Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport-diffusion equation

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Abstract

In this paper, we prove explicit lower bounds for the cost of fast boundary controls for a class of linear equations of parabolic or dispersive type involving the spectral fractional Laplace operator. We notably deduce the following striking result: in the case of the heat equation controlled on the boundary, the Miller's conjecture formulated in [Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time, J. Differential Equations, 204 (2004), pp. 202-226] is not verified. Moreover, we also give a new lower bound for the minimal time needed to ensure the uniform controllability of the one-dimensional convection-diffusion equation with negative speed controlled on the left boundary, proving that the conjecture formulated in [J.-M. Coron and S. Guerrero, Singular optimal control: A linear 1-D parabolic-hyperbolic example, Asymptot. Anal., 44 (2005), pp. 237-257] concerning this problem is also not verified at least for negative speeds.

The proof is based on complex analysis, and more precisely on a representation formula for entire functions of exponential type, and is quite related to the *moment method*.

1 Introduction

1.1 Presentation of the problems

Let us consider the 1-D Laplace operator Δ with domain $D(\Delta) := H^1_0(0, L)$ and state space $H := H^{-1}(0, L)$. It is well-known that $-\Delta : D(\Delta) \to H^{-1}(0, L)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is

$$\lambda_k = \frac{k^2\pi^2}{L^2},$$

with eigenvector

$$e_k(x) := \sin\left(\frac{k\pi x}{L}\right).$$

Thanks to the continuous functional calculus for positive self-adjoint operators, one can define any positive power of $-\Delta$. Let us consider here some $\alpha > 1$.

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In what follows, we will consider two types of controlled equation on $(0,T)\times(0,L)$, one of parabolic type, that we write as

$$\begin{cases} \quad y_t = -(-\Delta)^{\alpha/2}y + bu & \text{in } (0,T)\times(0,L), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{1}$$

and one of dispersive type, that we write as

$$\begin{cases} \quad y_t = -i(-\Delta)^{\alpha/2}y + bu & \text{in } (0,T)\times(0,L), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{2}$$

where $b \in (\mathcal{D}(-(-\Delta)^{\alpha/2}))'$ is defined, for every $\varphi \in \mathcal{D}(-(-\Delta)^{\alpha/2})$, by

$$b(\varphi) = -\partial_x(\Delta^{-1}\varphi)(0),$$

i.e.

$$b := (\partial_x \delta_0) \circ \Delta^{-1},$$

and $u \in L^2((0,T),\mathbb{K})$, $\mathbb{K} := \mathbb{R}$ (for (1)) or $\mathbb{C}$ (for (2)) (see for example [20, Section 4.2] for complementary explanations on control operators for semigroups).

Equation (1) can model anomaly fast or slow diffusion (see for example [14]), whereas (2) can be used to study the energy spectrum of a 1-D fractional oscillator or for some fractional Bohr atoms (see for example [9]). For both equations, the most interesting case for physicists is $\alpha \in (1,2]$.

If $\alpha \in 2\mathbb{N}^*$, one can observe, using integrations by parts, that b corresponds to a boundary control on the left side on the $(\alpha/2-1)$-th derivative of y, so that b can be considered as a natural extension of the boundary control to the case of non-even α. This kind of controls has already been introduced in [17, Section 3.3] to give some negative results about the control of fractional diffusion equations with $\alpha \leqslant 1$ and in [13, Sections 3.2 and 3.3] as an application of some results about the cost of fast controls for some classes of abstract parabolic or dispersive equations.

One can prove, using the result of [8] for diagonal semigroups and scalar control, that b is an admissible control operator (see also [13, Sections 3.2 and 3.3]). Moreover, it is well-known that these equations are null-controllable in arbitrary small time (see [4] for the parabolic case and for example [13] for the dispersive case). Hence, one can easily prove (see for example [1, Chapter 2, Section 2.3]) that for every $y^0 \in H$, there exists a unique optimal (for the $L^2((0,T),\mathbb{K})$-norm) control $u_{opt} \in L^2((0,T),\mathbb{K})$ bringing y^0 to the equilibrium state 0, the map $y^0 \mapsto u_{opt}$ is then linear continuous. The norm of this operator is called the optimal null control cost at time T (or in a more concise form the cost of the control), denoted $C_H(T,L,\alpha)$ for equation (1) and $C_S(T,L,\alpha)$ for equation (2). Let us recall that these constants are also the smallest constants $C > 0$ such that for every $y^0 \in H$, there exists some control u driving y^0 to 0 at time T with

$$||u||_{L^2((0,T),\mathbb{K})} \leqslant C||y^0||_H.$$

Our first goal is mainly to continue the study done in [13]. In this article, the author proved precise upper bounds concerning the cost of the control for some large classes of linear parabolic or dispersive equations (including notably (1) and (2) for $\alpha \geqslant 2$) when the time T goes to 0, where the underlying "elliptic" operator was chosen to be self-adjoint or skew-adjoint with eigenvalues roughly as k^α or ik^α for some $\alpha \geqslant 2$ when $k \to +\infty$. The author also proved some lower bounds that were optimal concerning the power of T involved, but these estimates were not precise enough to understand what was the dependence of the cost of the control with respect to L and α. Here, we will in fact be able to give *precise* lower-bounds for equations (1) and (2) as soon as $\alpha > 1$ (and not only $\alpha \geqslant 2$), which will then generalize a little bit the study of lower bounds initiated in [13].

Moreover, in the dispersive case (2), we will see that in the particular case $\alpha = 2$ (i.e. the classical Schrödinger equation controlled on one side of the boundary), we will find again the lower bound that is conjectured to be the optimal one by Miller in [15], but a very surprising result is that in the case of the heat equation controlled on one side of the boundary (i.e. (1) with $\alpha = 2$), our lower bound will be *twice bigger* than the one expected according to the conjecture done by Miller in [16], and commonly accepted up to now (see notably [3], [13] or [19]). We will then formulate a new conjecture for this problem.

Remark 1 *Here, for the sake of simplicity (and because we think that it is enough for our purpose), we chose to treat only the case of equations* (1) *and* (2). *However, the results given below might be adapted to the following more general cases*

$$y_t + Ay = bu$$

or

$$y_t + iAy = bu,$$

where A is a positive self-adjoint operator on some Hilbert Space H with eigenvalues λ_n (the corresponding eigenvector being denoted e_n), with the assumption that $(\lambda_n)_{n\geqslant 1}$ is a regular increasing sequence of positive numbers verifying moreover that there exist some $\alpha > 1$ and some $R > 0$ such that

$$\lambda_n = Rn^\alpha + \underset{n\to\infty}{O}(n^{\alpha-1}),$$

and b is a scalar control input, i.e. $b \in \mathcal{D}(A)'$ and $u \in L^2((0,T),\mathbb{K})$, where $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$), and the sequence $(| < b, e_k >_{(D(A)',D(A))} |)_{k\in\mathbb{N}^*}$ is bounded from above and below (see [13]).

Understanding the behavior of fast controls is of interest in itself but it may also be applied (at least in some cases) to study the uniform controllability of transport-diffusion equations in the vanishing viscosity limit as explained in [11] and [12] because of the strong connection existing between these problems and highlighted in these references. In fact, the technique of the proof we will give here to estimate the cost of fast controls for equations (1) and (2) can also be used to obtain a new result for the transport-diffusion problem that we introduce now.

Let us consider some constant $M > 0$ and some viscosity coefficient $\varepsilon > 0$. We are interested in the following family of transport-diffusion equations

$$\begin{cases} y_t - \varepsilon y_{xx} - My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(\cdot,0) = v(t) & \text{in } (0,T), \\ y(\cdot,L) = 0 & \text{in } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{3}$$

with initial condition $y^0 \in H^{-1}(0,L)$ and control $v \in L^2(0,L)$ (remark that the speed $-M$ of the convection term is **negative**). If ε is taken equal to 0 and if the initial condition y^0 is taken in $L^2(0,L)$, we obtain a transport equation at constant speed:

$$\begin{cases} y_t - My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(\cdot,L) = 0 & \text{in } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{4}$$

where the control u has disappeared. The system evolves then freely and one has $y(T,\cdot) \equiv 0$ if and only if $T \geqslant L/M$. Hence, one can say that equation (4) is null-controllable if and only if $T \geqslant L/M$, the optimal control in L^2-norm being in this case the null function since we do not act on the equation. As before, one can define for equation (3) some cost of the control $C_{TD}(T,L,M,\varepsilon)$, and in the sequel we will precisely study its dependence with respect to ε at fixed T, L, M. Such a family of equations will be said uniformly controllable at time T if and only if $C_{TD}(T, L, M, \varepsilon) \to 0$ as $\varepsilon \to 0$ and non-uniformly controllable otherwise. As we will see later, the typical behavior of this kind of equations is that the cost of the control explodes for small enough T and decreases exponentially for large enough T when ε tends to 0. Our goal here will be to give a new lower bound for the minimal time needed to ensure uniform controllability.

1.2 State of the art

We will restrict here mainly to recall results in the 1-D case (the situation is far more complicated in the multidimensional case, see for example [15] and [16]). The first results concerning the cost of fast boundary controls have been obtained in the case of heat and Schrödinger equations. Concerning the one-dimensional heat equation on $(0, T) \times (0, L)$ with boundary control on one side, the time-dependence of the cost of the boundary control is $\simeq \exp(\beta^+/T)$ for some constant $\beta > 0$ (see [7] for the lower bound and [18] for the upper bound), where the notation β^+ means that we simultaneously have that the cost of the control is $\gtrsim \exp(\beta/T)$ and $\lesssim \exp(K/T)$ for every $K > \beta$ as close as β as we want (the implicit constant in front of the exponential may explode when we get closer to β because it seems to be a fraction of some power of T). The constant β verifies

$$L^2/4 \leqslant \beta \leqslant 3L^2/4.$$

The best upper bound was obtained in [19] and the lower bound in [16]. These estimates on β were the best that were known up to now. For the Schrödinger equation on $(0, T) \times (0, L)$ with boundary control on one side, one also has that the dependence in time of the cost of the boundary control is under the form $\simeq \exp(\tilde{\beta}^+/T)$ for some constant $\tilde{\beta} > 0$. The constant $\tilde{\beta}$ verifies

$$L^2/4 \leqslant \tilde{\beta} \leqslant 3L^2/2.$$

The upper bound is obtained in [19] and the lower bound in [15]. These estimates on $\tilde{\beta}$ are the best that are known up to now. In both cases, it was conjectured that the lower bound is optimal, i.e. that one can choose

$$\beta = \tilde{\beta} = L^2/4.$$

We will call from now on these conjectures on β and $\tilde{\beta}$ the *Miller's conjectures*.

Let us mention that, in the case of the heat equation, there exists another conjecture concerning sharp integral observability estimates and that is stronger than the previous one which concerns the observability of the heat equation, see [3] and [12]. More precisely, it was proved in [3] that there exists some constant $C_{int}(T, L)$ such that

$$\int_0^\infty \int_0^L e^{-\frac{L^2}{2t}} |\varphi(t, x)|^2 dx dt \leqslant C_{int}(T, L) \int_0^T |\partial_x \varphi(t, 0)|^2 dt, \tag{5}$$

where φ is a solution on the (forward) free heat equation

$$\begin{cases} \varphi_t - \varphi_{xx} = 0 \text{ in } (0, T) \times (0, L), \\ \varphi(\cdot, 0) = 0 \text{ in } (0, T), \\ \varphi(\cdot, L) = 0 \text{ in } (0, T), \\ \varphi(0, \cdot) = \varphi^0 \text{ in } (0, L). \end{cases} \tag{6}$$

with $\varphi \in L^2(0, L)$. Let us mention that, as explained in [3], this inequality is closely related to the study of the reachability set for the heat equation and notably to what is done in [4].

However, since (5) was obtained thanks to a reasoning by contradiction, the authors were unable to estimate precisely the constant $C_{int}(T, L)$.

A natural conjecture (cf. [3, Section 1.2, Section 3.2, Section 5] and also [12]) would be that the constant $C_{int}(T, L)$ does not blow up in a too violent way, in the following sense: For every $\delta > 0$ and $L > 0$, one can choose $C_{int}(T, L)$ such that

$$C_{int}(T, L) = \underset{T \to 0}{O}(e^{\frac{\delta}{T}}),$$

because this would notably give, after some easy computations, the Miller's conjecture (see [3] and [12]) and also the Coron-Guerrero conjecture for positive speeds (cf. [12]), with L^2 initial conditions (and not H^{-1} initial conditions, but it has only a neglecting impact on the cost of the control). From now on, we will call this conjecture the *Ervedoza-Zuazua conjecture*.

These results were later generalized to other self-adjoint or skew-adjoint elliptic operators by the author in [13]. More precisely, it was proved that if we consider some abstract linear control system with "boundary" control and where the elliptic operator associated to the system is skew-adjoint or self-adjoint with eigenvalues having a behavior roughly as Rk^α or iRk^α when $\alpha \geqslant 2$, then the cost of the control is bounded from above by $\exp\left(K/(RT)^{1/(\alpha-1)}\right)$ where K is some explicit constant depending on α, and is bounded from below by $\exp\left(C/T^{1/(\alpha-1)}\right)$, where C is some non-explicit constant independent of T (but depending on R and α). However, in this case, because of the lack of explicit lower bound and some lack of optimization in the computations of the upper bound, it was impossible to deduce some reasonable conjecture concerning the exact behavior of the cost of the control.

Concerning the transport-diffusion equation, let us recall the known results in the case of negative speed, which is interesting us here. Since one can prove (see [2, Appendix A]) that the solution of (3) with initial condition $y^0 \in L^2(0, L)$ converges in some sense to the one of (4) when $\varepsilon \to 0$, one might reasonably expect that $C_{TD}(T, L, M, \varepsilon) \to +\infty$ for $T < L/M$ and $C_{TD}(T, L, M, \varepsilon) \to 0$ for $T > L/M$ (the fact that we consider initial conditions in H^{-1} here is not a problem and only comes from the fact that we want to consider an admissible control operator, it has only a neglecting impact on the cost of the control).

However, it is proved in [2] that one has

$$C_{TD}(T, L, M, \varepsilon) \geqslant Ce^{\frac{K}{\varepsilon}}$$

for some constants C, K independent of ε if $T < 2L/M$ for $M > 0$. This surprising result led the authors to make the following conjecture concerning positive results for the uniform controllability of the family of equations (3) for large enough times:

$$C_{TD}(T, L, M, \varepsilon) \to 0$$

as $\varepsilon \to 0^+$ as soon as $T > 2L/M$. From now on, we will call this conjecture the *Coron-Guerrero* conjecture. In [2], it is proved the exponential decay of the cost of the control when $\varepsilon \to 0^+$ for sufficiently large time, the estimate on this time was improved in [5] and then [11], the later article making the link between this problem and the cost of fast controls for the heat equation. This study was also extended to varying in time and space (and regular enough) speed M and arbitrary space dimension in [6].

1.3 Main results and comments

In this section, we are going to give the main results of this paper and some additional comments.

The first result of this article is the following, which concerns equation (1):

Theorem 1.1 *For every $T > 0, L > 0$ and $\alpha > 1$, one has*

$$C_H(T,L,\alpha) \geqslant C \frac{\sqrt{L}(2\pi)^{2\alpha} T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}\left((2\pi)^{\alpha} T^{\frac{\alpha}{\alpha-1}} + \left(\frac{2L^\alpha}{\alpha \sin(\frac{\pi}{\alpha})}\right)^{\frac{\alpha}{\alpha-1}}\right)^2} \exp\left(\frac{2^{\frac{1}{\alpha-1}}(\alpha-1)L^{\frac{\alpha}{\alpha-1}}}{\left(\alpha \sin\left(\frac{\pi}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-1}} T^{\frac{1}{\alpha-1}}} - \frac{\pi^\alpha T}{L^\alpha}\right). \tag{7}$$

Notably, applying (7) for $\alpha = 2$, we have

$$C_H(T,L,2) \geqslant C \frac{8\sqrt{L}\pi^4 T^4}{\pi\sqrt{T}\left(16\pi^2 T^2 + L^4\right)^2} \exp\left(\frac{L^2}{2T} - \frac{\pi^2 T}{L^2}\right),$$

which is twice bigger than the usual conjecture. As a consequence, the usual Miller's conjecture made in [16], and the stronger Ervedoza-Zuazua conjecture made in [3] and studied in details in [12], are not verified.

Our second result concerns equation (2):

Theorem 1.2 *For every $T > 0, L > 0$ and $\alpha > 1$, one has*

$$C_S(T,L,\alpha) \geqslant C \frac{\sqrt{L}(2\pi)^{2\alpha} T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}\left((2\pi)^{2\alpha} T^{\frac{2\alpha}{\alpha-1}} + \left(\frac{L}{\alpha \sin(\frac{\pi}{2\alpha})}\right)^{\frac{2\alpha}{\alpha-1}}\right)} \exp\left(\frac{(\alpha-1)L^{\frac{\alpha}{\alpha-1}}}{2\left(\alpha \sin\left(\frac{\pi}{2\alpha}\right)\right)^{\frac{\alpha}{\alpha-1}} T^{\frac{1}{\alpha-1}}}\right). \tag{8}$$

Notably, applying (8) for $\alpha = 2$, we have

$$C_S(T,L,2) \geqslant C \frac{8\sqrt{L}\pi^4 T^4}{\pi\sqrt{T}(16\pi^2 T^2 + L^4/4)} \exp\left(\frac{L^2}{4T}\right),$$

and we find again the Miller's conjecture made in [15] for the Schrödinger equation.

The last result is the following, and concerns equation (3):

Theorem 1.3 *For every $M > 0, T > 0, L > 0$ and $\varepsilon > 0$, one has*

$$C_{TD}(T,L,M,\varepsilon) \geqslant \left(\frac{M^3+\varepsilon^3}{\varepsilon^3 L^3}\right)^{1/2} \frac{L^2}{2\pi\varepsilon\sqrt{T}\left(1 + \frac{(LM)^2}{8(\pi\varepsilon)^2}\right)} \exp\left(\frac{LM}{\sqrt{2}\varepsilon} - \frac{M^2 T}{4\varepsilon} - \frac{\pi^2 \varepsilon T}{L^2}\right). \tag{9}$$

Notably, $C_{TD}(T,L,M,\varepsilon)$ explodes when $\varepsilon \to 0$ as soon as

$$\frac{M^2 T}{4\varepsilon} < \frac{LM}{\sqrt{2}\varepsilon},$$

i.e.

$$T < \frac{2\sqrt{2}L}{M},$$

which is very surprising. As a consequence, the Coron-Guerrero conjecture given in [2] is also not verified for negative speeds.

Let us give additional remarks.

Remark 2 1. *The same computations for positive speed of propagation for* (3) *(which would correspond to $M < 0$ here in equation* (3)*) do not improve the existing result given in [2] (i.e. $T > L/|M|$ as a lower bound for the time needed for the uniform controllability).*

2. *If we compare the results given in Theorems 1.1 and 1.2 to the one given in [13] concerning upper bounds, we see that they do not really have the same shapes. In fact, the quantity* $\sin(\pi/(2\alpha))$ *which was in the parabolic case in [13] appears here in the dispersive case and conversely for the quantity* $\sin(\pi/\alpha)$. *The author was not able to understand deeply the reason of this lack of unity.*
3. *It is very surprising that the dispersive case gives a lower bound that is twice less that the one in the parabolic case. In fact, if we think a little bit about the computations done in many articles concerning the dispersive case ([15], [19] or [13] for example), we* always *obtain an upper bound for the dispersive case which is twice the one for the parabolic case (because of the study of the Weierstrass product that is used in the moment method, where the asymptotic upper bound at infinity is different in the two cases). Hence, it would seem more "logical" that the cost of the control for the dispersive case is the same or twice as for the parabolic case, and not half.*
4. *By using the results given in [11], we see that if we assume that we were able to prove that* $C_H(T,L,2) \simeq e^{L^2/(2T)}$, *then we would obtain new upper bound for the transport-diffusion problem* ($T > (2\sqrt{2})L/|M|$ *and* $T > (2\sqrt{2}+2)L/|M|$ *resp. for positive and negative speeds).*
5. *Since the Ervedoza-Zuazua conjecture is not verified, one can think on how to replace it. A natural substitution would be the following one: for every solution* φ *of* (6), *we have*

$$\int_0^\infty \int_0^L e^{-\frac{L^2}{t}} |\varphi(t,x)|^2 dxdt \leqslant C_{int}(T,L) \int_0^T |\partial_x \varphi(t,0)|^2 dt,$$

where $C_{int}(T,L)$ *is growing sub-exponentially in* $1/T$. *Unfortunately, this inequality is not verified. Let us prove it by contradiction. If this inequality where true, then, using the computations of [12, Page 101], we would obtain the uniform controllability of* (3) *as soon as* $T > (1+\sqrt{3})L/M \simeq 2.73L/M$, *which cannot be true because of Theorem 1.3 and the fact that* $2\sqrt{2} \simeq 2.82$.

The results and preceding remarks lead us to the following open questions:

Open Questions

Are the lower bounds given in Theorems 1.1, 1.2 and 1.3 optimal? Are the lower bounds in the case of the heat and Schrödinger equations (i.e. $\alpha = 2$) optimal?

The author believes that this might be true at least for the heat equation or more generally for equation (1), but is more skeptical concerning equation (2) and has no idea for equation (3). Moreover, according to the previous remark, the author thinks that it might not be possible to find some integral observability estimate similar to (5) with sub-exponential (in $1/T$) constant in the right-hand side.

2 Proofs of Theorems 1.1, 1.2 and 1.3

The proofs are based on the following idea: we are going to consider the optimal control associated to the first eigenfunction, and then we will study the Fourier transform of this control, which is an entire function of exponential type and with some prescribed zeros. In some sense, this idea comes from the moment method of [4], but we use it in a "reverse" way compared to what is done usually: we do not construct the control thanks to the Paley-Wiener Theorem (this will only give upper bounds) but we assume that the control exists and we see what we can deduce if we remark that it verifies the moment problem. After some rescaling and translations, we are then led to study an entire function of exponential type with some prescribed zeros, and we use a representation formula for functions of exponential type in order to make a link between the value and the functions and the repartition of its zeros on the upper half-plane. Let us mention that this idea has already been used in [2] to derive lower bounds for the problem of the uniform controllability of the transport-diffusion equation. The main differences here are that we were able to find a better result in the case of negative speed, and we also that were able to extend significantly the scope of the method to other cases than a singular limit, i.e. to the case of study of the cost of fast controls for (1) and (2), where the eigenvalues have a very different behavior from the ones of equation (3), which is interesting in itself and highlights one more time the connection between the uniform controllability and the cost of fast controls.

2.1 Proof of Theorem 1.1

In all what follows, C will always be a numerical constant independent of the parameters. We define $y^0 \in H^{-1}(0,L)$ as follows:

$$y^0(x) := \sin\left(\frac{\pi x}{L}\right). \tag{10}$$

One readily verifies that there exists some numerical constant C such that

$$||y^0||^2_{H^{-1}(0,L)} \leqslant CL^3. \tag{11}$$

We consider u the optimal control associated to this initial condition, which verifies by definition and thanks to estimate (11)

$$||u||_{L^2(0,L)} \leqslant C_H(T,L,\alpha)||y^0||_{H^{-1}(0,L)} \leqslant CC_H(T,L,\alpha)L^{3/2}. \tag{12}$$

Proceeding as in [1, Page 106-107], we obtain (because of the fact that $y(T,.)=0$ and the definition by transposition of the solutions of (1))

$$\frac{k\pi}{L}\int_0^T u(t)\exp\left(\frac{k^\alpha\pi^\alpha}{L^\alpha}t\right)dt = -\int_0^L \sin\left(\frac{\pi}{L}\right)\sin\left(\frac{k\pi}{L}\right)dx. \tag{13}$$

Let us define the complex function v by

$$v(z) := \int_{-T/2}^{T/2} u\left(t+\frac{T}{2}\right)\exp(-izt)dt. \tag{14}$$

Using (13) and (14), we deduce that

$$v\left(i\frac{\pi^\alpha}{L^\alpha}\right) = -\frac{L^2}{2\pi}\exp\left(-\frac{\pi^\alpha T}{2L^\alpha}\right), \tag{15}$$

and for every $k\in\mathbb{N}$ with $k>1$ we have

$$v\left(i\frac{k^\alpha\pi^\alpha}{L^\alpha}\right) = 0. \tag{16}$$

We deduce, using (14) and (12), that

$$\begin{aligned} |v(z)| &\leqslant \exp\left(\frac{T|Im(z)|}{2}\right)\int_0^T |u(t)|dt \\ &\leqslant C_H(T,L,\alpha)\sqrt{T}\exp\left(\frac{T|Im(z)|}{2}\right)||y_0||_{H^{-1}(0,L)} \\ &\leqslant CC_H(T,L,\alpha)\sqrt{T}\exp\left(\frac{T|Im(z)|}{2}\right)L^{3/2}. \end{aligned} \tag{17}$$

Let us consider some numerical parameter $\beta > 0$ to be chosen later. We introduce

$$f(z) := v\left(\frac{z - i\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{\alpha}{\alpha-1}}}\right). \tag{18}$$

Inequality (17) becomes

$$|f(z)| \leqslant CC_H(T,L,\alpha)\sqrt{T}\exp\left(\frac{|Im(z) - \beta L^{\frac{\alpha}{\alpha-1}}|}{2T^{1/(\alpha-1)}}\right)L^{3/2}. \tag{19}$$

One has, for $k \in \mathbb{N}$ and $k > 1$, and thanks to (16),

$$f(b_k) = 0, \tag{20}$$

where b_k verifies

$$\frac{b_k - iL^{\frac{\alpha}{\alpha-1}}\beta}{T^{\frac{\alpha}{\alpha-1}}} = \frac{ik^\alpha\pi^\alpha}{L^\alpha},$$

i.e.

$$b_k := i\left(L^{\frac{\alpha}{\alpha-1}}\beta + \frac{T^{\frac{\alpha}{\alpha-1}}k^\alpha\pi^\alpha}{L^\alpha}\right). \tag{21}$$

We also have, thanks to (15),

$$f(b_1) = -\frac{L^2}{2\pi}\exp\left(-\frac{\pi^\alpha T}{2L^\alpha}\right), \tag{22}$$

where

$$b_1 := i\left(L^{\frac{\alpha}{\alpha-1}}\beta + \frac{T^{\frac{\alpha}{\alpha-1}}\pi^\alpha}{L^\alpha}\right). \tag{23}$$

Using the usual representation of the functions of exponential type given for example in [10, Theorem p.56], we have, for every z such that $Im(z) > 0$,

$$\ln(|f(z)|) = \sum_1^\infty \ln\left(\frac{|z - a_l|}{|z - \overline{a_l}|}\right) + \sigma x_2 + \frac{x_2}{\pi}\int_{\mathbb{R}}\frac{\ln(|f(\tau)|)}{|\tau - z|^2}d\tau,$$

where the a_k are all the roots of f of positive imaginary part and σ is the type of f, which verifies thanks to (19) that

$$\sigma \leqslant \frac{1}{2T^{\frac{1}{\alpha-1}}}. \tag{24}$$

We apply this equality at point b_1, then we use (23) (remark that b_1 is a pure imaginary number) and (24) to obtain

$$\ln(|f(b_1)|) \leqslant \sum_1^\infty \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right) + \frac{L^{\frac{\alpha}{\alpha-1}}\beta}{2T^{\frac{1}{\alpha-1}}} + \frac{T\pi^\alpha}{2L^\alpha} + \frac{|b_1|}{\pi}\int_{\mathbb{R}}\frac{\ln(|f(\tau)|)}{\tau^2 + |b_1|^2}d\tau. \tag{25}$$

Let us study the right-hand side of this equality.

1. First term of the right-hand side: We study

$$I := \sum_{l=1}^\infty \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right).$$

It is easy to prove that, if $(z_1, z_2) \in \mathbb{C}^2$, then

$$\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|} \leqslant 1 \text{ if and only if } Im(z_1)Im(z_2) \geqslant 0. \tag{26}$$

Hence, we deduce that we can remove the a_l that are not of the form b_i ($i \in \mathbb{N}^*$) and obtain

$$\begin{aligned} I &\leqslant \sum_2^\infty \ln\left(\frac{|b_1 - b_l|}{|b_1 - \overline{b_l}|}\right) = \sum_2^\infty \ln\left(\frac{(k^\alpha - 1)T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/L^\alpha}{2L^{\frac{\alpha}{\alpha-1}}\beta + (k^\alpha + 1)T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/L^\alpha}\right) \\ &\leqslant \int_2^\infty \ln\left(\frac{x^\alpha T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/(2\beta L^{\frac{\alpha^2}{\alpha-1}})}{1 + x^\alpha T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/(2\beta L^{\frac{\alpha^2}{\alpha-1}})}\right) dx. \end{aligned} \tag{27}$$

We use the change of variables

$$\tau := \frac{\pi T^{\frac{1}{\alpha-1}}}{(2\beta)^{\frac{1}{\alpha}} L^{\frac{\alpha}{\alpha-1}}} x.$$

Hence we obtain

$$I \leqslant \frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{T^{\frac{1}{\alpha-1}}\pi} \int_{\frac{2\pi T^{\frac{1}{\alpha-1}}}{(2\beta)^{\frac{1}{\alpha}} L^{\frac{\alpha}{\alpha-1}}}}^\infty \ln\left(\frac{\tau^\alpha}{1+\tau^\alpha}\right) d\tau. \tag{28}$$

We call

$$A := \frac{2\pi T^{\frac{1}{\alpha-1}}}{(2\beta)^{\frac{1}{\alpha}} L^{\frac{\alpha}{\alpha-1}}}. \tag{29}$$

Using an integration by parts, we obtain

$$\int_A^\infty \ln\left(\frac{\tau^\alpha}{1+\tau^\alpha}\right) d\tau = -A \ln\left(\frac{A^\alpha}{1+A^\alpha}\right) - \alpha \int_A^\infty \frac{1}{1+\tau^\alpha} d\tau. \tag{30}$$

One can write

$$\int_A^\infty \frac{1}{1+\tau^\alpha} d\tau = \int_0^\infty \frac{1}{1+\tau^\alpha} d\tau - \int_0^A \frac{1}{1+\tau^\alpha} d\tau. \tag{31}$$

It is well-known that

$$\int_0^\infty \frac{1}{1+\tau^\alpha} d\tau = \Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(1 + \frac{1}{\alpha}\right).$$

Using the Euler reflection formula for the Γ function and the relation $\Gamma(z+1) = z\Gamma(z)$, we deduce that

$$\int_0^\infty \frac{1}{1+\tau^\alpha} d\tau = \frac{\pi}{\alpha \sin(\frac{\pi}{\alpha})}. \tag{32}$$

Concerning the second term of (31), we have

$$\int_0^A \frac{1}{1+\tau^\alpha} d\tau \leqslant A,$$

hence

$$\frac{2}{A} \int_0^A \frac{1}{1+\tau^\alpha} d\tau \leqslant 2. \tag{33}$$

Putting together (27), (29), (30), (31), (32) and (33), we deduce that

$$\sum_1^\infty \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right) \leqslant 2 \ln\left(1 + \frac{2\beta L^{\frac{\alpha^2}{\alpha-1}}}{(2\pi)^\alpha T^{\frac{\alpha}{\alpha-1}}}\right) - \frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{\sin(\frac{\pi}{\alpha}) T^{\frac{1}{\alpha-1}}} + 2. \tag{34}$$

2. Concerning the third term of the right-hand-side, an easy changing of variables gives

$$|b_1|\int_{\mathbb{R}}\frac{d\tau}{\tau^2+|b_1|^2}=\pi.$$

Hence, using the fact that τ is real and (19), we deduce that

$$\frac{b_1}{\pi}\int_{\mathbb{R}}\frac{\ln|f(\tau)|}{\tau^2+b_1^2}d\tau\leqslant\frac{\beta L^{\frac{\alpha}{\alpha-1}}}{2T^{\frac{1}{\alpha-1}}}+\ln(CC_H(T,L,\alpha)\sqrt{T}L^{3/2}). \tag{35}$$

Using (22), (25), (34) and (35), we deduce that

$$\begin{aligned}&\ln\left(\frac{L^2}{2\pi}\right)-\frac{\pi^\alpha T}{2L^\alpha}\\&\leqslant 2\ln\left(1+\frac{2\beta L^{\frac{\alpha^2}{\alpha-1}}}{(2\pi)^\alpha T^{\frac{\alpha}{\alpha-1}}}\right)-\frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{\sin(\frac{\pi}{\alpha})T^{\frac{1}{\alpha-1}}}+\frac{\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}}+2+\frac{\pi^\alpha T}{2L^\alpha}+\ln(CC_H(T,L,\alpha)\sqrt{T}L^{3/2}),\end{aligned} \tag{36}$$

hence there exists a numerical constant C such that

$$C_H(T,L,\alpha)\geqslant C\frac{L^{1/2}(2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}((2\pi)^\alpha T^{\frac{\alpha}{\alpha-1}}+2\beta L^{\frac{\alpha^2}{\alpha-1}})^2}\exp\left(\frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{\sin(\frac{\pi}{\alpha})T^{\frac{1}{\alpha-1}}}-\frac{\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}}-\frac{\pi^\alpha T}{L^\alpha}\right).$$

Now, we optimize β by trying to maximize what is inside the exponential. We find

$$\beta=2^{\frac{1}{\alpha-1}}\left(\frac{1}{\alpha\sin(\frac{\pi}{\alpha})}\right)^{\frac{\alpha}{\alpha-1}},$$

and we deduce

$$C_H(T,L,\alpha)\geqslant C\frac{\sqrt{L}(2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}\left((2\pi)^\alpha T^{\frac{\alpha}{\alpha-1}}+2^{\frac{\alpha}{\alpha-1}}\left(\frac{L^\alpha}{\alpha\sin(\frac{\pi}{\alpha})}\right)^{\frac{\alpha}{\alpha-1}}\right)^2}\exp\left(\frac{2^{\frac{1}{\alpha-1}}(\alpha-1)L^{\frac{\alpha}{\alpha-1}}}{(\alpha\sin(\pi/\alpha))^{\frac{\alpha}{\alpha-1}}T^{\frac{1}{\alpha-1}}}-\frac{\pi^\alpha T}{L^\alpha}\right).$$

2.2 Proof of Theorem 1.2

The computations are very similar to the one of the previous part, hence we are going to skip some details. We define $y^0\in H^{-1}(0,L)$ as in (10). We consider u the optimal control associated to this initial condition, which verifies by definition and thanks to estimate (11)

$$||u||_{L^2(0,L)}\leqslant C_S(T,L,\alpha)||y^0||_{H^{-1}(0,L)}\leqslant CC_S(T,L,\alpha)L^{3/2}. \tag{37}$$

Proceeding as before, we obtain

$$\frac{k\pi}{L}\int_0^T u(t)\exp\left(\frac{ik^\alpha\pi^\alpha}{L^\alpha}t\right)dt=-\int_0^L\sin\left(\frac{\pi}{L}\right)\sin\left(\frac{k\pi}{L}\right)dx. \tag{38}$$

Let us define the complex function v by

$$v(z):=\int_{-T/2}^{T/2}u(t+\frac{T}{2})\exp(-izt)dt. \tag{39}$$

Using (38) and (39), we deduce that

$$v\left(-\frac{\pi^\alpha}{L^\alpha}\right)=-\frac{L^2}{2\pi}\exp\left(-\frac{i\pi^\alpha T}{2L^\alpha}\right). \tag{40}$$

and for every $k \in \mathbb{N}$ with $k > 1$ we have

$$v\left(-\frac{k^\alpha \pi^\alpha}{L^\alpha}\right) = 0. \tag{41}$$

We also have, using (39) and (37), that

$$|v(z)| \leqslant \exp\left(\frac{T|Im(z)|}{2}\right)\int_0^T |u(t)|dt \leqslant CC_S(T,L,\alpha)\sqrt{T}\exp\left(\frac{T|Im(z)|}{2}\right)L^{3/2}. \tag{42}$$

Let us consider some numerical parameter $\beta > 0$ to be chosen later. We introduce

$$f(z) := v\left(\frac{-z + i\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{\alpha}{\alpha-1}}}\right). \tag{43}$$

Inequality (42) becomes

$$|f(z)| \leqslant CC_S(T,L,\alpha)\sqrt{T}\exp\left(\frac{|Im(z) - \beta L^{\frac{\alpha}{\alpha-1}}|}{2T^{\frac{1}{\alpha-1}}}\right)L^{3/2}. \tag{44}$$

One has, for $k \in \mathbb{N}$ and $k > 1$, and thanks to (41),

$$f(b_k) = 0, \tag{45}$$

where b_k verifies

$$b_k := \frac{T^{\frac{\alpha}{\alpha-1}} k^\alpha \pi^\alpha}{L^\alpha} + iL^{\frac{\alpha}{\alpha-1}}\beta. \tag{46}$$

We also have, thanks to (15),

$$f(b_1) = -\frac{L^2}{2\pi}\exp\left(-\frac{i\pi^\alpha T}{2L^\alpha}\right), \tag{47}$$

where

$$b_1 := \frac{T^{\frac{\alpha}{\alpha-1}}\pi^\alpha}{L^\alpha} + iL^{\frac{\alpha}{\alpha-1}}\beta. \tag{48}$$

Using the same representation theorem as in the proof of Theorem 1.1, we have for every z such that $Im(z) > 0$,

$$\ln(|f(z)|) = \sum_1^\infty \ln\left(\frac{|z - a_l|}{|z - \overline{a_l}|}\right) + \sigma x_2 + \frac{x_2}{\pi}\int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{|\tau - z|^2}d\tau,$$

where the a_k are all the roots of f of positive imaginary part and

$$\sigma \leqslant \frac{1}{2T^{\frac{1}{\alpha-1}}}. \tag{49}$$

We apply this equality at point b_1 and use (48) and (49) to obtain

$$\ln(|f(b_1)|) \leqslant \sum_1^\infty \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right) + \frac{L^{\frac{\alpha}{\alpha-1}}\beta}{2T^{\frac{1}{\alpha-1}}} + \frac{|b_1|}{\pi}\int_{-\infty}^{+\infty}\frac{\ln(|f(\tau)|)}{\tau^2 + |b_1|^2}d\tau. \tag{50}$$

Let us study the right-hand side of this equality.

1. First term of the right-hand side: We study

$$I := \sum_{l=1}^{\infty} \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right).$$

Using (26), one obtains has before that

$$\begin{aligned} I \leqslant \sum_{2}^{\infty} \ln\left(\frac{|b_1 - b_l|}{|b_1 - \overline{b_l}|}\right) &= \sum_{2}^{\infty} \ln\left(\frac{(k^\alpha - 1)T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/L^\alpha}{\sqrt{(2L^{\frac{\alpha}{\alpha-1}}\beta)^2 + ((k^\alpha+1)T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/L^\alpha)^2}}\right) \\ &\leqslant \int_2^{\infty} \ln\left(\frac{x^\alpha T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/(2\beta L^{\frac{\alpha^2}{\alpha-1}})}{\sqrt{1 + \left(x^\alpha T^{\frac{\alpha}{\alpha-1}}\pi^\alpha/(2\beta L^{\frac{\alpha^2}{\alpha-1}})\right)^2}}\right). \end{aligned} \tag{51}$$

We use the same change of variables

$$\tau := \frac{\pi T^{\frac{1}{\alpha-1}}}{(2\beta)^{1/\alpha} L^{\frac{\alpha}{\alpha-1}}} x,$$

so that we obtain

$$I \leqslant \frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{\pi T^{\frac{1}{\alpha-1}}} \int_{\frac{2\pi T^{\frac{1}{\alpha-1}}}{(2\beta)^{\frac{1}{\alpha}} L^{\frac{\alpha}{\alpha-1}}}}^{\infty} \ln\left(\frac{\tau^\alpha}{\sqrt{1+\tau^{2\alpha}}}\right) d\tau. \tag{52}$$

Using an integration by parts, we obtain

$$\int_A^{\infty} \ln\left(\frac{\tau^\alpha}{\sqrt{1+\tau^{2\alpha}}}\right) d\tau = -\frac{A}{2} ln\left(\frac{A^{2\alpha}}{1+A^{2\alpha}}\right) - \alpha \int_A^{\infty} \frac{1}{1+\tau^{2\alpha}} d\tau, \tag{53}$$

where A was defined in (29). We have

$$\int_A^{\infty} \frac{1}{1+\tau^{2\alpha}} d\tau = \int_0^{\infty} \frac{1}{1+\tau^{2\alpha}} d\tau - \int_0^{A} \frac{1}{1+\tau^{2\alpha}} d\tau. \tag{54}$$

Using (32), we deduce that

$$\int_0^{\infty} \frac{1}{1+\tau^{2\alpha}} d\tau = \frac{\pi}{2\alpha \sin(\frac{\pi}{2\alpha})}. \tag{55}$$

Concerning the second term of (54), we still have thanks to (33)

$$\frac{2}{A} \int_0^{A} \frac{1}{1+\tau^{2\alpha}} d\tau \leqslant 2. \tag{56}$$

Putting together (29), (51), (53), (54), (55) and (56), we deduce that

$$\sum_{1}^{\infty} \ln\left(\frac{|b_1 - a_l|}{|b_1 - \overline{a_l}|}\right) \leqslant \ln\left(1 + \left(\frac{2\beta L^{\frac{\alpha^2}{\alpha-1}}}{(2\pi)^\alpha T^{\alpha/(\alpha-1)}}\right)^2\right) - \frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{2\sin(\frac{\pi}{2\alpha}) T^{\frac{1}{\alpha-1}}} + 2. \tag{57}$$

2. Concerning the third term of the right-hand-side, we obtain exactly as before and according to (48)
$$\frac{|b_1|}{\pi}\int_{\mathbb{R}}\frac{\ln(|f(\tau)|)}{\tau^2+|b_1|^2}d\tau \leqslant \frac{\beta L^{\frac{\alpha}{\alpha-1}}}{2T^{\frac{1}{\alpha-1}}}+\ln(CC_S(T,L,\alpha)\sqrt{T}L^{3/2}). \tag{58}$$

Using (47), (50), (57) and (58), we deduce that
$$\ln\left(\frac{L^2}{2\pi}\right)$$
$$\leqslant \ln\left(1+\left(\frac{2\beta L^{\frac{\alpha^2}{\alpha-1}}}{(2\pi)^\alpha T^{\alpha/(\alpha-1)}}\right)^2\right)-\alpha\frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{1/\alpha}}{2\alpha\sin(\frac{\pi}{2\alpha})T^{\frac{1}{\alpha-1}}}+\frac{\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}}+2+\ln(CC_S(T,L,\alpha)\sqrt{T}L^{3/2}), \tag{59}$$

hence there exists a numerical constant C such that
$$C_S(T,L,\alpha)\geqslant C\frac{L^{1/2}(2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}((2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}+4\beta^2L^{\frac{2\alpha^2}{\alpha-1}})}\exp\left(\frac{L^{\frac{\alpha}{\alpha-1}}(2\beta)^{\frac{1}{\alpha}}}{2\sin(\frac{\pi}{2\alpha})T^{\frac{1}{\alpha-1}}}-\frac{\beta L^{\frac{\alpha}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}}\right).$$

Now, we optimize β by trying to maximize what is inside the exponential. We find
$$\beta=\frac{1}{2}\left(\frac{1}{\alpha\sin(\frac{\pi}{2\alpha})}\right)^{\frac{\alpha}{\alpha-1}},$$

and we deduce
$$C_S(T,L,\alpha)\geqslant C\frac{L^{1/2}(2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}}{2\pi\sqrt{T}\left((2\pi)^{2\alpha}T^{\frac{2\alpha}{\alpha-1}}+\left(\frac{L^\alpha}{\alpha\sin(\frac{\pi}{2\alpha})}\right)^{\frac{2\alpha}{\alpha-1}}\right)}\exp\left(\frac{(\alpha-1)L^{\frac{\alpha}{\alpha-1}}}{2(\alpha\sin(\frac{\pi}{2\alpha}))^{\frac{\alpha}{\alpha-1}}T^{\frac{1}{\alpha-1}}}\right).$$

2.3 Proof of Theorem 1.3

The computations are very similar to the ones done in [1, Pages 106-109]. Let us mention that the main difference is in the definition of f given in (65): in [1] and [2], the corresponding function f is (see [1, Page 108])
$$f(s):=v\left(\frac{s-iM^2}{4\varepsilon}\right),$$
that is to say, additionally to the rescaling argument, there is also a translation argument similar to the ones used in our previous proofs (see (18) and (43)). This translation is in fact optimal for positive speeds but is not for negative speeds, that is the reason why we are able to improve the result of [2] by a factor $\sqrt{2}$ in the case of negative speeds. Let us also remember that here by convention $M>0$ and the speed of propagation is $-M$ (see equation (3)).

First of all, we choose the initial condition as
$$y^0(x):=\sin\left(\frac{\pi x}{L}\right)\exp\left(\frac{-Mx}{2\varepsilon}\right).$$

Using [1, Pages 106-107], one has
$$||y^0||_{H^{-1}(0,L)}\leqslant C\frac{\varepsilon^3L^3}{M^3+\varepsilon^3}.$$

We consider u the optimal control associated to this initial condition, which verifies by definition
$$||u||_{L^2(0,L)}\leqslant C_{TD}(T,L,M,\varepsilon)||y^0||_{H^{-1}(0,L)}\leqslant CC_{TD}(T,L,M,\varepsilon)\frac{\varepsilon^3L^3}{M^3+\varepsilon^3}. \tag{60}$$

Following [1, Page 107], we see that if we consider

$$v(z) := \int_{-T/2}^{T/2} u\left(t + \frac{T}{2}\right) \exp(-izt)dt, \tag{61}$$

we have

$$v\left(i\left(\frac{M^2}{4\varepsilon} + \frac{\pi^2\varepsilon}{L^2}\right)\right) = -\frac{L^2}{2\pi\varepsilon}\exp\left(-\frac{\pi^2\varepsilon T}{2L^2} - \frac{M^2 T}{8\varepsilon}\right) \tag{62}$$

and for every $k \in \mathbb{N}$ with $k > 1$ we have

$$v\left(i\left(\frac{M^2}{4\varepsilon} + \frac{k^2\pi^2\varepsilon}{L^2}\right)\right) = 0. \tag{63}$$

We deduce, using (61) and (60), that

$$\begin{aligned} |v(z)| &\leqslant \exp\left(\frac{T|Im(z)|}{2}\right)\int_0^T |u(t)|dt \leqslant C_{TD}(T,L,M,\varepsilon)\sqrt{T}\exp\left(\frac{T|Im(z)|}{2}\right)||y_0||_{H^{-1}(0,L)} \\ &\leqslant C\left(\frac{\varepsilon^3 L^3}{M^3+\varepsilon^3}\right)^{1/2} C_{TD}(T,L,M,\varepsilon)\sqrt{T}\exp\left(\frac{T|Im(z)|}{2}\right). \end{aligned} \tag{64}$$

Let us introduce

$$f(s) := v\left(\frac{s}{4\varepsilon}\right). \tag{65}$$

Then inequality (64) becomes

$$|f(z)| \leqslant C\left(\frac{\varepsilon^3 L^3}{M^3+\varepsilon^3}\right)^{1/2} C_{TD}(T,L,M,\varepsilon)\sqrt{T}\exp\left(\frac{T|Im(z)|}{8\varepsilon}\right). \tag{66}$$

One has, for $k \in \mathbb{N}$ and $k > 1$ and thanks to (63)

$$f(b_k) = 0, \tag{67}$$

where b_k verifies

$$b_k := i\left(M^2 + \frac{4k^2\varepsilon^2\pi^2}{L^2}\right). \tag{68}$$

We also have, thanks to (62),

$$f(b_1) = -\frac{L^2}{2\pi\varepsilon}\exp\left(-\frac{\pi^2\varepsilon T}{2L^2} - \frac{M^2 T}{8\varepsilon}\right), \tag{69}$$

where

$$b_1 := i\left(M^2 + \frac{4\varepsilon^2\pi^2}{L^2}\right). \tag{70}$$

Using the same representation theorem, one has, for every z such that $Im(z) > 0$,

$$\ln(|f(z)|) = \sum_1^\infty \ln\left(\frac{|z-a_l|}{|z-\overline{a_l}|}\right) + \sigma x_2 + \frac{x_2}{\pi}\int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{|\tau - z|^2}d\tau,$$

that we apply at point b_1:

$$\ln(|f(b_1)|) = \sum_1^\infty \ln\left(\frac{|b_1-a_l|}{|b_1-\overline{a_l}|}\right) + \frac{TM^2}{8\varepsilon} + \frac{\varepsilon\pi^2}{2L^2} + \frac{|b_1|}{\pi}\int_{\mathbb{R}} \frac{\ln(|f(\tau)|)}{\tau^2 + |b_1|^2}d\tau. \tag{71}$$

Let us study separately the terms of the right-hand side.

1. First term of the right-hand side: we can proceed as we did before, and we obtain

$$\sum_1^\infty \ln\left(\frac{|b_1-a_l|}{|b_1-\overline{a_l}|}\right) \leqslant \sum_2^\infty \ln\left(\frac{\frac{(k^2-1)\varepsilon^2\pi^2}{L^2}}{M^2/2+\frac{(k^2+1)\varepsilon^2\pi^2}{L^2}}\right) \leqslant \int_2^\infty \ln\left(\frac{\varepsilon^2\pi^2x^2}{L^2M^2/2+\varepsilon^2\pi^2x^2}\right)dx.$$

We use the change of variables

$$\tau := \frac{\sqrt{2}\pi\varepsilon}{LM}x.$$

Hence we obtain

$$\sum_1^\infty \ln\left(\frac{|b_1-a_l|}{|b_1-\overline{a_l}|}\right) \leqslant \frac{LM}{\sqrt{2}\pi\varepsilon}\int_{\frac{2\sqrt{2}\pi\varepsilon}{LM}}^\infty \ln\left(\frac{\tau^2}{1+\tau^2}\right)d\tau.$$

Using an integration by parts, we easily obtain

$$\sum_1^\infty \ln\frac{|b_1-a_l|}{|b_1-\overline{a_l}|} \leqslant 2\ln\left(1+\frac{(LM)^2}{8(\pi\varepsilon)^2}\right) - \frac{LM}{\sqrt{2}\varepsilon}+2. \tag{72}$$

2. Third term: using the fact that τ is real, we have $Im(\tau) = 0$ and then by (66) and straightforward computations

$$\frac{|b_1|}{\pi}\int_{\mathbb{R}}\frac{\ln(|f(\tau)|)}{\tau^2+|b_1|^2}d\tau \leqslant \ln\left(\left(\frac{\varepsilon^3L^3}{M^3+\varepsilon^3}\right)^{1/2} CC_{TD}(T,L,M,\varepsilon)\right)\sqrt{T}). \tag{73}$$

Conclusion: by using (69), (71), (72) and (73), we deduce that

$$\begin{aligned}&\ln\left(\frac{L^2}{2\pi\varepsilon}\right)-\frac{\pi^2\varepsilon T}{2L^2}-\frac{M^2T}{8\varepsilon}\\ &\leqslant 2\ln\left(1+\frac{(LM)^2}{8(\pi\varepsilon)^2}\right)-\frac{LM}{\sqrt{2}\varepsilon}+2+\ln\left(\left(\frac{\varepsilon^3L^3}{(M^3+\varepsilon^3}\right)^{1/2} CC_{TD}(T,L,M,\varepsilon)\sqrt{T}\right)+\frac{TM^2}{8\varepsilon}+\frac{\varepsilon\pi^2T}{2L^2}.\end{aligned}$$

Hence, we obtain

$$C_{TD}(T,L,M,\varepsilon) \geqslant C\left(\frac{M^3+\varepsilon^3}{\varepsilon^3L^3}\right)^{1/2}\frac{L^2}{2\pi\varepsilon\left(1+\frac{(LM)^2}{8(\pi\varepsilon)^2}\right)^2\sqrt{T}}\exp\left(\frac{LM}{\sqrt{2}\varepsilon}-\frac{M^2T}{4\varepsilon}-\frac{\pi^2\varepsilon T}{L^2}\right).$$

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