,\infty}$}$$

where $\mathcal{A}_\infty(0,L) = \cup_{M>0} \mathcal{A}_M(0,L)$, has no solution.

We conclude this section by some remarks and comments.

Remark 2. The estimate (7) can be considered as sharp with respect to the parameter r, at least for r small enough (which is the most interesting case in view of the applications). Indeed, there holds $\frac{\pi r - \sin(\pi r)}{\pi} \sim \frac{\pi^2}{6} r^3$ as r tends to 0, which shows that the power of r in the left-hand side cannot be improved. The graphs of the quantities appearing in the left and right-hand sides of (7) with respect to r are plotted on Figure 1 below.

An interesting issue would thus consist in determining the optimal bounds in the estimate (7), namely

$$\ell_- = \inf_{r \in (0,1)} \frac{m(L,M,r)}{r^3} \qquad \text{and} \qquad \ell_+ = \sup_{r \in (0,1)} \frac{m(L,M,r)}{r^3}. \tag{$\mathcal{P}_{\ell_-,\ell_+}$}$$

It is likely that investigating this issue would rely on a refined study of the optimality conditions of the problems ($\mathcal{P}_{\ell_-,\ell_+}$), but also of the problem ($\mathcal{P}_{L,r,M}$). According to (7), we know that $\ell_- \in [\gamma(\frac{\sqrt{5}}{5} + \frac{\sqrt{10}}{10})^3, 1]$ and $\ell_+ \in [\gamma, \pi^2/6]$.

Remark 3. Let us highlight the interest of Theorem 1 for numerical investigations. Indeed, in view of providing numerical lower bounds of the quantity $J(a,\omega)$, Theorem 1 enables us to reduce the solving of the infinite-dimensional problems ($\mathcal{P}_{L,r,M}$) to the solving of finite ones, since one has just to choose the optimal $3j_0^* - 1$ switching points defining the best potential function a^* and to remark that necessarily ω^* is uniquely defined once a^* is defined (since it is defined in terms of a precise level set of e_{a^*,j_0^*}, see Proposition 1). We will strongly use this remark in Section 4.2.1, where illustrations and applications of Problem ($\mathcal{P}_{L,r,M}$) are developed.

Remark 4. The restriction on the range of values of the parameter M in the second point of Theorem 1 makes each eigenfunction $e_{a,j}$ convex or concave on each nodal domain. The upper bound on the parameter M, namely the real number π^2/L^2 correspond to the lowest eigenvalue of the Dirichlet Laplacian A_0. It constitutes a crucial element of the proofs of Theorem 1 and Proposition 4, that allows to compare each quantity $\int_\omega e_{a,j}(x)^2\, dx$ with the integral of the square of well-chosen piecewise affine functions. Unfortunately, the solving of Problem ($\mathcal{P}_{L,r,M}$) when $M > \pi^2/L^2$ appears much more intricate and cannot be handled with the same kinds of arguments. Some numerical experiments in the case $M > \pi^2/L^2$ will be presented in Section 4.2.

Remark 5. According to Section 4.2.1, numerical simulations seem to indicate that there exists a triple (j_0^*, ω^*, a^*) solving Problem ($\mathcal{P}_{L,r,M}$)such that $j_0^* = 1$, the set ω^* and the graph of a^* are symmetric with respect to $L/2$ and a^* is a *bang-bang* function having exactly two switching points. We were unfortunately unable to prove this assertion.

A first step would consist in finding an upper estimate of the optimal index j_0^*. Even this question appears difficult in particular since it is not obvious to compare the real numbers $\int_\omega e_{a,j}(x)^2\, dx$ for different indices j. One of the reasons of that comes from the fact that the cost functional we considered does not write as the minimum of an energy function, making the comparison between eigenfunctions of different orders on the subdomain ω intricate.

Remark 6. It can be noticed that the lower bound in Proposition 4 is independent of the parameter L. This can be justified by using an easy rescaling argument allowing in particular to restrict our numerical investigations to the case where $L = \pi$ (for instance).

Lemma 1. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$, $M > 0$ and $L > 0$. Then, there holds*

$$\inf_{a \in \mathcal{A}_M(0,\pi)} \inf_{\omega \in \Omega_r(0,\pi)} \int_\omega e_{a,j}(x)^2\,dx = \inf_{a \in \mathcal{A}_{\frac{M\pi^2}{L^2}}(0,L)} \inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\,dx, \tag{8}$$

Remark 7. One can show by using tedious computations inspired by those of Appendix B that the sequences $(\underline{m}_j)_{j\in\mathbb{N}^*}$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ are increasing. Moreover, straightforward computations show that $(\underline{m}_j)_{j\in\mathbb{N}^*}$ converges to $2/3$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ converges to 1 as j tends to $+\infty$.

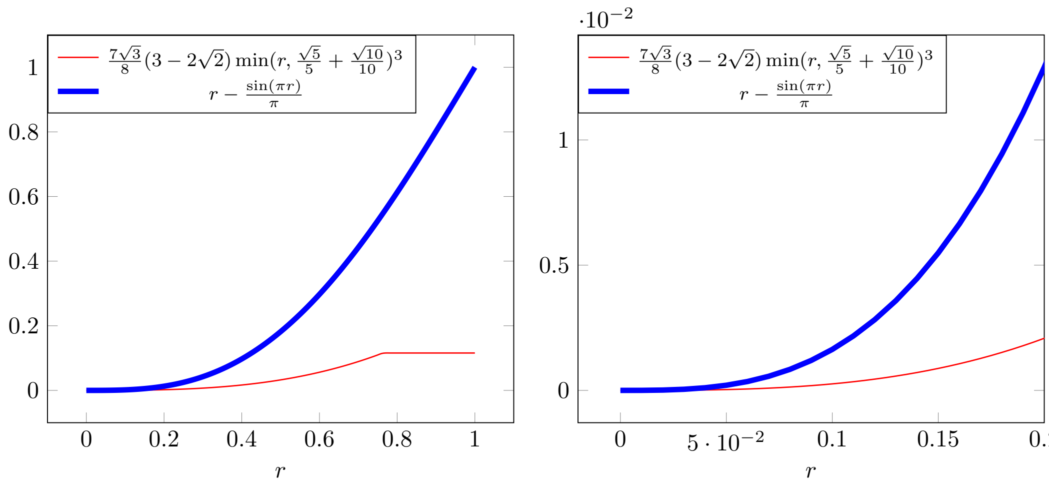

Figure 1: (Left) Plots of $r \mapsto \gamma \min(r, \frac{\sqrt{5}}{5}+\frac{\sqrt{10}}{10})^3$ (thin line) and $r \mapsto (\pi r - \sin(\pi r))/\pi$ (bold line). (Right) Zoom on the graph for the range of values $r \in [0, 0.2]$.

3 Proofs of Theorem 1 and Theorem 2

3.1 Preliminary material: existence results and optimality conditions

We gather in this section several results we will need to prove Theorem 1 and Theorem 2.

Let us first investigate the following simpler optimal design problem, where the potential $a(\cdot)$ is now assumed to be fixed, and which will constitute an important ingredient in the proof.

<u>**Auxiliary problem: fixed potential.**</u> *For a given $j \in \mathbb{N}^*$, $M > 0$, $r \in (0,1)$ and $a \in \mathcal{A}_M(0,L)$, we investigate the optimal design problem*

$$\inf_{\chi \in \mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\,dx, \tag{Aux-Pb}$$

where the set $\mathcal{U}_r$ is defined by

$$\mathcal{U}_r = \left\{ \chi \in L^\infty(0,L) \mid 0 \leqslant \chi \leqslant 1 \text{ a.e. in } (0,L) \text{ and } \int_0^L \chi(x)\,dx = rL \right\}. \tag{9}$$

We provide a characterization of the solutions of Problem (Aux-Pb).

Proposition 1. *Let $r \in (0,1)$. The optimal design problem Problem (Aux-Pb) has a unique solution that writes as the characteristic function of a measurable set ω^* of Lebesgue measure rL. Moreover, there exists a positive real number τ such that ω^* is a solution of Problem (Aux-Pb) if and only if*

$$\omega^* = \{e_{a,j}(x)^2 < \tau\}, \tag{10}$$

up to a set of zero Lebesgue measure.

In other words, any optimal set, solution of Problem (Aux-Pb), is characterized in terms of the level set of the function $e_{a,j}(\cdot)^2$. This result is a direct consequence of [34, Theorem 1] or [39, Chapter 1] and the fact that for every $c > 0$, the set $\{e_{a,j}^2 = c\}$ has zero Lebesgue measure, by using standard properties of eigenfunctions of Sturm-Liouville operators.

The following continuity result is standard. We refer to [32, Chap. 1, page 10] for a proof.

Lemma 2. *Let $M \in \mathbb{R}_+^*$ and $j \in \mathbb{N}^*$. Let us endow the space $\mathcal{A}_M(0,L)$ with the weak-$\star$ topology of $L^\infty(0,L)$ [2] and the space $H_0^1(0,L)$ with the standard strong topology inherited from the Sobolev norm $\|\cdot\|_{H_0^1}$. Then the function $a \in \mathcal{A}_M(0,L) \mapsto e_{a,j} \in H_0^1(0,L)$ is continuous.*

Another key point ouf our proof is the study of the following auxiliary optimal design Problem:

$$m_j(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\, dx, \tag{$\mathcal{P}_{j,L,r,M}$}$$

for a fixed $j \in \mathbb{N}^*$.

The next result is a direct consequence of Lemma 2.

Proposition 2. *Let $M \in \mathbb{R}_+^*$, $j \in \mathbb{N}^*$ and $r \in (0,1)$. The optimal design Problem ($\mathcal{P}_{j,L,r,M}$) has at least one solution (a_j^*, ω_j^*).*

Proof of Proposition 2. Let us consider a relaxed version of the optimal design Problem ($\mathcal{P}_{j,L,r,M}$), where the characteristic function of ω has been replaced by a function χ in $\mathcal{U}_r$ (defined in (9)). This relaxed version of ($\mathcal{P}_{j,L,r,M}$) writes

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\, dx.$$

Let us endow $\mathcal{A}_M(0,L)$ and $\mathcal{U}_r$ with the weak-$\star$ topology of $L^\infty(0,L)$. Thus, both sets are compact. Moreover, according to Lemma 2 and since it is linear in the variable χ, the functional $(a,\chi) \in \mathcal{A}_M(0,L)\times\mathcal{U}_r \mapsto \int_0^L \chi(x)e_{a,j}(x)^2\,dx$ is continuous. The existence of a solution (a_j^*, χ_j^*) follows for the relaxed problem. Finally, noting that

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\, dx = \inf_{\chi\in\mathcal{U}_r} \int_0^L \chi(x) e_{a_j^*,j}(x)^2\, dx,$$

there exists a measurable set ω_j^* of measure rL such that $\chi_j^* = \chi_{\omega_j^*}$, by Proposition 1.

The existence of a solution of Problem ($\mathcal{P}_{j,L,r,M}$) is then proved and there holds

$$\inf_{(a,\chi)\in\mathcal{A}_M(0,L)\times\mathcal{U}_r} \int_0^L \chi(x) e_{a,j}(x)^2\, dx = \inf_{a\in\mathcal{A}_M(0,L)} \int_{\omega^*} e_{a,j}(x)^2\, dx.$$

■

[2] Recall that a sequence $(a_n)_{n\in\mathbb{N}^*}$ of $L^\infty(0,L)$ converges to a for the weak-$\star$ topology of $L^\infty(0,L)$ whenever $\int_0^L a_n(x)\varphi(x)$ dx converge to $\int_0^L a(x)\varphi(x)\,dx$ for every $\varphi \in L^1(0,L)$.

We now state necessary first order optimality conditions that enable us to characterize every critical point (a_j^*, ω_j^*) of the optimal design problem $(\mathcal{P}_{j,L,r,M})$.

Proposition 3. *Let $j \in \mathbb{N}^*$, $r \in (0,1)$ and $M > 0$. Let (a_j^*, ω_j^*) be a solution of the optimal design problem $(\mathcal{P}_{j,L,r,M})$ and let*

$$0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j \tag{11}$$

be the $j+1$ zeros[3] of the j-th eigenfunction $e_{a_j^*,j}$.

For $i \in \{1 \dots j\}$, let us denote by $a_{j,i}^*$ the restriction of the function a_j^* to the interval (x_j^{i-1}, x_j^i) and by $\omega_{j,i}^*$ the set $\omega_j^* \cap (x_j^{i-1}, x_j^i)$. Then, necessarily, there exists $\tau \in \mathbb{R}_+^*$ such that

- *one has $\omega_j^* = \{e_{a_j^*,j}(x)^2 < \tau\}$ up to a set of zero Lebesgue measure,*
- *one has*
$$M\chi_{\{p_j(x)e_{a_j^*,j}(x)>0\}}(x) \leqslant a_j^*(x) \leqslant M\chi_{\{p_j(x)e_{a_j^*,j}(x)\geqslant 0\}}(x), \tag{12}$$
for almost every $x \in (0, L)$, where p_j is defined piecewisely as follows: for $i \in \{1 \dots j\}$, the restriction of p_j to the interval (x_j^{i-1}, x_j^i) is denoted $p_{j,i}$, and $p_{j,i}$ is the (unique) solution of the adjoint system
$$\begin{cases} -p_{j,i}''(x) + (a_{j,i}^*(x) - \lambda_{a_j^*,j}^2)p_{j,i}(x) = (\chi_{\omega_{j,i}^*} - \tilde{c}_{j,i})e_{a_j^*,j}, & x \in (x_j^{i-1}, x_j^i), \\ p_{j,i}(x_j^{i-1}) = p_{j,i}(x_j^i) = 0, \end{cases} \tag{13}$$
verifying moreover
$$\int_{x_j^{i-1}}^{x_j^i} p_{j,i}(x)e_{a_{j,i}^*,j}(x)\,dx = 0, \tag{14}$$
where $\tilde{c_{j,i}}$ is given by
$$\tilde{c}_{j,i} = \frac{\int_{x_j^{i-1}}^{x_j^i} \chi_{\omega_{j,i}^*}(x)e_{a_j^*,j}^2(x)\,dx}{\int_{x_j^{i-1}}^{x_j^i} e_{a_j^*,j}^2(x)\,dx}. \tag{15}$$

In other words, any optimal set solution of Problem $(\mathcal{P}_{j,L,r,M})$ is characterized in terms of a level set of the function $e_{a_j^*,j}(\cdot)^2$ and the optimal potential a_j^* is characterized in terms of a level set of the function $p_j(\cdot)e_{a_j^*,j}(\cdot)$.

Remark 8. i According to Fredholm's alternative (see for example [15]), System (13)-(14) has a unique solution.

ii Another presentation of the first order optimality conditions gathered in Proposition 3 could have been obtained by applying the so-called Pontryagin Maximum Principle (see e.g. [27]).

Before proving this proposition, let us state a technical lemma about the differentiability of the eigenfunctions $e_{a,j}$ with respect to a.

[3] Recall that the family $\{x_j^k\}_{0\leqslant k\leqslant j}$ is the set of nodal points that are known to be of cardinal j and to be simple roots of the eigenfunction $e_{a^*,j}$ (see [32, Chap. 2, Thm 6]).

Lemma 3. *Let us endow the space $\mathcal{A}_M(0,L)$ with the weak-$\star$ topology of $L^\infty(0,L)$ and the space $H_0^1(0,L)$ with the standard strong topology inherited from the Sobolev norm $\|\cdot\|_{H^1}$. Let $a \in \mathcal{A}_M(0,L)$ and $\mathcal{T}_{a,\mathcal{A}_M(0,L)}$ be the tangent cone*[4] *to the set $\mathcal{A}_M(0,L)$ at point a. For every $h \in \mathcal{T}_{a,\mathcal{A}_M(0,L)}$, the mapping $a \mapsto e_{a,j}$ is Gâteaux-differentiable in the direction h, and its derivative, denoted $\dot{e}_{a,j}$, is the (unique) solution of*

$$\begin{cases} -\dot{e}''_{a,j}(x) + a(x)\dot{e}_{a,j}(x) + h(x)e_{a,j}(x) = \left(\dot{\lambda^2_{a,j}}\right)e_{a,j}(x) + \lambda^2_{a,j}\dot{e}_{a,j}(x), \ x \in (0,L), \\ \dot{e}_{a,j}(0) = \dot{e}_{a,j}(L) = 0, \end{cases} \tag{16}$$

with $\left(\dot{\lambda^2_{a,j}}\right) = \int_0^L h(x)e_{a,j}(x)^2\,dx$.

The proof of the differentiability is completely standard and is based on the fact that the eigenvalues $\lambda_{a,j}$ are simple. For this reason, we skip it and refer to [21, pages 375 and 425].

Proof of Proposition 3. The first point results from Proposition 1.

Let us prove the second point.We compute the Gâteaux-derivative of the cost functional $a \mapsto J_{\omega_j^*}(a)$, where

$$J_{\omega_j^*}(a) = J(a,\omega_j^*) = \int_{\omega_j^*} e_{a,j}(x)^2\,dx,$$

at $a = a_j^*$ in the direction $h_j \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. We denote it by $\langle dJ_{\omega_j^*}(a_j^*), h\rangle$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = 2\int_{\omega^*} \dot{e}_{a_j^*,j}(x)e_{a_j^*,j}(x)\,dx,$$

where $\dot{e}_{a_j^*,j}$ is the solution of (16).

Let us write this quantity in a more convenient form to state the necessary first order optimality conditions. Let h_j be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$. Let us write $h_j = \sum_{i=1}^{j-1} h_{j,i}$ where $h_{j,i} = h_j\chi_{(x_j^{i-1},x_j^i)}$ for all $i \in \{1\dots j\}$. Hence, $h_{j,i} \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$ and there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = \sum_{i=1}^{j}\langle dJ_{\omega_j^*}(a_j^*), h_{j,i}\rangle.$$

It follows that it is enough to consider perturbations with compact support contained in each nodal domain in order to compute the Gâteaux-derivative of $J_{\omega_j^*}$. We will use for that purpose the adjoint state p_j piecewisely defined by (13)-(14).

Fix $i \in \{1\dots j\}$ and let $h_{j,i}$ be an element of the tangent cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(x_j^{i-1},x_j^i)}$. Let us multiply the first line of (13) by $\dot{e}_{a_{j,i}^*,1}$ and then integrate by parts. We get

$$\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a_{j,i}^*,1}(x)p'_{j,i}(x) + (a_{j,i}^*(x) - \lambda^2_{a_{j,i}^*,1})\dot{e}_{a_{j,i}^*,1}(x)p_{j,i}(x)\,dx = \frac{1}{2}\langle dJ(a_j^*), h_{j,i}\rangle. \tag{17}$$

Similarly, let us multiply the first line of (16) by $p_{j,i}$ and then integrate by parts. We get

$$\begin{aligned} &\int_{x_j^{i-1}}^{x_j^i} \dot{e}'_{a_{j,i}^*,1}(x)p'_{j,i}(x) + (a_{j,i}^*(x) - \lambda^2_{a_{j,i}^*,1})\dot{e}_{a_{j,i},1}(x)p_{j,i}(x)\,dx \\ &= \left(\dot{\lambda^2_{a_j^*,j}}\right)\int_{x_j^{i-1}}^{x_j^i} e_{a_{j,i},1}(x)p_{j,i}(x)\,dx - \int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a_{j,i},1}(x)p_{j,i}(x)\,dx. \end{aligned} \tag{18}$$

[4] That is the set of functions $h \in L^\infty(0,L)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(0,L)$ converging to h as $n \to +\infty$, and $a + \varepsilon_n h_n \in \mathcal{A}_M(0,L)$ for every $n \in \mathbb{N}$ (see for instance [16, chapter 7]).

Combining (17) with (18) and using the condition (14) yields

$$\langle dJ_{\omega_j^*}(a_j^*), h_{j,i}\rangle = -2\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)e_{a_{j,i}^*,1}(x)p_{j,i}(x)\,dx. \tag{19}$$

As a result, for a general $h_j \in \mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$, there holds

$$\langle dJ_{\omega_j^*}(a_j^*), h_j\rangle = -2\sum_{i=1}^{j}\int_{x_j^{i-1}}^{x_j^i} h_{j,i}(x)p_{j,i}(x)e_{a_{j,i}^*,1}\,dx = -2\int_0^L h_j(x)e_{a_j^*,j}(x)p_j(x)\,dx.$$

Let us state the first order optimality conditions. For every perturbation h_j in the cone $\mathcal{T}_{a_j^*,\mathcal{A}_M(0,L)}$, there holds $\langle dJ(a_j^*), h_j\rangle \geqslant 0$, which writes

$$-2\int_0^L h_j(x)e_{a_j^*,j}(x)p_j(x)\,dx \geqslant 0. \tag{20}$$

The analysis of such optimality condition is standard in optimal control theory (see for example [27]) and permits to recover easily (12). ■

3.2 Proof of Theorem 1

Step 1: existence and bang-bang property of the minimizers (first point of Theorem 1). Notice first that each of the infima defining Problem ($\mathcal{P}_{L,r,M}$) can be inverted with each other. As a result, and according to Proposition 2, there exists an optimal pair (a_j^*, ω_j^*) such that

$$\begin{aligned} m(L,M,r) &= \inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)} J(a,\omega) = \inf_{j\in\mathbb{N}^*}\inf_{a\in\mathcal{A}_M(0,L)}\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx \\ &= \inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx. \end{aligned}$$

It remains then to prove that the last infimum is reached by some index $j_0 \in \mathbb{N}^*$. We will use the two following lemmas.

Lemma 4. *Let $M > 0$ and $(a_j)_{j\in\mathbb{N}^*}$ be a sequence of $\mathcal{A}_M(0,L)$. The sequence $(e_{a_j,j}^2)_{j\in\mathbb{N}^*}$ converges weakly-$\star$ in $L^\infty(0,L)$ to the constant function $\frac{1}{L}$.*

The proof of Lemma 4 is standard and is postponed to Appendix A for the sake of clarity. The proof of the next lemma can be found in [12, 31, 35].

Lemma 5. *Let $L > 0$ and $V_0 \in (0,L)$. There holds*

$$\inf_{\substack{\rho\in L^\infty(0,L;[0,1])\\ \int_0^L \rho(x)\,dx = V_0}} \int_0^L \rho(x)\sin^2\left(\frac{j\pi}{L}x\right)dx = \frac{1}{2}\left(V_0 - \frac{L}{\pi}\sin\left(\frac{\pi}{L}V_0\right)\right),$$

for every $j \in \mathbb{N}^*$. Moreover, this problem has a unique solution ρ that writes as the characteristic function of a measurable subset ω_j^* defined by $\omega_j^* = \{\sin^2(\frac{j\pi}{L}\cdot) \leqslant \eta_j\}$ for some well-chosen positive number η_j determined in such a way that the constraint $\int_0^L \rho(x)\,dx = V_0$ is satisfied.

As a consequence of Lemma 5 (which gives the last equality) and by minimality of $m(L,M,r)$ (note that $0\in\mathcal{A}_M(0,L)$), there holds

$$\begin{aligned} m(L,M,r) &= \inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx = \inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_\omega e_{a_j^*,j}(x)^2\,dx \\ &\leqslant \inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_\omega e_{0,j}(x)^2\,dx = \frac{2}{L}\inf_{\omega\in\Omega_r(0,L)}\inf_{j\in\mathbb{N}^*}\int_\omega \sin^2\left(\frac{j\pi}{L}x\right)dx \\ &= r-\frac{1}{\pi}\sin(r\pi). \end{aligned}$$

Using Lemma 4 (the weak-$\star$ convergence being used with the "test" function $\chi_{\omega^*}\in L^1(0,L)$) we have $r=\lim_{j\to+\infty}\int_{\omega^*} e_{a_j^*,j}(x)^2\,dx$. Thus, $m(L,M,r)<\lim_{j\to+\infty}\int_{\omega^*} e_{a_j^*,j}(x)^2\,dx$. As a consequence, the infimum $\inf_{j\in\mathbb{N}^*}\int_{\omega_j^*} e_{a_j^*,j}(x)^2\,dx$ is reached by a finite integer j_0^*. The existence result follows.

From now on and for the sake of clarity, we will denote by (j_0,ω^*,a^*) instead of $(j_0,\omega_{j_0}^*,a_{j_0}^*)$ a solution of Problem ($\mathcal{P}_{L,r,M}$). We now prove that the solution a^* of Problem ($\mathcal{P}_{L,r,M}$) is *bang-bang*. Let us write the necessary first order optimality conditions of Problem ($\mathcal{P}_{L,r,M}$). To simplify the notations, the adjoint state introduced in Proposition 3 will be denoted by p (resp. p_i) instead of p_{j_0} (resp. $p_{j_0,i}$). For $0<\alpha<\beta<L$, introduce the sets

- $\mathcal{I}_{0,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $a^*(x)=0$ a.e.;
- $\mathcal{I}_{M,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $a^*(x)=M$ a.e.;
- $\mathcal{I}_{\star,a^*}(\alpha,\beta)$: any element of the class of subsets of $[\alpha,\beta]$ in which $0<a^*(x)<M$ a.e., that writes also

$$\mathcal{I}_{\star,a^*}(\alpha,\beta)=\bigcup_{k=1}^{+\infty}\left\{x\in(\alpha,\beta):\frac{1}{k}<a^*(x)<M-\frac{1}{k}\right\}=:\bigcup_{k=1}^{+\infty}\mathcal{I}_{\star,a^*,k}(\alpha,\beta).$$

We will prove that the set $\mathcal{I}_{\star,a^*,k}(0,L)=\bigcup_{i_0=1}^{j}\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ has zero Lebesgue measure, for every nonzero integer k. Let us argue by contradiction, assuming that one of these sets $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ is of positive measure. Let $x_0\in\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and let $(G_{k,n})_{n\in\mathbb{N}}$ be a sequence of measurable subsets with $G_{n,k}$ included in $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ and containing x_0, the perturbations a^*+th and a^*-th are admissible for t small enough. Choosing $h=\chi_{G_{k,n}}$ and letting t go to 0, it follows that

$$\langle dJ(a^*),h\rangle=\int_{x_j^{i_0-1}}^{x_j^{i_0}} h(x)\left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)dx=0\Longleftrightarrow\int_{G_{k,n}}\left(-e_{a_j^*,j}(x)p_{i_0}(x)\right)dx=0.$$

Dividing the last equality by $|G_{k,n}|$ and letting $G_{k,n}$ shrink to $\{x_0\}$ as $n\to+\infty$ shows that $e_{a_j^*,j}(x_0)p_{i_0}(x_0)=0$ for almost every $x_0\in\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$, according to the Lebesgue density Theorem. Since $e_{a_j^*,j}$ does not vanish on $(x_j^{i_0-1},x_j^{i_0})$ we then infer that $p_{i_0}(x)=0$ for almost every $x\in\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$. Let us prove that such a situation cannot occur. The variational formulation of System (13)-(14) writes: find $p_{i_0}\in H_0^1(x_j^{i_0-1},x_j^{i_0})$ such that for every test function $\varphi\in H_0^1(x_j^{i_0-1},x_j^{i_0})$, there holds

$$-\int_{x_j^{i_0-1}}^{x_j^{i_0}} p_{i_0}(x)\varphi''(x)+(a_{i_0}(x)-\lambda_{a_{i_0},1}^2)p_{i_0}(x)\varphi(x)\,dx=\int_{x_j^{i_0-1}}^{x_j^{i_0}}(\chi_{\omega_{j,i_0}^*}-\tilde{c}_{j,i_0})e_{a_j^*,j}\varphi(x)\,dx.$$

Since $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$ is assumed to be of positive measure, let us choose test functions φ whose support is contained in $\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})$. There holds

$$\int_0^L (\chi_{\omega^*_{j,i_0}} - \tilde{c}_{j,i_0}) e_{a_j^*,j}(x)\varphi(x)\, dx = 0,$$

for such a choice of test functions, whence the contradiction by using that $\tilde{c}_{j,i_0} \in (0,1)$. We then infer that $|\mathcal{I}_{\star,a^*,k}(x_j^{i_0-1},x_j^{i_0})| = 0$ and necessarily, a^* is bang-bang.

Step 2: counting the switching points of a^* and the number of connected components of ω^* (first part of the second point of Theorem 1). For the sake of simplicity, we first give the argument in the case where the infimum $m(L,M,r)$ is reached at $j_0 = 1$ and we will then generalize our analysis to any $j_0 \in \mathbb{N}^*$.

At this step, we know that a_1^* is *bang-bang* and ω_1^* is characterized in terms of the level set of the function $e_{a_1^*,1}(\cdot)^2$. According to (12), the number of switching points of a_1^* corresponds to the number of zeros of the function $x \mapsto p_1(x)e_{a_1^*,1}(x)$. Let us evaluate this number.

Since $M \leqslant \pi^2/L^2$, there holds $\|a_1^*\|_\infty \leqslant \frac{\pi^2}{L^2}$. As a consequence and using (2), one deduces that the eigenfunction $e_{a_1^*,1}$ is concave and reaches its maximum at a unique point $x_{\max} \in (0,L)$. Moreover, since $e_{a_1^*,1}$ is increasing on $(0,x_{\max})$ and decreasing on $(x_{\max},L)$ with $e_{a_1^*,1}(0) = e_{a_1^*,1}(\pi) = 0$, from Proposition 1, there exists $(\alpha,\beta) \in (0,L)^2$ such that $\alpha < \beta$ and

$$\chi_{\omega_1^*} = 1 - \chi_{(\alpha,\beta)}, \tag{21}$$

with $\alpha < x_{\max} < \beta$.

Let us provide a precise description of the function p_1. One readily checks by differentiating two times the function $p_1/e_{a_1^*,1}$ that the function p_1 may be written as

$$p_1(x) = f(x)e_{a_1^*,1}(x) \tag{22}$$

for every $x \in (0,L)$, where the function f is defined by

$$f(x) = -\int_0^x g(t)dt + f(0), \qquad \text{with} \qquad f(0) = \int_0^L \left(\int_0^x g(t)dt\right) e^2_{a_1^*,1}(x)\, dx \tag{23}$$

and the function g is defined by

$$g(t) = \frac{\int_0^t (\chi_{\omega_1^*}(s) - \tilde{c})e^2_{a_1^*,1}(s)\, ds}{e^2_{a_1^*,1}(t)}, \tag{24}$$

where here and in the rest of the proof, we will call $\tilde{c}$ the number $\tilde{c}_{1,1}$ (whose definition was given in (15)). In the following result, we provide a description of the function g.

Lemma 6. *The function g defined by (24) verifies*

$$g(0) = g(L) = 0, \tag{25}$$

$$\text{there exists a unique real number } o_g \text{ in } (0,L) \text{ such that } g(o_g) = 0, \tag{26}$$

$$g > 0 \text{ in } (0,o_g) \text{ and } g < 0 \text{ in } (o_g,L), \tag{27}$$

$$g \text{ is decreasing on } (\alpha, \min(o_g,x_{max})) \text{ and } (\max(o_g,x_{max}),\beta). \tag{28}$$

Proof of Lemma 6. Let us first prove (26). We consider the function $\tilde{g}$ defined by

$$\tilde{g}(t) = \int_0^t (\chi_{\omega_1^*}(s) - \tilde{c}) e^2_{a_1^*,1}(s)\, ds, \tag{29}$$

so that the function g writes

$$g = \frac{\tilde{g}}{e_{a_1^*,1}(\cdot)^2}. \tag{30}$$

According to (21) and remarking that $0 < \tilde{c} < 1$ according to (15), the function $\tilde{g}$ is strictly increasing on $(0, \alpha)$ and (β, L), and strictly decreasing on (α, β). Besides, using (15), there holds

$$\tilde{g}(0) = \tilde{g}(L) = 0. \tag{31}$$

Hence, using the variations of $\tilde{g}$ given before and (31) that $\tilde{g}$ (and hence g) has a unique zero on $(0, L)$ that we call o_g from now on. Moreover, clearly $\tilde{g} > 0$ on $(0, o_g)$ and $\tilde{g} < 0$ on (o_g, L), hence, using (30), we deduce the same property for g and (27) is proved.

Equality (25) is readily obtained by making a Taylor expansion of $e^2_{a_1^*,1}$ and $\tilde{g}$ around 0 and L and using (30). Indeed, there holds

$$e^2_{a_1^*,1}(x) \sim x^2 (e^2_{a_1^*,1})'(0) \quad \text{and} \quad \tilde{g}(x) \sim (\chi_{\omega_1^*}(0) - \tilde{c}) x^3 (e^2_{a_1^*,1})'(0)/3 \qquad \text{as } x \to 0,$$

$$e^2_{a_1^*,1}(x) \sim \frac{(x-\pi)^2}{2} (e^2_{a_1^*,1})'(\pi) \quad \text{and} \quad \tilde{g}(x) \sim (\chi_{\omega_1^*}(\pi) - \tilde{c})(x-\pi)^3 (e^2_{a_1^*,1})'(\pi)/3 \qquad \text{as } x \to \pi.$$

To conclude, it remains to prove (28). From (24), one observes that g is differentiable almost everywhere on $(0, L)$ and

$$g'(x) = \chi_{\omega_1^*}(x) - \tilde{c} - \frac{2e'_{a_1^*,1}(x) g(x)}{e_{a_1^*,1}(x)}, \tag{32}$$

for almost every $x \in (0, L)$. Using the variations of $e_{a_1^*,1}$, (21) and (32), we infer that g' is negative almost everywhere on $(\alpha, \min(o_g, x_{\max})) \cup (\max(o_g, x_{\max}), \beta)$, from which we deduce (28). ∎

As a consequence of (27) and (23), f is strictly decreasing on $(0, o_g)$ and strictly increasing on (o_g, L) where o_g is defined in (26). We conclude that f has at most two zeros in $(0, L)$. Moreover, thanks to (14) and (22), f has at least one zero in $(0, L)$. Since $e_{a_1^*,1}(\cdot)$ does not vanish in $(0, L)$, one infers that a_1^* has at least 1 and at most 2 switching points.

To generalize our argument to any order $j \geqslant 2$, notice that using the notations of Proposition 3 and its proof, one has $(-1)^{i_0+1} e_{a^*_{j,i_0},1}(x) > 0$ for all $x \in (x_j^{i_0-1}, x_j^{i_0})$ with $i_0 \in \{1 \cdots j\}$. Then, mimicking the argument above in the particular case where $j_0 = 1$, one shows that the function a_j^* has at most two switching points in $(x_j^{i_0-1}, x_j^{i_0})$ and at least one. Besides, since the nodal points $\{x_j^i\}_{i \in \{1, \cdots j-1\}}$ can also be switching points, we conclude that the function a_j^* has at most $3j_0 - 1$ and at least j_0 switching points in $(0, L)$.

Step 3: proof of the estimate (7) (last part of the second point of Theorem 1). Let us first show the easier inequality, in other word the right one. It suffices to write that $m(L, M, r)$ is bounded from above by $\inf_{\omega \in \Omega_r(0,L)} J(0, \omega)$. Inverting the two infima and using Lemma 5 leads immediately to the desired estimate.

The left inequality appears more intricate to establish. It is in fact inferred from a more precise estimate for the optimal design problem $(\mathcal{P}_{j,L,r,M})$. Because of its intrinsic interest, we state this estimate in the following proposition, which constitutes therefore an essential ingredient for the proof of the last point of Theorem 1.

Proposition 4. *Let $r \in (0,1)$ and let us assume that $M \in (0, \pi^2/L^2]$. Then, there holds*

$$m_j(L,M,r) \geqslant \min(r, \underline{r}_j)^3 \underline{m}_j, \tag{33}$$

for every $j \in \mathbb{N}^*$, where the sequences $(\underline{m}_j)_{j\in\mathbb{N}^*}$ and $(\underline{r}_j)_{j\in\mathbb{N}^*}$ are defined by

$$\underline{m}_j = \begin{cases} \frac{1}{2} & \text{if } j = 1, \\ \dfrac{(2j^2-1)(j^2-1)^{\frac{3}{2}}\left(\sqrt{\frac{j^2}{j^2-2}}-1\right)^2}{3j^3\left(\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}}-1\right)^2} & \text{if } j \geqslant 2, \end{cases}$$

and

$$\underline{r}_j = \begin{cases} 1 & \text{if } j = 1, \\ \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right) & \text{if } j \geqslant 2. \end{cases}$$

The proof of this proposition is quite long and technical but the method is elementary and interesting. For this reason, we will temporarily admit this result and postpone its proof to Section 3.3. Let us assume for the moment that $r \leqslant \underline{r}_2$. Let us notice that $\underline{m}_2 = \frac{7}{8}\sqrt{3}(3-2\sqrt{2})$ and $\underline{r}_2 = \frac{\sqrt{5}}{5} + \frac{\sqrt{10}}{10}$. Then, it just remains to prove that $\underline{m}_j \geqslant \underline{m}_2$ and $\underline{r}_j \geqslant \underline{r}_2$ for every $j \in \mathbb{N}\backslash\{0,1\}$.

Proof of $\underline{m}_j \geqslant \underline{m}_2$: We introduce the function F defined on $[2,+\infty)$ by

$$F(x) = \frac{(2x^2-1)(x^2-1)^{3/2}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{x^3\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2}.$$

Notice that $F(j) = 3\underline{m}_j$ for every $j \in \mathbb{N}^*$. Let us write $F(x) = u(x)v(x)$ with

$$u(x) = \frac{(2x^2-1)\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right)^2}{\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right)^2} \quad \text{and} \quad v(x) = \frac{(x^2-1)^{3/2}}{x^3}.$$

Let us show that $u(x) \geqslant u(2)$ for every $x \geqslant 2$. This comes to show that $\psi(x) \leqslant 0$, where

$$\psi(x) = \gamma\left(\left(\frac{x^2}{x^2-2}\right)^{x/2}-1\right) - \sqrt{2x^2-1}\left(\left(\frac{x^2}{x^2-2}\right)^{1/2}-1\right),$$

with $\gamma = \sqrt{u(2)} = \sqrt{7}(\sqrt{2}-1)$, for every $x \geqslant 2$. The derivative of ψ writes

$$\psi'(x) = \gamma\left(\frac{x^2}{x^2-2}\right)^{x/2} w(x) - \frac{2x}{\sqrt{2x^2-1}}\left(\sqrt{\frac{x^2}{x^2-2}}-1\right) - \frac{\sqrt{2x^2-1}}{(x^2-2)^{3/2}},$$

with $w(x) = \ln\left(\sqrt{\frac{x^2}{x^2-2}}\right) - \frac{1}{x^2-2}$. The derivative of w writes $w'(x) = \frac{4}{x(x^2-2)^2}$, and therefore, the function w is increasing and negative since $\lim_{x\to+\infty} w(x) = 0$. One then infers that $\psi'(x)$ writes as the sum of three negative terms and is thus negative on $[2,+\infty)$. Hence, the function

ψ decreases on this interval and therefore, $\psi(x) \leqslant \psi(2) = 0$ for every $x \in [2, +\infty)$. The expected result on u follows.

Let us now show that $v(x) \geqslant v(2)$ for every $x \in [2, +\infty)$. The derivative of v writes

$$v'(x) = \frac{3\sqrt{x^2-1}}{x^4},$$

is therefore positive on $[2, +\infty)$, and the expected conclusion follows.

Proof of $\underline{r}_j \geqslant \underline{r}_2$: Let us write $\underline{r}_j = u(j)v(j)$ with

$$u(j) = \frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}} \quad \text{and} \quad v(j) = j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}.$$

We claim that $j \mapsto u(j)$ is a increasing function for every $j \geqslant 2$ and $j \mapsto v(j)$ is a increasing function for every $j \geqslant 4$. Indeed, straightforward computations show that

$$\forall j \geqslant 2, \qquad \frac{du}{dj}(j) = \frac{(j+\sqrt{j^2-2})(j^2+1-j\sqrt{j^2-2})}{2\sqrt{j^2-2}(j^2+1)^{3/2})} \geqslant 0$$

and

$$\forall j \geqslant 4, \qquad \frac{dv}{dj}(j) = \frac{(j^{j-1}+(j^2-2)^{j/2}(\ln\left(\sqrt{\gamma(j)}\right) - \gamma(j) + 1 - \frac{1}{j})}{j^{j-1}} \geqslant 0$$

We thus infer that for every $j \geqslant 4$, there holds $g(j-1,j) \geqslant g(3,4)$. Dealing separately with the cases $j=2$ and $j=3$ leads to the desired conclusion for $r \leqslant \underline{r}_2$.

Let us now treat the case $r \geqslant \underline{r}_2$. It is obvious that the solution $(\tilde{a}, \tilde{\omega})$ of the following problem

$$\inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\tilde{\Omega}_r(0,L)} J(a,\omega), \qquad (\tilde{\mathcal{P}}_{L,r,M})$$

where

$$\tilde{\Omega}_r(\alpha,\beta) = \{\text{Lebesgue measurable subset } \omega \text{ of } (\alpha,\beta) \text{ such that } |\omega| \geqslant r(\beta-\alpha)\},$$

satisfies in particular that $\tilde{\omega} \in \Omega_r(0,L)$ (in other words, the inequality constraint is reached at the optimum). Therefore, the problems ($\mathcal{P}_{L,r,M}$) and ($\tilde{\mathcal{P}}_{L,r,M}$) are the same.

Taking into account this new expression, we remark that the quantity $m(L,M,r)$ is nondecreasing with respect to r and so for $r \geqslant \underline{r}_2$ we have $m(L,M,r) \geqslant m(L,M,\underline{r}_2) \geqslant \underline{m}_2\underline{r}_2^3$, which concludes the proof. ∎

3.3 Proof of Proposition 4

The proof, although quite technical, is based on a simple idea: by using the concavity of the eigenfunction $e_{a,j}$ on each nodal domain, we will determine a piecewise affine function Δ_j such that $e_{a,j} \geqslant \Delta_j$ for every $a \in \mathcal{A}_M(0,L)$. The construction of such a function is not obvious since one has to control at the same time the slope of the graph of Δ_j on each interval on which it is affine, and its maximum, as the potential a describes the set $\mathcal{A}_M(0,L)$.

As previously, we will first consider the case where $j=1$. In other words, we will first provide a lower estimate of the quantity $m_1(L,r)$. We will then generalize this estimate to any $j \in \mathbb{N}^*$ by using that the j-th eigenfunction $e_{a,j}$ of A_a coincides with the first eigenfunction of the restriction of A_a on each nodal domain.

Notice that, proving that the estimate (33) holds for every $M \in (0, \pi^2/L^2]$ is equivalent to showing it for the particular value

$$M = \pi^2/L^2,$$

which will be assumed from now on. Let a be a generic element of $\mathcal{A}_M(0,L)$.

First step: proof of Proposition 4 in the case "$j = 1$". By using the concavity of $e_{a,1}$ we will construct an affine function Δ_1 (see Figure 2) such that $e_{a,1} \geqslant \Delta_1$ pointwisely on $[0, L]$. We will infer a lower bound of $m_1(L, M, r)$ by computing explicitly the minimum of the quantity $\int_\omega \Delta_1^2(x)\, dx$ over the class of measurable subsets ω of $(0, L)$ such that $|\omega| = rL$. For that purpose, let us first provide an estimate of the $\max_{(0,L)} e_{a,1}$ in terms of the L^2 norm of $e_{a,1}$ and the derivatives of $e_{a,1}$ at $x = 0$ and $x = L$.

Lemma 7. *With the assumptions of Proposition 4, there holds*

$$e_{a,1}^2(x_{max}) \geqslant \max\left\{\frac{3}{2L}\int_0^L e_{a,1}^2(x)\, dx, \frac{L^2}{2\pi^2}\max\{e_{a,1}'(0)^2, e_{a,1}'(L)^2\}\right\}. \tag{34}$$

Proof of Lemma 7. Since $e_{a,1}'(x_{max}) = 0$, multiplying Equation (2) by e_1' and integrating on (x, x_{max}) (with possibly $x > x_{max}$) leads to

$$e_{a,1}'(x)^2 = \int_x^{x_{max}} (\lambda_{a,1}^2 - a(x))\frac{d}{dx}(e_{a,1}^2(x))\, dx \tag{35}$$

for every $x \in (0, L)$. Besides, according to the Courant-Fischer minimax principle (see for instance [Section C, (90)]) , one has

$$0 \leqslant \lambda_{a,1}^2 - a(\cdot) \leqslant \frac{2\pi^2}{L^2} \qquad \text{in } (0, L). \tag{36}$$

Therefore, combining (35) and (36) yields

$$e_{a,1}'(x)^2 \leqslant \frac{2\pi^2}{L^2}(e_{a,1}^2(x_{max}) - e_{a,1}^2(x)), \tag{37}$$

for every $x \in (0, L)$. In particular, applying (37) at $x = L$ and $x = 0$, we obtain

$$\max\{e_{a,1}'(L)^2, e_{a,1}'(0)^2\} \leqslant \frac{2\pi^2}{L^2} e_{a,1}^2(x_{max}), \tag{38}$$

Moreover, by integrating (37) between 0 and L, we obtain

$$\int_0^L e_{a,1}'(x)^2\, dx + \frac{2\pi^2}{L^2}\int_0^L e_{a,1}^2(x)\, dx \leqslant \frac{2\pi^2}{L^2} e_{a,1}^2(x_{max})L. \tag{39}$$

We obtain (34) from (38) and by combining (39) with Poincaré's inequality

$$\int_0^L e_{a,1}(x)^2\, dx \leqslant \frac{L^2}{\pi^2}\int_0^L e_{a,1}'(x)^2\, dx.$$

■

According to (34), and assuming since the eigenfunction $e_{a,1}$ is normalized in $L^2(0, L)$, there holds

$$e_{a,1}^2(x_{max}) \geqslant \frac{3}{2L}. \tag{40}$$

Since $e_{a,1}$ is concave and according to (40), one has the successive inequalities

$$e_{a,1}(x) \geqslant Tr_{a,1}(x) \geqslant \Delta_1(x), \tag{41}$$

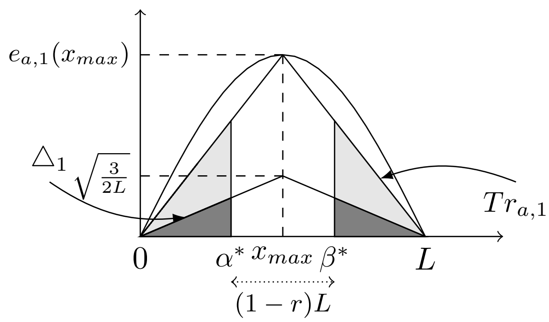

Figure 2: Graphs of the functions $e_{a,1}$, $Tr_{a,1}$ and Δ_1.

for every $x \in [0, L]$, where $Tr_{a,1}$ and Δ_1 denote the piecewise affine functions defined by

$$Tr_{a,1}(x) = \begin{cases} \frac{e_{a,1}(x_{max})x}{x_{max}} & \text{on } (0, x_{max}) \\ \frac{e_{a,1}(x_{max})(L-x)}{L-x_{max}} & \text{on } (x_{max}, L) \end{cases} \quad \text{and} \quad \Delta_1(x) = \begin{cases} \frac{\sqrt{3}x}{\sqrt{2L}x_{max}} & \text{on } (0, x_{max}) \\ \frac{\sqrt{3}(L-x)}{\sqrt{2L}(L-x_{max})} & \text{on } (x_{max}, L). \end{cases}$$

Combining (40) with (41) and according to Proposition 1, we readily obtain

$$\inf_{\omega \in \Omega_r(0,L)} \int_\omega e_{a,1}(x)^2\, dx \geqslant \inf_{\omega \in \Omega_r(0,L)} \int_\omega \Delta_1(x)^2 dx = \int_{\widehat{\omega}} \Delta_1(x)^2 dx, \tag{42}$$

with $\widehat{\omega} = (0, \alpha^*) \cup (\beta^*, L)$ verifying

$$\Delta_1(\alpha^*) = \Delta_1(\beta^*) \qquad \text{and} \qquad |\widehat{\omega}| = L - \beta^* + \alpha^* = rL.$$

It follows that $\alpha^* = rx_{max}$, $\beta^* = (1-r)L + rx_{max}$ and one obtains

$$\int_{\widehat{\omega}} \Delta_1(x)^2\, dx = \int_0^{rx_{max}} \Delta_1(x)^2\, dx + \int_{(1-r)L+rx_{max}}^L \Delta_1(x)^2\, dx = \frac{r^3}{2} = r^3 \underline{m}_1. \tag{43}$$

We have then proved Proposition 4 in the case where $j = 1$.

Second step: proof of Proposition 4 in the general case. We now assume that $j \geqslant 2$. Let $0 = x_j^0 < x_j^1 < x_j^2 < \cdots < x_j^{j-1} < L = x_j^j$ be the $j+1$ zeros of the j-th eigenfunction $e_{a,j}$. Introduce $\gamma = (\gamma_1, \cdots, \gamma_{j-1}) \in (0,1)^j$ such that

$$\gamma_i = \int_{x_j^i}^{x_j^{i+1}} e_{a,j}^2(x)\, dx, \qquad i = 0, \ldots, j-1. \tag{44}$$

Note that, because of the normalization condition on the function $e_{a,j}$, there holds

$$\sum_{i=0}^{j-1} \gamma_i = 1. \tag{45}$$

In the sequel, we will use the following notations, for all $i \in \{0, \cdots, j-1\}$,

$$\Omega_i = (x_j^i, x_j^{i+1}), \qquad \eta_i = \frac{|\Omega_i|}{L} \in (0,1), \qquad \text{and} \qquad e_{a,j}(x_{max}^i) = \max_{x \in \Omega i} e_{a,j}(x). \tag{46}$$

Now, assume that $r \leqslant \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$. We will distinguish between several cases, depending on the value of the first integer $i_0 \in \{0, \cdots, j-1\}$ (that exists thanks to (45)) such that

$$\gamma_{i_0} \geqslant \frac{1}{j}. \tag{47}$$

By exploiting condition (47), we will yield a lower bound A_{i_0} of the positive number $|e_{a,j}(x_{max}^{i_0})|$. Then we will derive an estimate of $e_{a,j}(x_{max}^{i_0-1})$ and $e_{a,j}(x_{max}^{i_0+1})$ in terms of $e_{a,j}(x_{max}^{i_0})$. Hence, step by step we will get lower bounds of all j maxima needed to construct the piecewise affine function Δ_j. For that purpose, we have to distinguish between several cases, depending on the value of the integer i_0

First case: assume that $i_0 = 0$. Since the function $e_{a,j}(\cdot)$ is concave, we claim that

$$\int_\omega e_{a,j}(x)^2\,dx \geqslant \int_\omega Tr_{a,j}(x)^2 dx, \tag{48}$$

for every $\omega \in \Omega_r(0,L)$ where the function $Tr_{a,j}$ is piecewise affine, defined on each interval (x_j^i, x_j^{i+1}), with $i \in \{0,\dots,j-1\}$, by

$$Tr_{a,j}(x) = \begin{cases} \frac{x - x_j^i}{x_{max}^i - x_j^i} e_{a,j}(x_{max}^i) & \text{on } (x_j^i, x_{max}^i), \\ \frac{x_j^{i+1} - x}{x_j^{i+1} - x_{max}^i} e_{a,j}(x_{max}^i) & \text{on } (x_{max}^i, x_j^{i+1}), \end{cases}$$

for every $x \in (x_j^i, x_j^{i+1})$.

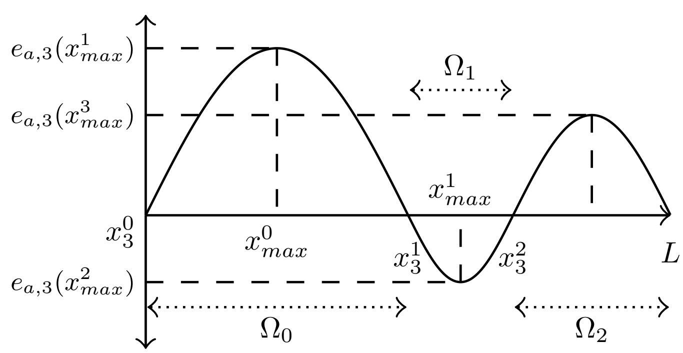

Figure 3: Illustration of the case "$i_0 = 0$" with 3 nodal domains ($j = 3$).

Since the j-th eigenfunction $e_{a,j}$ coincides with the first eigenfunction of $-\partial_{xx} + a(\cdot)$ with Dirichlet conditions on (x_j^i, x_j^{i+1}), the method of the first step can be adapted. By reproducing the proof of Lemma 7 to show (34), we obtain

$$e_{a,j}(x_{max}^i)^2 \geqslant \max\left\{ \frac{\frac{2\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}}{\left(\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}\right)|\Omega_i|} \int_{x_j^i}^{x_j^{i+1}} e_{a,j}(x)^2\,dx, \frac{1}{\frac{\pi^2}{|\Omega_i|^2} + \frac{\pi^2}{L^2}} \max\{e'_{a,j}(x_j^i)^2, e'_{a,j}(x_j^{i+1})^2\} \right\}. \tag{49}$$

Notice that one recovers (34) by substituting $|\Omega_i|$ by L in (49). One derives from the equivalent of (37) in this case the estimates

$$\frac{\pi^2}{|\Omega_i|^2}-\frac{\pi^2}{L^2}\leqslant\frac{e'_{a,j}(x_j^i)^2}{e_{a,j}(x^i_{max})^2}\leqslant\frac{\pi^2}{|\Omega_i|^2}+\frac{\pi^2}{L^2}\quad\text{and}\quad\frac{\pi^2}{|\Omega_i|^2}-\frac{\pi^2}{L^2}\leqslant\frac{e'_{a,j}(x_j^{i+1})^2}{e_{a,j}(x^i_{max})^2}\leqslant\frac{\pi^2}{|\Omega_i|^2}+\frac{\pi^2}{L^2}.\quad(50)$$

Applying (49) for $i=0$ and using (47) yields

$$e_{a,j}(x^0_{max})^2\geqslant A_0\quad\text{with}\quad A_0=\frac{\frac{2\pi^2}{|\Omega_0|^2}+\frac{\pi^2}{L^2}}{j\left(\frac{\pi^2}{|\Omega_0|^2}+\frac{\pi^2}{L^2}\right)|\Omega_0|}=\frac{1}{Lj}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right).\quad(51)$$

Let us now provide an estimate of $e_{a,1}(x^1_{max})^2$. Combining the inequalities (49) with $i=1$ and (50) with $i=0$, we get

$$e_{a,1}(x^1_{max})^2\geqslant\frac{e'_{a,j}(x_j^1)^2}{\frac{\pi^2}{|\Omega_1|^2}+\frac{\pi^2}{L^2}}\geqslant\frac{(\frac{\pi^2}{|\Omega_0|^2}-\frac{\pi^2}{L^2})e_{a,1}(x^0_{max})^2}{\frac{\pi^2}{|\Omega_1|^2}+\frac{\pi^2}{L^2}}.\quad(52)$$

Combining (51) and (52) yields

$$e_{a,j}(x^1_{max})^2\geqslant A_1\quad\text{with}\quad A_1=\frac{(L^2-|\Omega_0|^2)|\Omega_1|^2}{(L^2+|\Omega_1|^2)|\Omega_0|^2}A_0=\frac{(1-\eta_0^2)\eta_1^2}{(1+\eta_1^2)\eta_0^2}A_0.$$

By induction, it follows that

$$e_{a,j}(x^i_{max})^2\geqslant A_i\quad\text{with}\quad A_i=\left(\prod_{k=1}^{i}\frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right)A_0,\quad(53)$$

for every $i\in\{1,\cdots,j-1\}$. Hence, (53) together with (48) allows us to write

$$\int_\omega e_{a,j}(x)^2\,dx\geqslant\int_\omega\Delta_j(x)^2\,dx,$$

where Δ_j is the piecewise affine function defined on $(0,L)$ by

$$\Delta_j(x)=\begin{cases}\frac{(x-x_j^i)}{(x^i_{max}-x^i_j)}\sqrt{A_i} & \text{on } (x_j^i,x^i_{max}),\\ \frac{(x_j^{i+1}-x)}{(x_j^{i+1}-x^i_{max})}\sqrt{A_i} & \text{on } (x^i_{max},x_j^{i+1}),\end{cases}\quad(54)$$

for every $i\in\{0,\cdots,j-1\}$ and $x\in(x^i_j,x^{i+1}_j)$ (see Fig. 4 below).

According to Proposition 1, we obtain

$$\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx\geqslant\inf_{\omega\in\Omega_r(0,L)}\int_\omega\Delta_j(x)^2\,dx=\int_{\hat\omega}\Delta_j(x)^2\,dx,\quad(55)$$

where $\hat\omega=\{\Delta_j(x)^2<\tau\}$ up to a set of zero Lebesgue measure and $|\hat\omega|=rL$. Let $i\in\{0,\cdots,j-1\}$ and let us introduce $\omega_i=\hat\omega\cap(x^i_j,x^{i+1}_j)$.

The following technical Lemma allows to deal with small values of the parameter r.

Lemma 8. *Let $j\geqslant 1$, $L,\tau>0$, let $r\in(0,1)$ such that $|\hat\omega|=rL$, where $\hat\omega=\{\Delta_j^2(\cdot)<\tau\}$, with Δ_j defined by (54). Recall that A_0 is defined by (51) and A_i is defined by (53) for every $i\in\{1,\cdots,j-1\}$. There holds*

$$r<\left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j-\frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)\quad\Longrightarrow\quad\tau\leqslant\min_{i\in\{0,\cdots,j-1\}}A_i.$$

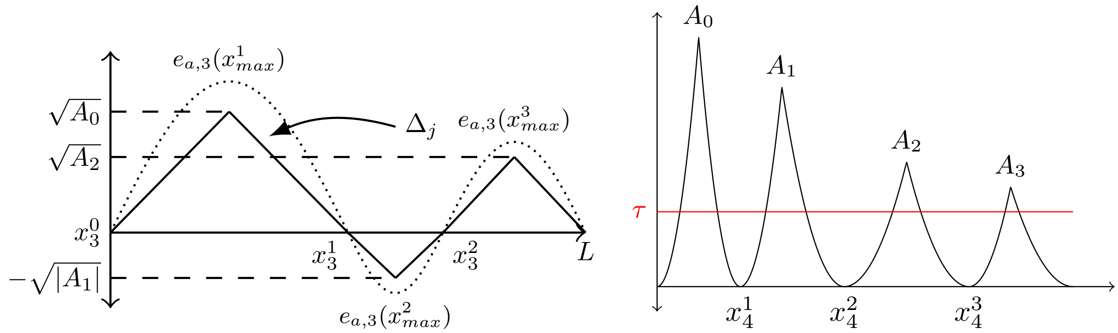

Figure 4: Left: graphs of the functions $e_{a,3}$ and Δ_3. Right: $j=4$. Graph of Δ_4^2 with respect to x.

The proof Lemma 8 is postponed to Section B. As a consequence of this lemma, there exist α_i^*, β_i^* and $r_i \in (0,1)$ such that $\omega_i = (x_j^i, \alpha_i^*) \cup (\beta_i^*, x_j^{i+1})$, $|\omega_i| = r_i(x_j^{i+1} - x_j^i)$ (see Figure 4 for an illustration), and therefore

$$\sum_{i=0}^{j-1} r_i(x_j^{i+1} - x_j^i) = rL. \tag{56}$$

By definition of $\hat{\omega}$, one has $\Delta_j^2(\alpha_i^*) = \Delta_j^2(\beta_i^*) = \Delta_j^2(\alpha_{i+1}^*)$, consequently there holds

$$\alpha_i^* = r_i(x^i_{max} - x_j^i) + x_j^i \quad , \quad \beta_i^* = x_j^{i+1} - r_i(x_j^{i+1} - x^i_{max}), \tag{57}$$

and

$$r_{i+1}^2 A_{i+1} = r_i^2 A_i = \cdots = r_0^2 A_0. \tag{58}$$

As a result, one obtains

$$\int_{\omega^*} \Delta_j(x)^2\, dx = \sum_{i=0}^{j-1} \int_{\omega_i} \Delta_j(x)^2\, dx = \frac{1}{3} \sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i. \tag{59}$$

To compute the numbers r_i, we use (58) together with (53), which yields to

$$r_i = \sqrt{\frac{A_0}{A_i}} r_0 = \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}\, r_0. \tag{60}$$

Since $\sum_{i=0}^{j-1} r_i \frac{|\Omega_i|}{L} = \sum_{i=0}^{j-1} r_i \eta_i = r$, one infers

$$r_0 \quad = \quad \frac{r}{\eta_0 + \sum_{i=1}^{j-1} \eta_i \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}} = \frac{r}{\eta_0 \left(1 + \sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}}\right)}. \tag{61}$$

Besides, using (60), there holds

$$\sum_{i=0}^{j-1} r_i^3 |\Omega_i| A_i = r_0^2 A_0 \sum_{i=0}^{j-1} r_i |\Omega_i| = r_0^2 A_0 r L. \tag{62}$$

We conclude by combining (62) with (59) that

$$\int_{\hat{\omega}} \Delta_j(x)^2\, dx = \frac{1}{3} A_0 r_0^2 r L, \tag{63}$$

where A_0 and r_0 are respectively given by (51) and (61).

Since our goal is to estimate $\int_{\hat\omega} \Delta_j(x)^2\,dx$ from below, regarding (63), we need to find a lower bound on r_0 and consequently on the numbers $|\Omega_i|$ according to (62). We will use the following Lemma.

Lemma 9. *For every $i \in \{0, \cdots, j-1\}$, there holds*

$$\frac{L}{\sqrt{j^2+1}} \leqslant |\Omega_i| \leqslant \frac{L}{\sqrt{j^2-1}}. \tag{64}$$

Proof of Lemma 9. According to the Courant-Fischer minimax principle, one has

$$\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}.$$

Since the j-th eigenfunction $e_{a,j}$ is also the first eigenfunction of the operator $-\partial_{xx} + a(\cdot)$ with Dirichlet boundary conditions on Ω_i, we also have

$$\frac{\pi}{|\Omega_i|} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{\pi}{|\Omega_i|}\right)^2 + \frac{\pi^2}{L^2}}.$$

We then infer

$$\frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 + \frac{\pi^2}{L^2}}} \leqslant |\Omega_i| \leqslant \frac{\pi}{\sqrt{\left(\frac{j\pi}{L}\right)^2 - \frac{\pi^2}{L^2}}}.$$

■

It follows from Lemma 9 that $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ and therefore

$$\sum_{i=1}^{j-1} \sqrt{\prod_{k=1}^{i} \frac{(1+\eta_k^2)}{(1-\eta_{k-1}^2)}} \leqslant g_1(j),$$

where $g_1(j) = \sum_{i=1}^{j-1} \left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}} = \frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}} - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}} - 1}$. According to (61), one has

$$r_0 \geqslant \frac{r}{\eta_0(1+g_1(j))}. \tag{65}$$

Combining (51), (63) and (65), we obtain

$$\int_{\hat\omega} \Delta_j(x)^2\,dx \geqslant r^3 \inf_{\eta_0 \in \left(\frac{1}{\sqrt{j^2+1}}, \frac{1}{\sqrt{j^2-1}}\right)} g_2(\eta_0, j), \tag{66}$$

with

$$g_2(\eta_0, j) = \frac{1}{3j}\left(\frac{2+\eta_0^2}{\eta_0(1+\eta_0^2)}\right)\left(\frac{1}{\eta_0 + \eta_0 g_1(j)}\right)^2.$$

Since for every $\eta_0 > 0$, we have

$$\frac{\partial g_2}{\partial \eta_0}(\eta_0, j) = -\frac{\left(\sqrt{\frac{j^2}{j^2-1}} - 1\right)^2 (11\eta_0^2 + 3\eta_0^4 + 6)}{\left((\frac{j^2}{j^2-1})^{\frac{j}{2}} - 1\right)^2 \eta_0^4 (1+\eta_0^2)^2 j} \leqslant 0,$$

the function $\eta_0 \mapsto g_2(\eta_0, j)$ is decreasing, so that (66) becomes

$$\int_{\hat{\omega}} \Delta_j(x)^2 \, dx \geqslant r^3 g_2\left(\sqrt{\frac{1}{j^2-1}}, j\right) = r^3 \underline{m}_j, \tag{67}$$

and the expected result is proved for $r \leqslant \underline{r}_j := \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$.

Noticing that $r \mapsto \inf_{\omega\in\Omega_r(0,L)} \int_\omega \Delta_j(x)^2 dx$ is an increasing function, we infer that for every $r \in [\underline{r}_j, 1]$, there holds

$$\inf_{\omega\in\Omega_r(0,L)} \int_\omega \Delta_j(x)^2 \, dx \geqslant \inf_{\omega\in\Omega_{\underline{r}_j}(0,L)} \int_\omega \Delta_j(x)^2 dx \geqslant \underline{r}_j^3 \underline{m}_j,$$

and the expected result is proved for $r \in [0, 1]$.

<u>*Second case:*</u> assume now that $i_0 = 1$. We will prove that the estimate choosing $i_0 = 0$ is worst than the estimate that we obtain with $i_0 = 1$. Using (49) with $i = 1$, we have

$$e^2_{a,j}(x^1_{max}) \geqslant A_1 \quad \text{with} \quad A_1 = \frac{1}{Lj}\left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right). \tag{68}$$

Combining the inequality (49) with $i = 0$, (50) with $i = 1$ and (68) we get

$$A_0 = \frac{(1-\eta_1^2)\eta_0^2}{(1+\eta_0^2)\eta_1^2} A_1. \tag{69}$$

Using (49) with $i = 2$, (50) with $i = 1$ and (68) we have

$$e^2_{a,j}(x^2_{max}) \geqslant A_2 \quad \text{with} \quad A_2 = \frac{(1-\eta_1^2)\eta_2^2}{(1+\eta_2^2)\eta_1^2} A_1.$$

By induction, for every $i \in \{2, \cdots, j-1\}$ we have

$$e^2_{a,j}(x^i_{max}) \geqslant A_i \quad \text{with} \quad A_i = \left(\prod_{k=2}^{i} \frac{(1-\eta_{k-1}^2)\eta_k^2}{(1+\eta_k^2)\eta_{k-1}^2}\right) A_1 \tag{70}$$

Let us state the equivalent of Lemma 8 for the case considered here.

Lemma 10. *Let $j \geqslant 1$, $L, \tau > 0$, let $r \in (0, 1)$ such that $|\hat{\omega}| = rL$, where $\hat{\omega} = \{\Delta_j^2(\cdot) < \tau\}$, with Δ_j is defined by (54). Recall that A_1 is defined by (68), A_0 is defined by (69) and A_i is defined by (70) for every $i \in \{2, \cdots, j-1\}$. There holds*

$$r < \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right) \implies \tau \leqslant \min_{i\in\{0,\cdots,j-1\}} A_i.$$

The proof of this lemma is postponed to Section B.

Hence, we conclude that $r_i^2 A_i = r_1^2 A_1$ for all $i \in \{0, \cdots, j-1\}$. Since $\sum_{i=0}^{j-1} r_i \eta_i = r$, one computes by using (70) and (69)

$$r_1 = \frac{r}{\eta_1 + \eta_0\sqrt{\frac{A_1}{A_0}} + \sum_{i=2}^{j-1}\eta_i\sqrt{\frac{A_1}{A_i}}} = \frac{r}{\eta_1 + \eta_0\sqrt{\frac{1+\eta_0^2}{1-\eta_1^2}} + \sum_{i=2}^{j-1}\eta_i\sqrt{\prod_{k=2}^{i}\frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}}.$$

Moreover, one has

$$\sum_{i=2}^{j-1}\eta_i\sqrt{\prod_{k=2}^{i}\frac{(1+\eta_k^2)\eta_{k-1}^2}{(1-\eta_{k-1}^2)\eta_k^2}}=\eta_1\sum_{i=2}^{j-1}\sqrt{\prod_{k=2}^{i}\frac{1+\eta_k^2}{1-\eta_{k-1}^2}}$$

and since $\frac{1}{\sqrt{j^2+1}}\leqslant\eta_i\leqslant\frac{1}{\sqrt{j^2-1}}$ according to Lemma 9, there holds

$$r_1\geqslant\frac{r}{\eta_1+\eta_1\sqrt{\frac{j^2}{j^2-2}}+\eta_1\sum_{i=2}^{j-1}\left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}}}.$$

Since $j\geqslant 2$, we have

$$\sqrt{\frac{j^2}{j^2-2}}+\sum_{i=2}^{j-1}\left(\frac{j^2}{j^2-2}\right)^{\frac{i-1}{2}}-\sum_{i=1}^{j-1}\left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}}=\sqrt{\frac{j^2}{j^2-2}}-\left(\frac{j^2}{j^2-2}\right)^{\frac{j-1}{2}}\leqslant 0,$$

and it follows that

$$r_1\geqslant\frac{r}{\eta_1+\eta_1\sum_{i=1}^{j-1}\left(\frac{j^2}{j^2-2}\right)^{\frac{i}{2}}}.$$

As a consequence, using the same approach as the one used for the case where $i_0=0$, we infer that

$$\inf_{\omega\in\Omega_r(0,L)}\int_\omega\Delta_j(x)^2\,dx\geqslant r^3\inf_{\eta_1\in\left(\frac{1}{\sqrt{j^2+1}},\frac{1}{\sqrt{j^2-1}}\right)}g_2(\eta_1,j)$$

with

$$g_2(\eta_1,j)=\frac{1}{3j}\left(\frac{2+\eta_1^2}{\eta_1(1+\eta_1^2)}\right)\left(\frac{1}{\eta_1+\eta_1 g_1(j)}\right)^2\quad\text{and}\quad g_1(j)=\frac{\left(\frac{j^2}{j^2-2}\right)^{\frac{j}{2}}-\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}{\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}-1}.$$

Noticing that the functions g_1 and g_2 are exactly the same as in the case $i_0=1$, we conclude similarly to the first case.

Finally, mimicking this proof and adapting it for every $i_0\in\{2,\cdots,j-1\}$, we prove that the estimate with $i_0=0$ is the worst one. We then obtain the same conclusion.

■

3.4 Proof of Theorem 2

We argue by contradiction, assuming that the optimal design problem ($\mathcal{P}_{L,r,\infty}$) has a solution $a^*\in L^\infty(0,L)$. Then, denoting $M_0=\|a^*\|_{L^\infty(0,L)}$ and noting that $\mathcal{A}_{M_0}(0,L)$ is included in $\mathcal{A}_\infty(0,L)$, it follows that a^* is a solution of the problem

$$\inf_{a\in\mathcal{A}_\infty(0,L)}\inf_{\omega\in\Omega_r(0,L)}J(a,\omega)=\inf_{a\in\mathcal{A}_{M_0}(0,L)}\inf_{\omega\in\Omega_r(0,L)}\int_\omega e_{a,j}(x)^2\,dx,$$

for some given nonzero integer j, by using the same argument as in the first step of the proof of Theorem 1 to show the existence of a minimizing integer.

We will use the notations of Proposition 3 and Section 3.2. The contradiction will be obtained by constructing a perturbation a_n^* of a^* such that $J(a_n^*)<J(a^*)$. According to Proposition 3, a^*

is non-trivial and *bang-bang*, equal to 0 and M_0 almost everywhere in $(0,L)$ so that there exists $i_0\in\{1,\cdots,j\}$ such that the set $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})$ is measurable of positive measure.

Thanks to the regularity of the Lebesgue measure, there exists an increasing sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ strictly included in $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})$ satisfying

$$\lim_{n\to\infty}|K_n|=|\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})|,$$

where $|\cdot|$ denotes the Lebesgue measure. In what follows, we will use the notation I^c to denote the complement of any set $I\subset[0,L]$ in $[0,L]$. We introduce

$$a_n^*(x)=\begin{cases} M_0+\frac{1}{\varphi(n)} & \text{on } K_n,\\ 0 & \text{on } K_n^c,\end{cases}\qquad \text{with}\quad \varphi(n)=|\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})\cap K_n^c|.$$

Let us remark that

$$a_n^*(x)-a(x)=\begin{cases}\frac{1}{\varphi(n)} & \text{on } K_n,\\ -M_0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})\cap K_n^c,\\ 0 & \text{on } \mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})^c.\end{cases}$$

Hence, we get

$$\begin{aligned}\langle dJ(a),\varphi(n)(a_n^*-a^*)\rangle &= -2\int_{x_j^{i_0-1}}^{x_j^{i_0}}\varphi(n)(a_n^*(x)-a^*(x))e_{a_{i_0},1}(x)p_{i_0}(x)\,dx\\ &= 2M_0\varphi(n)\int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})\cap K_n^c}e_{a_{i_0},1}(x)p_{i_0}(x)\,dx\\ &\quad -\int_{K_n}e_{a_{i_0},1}(x)p_{i_0}(x)\,dx,\end{aligned}$$

for $n\in\mathbb{N}$. Using (12), we have $e_{a_{i_0},1}p_{i_0}\geqslant 0$ on $\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})\cap K_n^c$ and $e_{a_{i_0},1}p_{i_0}>0$ on K_n for all $n\in\mathbb{N}$. Since $\lim_{n\to+\infty}\varphi(n)=0$ and according to the Lebesgue density theorem,

$$\lim_{n\to+\infty}2M_0\varphi(n)\int_{\mathcal{I}_{M_0,a^*}(x_j^{i_0-1},x_j^{i_0})\cap K_n^c}e_{a_{i_0},1}(x)p_{i_0}(x)\,dx=0.$$

As a consequence, there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$ $\langle dJ(a),\varphi(n)(a_n^*-a^*)\rangle<0$. Thus, there exists $n_1\in\mathbb{N}$ verifying $J(a_{n_1}^*)<J(a^*)$, whence the contradiction. We then infer that the optimal design problem $(\mathcal{P}_{L,r,\infty})$ has no solution.

4 Applications and numerical investigations

4.1 Controllability issues for the wave equation

4.1.1 The cost of the control in large time

Let us fix $T>0$ and consider the one dimensional wave equation with potential

$$\begin{aligned}\partial_{tt}\varphi(t,x)-\partial_{xx}\varphi(t,x)+a(x)\varphi(t,x)=0, &\qquad (t,x)\in(0,T)\times(0,L),\\ \varphi(t,0)=\varphi(t,L)=0, &\qquad t\in[0,T],\\ (\varphi(0,x),\partial_t\varphi(0,x))=(\varphi_0(x),\varphi_1(x)), &\qquad x\in[0,L],\end{aligned}\tag{71}$$

where the potential $a(\cdot)$ is a nonnegative function belonging to $L^\infty(0,L)$. It is well known that for every initial data $(\varphi_0,\varphi_1)\in H^1_0(0,L)\times L^2(0,L)$, there exists a unique solution φ in $C^0(0,T;H^1_0(0,L))\cap C^1(0,T;L^2(0,L))$ of the Cauchy problem (71).

Let ω be a given measurable subset of $(0,L)$ of positive Lebesgue measure. The system (71) is said to be *observable* on ω in time T if there exists a positive constant C such that

$$CE_a(0)\leqslant \int_0^T\int_\omega \partial_t\varphi(t,x)^2\,dxdt \tag{72}$$

for all $(\varphi_0,\varphi_1)\in H^1_0(0,L)\times L^2(0,L)$ where

$$E_a(t)=\int_0^L\left(\partial_t\varphi(t,x)^2+\partial_x\varphi(t,x)^2+a(x)\varphi(t,x)^2\right)\,dx$$

for all $t\geqslant 0$. Notice moreover that the function $E_a(\cdot)$ is constant[5]. We denote by $C_{T,\mathrm{obs}}(a,\omega)$ the largest constant in the previous inequality, that is

$$C_{T,\mathrm{obs}}(a,\omega)=\inf_{\substack{(\varphi_0,\varphi_1)\in H^1_0(0,L)\times L^2(0,L)\\(\varphi_0,\varphi_1)\neq(0,0)}}\frac{\int_0^T\int_\omega\partial_t\varphi(t,x)^2\,dxdt}{E_a(0)}. \tag{73}$$

This constant can be interpreted a quantitative measure of the well-posed character of the inverse problem of reconstructing the solutions from measurements over $[0,T]\times\omega$. Moreover, this constant also plays a crucial role in the frameworks of control theory. Indeed, consider the internally controlled wave equation on $(0,L)$ with Dirichlet boundary conditions

$$\begin{cases}\partial_{tt}y(t,x)-\partial_{xx}y(t,x)+a(x)y(t,x)=h_{a,\omega}(t,x), & (t,x)\in(0,T)\times(0,L),\\ y(t,0)=y(t,\pi)=0, & t\in[0,T],\\ (y(0,x),\partial_t y(0,x))=(y^0(x),y^1(x)), & x\in(0,L),\end{cases} \tag{74}$$

where $h_{a,\omega}$ is a control supported by $[0,T]\times\omega$ and ω is a Lebesgue measurable subset of $(0,L)$. Recall that for every initial data $(y^0,y^1)\in L^2(0,L)\times H^{-1}(0,L)$ and every $h_{a,\omega}\in L^2((0,T)\times(0,L))$, the problem (74) has a unique solution y verifying moreover $y\in C^0(0,T;L^2(0,L))\cap C^1(0,T;H^{-1}(0,L))$. This problem is said to be *null controllable at time* T if and only if for every initial data $(y^0,y^1)\in L^2(0,L)\times H^{-1}(0,L)$, one can find a control $h_{a,\omega}\in L^2((0,T)\times(0,L))$ supported by $[0,T]\times\omega$ such that the solution y of (74) verifies $y(T,\cdot)=\partial_t y(T,\cdot)=0$.

Let us assume that (74) is null controllable. At fixed $(y^0,y^1)\in L^2(0,L)\times H^{-1}(0,L)$, since the set of all controls $h_{a,\omega}$ steering (y^0,y^1) to $(0,0)$ is a closed vector space of $L^2((0,T)\times(0,L))$, there exists a unique control of minimal $L^2((0,T)\times\omega)-$norm (see e.g. [5, Chap.2, Section 2.3] and [28]) that we denote $h^{opt}_{a,\omega}$, which can be constructed "explicitly" as the minimum of a functional according to the Hilbert Uniqueness Method. Thus, we can define the HUM operator $\Gamma^T_{a,\omega}$ by

$$\begin{array}{rrcl}\Gamma^T_{a,\omega}: & H^1_0(0,L)\times L^2(0,L) & \longrightarrow & L^2((0,T)\times(0,L))\\ & (y^0,y^1) & \longmapsto & h^{opt}_{a,\omega}.\end{array}$$

$\Gamma^T_{a,\omega}$ is linear and continuous and we define its norm

$$\|\Gamma^T_{a,\omega}\|=\sup\left\{\frac{\|h_{a,\omega}\|_{L^2((0,T)\times(0,L)}}{\|(y^0,y^1)\|_{L^2(0,L)\times H^{-1}(0,L)}}\mid (y^0,y^1)\in L^2(0,L)\times H^{-1}(0,L)\setminus\{(0,0)\}\right\},$$

[5] Note that, according to [8, Section 2.4.3], one has

$$E_a(0)=\int_0^L\left(\varphi_1(x)^2+\varphi_0'(x)^2+a(x)\varphi_0(x)^2\right)\,dx$$

which is called *the cost of the control at time* T (because it measures the minimal energy needed to bring an initial condition to $(0,0)$). Using a standard duality argument, it can be showed that (74) is null controllable if and only if (71) is observable, and in this case the cost of the control is

$$\|\Gamma_{a,\omega}^T\|^2 = C_{T,\text{obs}}(a,\omega)^{-1},$$

with $C_{T,\text{obs}}(\omega)^{-1}$ the optimal constant in the observability inequality (72), defined by (73).

The dependence of $C_{T,\text{obs}}(a,\omega)^{-1}$ with respect to different parameters (the observability time T, the potential a, the observability set ω) has been studied by many authors (see [41], where an application to the controllability of semilinear wave equations is given, [7] for some results in the multi-dimensional case obtained thanks to Carleman estimates and [12] for precise lower bounds obtained through different methods) but its exact behavior is not known.

In the following result, one provides several estimates of $C_{T,\text{obs}}(a,\omega)$ (and then $\|\Gamma_{a,\omega}^T\|$) and constitutes another justification of the interest of the problems introduced in Section 1.2, in particular of the issue of obtaining a lower bound estimate of the quantity $J(a,\omega)$.

Theorem 3. *Let $L>0$ and let a be a nonnegative function in $L^\infty(0,L)$.*

i There holds

$$C_{T,\text{obs}}(a,\omega) \sim \frac{T}{2} J(a,\omega) \qquad \text{as} \qquad T\to+\infty.$$

ii Let $a\in\mathcal{A}_M(0,L)$ with $M<3\pi^2/L^2$ and define $T(M)=\frac{2\pi}{\gamma_M}$ with $\gamma_M=\frac{\frac{3\pi^2}{L^2}-M}{\frac{2\pi}{L}+\sqrt{\frac{\pi^2}{L^2}+M}}$. For all $T>T(M)$, there holds

$$0<\frac{c_1(T,\gamma_M)}{2} \leqslant \frac{C_{T,\text{obs}}(a,\omega)}{J(a,\omega)} \leqslant \frac{T}{2}$$

with $c_1(T,\gamma_M)=\frac{2}{\pi}\left(T-\frac{4\pi^2}{\gamma_M^2 T}\right)$.

The proof of this theorem is postponed to Section C. Combining Theorem 3 with Theorem 1, we infer the following result.

Corollary 1. *Let $a\in\mathcal{A}_M(0,L)$ with $M\leqslant\pi^2/L^2$ and define $T(M)=\frac{2\pi}{\gamma_M}$ with $\gamma_M=\frac{\frac{3\pi^2}{L^2}-M}{\frac{2\pi}{L}+\sqrt{\frac{\pi^2}{L^2}+M}}$. For all $T>T(M)$, there holds*

$$\frac{7\sqrt{3}}{16L^3}(3-2\sqrt{2})c_1(T,\gamma_M)\min(|\omega|,\underline{r}_2)^3 \leqslant C_{T,\text{obs}}(a,\omega),$$

with $\underline{r}_2=\frac{\sqrt{5}}{5}+\frac{\sqrt{10}}{10}$ and $c_1(T,\gamma_M)=\frac{2}{\pi}\left(T-\frac{4\pi^2}{\gamma_M^2 T}\right)$.

4.1.2 Decay rate for a damped wave equation

From the estimates of the observability constant $C_{T,\text{obs}}(a,\omega)$, we can also deduce estimates of the rate at which energy decays in a damped string. Consider the damped wave equation on $(0,\pi)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_{tt}y(t,x)-\partial_{xx}y(t,x)+a(x)y(t,x)+2k\chi_\omega(x)\partial_t y(t,x)=0, & (t,x)\in(0,T)\times(0,\pi),\\ y(t,0)=y(t,\pi)=0, & t\in[0,T],\\ (y(0,x),\partial_t y(0,x))=(y^0(x),y^1(x)), & x\in(0,\pi), \end{cases} \tag{75}$$

with $k > 0$. Recall that for all initial data $(y_0, y_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$, the problem (75) is well posed and its solution y belongs to $C^0(0,T;H_0^1(0,\pi)) \cap C^1(0,T;L^2(0,\pi))$.

The energy associated to System (75) is defined by

$$E_{a,\omega}(t) = \int_0^\pi \left(\partial_t y(t,x)^2 + \partial_x y(t,x)^2 + a(x)y(t,x)^2\right)\, dx.$$

According to Theorem 3 and to [12, Section 3.3], by using the same notations as in the statement of Theorem 3, if ω is a measurable subset of $(0,\pi)$, $a \in \mathcal{A}_M(0,\pi)$ with $M < 3$, there holds for every $(y_0, y_1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and $t \geqslant 2T(M)$,

$$E_{a,\omega}(t) \leqslant E_{a,\omega}(0)e^{-\delta(a,\omega)t},$$

with

$$\ln\left(\frac{1+(1+T(M)^2)c_1(T,\gamma_M)J(a,\omega)}{(1+T(M)^2)c_1(T,\gamma_M)J(a,\omega)}\right) \leqslant 2T(M)\delta(a,\omega) \leqslant \ln\left(\frac{1+(1+T(M)^2)c_2(T,\gamma_M)J(a,\omega)}{(1+T(M)^2)c_2(T,\gamma_M)J(a,\omega)}\right),$$

where $c_1(T,\gamma_M) = \frac{2}{\pi}\left(T - \frac{4\pi^2}{\gamma_M^2 T}\right)$ and $c_2(T,\gamma_M) = \frac{10T}{\pi}$.

Notice that a close problem has been investigated in [6] in the very case where $a(\cdot) = 0$ and with a general positive damping term. The authors provide a simple expression of the decay rate in the case where the damping term is bounded, and an explicit lower bound on the decay rate in the general case.

Let us also mention the related works [14, 26, 30] where the authors aim at determining either the damping term or the shape and location of its support in order to stabilize the more efficiently the damped wave equation. However, our result is of different nature since we provide explicit upper and lower bound of the decay rate for any mesurable set and any small enough potentials.

4.2 Numerical investigations

In what follows, we will consider for the sake of clarity that $L = \pi$ according to Lemma 1, and two given numbers $r \in (0,1)$ and $M \in (0,1]$. As pointed out in Remark 5, the study reduces to determining a finite number of switching points. A difficulty of this approach is to deal with the fact that no upper bound of the optimal index j_0^* introduced in Theorem 1 is known. For this reason, we have adopted the following numerical strategy, using the real numbers $m_j(L,M,r)$ (already used in the proof of Theorem 1), defined for $j \in \mathbb{N}^*$ by

$$m_j(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} \int_\omega e_{a,j}(x)^2\, dx.$$

For $N \in \mathbb{N}^*$, we also introduce the "truncated criterion"

$$m^N(L,M,r) = \inf_{a\in\mathcal{A}_M(0,L)} \inf_{\omega\in\Omega_r(0,L)} J_N(a,\omega) \quad \text{with} \quad J_N(a,\omega) = \inf_{1\leqslant j\leqslant N} \int_\omega e_{a,j}^2(x)\, dx.$$

It follows easily from Theorem 1 that the sequence $(m^N(L,M,r))_{N\in\mathbb{N}^*}$ is non-increasing, stationary and converges to $m(L,M,r)$ as $N \to +\infty$.

In the numerical procedure below, we will say that the sequence $(m^N(L,M,r))_{N\in\mathbb{N}^*}$ satisfies the stationarity property if this sequence takes equal values for at least N_0 consecutive indices, where N_0 is a fixed nonzero integer.

Numerical solving of Problem ($\mathcal{P}_{L,r,M}$)

Let $L>0$, $M\in(0,\pi^2/L^2]$, $r\in(0,1)$, $N_0\in\mathbb{N}^*$.
For $j=1,\dots,N$,

i compute the real number $m_j(L,M,r)$ (by solving a finite dimensional optimization problem, see the explanations below);

ii compute the real number $m^N(L,M,r)$.

Stop at the first integer N such that $(m^N(L,M,r))_{N\in\mathbb{N}^*}$ satisfies the stationarity property.

Let us provide some explanations about the first step of the algorithm.

Let $j\in\mathbb{N}\backslash\{0,1\}$ and $M\in(0,1]$. We fix $o_0=0$ and $o_{3j}=\pi$. According to Theorem 1, we reduce the computation of $m_j(L,M,r)$ to the resolution of a $(3j-1)$-dimensional optimization problem. More precisely, we are led to minimize the function

$$(0,\pi)^{3j-1}\ni o=(o_1,\cdots,o_{3j-1})\longmapsto \inf_{\substack{\omega\subset(0,\pi)\\ \text{s.t. } |\omega|=r\pi}}\int_\omega e_{a_o,1}(x)^2\,dx,$$

where $a_o(\cdot)$ denotes the potential function defined on $(0,\pi)$ by

$$a_o(x)=\left\{\begin{array}{ll} M & \text{on } (o_i,o_{i+1}),\\ 0 & \text{on } (o_{i+1},o_{i+2}),\end{array}\right.$$

for every even integer $i\in\{0,\cdots,3j-2\}$ and every $x\in(o_i,o_{i+2})$. Notice that, when $3j-2$ is a odd number then $a(x)=M$ on (o_{3j-1},o_{3j}).

Thus, given the switching points $o\in(0,\pi)^{3j-1}$, one computes the eigenfunction $e_{a_o,j}(\cdot)$ by using a shooting method combined with a Runge-Kutta method. The eigenvalue λ is determined by solving $e_{a,\lambda}(\pi)=0$ with a Newton method.

According to Proposition 1, the set ω coincides with $\{e_{a,j}^2\leqslant\tau\}$ for some parameter τ chosen in such a way that $|\omega|=r\pi$. We are then driven to find an estimate of τ, which is done by computing the decreasing rearrangement $\left(e_{a,j}^2\right)^*$ of $e_{a,j}^2$ (see, e.g., [16, 22, 39]) and using that $\tau=\left(e_{a,j}^2\right)^*(r\pi)$.

These considerations allow to rewrite the cost functional as a function of $(3j-1)$ variables. The resulting finite-dimensional problem is then solved numerically by using a Nelder-Mead simplex search method on a standard desktop machine, which provides a global minimizer.

We present below some numerical simulations to compute the numbers $m_j(L,M,r)$. In what follows and when no confusion is possible, we will simply denote by $a(\cdot)$ the optimal potential associated to $m_j(L,M,r)$.

4.2.1 Computation of $m_1(L,M,r)$

Let $M\in(0,1]$. According to Theorem 1, there exist at most two switching points in $(0,\pi)$ denoted o_1 and o_2 such that

$$0\leqslant o_1\leqslant o_2\leqslant\pi \qquad\text{and}\qquad a(x)=M\chi_{(0,o_1)\cup(o_2,\pi)}. \tag{76}$$

Note that, if $o_1=o_2$, there is only one switching point. Therefore, the issue of determining the optimal potential $a(\cdot)$ comes to minimize the function $(0,\pi)^2\ni(o_1,o_2)\mapsto\inf_{\substack{\omega\subset(0,\pi)\\ \text{s.t. } |\omega|=r\pi}}\int_\omega e_{a,1}(x)^2\,dx$, where $a(\cdot)$ is given by (76). Fixing $\tau_a = \sqrt{\lambda_{a,1}^2 - M}$, one computes

$$e_{a,1}(x) = \begin{cases} \sin(\tau_a x) & x \in (0, o_1), \\ \sin(\tau_a o_1)\cos(\lambda_{a,1}(x - o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(x - o_1)) & x \in (o_1, o_2), \\ \frac{\sin(\tau_a o_1)\cos(\lambda_{a,1}(o_2 - o_1)) + \frac{\tau_a}{\lambda_{a,1}}\cos(\tau_a o_1)\sin(\lambda_{a,1}(o_2 - o_1))}{\sin(\tau_a(\pi - o_2))}\sin(\tau_a(\pi - x)) & x \in (o_2, \pi), \end{cases}$$

up to a multiplicative normalization constant, where the eigenvalue $\lambda_{a,1}$ solves the transcendental equation

$$\lambda_{a,1}\tau_a \frac{\tan(\tau_a(\pi - o_2 + o_1))}{\tan(\lambda_{a,1}(o_2 - o_1))} - \tau_a^2 = M \frac{\sin(\tau_a(\pi - o_2))\sin(\tau_a o_1)}{\cos(\tau_a(\pi - o_2 + o_1))}.$$

This last equation is solved numerically by using a Newton method. Since $M \in (0, 1]$, the eigenfunction $e_{a,1}$ is concave on $(0, \pi)$. As a consequence, the optimal set ω, as level set of the function $e_{a,1}^2$, writes $\omega = (0, \alpha) \cup (\beta, \pi)$ with $\alpha < \beta$, according to Proposition 1. In that case, α and β are determined with the help of a Newton method, using that $\beta = (1-r)\pi + \alpha$ and $e_{a,1}(\alpha)^2 = e_{a,1}(\beta)^2$. The numerical results are gathered on Fig. 5.

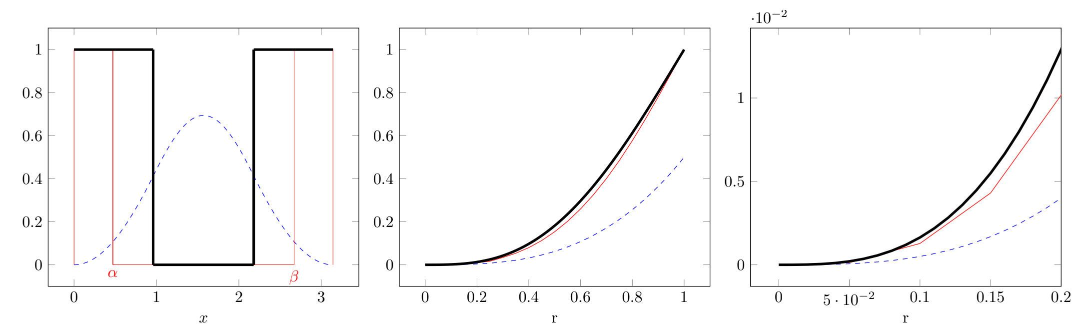

Figure 5: Left: $L = \pi$ and $M = 1$. Plots of the optimal set $\omega_1^*(-)$, $a_1^*(\text{-})$ and $e^2_{a_1^*,1}$ $(\dots)$ w.r.t. the space variable with $r = 0.3$. Middle and right: $L = \pi$ and $M = 1$. Comparison of the numerical results with the bounds obtained in Theorem 1: plots of $r \mapsto m_1(\pi, 1, r)$ (—), $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ (——) and $r \mapsto r^3/2$ $(- - -)$.

4.2.2 Computation of $m_j(L, M, r)$ for $j \geqslant 2$.

Figures 6 and 7 illustrate the cases $j = 2, 3, 4$. The parameters r and M are running over the interval $[0, 1]$. On Figure 6, the optimal value of the criterion (w.r.t. r), obtained by using a Nelder-Mead simplex search method, is compared to the estimate obtained in Theorem 4 for the parameter values $j \in \{2, 3, 4\}$. Recall that the numbers $\underline{m}_j$ are defined in Proposition 4. On Figure 7, the graph of the optimal value with respect to r is plotted for the parameter values $j \in \{1, 2, 6\}$. Notice that the mapping $j \mapsto m_j(L, M, r)$ seems to be increasing, although we did not manage to prove it. This seems to indicate that the optimal index j_0 introduced in Theorem 1 is equal to 1.

5 Concluding remarks

In this article, we have investigated the optimization problem $(\mathcal{P}_{L,r,M})$ which allows to provide a quantitative estimate of the "non-localization" property of Sturm-Liouville eigenfunctions related to the operator A_a defined by (1).

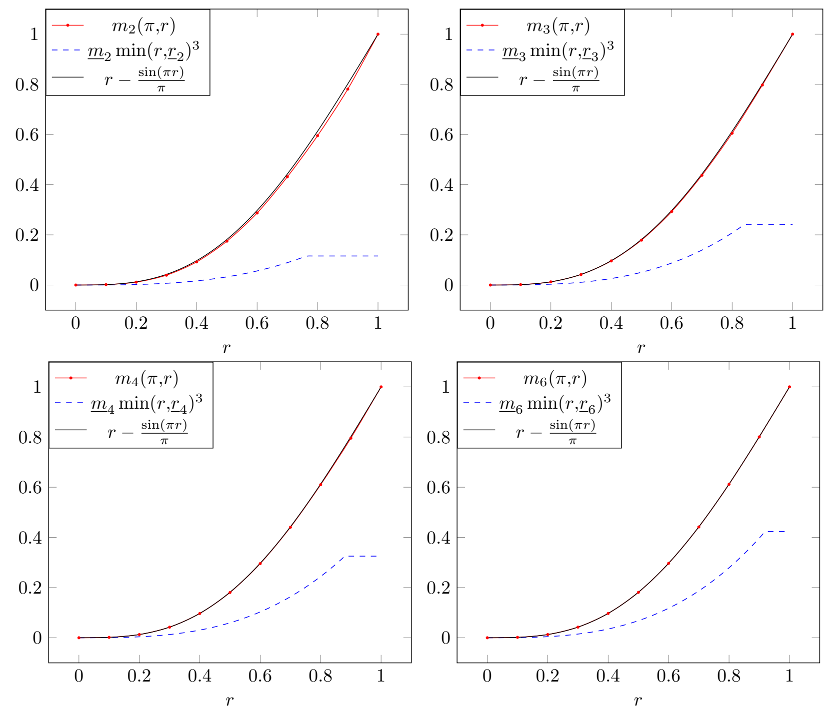

Figure 6: $L = \pi$ and $M = 1$. Plots of m_j $(-o-)$, $\underline{m}_j \min(r, \underline{r}_j)^3$ $(\cdots)$ and $r \mapsto r - \frac{\sin(\pi r)}{\pi}$ $(-)$ with respect to r for $j \in \{2, 3, 4\}$.

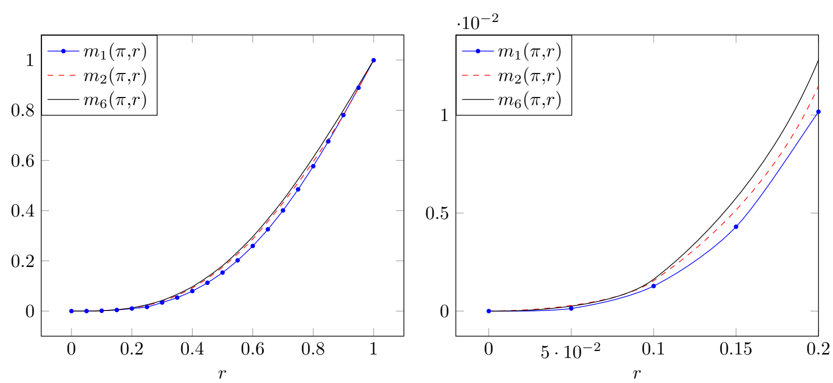

Figure 7: $L = \pi$ and $M = 1$. Plots of $m_j(\pi, 1, r)$ for $j = 1$(o), $j = 2$(- -) and $j = 6$ with respect to r.

We have showed that it is relevant to consider potential functions $a(\cdot)$ whose essential-supremum is uniformly bounded by a positive constant M, and we have provided sharp estimates of the optimal value under smallness assumptions on the parameter M. It is notable that our estimates only depend in that case on the measure of the observation subset ω.

Several issues remain open. Let us mention two of them:

- the investigation of the same problem for larger values of M (characterization of minimizers, sharp estimate of the optimal value). Indeed, our approach was based on particular properties of eigenfunctions holding only whenever M is small enough. Obtaining new estimates would require to develop a new approach.

- the development of efficient numerical methods to solve $(\mathcal{P}_{L,r,M})$. On Fig. 8, we have plotted the quantity $m_j(\pi, M, r)$, $j = 1, 2, 3, 5$ with respect to the parameter r, for several values of M greater than the critical value $M = 1$. These simulations drive us to formulate, as previously, the conjecture that the optimal index is $j_0 = 1$. Note that the computation of these quantities need to solve optimization problems for which the objective function enjoys plenty of local minimizers. This is why we chose to solve this problem with the help of a genetic algorithm, quite efficient but very costly in terms of computing time, even for small values of j.

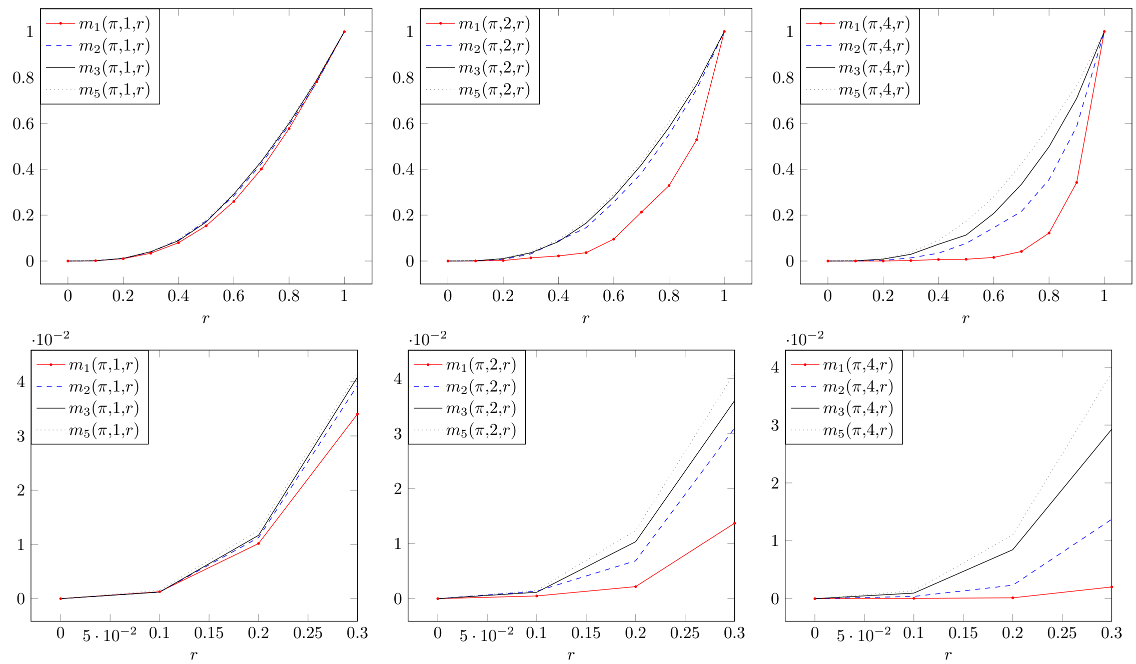

Figure 8: $L = \pi$. (Top) Plots of $m_j(\pi, M, r)$ w.r.t. r for $M = 1, 2, 4$. (Bottom) Zoom on the previous plots around $r = 0$.

Appendix

A Proof of Lemma 4

Let us define $\phi_j = \frac{e_{a_j,j}(\cdot)}{e'_{a_j,j}(0)}$. The function ϕ_j solves the Cauchy system

$$\begin{cases} -\phi_j''(x) + a_j(x)\phi_j(x) = \lambda_{a,j}^2 \phi_j(x), & x \in (0, L), \\ \phi_j(0) = 0, \quad \phi_j'(0) = 1. \end{cases}$$

Let us notice that, according to the Courant-Fischer minimax principle, there holds $\lambda_{a_j,j} \geqslant \frac{\pi}{L}$ for every $j \in \mathbb{N}^*$ and $\lim_{j\to+\infty} \lambda_{a_j,j} = +\infty$. According to [32, Chapter 1, Theorem 3] and using a rescaling argument, we infer $\phi_j(x) = \frac{\sin(\lambda_{a_j,j}x)}{\lambda_{a_j,j}} + \mathrm{O}\left(\frac{1}{\lambda^2_{a_j,j}}\right)$. As a consequence, there holds $\phi_j^2(x) = \frac{\sin^2(\lambda_{a_j,j}x)}{\lambda^2_{a_j,j}} + \mathrm{O}\left(\frac{1}{\lambda^3_{a_j,j}}\right)$, where the remainder term does not depend on x. Therefore, using the Riemann-Lebesgue lemma, one gets that $\int_0^L \phi_j^2(x)dx = \frac{L}{2\lambda^2_{a,j}} + o\left(\frac{1}{\lambda^2_{a,j}}\right)$, and since $e_{a_j,j} = \frac{\phi_j}{\|\phi_j\|_2}$, the combination of two last equalities yields

$$e^2_{a_j,j}(x) = \frac{2}{L}\sin^2(\lambda_{a_j,j}x) + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right). \tag{77}$$

Let $\varphi \in L^1(0,L)$. Using (77), one shows that

$$\int_0^L e_{a_j,j}(x)^2\varphi(x)\,dx = \frac{2}{L}\int_\alpha^\beta \sin^2(\lambda_{a_j,j}x)\varphi(x)\,dx + \mathrm{O}\left(\frac{1}{\lambda_{a_j,j}}\right).$$

The expected result follows by linearizing $\sin^2(\lambda_{a_j,j}x)$ and using the Riemann-Lebesgue lemma. ■

B Proofs of Lemmas 8 and 10

The proofs are based on the following Lemma.

Lemma 11. *Let $j \in \mathbb{N}^*$ and $i_0 \in \mathbb{N}^*$ such that $i_0 \leqslant j-1$. Define*

$$g(i_0,j) = \frac{j-i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}}\left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1-\left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}}\right).$$

There holds

$$g(i_0,j) \geqslant \underline{r}_j := \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right).$$

Proof. Let $\gamma : \mathbb{N}\backslash\{0,1\} \ni j \mapsto \left(\frac{j^2}{j^2-2}\right)$. Notice that $\gamma(j) \in (1,2)$ for every $j \in \mathbb{N}\backslash\{0,1\}$. The derivative of g with respect to i_0 writes

$$\partial_{i_0} g(i_0,j) = \frac{-\left(\frac{1}{\gamma(j)}\right)^{\frac{i_0}{2}}\ln(\frac{1}{\gamma(j)}) + 2(\sqrt{\gamma(j)}-1)}{2\sqrt{j^2+1}\left(\sqrt{\gamma(j)}-1\right)} \leqslant 0,$$

and therefore

$$g(i_0,j) \geqslant g(j-1,j) = \frac{\sqrt{\gamma(j)}(\gamma(j)^{\frac{j}{2}}-1)}{\sqrt{j^2+1}(\sqrt{\gamma(j)}-1)\gamma(j)^{\frac{j}{2}}}.$$

Straightforward computations show that

$$\frac{\sqrt{\gamma(j)}(\gamma(j)^{\frac{j}{2}}-1)}{\sqrt{j^2+1}(\sqrt{\gamma(j)}-1)\gamma(j)^{\frac{j}{2}}} = \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right),$$

which concludes the proof. □

Proof of Lemma 8. In the sequel we will use the notations in (46), the definition of A_i in (53) and the definition of Δ_j in (54).

From (53) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$, we notice that

$$\frac{A_{i+1}}{A_i} = \frac{(1-\eta_i^2)\eta_{i+1}^2}{(1+\eta_{i+1}^2)\eta_i^2} \leqslant 1$$

Let $i_0 \in \{1, \cdots, j-1\}$ such that for every $i \leqslant i_0 - 1$, $\tau < A_i$ and for every $i \geqslant i_0$, $\tau \geqslant A_i$ (see Figure **??** with $j = 4$ and $i_0 = 2$). By definition of $\hat{\omega}$, one has

$$|\hat{\omega}| = |\{\Delta_j(x)^2 < \tau\}| \geqslant |\hat{\omega}_{i_0}|$$

with $\hat{\omega}_{i_0} = \{\Delta_j(\cdot)^2 \leqslant A_{i_0}\}$. Let us denote by $\bar{r}_{i_0} \in (0,1)$ the real number defined by $\bar{r}_{i_0} = \frac{|\{\Delta_j(\cdot)^2 \leqslant A_{i_0}\}|}{L}$. Note that there holds obviously $r \geqslant \bar{r}_{i_0}$.

Let us now find a lower bound estimate of $\bar{r}_{i_0}$. Let $i \in \{0, \cdots, j-1\}$ and let $\omega_i^{i_0} = \hat{\omega}_{i_0} \cap (x_j^i, x_j^{i+1})$. There exist $\alpha_i^{i_0}$, $\beta_i^{i_0}$ and $r_i \in [0,1]$ such that $\omega_i^{i_0} = (x_j^i, \alpha_i^{i_0}) \cup (\beta_i^{i_0}, x_j^{i+1})$, $|\omega_i^{i_0}| = r_i(x_j^{i+1} - x_j^i)$, and therefore $\sum_{i=0}^{j-1} r_i(x_j^{i+1} - x_j^i) = \bar{r}_{i_0} L$.

By definition of $\hat{\omega}_{i_0}$ one has $\Delta_j(\alpha_i^{i_0}) = \Delta_j(\beta_i^{i_0}) = \sqrt{A_{i_0}}$, for every $i \leqslant i_0 - 1$ and $\hat{\omega}_i^{i_0} = (x_j^i, x_j^{i+1})$ for every $i \geqslant i_0$. Consequently there holds

$$\alpha_i^{i_0} = \sqrt{\frac{A_{i_0}}{A_i}}(x_{max}^i - x_j^i) + x_j^i \quad , \quad \beta_i^{i_0} = x_j^{i+1} - \sqrt{\frac{A_{i_0}}{A_i}}(x_j^{i+1} - x_{max}^i).$$

for all $i \leqslant i_0 - 1$. As a result, one gets

$$\bar{r}_{i_0} L \;=\; \textstyle\sum_{i=0}^{i_0-1}(\alpha_i^{i_0} - x_j^i + x_j^{i+1} - \beta_i^{i_0}) + (L - x_{i_0}) = \sum_{i=0}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}}(x_j^{i+1} - x_j^i) + (L - x_{i_0}).$$

Dividing by L, we have

$$\bar{r}_{i_0} = \sum_{i=0}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}}\eta_i + \sum_{i=i_0}^{j-1} \eta_i, \tag{78}$$

where the numbers η_i are defined by (46).

This last expression also rewrites

$$\bar{r}_{i_0} = \eta_{i_0} \sum_{i=0}^{i_0-1} \sqrt{\prod_{k=i+1}^{i_0} \left(\frac{1-\eta_{k-1}^2}{1+\eta_k^2}\right)} + \sum_{i=i_0}^{j-1} \eta_i,$$

in terms of the real numbers η_i. Since $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$ for every $j \in \mathbb{N}^*$ and $i \in \{0, \cdots, j-1\}$, one infers that

$$\bar{r}_{i_0} \;\geqslant\; \eta_{i_0} \textstyle\sum_{i=0}^{i_0-1} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}} + \sum_{i=i_0}^{j-1} \eta_i \geqslant g(i_0, j),$$

with

$$g : (i_0, j) \mapsto \frac{j - i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}} \left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1 - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}} \right)$$

By Lemma 11, we conclude that for every $i_0 \geqslant 1$, $r \geqslant \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$. As a result, if $r < \left(\frac{j+\sqrt{j^2-2}}{2\sqrt{j^2+1}}\right)\left(j - \frac{(j^2-2)^{\frac{j}{2}}}{j^{j-1}}\right)$, one has $\tau \leqslant A_i$ for every $i \in \{0, \cdots, j-1\}$.

Proof of Lemma 10. In the sequel we will use the notations in (46) and the definition of Δ_j in (54) where A_1 is defined in (68), A_0 is defined in (69) and for all $i \in \{2, \cdots, j-1\}$ A_i is defined in (70) .

Let $i_0 \in \{1, \cdots, j-1\}$ and let us introduce the sets

$$I_k = \{i \in \{0, \cdots, j-1\} \text{ such that } A_i > A_k\} \quad \text{and} \quad J_k = \{i \in \{0, \cdots, j-1\} \text{ such that } A_i \leqslant A_k\}.$$

From (68), (69), (70) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$, we notice that

$$\frac{A_{i+1}}{A_i} = \frac{(1-\eta_i^2)\eta_{i+1}^2}{(1+\eta_{i+1}^2)\eta_i^2} \leqslant 1, \quad \text{for all} \quad i \in \{1, \cdots, j-1\} \quad \text{and} \quad \frac{A_0}{A_1} \leqslant 1. \tag{79}$$

<u>*Case 1:* $0 \in J_{i_0}$.</u> Thanks to (79), one has $J_{i_0} = \{0, i_0+1, \cdots, j-1\}$. By mimicking the proof of Lemma 8 (notably the equality (78)), we have

$$\begin{aligned} r &= \sum_{i=1}^{i_0} \sqrt{\frac{A_{i_0+1}}{A_i}}\eta_i + \sum_{i \in J_{i_0}} \eta_i \geqslant \eta_{i_0} \sum_{i=1}^{i_0} \sqrt{\prod_{k=i+1}^{i_0+1} \left(\frac{1-\eta_{k-1}^2}{1+\eta_k^2}\right)} + \sum_{i=i_0+1}^{j-1} \eta_i + \eta_0, \\ &\geqslant \eta_{i_0} \sum_{i=1}^{i_0} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0+1-i}{2}} + \sum_{i=i_0+1}^{j-1} \eta_i + \eta_0 \geqslant g(i_0, j), \end{aligned}$$

with $g : (i_0, j) \mapsto \frac{j-i_0}{\sqrt{j^2+1}} + \frac{1}{\sqrt{j^2+1}} \left(\frac{\left(\frac{j^2-2}{j^2}\right)^{\frac{i_0}{2}} - 1}{1 - \left(\frac{j^2}{j^2-2}\right)^{\frac{1}{2}}} \right)$, which leads to the desired conclusion using Lemma 11.

<u>*Case 2:* $0 \in I_{i_0}$.</u> Thanks to (79), one has $J_{i_0} = \{i_0, \cdots, j-1\}$. By mimicking the proof of Lemma 8, we have

$$r_{i_0} = \sqrt{\frac{A_{i_0}}{A_0}}\eta_0 + \sum_{i=1}^{i_0-1} \sqrt{\frac{A_{i_0}}{A_i}}\eta_i + \sum_{i=i_0}^{j-1} \eta_i$$

Thanks to (69), (70) and $\frac{1}{\sqrt{j^2+1}} \leqslant \eta_i \leqslant \frac{1}{\sqrt{j^2-1}}$,

$$\sqrt{\frac{A_{i_0}}{A_0}}\eta_0 \geqslant \eta_{i_0} \sqrt{\frac{(j^2+2)(j^2-1)}{j^2(j^2+1)}} \left(\frac{j^2}{j^2-2}\right)^{\frac{i_0-2}{2}} \qquad \text{and} \qquad \sqrt{\frac{A_{i_0}}{A_i}}\eta_i \geqslant \eta_{i_0} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}}.$$

Since the inequality $\frac{(j^2+2)(j^2-1)}{j^2(j^2+1)} \geqslant \left(\frac{j^2}{j^2-2}\right)^2$ is true for every $j \geqslant 1$, we thus infer

$$\bar{r}_{i_0} \geqslant \eta_{i_0} \sum_{i=0}^{i_0-1} \left(\frac{j^2-2}{j^2}\right)^{\frac{i_0-i}{2}} + \sum_{i=i_0}^{j-1} \eta_i \geqslant g(i_0, j).$$

By using Lemma 11, we conclude the proof. ∎

C Proof of Theorem 3

Before proving this theorem, let us recall some basic facts on Ingham's inequality (see [17]), an inequality for nonharmonic Fourier series much used in control theory.

Proposition 5. *For every $\gamma > 0$ and every $T > \frac{2\pi}{\gamma}$, there exist two positive constants $C_1(T,\gamma)$ and $C_2(T,\gamma)$ such that for every sequence of real numbers $(\mu_n)_{n\in\mathbb{N}^*}$ satisfying*

$$\forall n \in \mathbb{N}^* \quad |\mu_{n+1} - \mu_n| \geqslant \gamma, \tag{80}$$

there holds

$$C_1(T,\gamma) \sum_{n\in\mathbb{Z}^*} |a_n|^2 \leqslant \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\mu_n t} \right|^2 dt \leqslant C_2(T,\gamma) \sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{81}$$

for every $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$.

Denoting by $C_1(T,\gamma)$ and $C_2(T,\gamma)$ the optimal constants in (81), several explicit estimates of these constants are provided in [17]. For example, it is proved in the article cited above that

$$C_1(T,\gamma) \geqslant 2\left(\frac{T}{\pi} - \frac{4\pi}{\gamma^2 T}\right) \quad \text{and} \quad C_2(T,\gamma) \leqslant \frac{10T}{\pi}.$$

The idea to use Ingham inequalities in control theory is a long story (see for instance [2, 11, 18, 19, 23]).

Notice that, up to our knowledge, the best constants in [17] are not known. In the particular case where $\mu_n = \pi n/L$ for every $n \in \mathbb{N}^*$, one shows easily that for every $T > 2L$, $C_1(T,\gamma) = 2\pi \left\lfloor \frac{T}{2\pi} \right\rfloor$ and $C_2(T,\gamma) = C_1(T,\gamma) + 1$, the bracket notation standing for the integer floor.

The following result on the asymptotic as $T \to +\infty$ of optimal constants Ingham's inequalities will be a crucial tool to prove Theorem 3.

Proposition 6. *Assume that the sequence $(\mu_n)_{n\in\mathbb{N}^*}$ satisfies (80). Then, there holds*

$$\lim_{T\to+\infty} \frac{C_1(T,\gamma)}{T} = \lim_{T\to+\infty} \frac{C_2(T,\gamma)}{T} = 1.$$

Proof of Proposition 6. Let $(a_n)_{n\in\mathbb{N}^*} \in \ell^2(\mathbb{C})$ be such that $\|a\|_{\ell^2} = 1$. Introduce the quantity

$$Q_T(a,\mu) = \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{i\mu_n t} \right|^2 dt.$$

We write

$$\begin{aligned} Q_T(a,\mu) &= \int_0^T \sum_{n\in\mathbb{Z}^*} |a_n|^2 \, dt + \int_0^T \sum_{n\neq m} a_n \bar{a}_m e^{i(\mu_n-\mu_m)t} \, dt \\ &= T - i \sum_{n\neq m} \frac{a_n \bar{a}_m (e^{i(\mu_n-\mu_m)T} - 1)}{\mu_n - \mu_m} = T - i \sum_{n\neq m} \frac{b_n \bar{b}_m}{\mu_n - \mu_m} + i \sum_{n\neq m} \frac{a_n \bar{a}_m}{\mu_n - \mu_m}, \end{aligned}$$

with $b_n = a_n e^{i\mu_n T}$ for every $n \in \mathbb{Z}^*$.

According to [29, Theorem 2], one has

$$\left| \sum_{n\neq m} \frac{b_n \bar{b}_m}{\mu_n - \mu_m} \right| \leqslant \frac{\pi}{\gamma} \|a\|_{\ell^2}^2 \quad \text{and} \quad \left| \sum_{n\neq m} \frac{a_n \bar{a}_m}{\mu_n - \mu_m} \right| \leqslant \frac{\pi}{\gamma} \|a\|_{\ell^2}^2,$$

where $\gamma = \inf_{n\in\mathbb{Z}^*} \mu_{n+1} - \mu_n$. Then, it follows that

$$1 - \frac{\pi}{\gamma T} \leqslant \frac{1}{T} \inf_{a\in\ell^2 | \|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant \frac{1}{T} \sup_{a\in\ell^2 | \|a\|_{\ell^2}=1} Q_T(a,\mu) \leqslant 1 + \frac{\pi}{\gamma T},$$

whence the result. ∎

Decomposing the solution φ of (71) in the spectral basis $\{e_{a,j}\}_{j\in\mathbb{N}^*}$ allows to write that

$$\varphi(t,x)=\sum_{j=1}^{+\infty}\left(\alpha_j\cos(\lambda_{a,j}t)+\beta_j\sin(\lambda_{a,j}t)\right)e_{a,j}(x), \tag{82}$$

where

$$\alpha_j=\int_0^L\varphi_0(x)e_{a,j}(x)\,dx,\qquad \beta_j=\frac{1}{\lambda_{a,j}}\int_0^L\varphi_1(x)e_{a,j}(x)\,dx, \tag{83}$$

for every $j\in\mathbb{N}^*$. We are now ready to prove Theorem 3.

Proof of Theorem 3. Introduce the spectral gap $\gamma=\inf_{j\in\mathbb{N}^*}\lambda_{a,j+1}-\lambda_{a,j}$. It is well-known that $\gamma>0$ for every $a\in L^\infty(0,L)$. Let us first prove point (i). Using (81), one has for $T\geqslant 2\pi/\gamma$,

$$\begin{aligned}\int_0^T\int_\omega|\partial_t\varphi(t,x)|^2\,dxdt &= \frac14\int_0^T\int_\omega\left|\sum_{k\in\mathbb{Z}^*}i\operatorname{sgn}(k)\lambda_{a,|k|}\sqrt{\alpha_{|k|}^2+\beta_{|k|}^2}e^{i\operatorname{sgn}(k)(\lambda_{a,|k|}t-\theta_{|k|})}e_{a,|k|}(x)\right|^2dxdt\\ &\geqslant \frac{C_1(T,\gamma)}{4}\sum_{k\in\mathbb{Z}^*}(\alpha_{|k|}^2+\beta_{|k|}^2)\lambda_{a,|k|}^2\int_\omega e_{a,|k|}(x)^2\,dx\\ &= \frac{C_1(T,\gamma)}{2}\sum_{j=1}^{+\infty}(\alpha_j^2+\beta_j^2)\lambda_{a,j}^2\int_\omega e_{a,j}(x)^2\,dx,\end{aligned} \tag{84}$$

where $(\theta_j)_{j\in\mathbb{N}^*}$ denotes the sequence defined by $e^{i\theta_j}=\frac{\alpha_j+i\beta_j}{\sqrt{\alpha_j^2+\beta_j^2}}$ for every $j\in\mathbb{N}^*$. Combining the energy identity

$$\int_0^L\left(\varphi_t^2(t,x)+\varphi_x^2(t,x)+a(x)\varphi^2(t,x)\right)\,dx=\sum_{j=1}^{+\infty}\lambda_{a,j}^2(\alpha_j^2+\beta_j^2), \tag{85}$$

with (73), (82), (83) and (85), one gets

$$C_{T,obs}(\omega)=\inf_{(\lambda_{a,j}\alpha_j,\lambda_{a,j}\beta_j)\in l^2(\mathbb{R})^2}\frac{\int_0^T\int_\omega|\sum_{j=1}^{+\infty}\left(-\lambda_{a,j}\alpha_j\sin(\lambda_{a,j}t)+\lambda_{a,j}\beta_j\cos(\lambda_{a,j}t)\right)e_{a,j}(x)|^2\,dxdt}{\sum_{j=1}^{+\infty}\lambda_{a,j}^2(\alpha_j^2+\beta_j^2)}, \tag{86}$$

According to (84), it follows that

$$C_{T,obs}(\omega)\geqslant\frac{C_1(T,\gamma)}{2}\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx. \tag{87}$$

Taking now $\alpha_j=(\delta_{kj})_{k\in\mathbb{N}^*}$ and $\beta_j=(\delta_{k'j})_{k'\in\mathbb{N}^*}$ in (86) where δ_{kj} denotes the Kronecker delta, we obtain

$$C_{T,obs}(\omega)\leqslant\frac{T}{2}\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx. \tag{88}$$

Combining (87), (88) with the asymptotic of optimal constant $C_1(T,\gamma)$ in Ingham's inequalities stated in Proposition 6 leads to the desired result. Let us now prove point ii. According to (85), (87) and (88), there holds

$$0<\frac{C_1(T,\gamma)}{2}\leqslant\frac{C_{T,\text{obs}}(\omega)}{\inf_{j\in\mathbb{N}^*}\int_\omega e_{a,j}(x)^2\,dx}\leqslant\frac{T}{2} \tag{89}$$

with $C_1(T,\gamma)\geqslant 2\left(\frac{T}{\pi}-\frac{4\pi}{\gamma^2T}\right)$. To conclude, it remains to provide an estimate of the spectral gap γ.

Lemma 12. *Let $a \in \mathcal{A}_M(0,L)$ with $M \in (0, 3\pi^2/L^2)$. There holds*

$$\forall j \in \mathbb{N}^*, \qquad \lambda_{a,j+1} - \lambda_{a,j} \geqslant \frac{\frac{3\pi^2}{L^2} - M}{\frac{2\pi}{L} + \sqrt{\frac{\pi^2}{L^2} + M}}.$$

Proof. The Courant-Fischer minimax principle writes

$$\lambda_{a,j}^2 = \min_{\substack{V \subset H_0^1(0,\pi) \\ \dim V = j}} \max_{u \in V \setminus \{0\}} \frac{\int_0^\pi (u'(x)^2 + a(x)u(x)^2)dx}{\int_0^\pi u(x)^2\, dx}. \tag{90}$$

Using that $0 \leqslant a(x) \leqslant M$ for almost every $x \in (0,L)$ yields $\frac{j\pi}{L} \leqslant \lambda_{a,j} \leqslant \sqrt{\left(\frac{j\pi}{L}\right)^2 + M}$, for every $j \in \mathbb{N}^*$. It suffices indeed to compare $\lambda_{a,j}^2$ with the j-th eigenvalue of a Sturm-Liouville operator with constant coefficients. We infer

$$\lambda_{a,j+1} - \lambda_{a,j} \quad \geqslant \quad \frac{(j+1)\pi}{L} - \sqrt{\left(\frac{j\pi}{L}\right)^2 + M} = \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right) + M}}$$

for every $j \in \mathbb{N}^*$. The sequence $j \mapsto \frac{(1+2j)\frac{\pi^2}{L^2} - M}{(j+1)\frac{\pi}{L} + \sqrt{\left(\frac{j^2\pi^2}{L^2}\right) + M}}$ being increasing, the expected estimate follows. □

Therefore, according to Lemma 12, one has $\gamma \geqslant \gamma_M$. Hence, choosing $\gamma = \gamma_M$ in (89) for all $T > T(M) = \frac{2\pi}{\gamma_M}$ yields the expected conclusion. ■

References

[1] Nalini Anantharaman and Stéphane Nonnenmacher, *Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold*, Ann. Inst. Four. (Grenoble), **57** (2007), no. 7, 2465–2523.

[2] John M. Ball and Marshall Slemrod, *Nonharmonic Fourier series and the stabilization of distributed semilinear control systems*, Comm. Pure Appl. Math. **32** (1979), no. 4, 555–587.

[3] Nicolas Burq and Maciej Zworski, *Bouncing ball modes and Quantum chaos*, SIAM Rev., **47** (2005), no. 1, 43–49.

[4] Yves Colin de Verdière, *Ergodicité et fonctions propres du Laplacien*, Comm. Math. Phys. **102** (1985), 497–502.

[5] Jean-Michel Coron, Control and nonlinearity, Volume 136 of Mathematical Surveys and Monographs. American Mathematical Society, Providence (2007).

[6] Steven Cox and Enrique Zuazua, *The rate at which energy decays in a damped string*, Comm. in partial differential equations **19**, (1994), 213–243.

[7] Thomas Duyckaerts, Xu Zhang and Enrique Zuazua, *On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 1–41.

[8] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, **19**, AMS (1998).

[9] Patrick Gérard and Éric Leichtnam, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J., 71 (1993), 559-607.

[10] Denis Grebenkov and Binh T. Nguyen *Geometrical structure of Laplacian eigenfunctions*, SIAM Rev. **55** (2013), no. 4, 601–667.

[11] Alain Haraux, *Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. (French. English summary) [Lacunary series and semi-internal control of the vibrations of a rectangular plate]*, J. Math. Pures Appl. (9) **68** (1989), no. 4, 457–465 (1990).

[12] Alain Haraux, Thibault Liard and Yannick Privat, *How to estimate observability constants of one-dimensional wave equations? Propagation versus Spectral methods*, To appear in J. Evol. Eq.

[13] Andrew Hassell and Steve Zelditch, *Quantum ergodicity of boundary values of eigenfunctions*, Comm. Math. Phys. **248** (2004), no. 1, 119–168.

[14] Pascal Hébrard and Antoine Henrot, *Optimal shape and position of the actuators for the stabilization of a string*, Syst. Cont. Letters **48** (2003), 199–209.

[15] Antoine Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[16] Antoine Henrot and Michel Pierre, *Variation et optimisation de formes (French) [Shape variation and optimization] Une analyse géométrique. [A geometric analysis]*, Math. & Appl. **48**, Springer, Berlin, 2005.

[17] Albert E. Ingham, *Some trigonometrical inequalities with applications to the theory of series*, Math. Zeitschrift **41** (1936), 367–379.

[18] Stéphane Jaffard and Sorin Micu, *Estimates of the constants in generalized Ingham's inequality and applications to the control of the wave equation*, Asymptot. Anal. **28** (2001), no. 3-4, 181–214.

[19] Stéphane Jaffard, Marius Tucsnak and Enrique Zuazua, *On a theorem of Ingham*, J. Fourier Anal. Appl. **3** (1997), 577–582.

[20] Dmitry Jakobson and Steve Zelditch, *Classical limits of eigenfunctions for some completely integrable systems*, Emerging applications of number theory (Minneapolis, MN, 1996), 329–354, IMA Vol. Math. Appl., 109, Springer, New York, 1999.

[21] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[22] Bernd Kawohl, *rearrangements and convexity of level sets in PDE*, Springer Lecture Notes in Math. 1150, (1985), 1–134.

[23] Vilmos Komornik and Paola Loreti, *Fourier Series in Control Theory*, Springer-Verlag, New York, 2005.

[24] Vilmos Komornik and Bernadette Miara, *Cross-like internal observability of rectangular membranes*, Evolution Equ. Control Theory **3** (2014), no.1, 135–146.

[25] John Lagnese, *Control of wave processes with distributed controls supported on a subregion*, SIAM J. Control Optim. (1983)

[26] Tony Lassila, *Optimal damping of a membrane and topological shape optimization*, Struct. Multidiscip. Optim. **38** (2009), no. 1, 43?52.

[27] Xunjing Li and Jiongmin Yong, *Necessary conditions for optimal control of distributed parameter systems*, SIAM J. Control Optim. **29** (1991), no. 4, 895–908.

[28] Jacques-Louis Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Tomes 1 & 2, Rech. Math. Appl. [Research in Applied Mathematics], Masson (1988).

[29] Hugh L. Montgomery and Robert C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. **2** (1974), no. 8, 73–82.

[30] Arnaud Münch, Pablo Pedregal and Francisco Periago *Optimal design of the damping set for the stabilization of the wave equation*, J. Differential Equations **231** (2006), no. 1, 331–358.

[31] Francisco Periago, *Optimal shape and position of the support for the internal exact control of a string*, Syst. Cont. Letters **58** (2009), no. 2, 136–140.

[32] Jürgen Pöschel, Eugene Trubowitz *Inverse spectral theory*, Academic Press Orlando, 1987.

[33] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains*, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 5, 1043?1111.

[34] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Complexity and regularity of optimal observation domains for the wave equation with fixed initial data*, Discrete Cont. Dynam. Syst. **35** (2015), no. 12, 6133–6153.

[35] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 1097–1126.

[36] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Optimal observation of the one-dimensional wave equation*, J. Fourier Anal. Appl. **19** (2013), no. 3, 514–544.

[37] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Optimal shape and location of sensors for parabolic equations with random initial data*, *Arch. Ration. Mech. Anal.,* **216** (2015), 921–981.

[38] Yannick Privat, Emmanuel Trélat and Enrique Zuazua, *Randomised observation, control and stabilisation of waves*, ZAMM Z. Angew. Math. Mech. 96 (2016), no. 5, 538?549.

[39] Jean-Michel Rakotoson, *Réarrangement relatif*, Vol. 64 of Math. & Appl. (Berlin) [Mathematics & Applications]. Springer, Berlin, 2008.

[40] Alexander I. Shnirelman, *Ergodic properties of eigenfunctions*, Uspenski Math. Nauk **29/6** (1974), 181–182.

[41] Enrique Zuazua, *Exact controllability for semilinear wave equations in one space dimension*, Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 109–129.