THE FICTITIOUS CONTROL METHOD FOR THE INTERNAL CONTROLLABILITY OF UNDERACTUATED SYSTEMS OF PDES

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Abstract. These are lecture notes for a short winter course at the Department of Mathematics, University of Coimbra, Portugal, December 6-8, 2018. The course was part of the 13th International Young Researchers Workshop on Geometry, Mechanics and Control. The aim of the lectures was to explain how some ideas initially developed by Gromov in his famous book "Partial Differential Equations" (1986, Springer), in order to find non-holonomic solutions to underdetermined partial differential equations, can be used in the context of control theory of coupled systems.

1. Introduction

1.1. **Aim of the lecture notes.** Studying the controllability for linear or nonlinear coupled systems of partial differential equations has been an intensive subject of interest these last years. The main issue is to try to control many equations with less controls than equations. We hope to act indirectly on the equations that are not directly controlled thanks to the coupling terms.

A typical situation is when the control appears as a source term (possibly localized in space and time) in our partial differential equation. Such a control is called an *internal control.* In this particular case, one way to prove a controllability property can be described as follows.

(1) Firstly, control the system with a control acting on each equation. This is in general simpler than the original underactuated problem and may be performed by using classical tools. This first step has firstly been used in [19] in the context of coupled systems of partial differential equations.

2010 *Mathematics Subject Classification.* 93B05, 93B25, 34H05, 35E99.
Key words and phrases. Coupled systems, Controllability, Fictitious control method, Algebraic solvability.

(2) Secondly, try to find a way to get rid of the control that should not appear in order to obtain the desired result. This can be done by using the notion of "algebraic solvability", which relies on a particular construction of a solution for a related underdetermined linear partial differential equation with a source term: the solution will be written as a linear combination of the source term and its derivatives. This property is strongly related to the inversion of differential operators as presented in [20, Section 2.3.8].

The above strategy has been firstly introduced in [9], in the context of stabilization of ordinary differential equations, and has been extended in [11] in the context of controllability of partial differential equations. The goal of these lecture notes is to explain the spirit of this "fictitious control + algebraic solvability" method (that we will shorten in "fictitious control method"), in order to highlight the main features, advantages, and drawbacks of this method.

The lecture notes are as self-contained as possible, up to two points: in Section 2.1, the proof of the well-posedness of our abstract setting is skipped, and in Section 3.1, the proofs of two controllability results on the scalar Schrödinger equation are also skipped. We hope that it should not cause difficulties for understanding the main ideas. They are organized as follows. Firstly, in Section 1.2, we give a simple example of the algebraic solvability of a differential system, in order to explain the spirit of the method we will use constantly in these lecture notes. Then, in Section 1.3, we explain the link with the h-principle. A very short introduction to controllability problems is provided in Section 2.1. In section 2.2, we will explain the main ideas on a very simple system of ordinary differential equations, following closely the presentation of [8, Second proof of Theorem 1.18]. In Section 2.3, we make some comments on the method. Section 3 is devoted to proving a result of controllability for a system of linear partial differential equations of Schrödinger type, which is a simplified version of the results proved in [26]. We give some concluding remarks and perspectives in Section 4.

1.2. **An introducing example.** In order to explain the main ideas of the following lecture notes, let us consider the following question. Let $f \in C_0^\infty(\mathbb{R}, \mathbb{R})$ (that we denote from now on $C_0^\infty(\mathbb{R})$) and $a_1, a_2, a_3, b_1, b_2, b_3$ some real numbers.

Question: is it possible to find two functions x_1 and x_2, that are also in $C_0^\infty(\mathbb{R})$, verifying the equation

$$a_1 x_1 - a_2 {x_1}' + a_3 x_1'' + b_1 x_2 - b_2 {x_2}' + b_3 {x_2}'' = f? \tag{1.1}$$

From an analytic point of view, this equation is underdetermined (there are more unknowns (2, the functions x_1 and x_2) than equations (only 1)). We follow here closely the ideas developed in [20, Section 2.3.8].

We rewrite (1.1) under the form $\mathcal{L}(x_1, x_2) = f$ and

$$\mathcal{L} = \big(a_1 - a_2\partial_t + a_3\partial_{tt}, \quad b_1 - b_2\partial_t + b_3\partial_{tt}\big) : C^\infty(\mathbb{R})^2 \to C^\infty(\mathbb{R}).$$

Since we impose the solution (x_1, x_2) to be compactly supported, the following remark is essential: the set $C_0^\infty(\mathbb{R})$ is stable by differentiation and linear combination. Moreover, for any $\psi \in C_0^\infty(\mathbb{R})$, the support of any derivative $\psi^{(k)}$ is included in the support of ψ. This suggests the following procedure: we aim to find a solution to (1.1) that can be written as a linear combination of the source term f and its derivatives. Such a solution will then be automatically with compact support. In other words, we would like to find a solution (x_1, x_2) under the form $(x_1, x_2) = \mathcal{M}f$, where $\mathcal{M} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})^2$ is a linear differential operator. In an equivalent way, we would like to find $\mathcal{M}$ such that the relation $\mathcal{L} \circ \mathcal{M} = \mathrm{Id}_{C^\infty(\mathbb{R})}$ holds. In order to solve this equation, we need to introduce the following tool.

Definition 1.1. Let $(p, q, r) \in (\mathbb{N}^*)^3$. For $i \in [|1, r|]$, we consider $C_i \in \mathcal{M}_{p,q}(\mathbb{R})$. We also call ${}^tC_i \in \mathcal{M}_{q,p}(\mathbb{R})$ the transpose of C_i.

We consider the following linear differential operator of order r with constant coefficients

$$\mathcal{P} : \varphi = (\varphi_1, \dots \varphi_q) \in C^\infty(\mathbb{R})^q \mapsto \sum_{i=1}^{r} C_r(\varphi^{(r)}) \in C^\infty(\mathbb{R})^p.$$

We call the formal adjoint of $\mathcal{P}$, that we denote by $\mathcal{P}^*$, the following operator:

$$\mathcal{P}^* : \psi = (\psi_1, \dots \psi_p) \in C^\infty(\mathbb{R})^p \mapsto \sum_{i=1}^{r} (-1)^{r\,t}C_r(\psi^{(r)}) \in C^\infty(\mathbb{R})^q.$$

The name "formal adjoint" comes from the fact that the following property is verified: for any $\varphi \in C_0^\infty(\mathbb{R})^q$ and any $\psi \in C_0^\infty(\mathbb{R})^p$, we have, using the definition of ${}^tC_i$ on the second line and an integration by parts on the third line (there are no boundary terms since the functions we consider are compactly supported),

$$\begin{aligned}
\int_{\mathbb{R}} \langle \mathcal{P}\varphi, \psi\rangle_{\mathbb{R}^p} &= \sum_{i=1}^{r} \int_{\mathbb{R}} \langle C_r(\varphi^{(r)}), \psi\rangle_{\mathbb{R}^p} \\
&= \sum_{i=1}^{r} \int_{\mathbb{R}} \langle \varphi^{(r)}, {}^t C_r\psi\rangle_{\mathbb{R}^q} \\
&= \sum_{i=1}^{r} \int_{\mathbb{R}} \langle \varphi, ((-1)^{r\,t} C_r\psi)^{(r)}\rangle_{\mathbb{R}^q} \\
&= \sum_{i=1}^{r} \int_{\mathbb{R}} \langle \varphi, (-1)^{r\,t} C_r(\psi^{(r)})\rangle_{\mathbb{R}^q} \\
&= \int_{\mathbb{R}} \langle \varphi, \mathcal{P}^*\psi\rangle_{\mathbb{R}^q}.
\end{aligned}$$

As for the usual adjoint of an operator, we can verify the involution property $(\mathcal{P})^{**} = \mathcal{P}$ and it is very easy to compute the formal adjoint of a composition $\mathcal{P} \circ \mathcal{Q}$ (where $\mathcal{Q}$ is another linear differential operator with appropriate size) and obtain the formula $(\mathcal{P} \circ \mathcal{Q})^* = \mathcal{Q}^* \circ \mathcal{P}^*$.

Coming back to (1.1), our problem can be reduced to find some operator $\mathcal{N} : C^\infty(\mathbb{R})^2 \to C^\infty(\mathbb{R})$ such that

$$\mathcal{N} \circ \mathcal{L}^* = \mathrm{Id}_{C^\infty(\mathbb{R})}. \tag{1.2}$$

Using the involution property of the formal adjoint, one can then recover $\mathcal{M}$ by posing $\mathcal{M} = \mathcal{N}^*$. Obviously, the identity (1.2) implies the following injectivity property: for any $\psi \in C^\infty(\mathbb{R})$,

$$\mathcal{L}^*\psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \psi(= \mathcal{M}^*(\mathcal{L}^*\psi)) = 0.$$

Now, we compute $\mathcal{L}^* : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})^2$ according to the previous rule:

$$\mathcal{L}^* = \begin{pmatrix} a_1 + a_2\partial_t + a_3\partial_{tt} \\ b_1 + b_2\partial_t + b_3\partial_{tt} \end{pmatrix}.$$

Let us solve $\mathcal{L}^*\psi = 0_{\mathbb{R}^2}$. This system is now overdetermined from an analytic point of view (we have 2 equations and 1 unknown ψ). However, from an algebraic point of view, if we forget that ψ' and ψ'' are derivatives of ψ and we consider ψ, ψ', ψ'' as independent algebraic unknowns, the system becomes underdetermined because we have 2 equations and 3 unknowns. Now, we write down explicitely what is the system of equations $\mathcal{L}^*\psi = 0_{\mathbb{R}^2}$:

$$\begin{cases} a_1\psi + a_2\psi' + a_3\psi'' = 0, \\ b_1\psi + b_2\psi' + b_3\psi'' = 0. \end{cases} \tag{1.3}$$

In order to make this system being well-posed in an algebraic point of view (*i.e.* we would like to have as many equations as independent algebraic unknowns), we differentiate these two equations. We obtain

$$\begin{cases} a_1\psi' + a_2\psi'' + a_3\psi''' = 0, \\ b_1\psi' + b_2\psi'' + b_3\psi''' = 0. \end{cases} \tag{1.4}$$

If we are interested in the system formed by (1.3) et (1.4), we remark that if we see ψ, ψ', ψ'' and ψ''' as independent algebraic unknowns, we obtain a system with 4 equations and 4 unknowns, that we can write under the form $C(\psi, \psi', \psi'', \psi''') = 0$ with

$$C = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Hence, under some generic algebraic relation between the coefficients of C (so that C is invertible, *i.e.* the determinant of C is nonzero), necessarily, we have $\psi \equiv 0$. Moreover, inverting the previous idea, the matrix C^{-1} can also be seen as a differential operator, in the sense that $C^{-1}(y, y', y'', y''') = \mathcal{P}(y)$ for some linear differential operator $\mathcal{P} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})^4$. More precisely, let us write C^{-1} under the form

$$C^{-1} := \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}.$$

We have the identity

$$C^{-1}C \begin{pmatrix} \psi \\ \psi' \\ \psi'' \\ \psi''' \end{pmatrix} = \begin{pmatrix} \psi \\ \psi' \\ \psi'' \\ \psi''' \end{pmatrix}.$$

If we consider the first line of this system of 4 equations, we obtain notably

$$\psi = c_{11}(a_1\psi + a_2\psi' + a_3\psi'') + c_{12}(b_1\psi + b_2\psi' + b_3\psi'') \\ + c_{13}(a_1\psi' + a_2\psi'' + a_3\psi''') + c_{14}(b_1\psi' + b_2\psi'' + b_3\psi''').$$

Since we want to write it under the form $\mathcal{N}(\mathcal{L}^*\psi) = \psi$, where $\mathcal{L}^*\psi$ is given by

$$\mathcal{L}^*\psi = \begin{pmatrix} a_1\psi + a_2\psi' + a_3\psi'' \\ b_1\psi + b_1\psi' + b_3\psi'' \end{pmatrix},$$

we choose

$$\mathcal{N} := \begin{pmatrix} c_{11}\mathrm{Id} + c_{13}\partial_t, & c_{12}\mathrm{Id} + c_{14}\partial_t \end{pmatrix} : C^\infty(\mathbb{R})^2 \to C^\infty(\mathbb{R}),$$

so that

$$\mathcal{N}^* = \mathcal{M} = \begin{pmatrix} c_{11}\mathrm{Id} - c_{13}\partial_t \\ c_{12}\mathrm{Id} - c_{14}\partial_t \end{pmatrix} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})^2.$$

Here, we say that we have solved *algebraically* the system $\mathcal{L}(x_1, x_2) = f$. The use of this notion of algebraic solvability and its link with the controllability properties of underdetermined systems will be made more precise later on.

1.3. **Link with the h-principle.** In this section, we aim to explain briefly in a more abstract way where the ideas that underly the approach of the previous example come from. Interested readers may consult [16, Part 2] and [20, Section 2.3.8].

Consider some (linear or nonlinear, ordinary or partial) differential equation that is written as

$$\Phi(\varphi, J^k_\varphi) = 0, \tag{1.5}$$

where $\varphi = (\varphi_1, \dots \varphi_n)$ is some smooth function (defined for instance on $\mathbb{R}^d$ with $d \in \mathbb{N}^*$) and J^k_φ represents all the derivatives up to order $k \in \mathbb{N}^*$. Note that it is a particular case of a closed partial differential relation, in the sense of [16, Sections 5.1 and 5.2]. Clearly, one necessary condition for (1.5) to have a solution is that the *algebraic equation*

$$\Phi(\varphi, Y) = 0, \tag{1.6}$$

where $Y = (y_1, \dots y_{N_k})$, has a solution (N_k is the number of derivatives of the components of φ up to order k). This leads to the following definitions.

Definition 1.2. A solution (φ, Y) of (1.6) is called a *formal* (or *non-holonomic*) solution of (1.5). A solution φ to the original problem (1.5) is called a *genuine* (or *holonomic*) solution of (1.5).

Assume that (1.5) is an underdetermined system, *i.e.* there are more equations than "unknowns" (which are just the components of φ). In general, the algebraic version (1.6) will be *overdetermined* if we consider all the algebraic unknowns $(\varphi, y_1, \dots y_{N_k})$. To bypass this difficulty, as in the previous example, we differentiate (with respect to all the variables) as many times as needed the equations of (1.5). Since new derivatives of φ appear, we make some new unknowns and new equations appear.

In the case of a underdetermined linear partial differential equation with constant coefficients, it can be proved theoretically (see [20, Section 2.3.8]) that we can differentiate enough times in order to have as many equations as "algebraic unknowns" (meaning as before that φ and all its derivatives are seen as independent algebraic unknowns). Then, under generic conditions on the coefficients of Φ, we can invert this well-posed algebraic system in order to solve (1.6). Hence, as in the previous example, we have found a genuine solution to (1.5) just by selecting an appropriate equation between the ones at our disposal. This leads to the following definition.

Definition 1.3. The system (1.5) satisfies the *h-principle* if any formal solution (φ, Y) of (1.5) can be deformed into a genuine solution g of (1.5) in the space of formal solutions of (1.5), in the following sense: one can find an homotopy (*i.e.* a continuous deformation) that brings (φ, Y) to g, in the class of non-holonomic solutions.

Note that all these definitions can be extended to the more general setting of partial differential relations.

2. Controllability of a coupled system of ordinary differential equations

2.1. **Controllability.**

Consider some linear operator A (for instance, a matrix, or a partial differential operator like $-\Delta$, where Δ denotes the Dirichlet-Laplace operator on some bounded domain of $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$), posed on some real or complex Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$ (which is called the *state space*). We consider a linear dynamical system of the form

$$\begin{cases} y'(t) = Ay(t),\ t \in [0,T], \\ y(0) = y^0 \in H, \end{cases} \tag{2.7}$$

where for any $t \in [0,T]$, $y(t) \in H$. y is the *state* of the system: one has to imagine some physical, chemical or biological quantity (*e.g.* the temperature, the concentration of some substance), that evolves during the time.

Under reasonable assumptions on A (see Remark 2.2), this system is well-posed: for any $y^0 \in H$, there exists a unique solution to (2.7) in the space $C^0([0,T], H)$. This solution verifies: for any $T > 0$, there exists a constant $C(T) > 0$ such that for any $y^0 \in H$, we have

$$||y||_{C^0([0,T],H)} \leqslant C(T) ||y^0||_H.$$

This is a *free* evolution: As soon as y^0 is fixed, we cannot choose the value of $y(T)$. We cannot *act* on the dynamical system. Hence, one natural question is to understand how we can introduce some way to influence the dynamics of (2.7).

A good model is the following. Let us consider some other Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ (which will be called the *control space*) and some $B \in \mathcal{L}_c(U, H)$ (the space of linear continuous maps from U to H). B is called a *linear bounded control operator.*

Definition 2.1. A *linear control system* is an equation of the form

$$\begin{cases} y'(t) = Ay(t) + Bu(t),\ t \in [0, T], \\ y(0) = y^0 \in H, \end{cases} \tag{2.8}$$

where $u \in L^2((0, T), U)$. u is called *the control.*

Remind that the definition of $L^2((0, T), U)$ is given by

$$L^2((0,T),U) := \{u : (0,T) \to U \ \ measurable\ s.t. \int_0^T ||u(t)||_U^2 dt < \infty\}.$$

(We could have introduced other functional spaces, but this choice is justified by the fact that it is more convenient to work in Hilbert spaces). We assume that we also have well-posedness, in the following sense: for fixed $u \in L^2((0, T), U)$, there exists a unique solution y to (2.8) in the space $C^0([0, T], H)$, verifying moreover: for any $T > 0$, there exists a constant $C(T) > 0$ such that for any $y^0 \in H$, we have

$$||y||_{C^0([0,T],H)} \leqslant C(T) \left(||y^0||_H + ||u||_{L^2((0,T),U)}\right).$$

This abstract setting is very useful to cover a wide range of situations, but not all possible situations (*e.g.* nonlinear control systems).

Remark 2.2. Concerning the well-posedness of (2.7), a good framework is the semigroup theory: we assume that A is a close unbounded operator on H with dense domain, that verifies two additional properties:

- A is dissipative, *i.e.*, for any $x \in H$, $\mathrm{Re}\left(\langle x, Ax \rangle\right) \leqslant 0$,
- A is maximal, *i.e.*, for any $\lambda > 0$, $\mathrm{Id} - \lambda A$ is surjective.

We refer for instance to [28, Theorem 4.3, Page 14]).

Concerning the well-posedness of (2.8) under the above assumptions on A, we refer to [8, Section 2.3] or [32, Chapter 4] for more explanations.

Now, if we consider (2.8) without fixing u, *i.e.* meaning that the pair (y, u) is considered as an unknown, (2.8) becomes what we have already called an *underdetermined system*: if we imagine that a well-posed system like (2.7) requires to have "as many equations as unknowns", in a system like (2.8), we have "more unknowns" than equations (u also appears at least in one equation). Moreover, it is likely that $y(T)$ depends strongly on the choice of our control u. We have gained some "degree of freedom" to act on the equation.

One very natural question associated to (2.8) is the following: for any initial condition, can we find a control that put the system at rest at some time $T > 0$? This is exactly the purpose of the following definition.

Definition 2.3. The linear control system (2.8) is said to be *null-controllable* if, for any $y^0 \in H$, there exists some $u \in L^2((0,T), H)$ such that $y(T) = 0$ in H.

This notion is particularly interesting: as soon as $y(T) = 0$, if we switch off the control (*i.e.*, we set $u = 0$ on $[T, \infty)$), up to a translation in time, we are back to the free system (2.7) with "initial condition" $0 \in H$. Since 0 is an equilibrium of (2.7), the only solution is 0 and the solution stays at rest forever for any $t \geqslant T$.

Question: what kind of condition we need on A and B to ensure that the null-controllability property holds?

In such a general setting, this question does not really make sense, so we will investigate some precise examples. To begin, let us consider the following trivial example.

Example 2.4. We consider $H = \mathbb{R}$, $A = a \in \mathbb{R}$, $B = \mathrm{Id}_{\mathbb{R}}$, and the following very simple scalar linear ordinary differential equation with constant coefficients

$$\begin{cases} y'(t) = ay(t) + u(t),\ t \in [0,T], \\ y(0) = y^0 \in \mathbb{R}. \end{cases} \tag{2.9}$$

Let us fix some $y^0 \in \mathbb{R}$ and some final time $T > 0$. Can we find u such that the solution of (2.9) verifies $y(T) = 0$?

The answer is yes, and it is very easy to construct "by hand" a control. Consider any function $y : \mathbb{R}^+ \to \mathbb{R}$ such that:

- $y \in C^\infty(\mathbb{R}^+, \mathbb{R})$,
- $y(0) = y^0$,
- The support (in the ambient space $\mathbb{R}^+$) of y is included in $[0, T]$.

We then introduce $u(t) = y'(t) - ay(t)$. It is clear that:

- $u \in C^\infty(\mathbb{R}^+, \mathbb{R})$ (so notably $u \in L^2((0,T), \mathbb{R})$),
- $(2.9)_1$ is verified by construction of u (here and in what follows, $(S)_i$ means that we consider the i-th line of system (S)),

- $(2.9)_2$ is also verified by construction of y,
- $y(T) = 0$ since y is supported in $[0, T]$.

Of course, the above example is quite trivial, and does not really highlight the deepness of the problems we will have to face in more complex situations.

Remark 2.5. In the above example, we have an infinite choice of functions for y. Hence, we do not have uniqueness of the control u and the trajectory y in the problem of null-controllability raised in Definition 2.1.

2.2. **Systems of two linear ODEs with one control.** Now, let us consider the following system of two linear ordinary differential equations with constant coefficients

$$
\begin{cases} y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t),\ t \in [0, T], \\ y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + u(t),\ t \in [0, T], \\ y_1(0) = y_1^0 \in \mathbb{R},\ \ y_2(0) = y_2^0 \in \mathbb{R}. \end{cases} \tag{2.10}
$$

Let us put this system in the abstract setting of (2.8). Here, $H = \mathbb{R}^2$, $U = \mathbb{R}$. The operator $A \in \mathcal{L}(\mathbb{R}^2)$ is the matrix of size 2×2 given by

$$
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
$$

and the control operator $B \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ is the matrix of size 2×1 given by

$$
B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

We ask the same question: let us fix some initial condition $(y_1^0, y_2^0) \in \mathbb{R}^2$ and some final time $T > 0$. Can we find u such that the solution of (2.10) verifies $y_1(T) = y_2(T) = 0$? Of course, here, the method we used for the scalar equation fails: if we choose any (y_1, y_2) regular enough verifying $(y_1(0), y_2(0)) = (y_1^0, y_2^0)$ and $(y_1(T), y_2(T)) = (0, 0)$, there is no reason for $(2.10)_1$ to be verified.

Equation (2.10) is the prototype of what we will call an *underactuated* system:

- We have *two* unknowns, y_1 and y_2, that are the states we would like to control.
- But we have only *one* control u, on the last equation.

In other worlds, we control only "directly" the quantity y_2, and we hope that thanks to the coupling terms, we can "indirectly" control y_1 "through" y_2 (for this reason, this situation is called a problem of "indirect control").

This phenomenon can be observed in real-life models. For instance, in the model of tumour growth presented in [6], three quantities are involved: the drug concentration, the healthy and the non-healthy cells. The resulting system consists of three equations, that are coupled with some nonlinear reaction terms. The control is a drug bolus inside a small subdomain. It only influences directly the drug concentration, so that the structure of the coupling terms is crucial to understand how we can kill all the non-healthy cells.

Question: What are reasonable conditions we need to impose on the coupling coefficients $a_{11}, \dots, a_{22}$ in order to ensure controllability?

One necessary condition is clearly the following.

Lemma 2.6. *In order that* (2.10) *is null-controllable, it is necessary to have* $a_{12} \neq 0$*.*

PROOF. If $a_{12} = 0$, then y_1 verifies the equation

$$y_1' = a_{11} y_1.$$

The control u has no influence on the dynamics of y_1. Hence, we can solve explicitly this very simple ordinary differential equation and find that

$$y_1(t) = y_1^0 e^{a_{11} t}.$$

As soon as $y_1^0 \neq 0$, obviously $y(T) \neq 0$, and we conclude that the system cannot be null-controllable in this case. □

In fact, it is remarkable that this condition is also *sufficient* to ensure null-controllability.

Theorem 2.7. *If* $a_{12} \neq 0$*, then* (2.10) *is null-controllable.*

PROOF OF THEOREM 2.7

In order to prove this Theorem, we will use a *fictitious control* argument, in the spirit of [8, Second proof of Theorem 1.18]. The strategy can be briefly described as folllows:

(1) Firstly, we control our system of equations with one control on each equation, *i.e.* we solve the control problem

$$\begin{cases} \widehat{y}_1'(t) = a_{11}\widehat{y}_1(t) + a_{12}\widehat{y}_2(t) + \widehat{u}_1(t),\ t \in [0,T], \\ \widehat{y}_2'(t) = a_{21}\widehat{y}_1(t) + a_{22}\widehat{y}_2(t) + \widehat{u}_2(t),\ t \in [0,T], \\ \widehat{y}_1(0) = y_1^0 \in \mathbb{R},\ \widehat{y}_2(0) = y_2^0 \in \mathbb{R}, \\ \widehat{y}_1(T) = 0,\ \widehat{y}_2(T) = 0. \end{cases}$$

This can be done by a direct construction, which is very similar to Example 2.4. Moreover, it is easy to construct $\widehat{u}_1$ and $\widehat{u}_2$ in such a way that they are smooth and compactly supported in $(0,T)$ (their support will be included in $[\frac{T}{3}, \frac{2T}{3}]$).

(2) Secondly, we solve the auxiliary control problem

$$\begin{cases} \tilde{y}_1'(t) = a_{11}\tilde{y}_1(t) + a_{12}\tilde{y}_2(t) + \widehat{u}_1(t),\ t \in [0,T], \\ \tilde{y}_2'(t) = a_{21}\tilde{y}_1(t) + a_{22}\tilde{y}_2(t) + \widehat{u}_2(t) + \tilde{u}(t),\ t \in [0,T], \\ \tilde{y}_1(0) = 0, \quad \tilde{y}_2(0) = 0, \\ \tilde{y}_1(T) = 0, \quad \tilde{y}_2(T) = 0. \end{cases}$$

$(\widehat{u}_1, \widehat{u}_2)$ is the same as the one introduced in the previous step and is seen as a source term. The unknowns are now $(\tilde{y}_1, \tilde{y}_2, \tilde{u})$. We are typically in a situation that is similar to the introducing example explained in Section 1.2: the system is underdetermined, and we will find a solution that can be written as a linear combination of the source term $(\widehat{u}_1, \widehat{u}_2)$ and its derivatives by applying the same procedure as in Section 1.2.

(3) To conclude, we investigate the system that is verified by $y_1 = \widehat{y}_1 - \tilde{y}_1$ and $y_2 = \widehat{y}_2 - \tilde{y}_2$, and we prove that it provides a solution to our desired control problem: there exists a control u such that (y_1, y_2) verifies (2.10) together with $y_1(T) = 0$ and $y_2(T) = 0$. Moreover, during this procedure, the “fictitious control” $(\widehat{u}_1, \widehat{u}_2)$ disappears, justifying the name of the method.

2.2.1. *First step: analytic part.* Let us introduce the following auxiliary control problem

$$\begin{cases} \widehat{y}_1'(t) = a_{11}\widehat{y}_1(t) + a_{12}\widehat{y}_2(t) + \widehat{u}_1(t),\ t \in [0,T], \\ \widehat{y}_2'(t) = a_{21}\widehat{y}_1(t) + a_{22}\widehat{y}_2(t) + \widehat{u}_2(t),\ t \in [0,T], \\ \widehat{y}_1(0) = y_1^0 \in \mathbb{R},\ \widehat{y}_2(0) = y_2^0 \in \mathbb{R}, \\ \widehat{y}_1(T) = 0,\ \widehat{y}_2(T) = 0, \end{cases} \tag{2.11}$$

where we have now two controls $\widehat{u}_1$ and $\widehat{u}_2$, each acting on one of the equations. This means that the control operator is now

$$\widehat{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As in the case of one equation described in Example 2.4, it is very easy to exhibit a solution to (2.11). However, we will require some additional properties (for some reasons that will become clear later on).

We introduce some function $\eta : [0,T] \to \mathbb{R}$ such that :

- η is of class C^∞ on $[0,T]$,
- $\eta = 1$ on $[0, \frac{T}{3}]$,
- $\eta = 0$ on $[\frac{2T}{3}, T]$.

Now, we consider the solution to the (free) equation

$$\left\{\begin{array}{l} y_1^{F\prime}(t) = a_{11}y_1^F(t) + a_{12}y_2^F(t),\ t \in [0,T], \\ y_2^{F\prime}(t) = a_{21}y_1^F(t) + a_{22}y_2^F(t),\ t \in [0,T], \\ y_1(0) = y_1^0 \in \mathbb{R},\ y_2(0) = y_2^0 \in \mathbb{R}. \end{array}\right. \tag{2.12}$$

We introduce

$$\begin{pmatrix} \widehat{y}_1 \\ \widehat{y}_2 \end{pmatrix} = \begin{pmatrix} \eta y_1^F \\ \eta y_2^F \end{pmatrix}$$

and

$$\begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix} = \begin{pmatrix} \widehat{y}_1' - a_{11}\widehat{y}_1 - a_{12}\widehat{y}_2 \\ \widehat{y}_2' - a_{21}\widehat{y}_1 - a_{22}\widehat{y}_2 \end{pmatrix}.$$

$(\widehat{u}_1, \widehat{u}_2)$ is called the *fictitious control* (for reasons that will appear at the end of the next step).

Then:

- $(\widehat{y}_1, \widehat{y}_2, \widehat{u}_1, \widehat{u}_2)$ verifies $(2.11)_{1-2}$ by construction,
- We also have $(2.11)_3$ by definition of $(\widehat{y}_1, \widehat{y}_2)$, since $\eta(0) = 1$,
- $(2.11)_4$ is also verified: $\eta = 0$ on $[\frac{2T}{3}, T]$, so this is also the case for $\widehat{y}_1$ and $\widehat{y}_2$,
- $\widehat{y}_1, \widehat{y}_2, \widehat{u}_1, \widehat{u}_2$ are C^∞ on $[0,T]$, since y_1^F, y_2^F are clearly in $C^\infty([0,T],\mathbb{R})$ and η is also in $C^\infty([0,T],\mathbb{R})$,
- The controls $(\widehat{u}_1, \widehat{u}_2)$ are compactly supported in $(0,T)$. Indeed, on $[0, \frac{T}{3}]$, by definition of $(\widehat{u}_1, \widehat{u}_2)$ and since $\eta = 1$, using (2.12), we have

$$\begin{pmatrix} \widehat{u}_1(t) \\ \widehat{u}_2(t) \end{pmatrix} = \begin{pmatrix} y_1^{F\prime}(t) - a_{11}y_1^F(t) - a_{12}y_2^F(t) \\ y_2^{F\prime}(t) - a_{21}y_1^F(t) - a_{22}y_2^F(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

  and on $[\frac{2T}{3}, T]$, by definition of $(\widehat{u}_1, \widehat{u}_2)$ and since $\eta = 0$, we also have $(\widehat{u}_1, \widehat{u}_2) = (0,0)$ on $[\frac{2T}{3}, T]$.

2.2.2. *Second step: algebraic part.* Now, we would like to solve the following auxiliary control problem:

$$\left\{\begin{array}{l} \tilde{y}_1'(t) = a_{11}\tilde{y}_1(t) + a_{12}\tilde{y}_2(t) + \widehat{u}_1(t),\ t \in [0,T], \\ \tilde{y}_2'(t) = a_{21}\tilde{y}_1(t) + a_{22}\tilde{y}_2(t) + \widehat{u}_2(t) + \tilde{u}(t),\ t \in [0,T], \\ \tilde{y}_1(0) = 0, \quad \tilde{y}_2(0) = 0, \\ \tilde{y}_1(T) = 0, \quad \tilde{y}_2(T) = 0. \end{array}\right. \tag{2.13}$$

Note that $(\widehat{u}_1, \widehat{u}_2)$ is the same as in (2.11), and has to be seen as a *source term.* The unknowns are now $(\tilde{y}_1, \tilde{y}_2, \tilde{u})$.

Remark that the system is now *underdetermined*: we have 2 equations and 3 unknowns. Hence, our intuition is that this system is likely to admit multiples solutions. We will select some particular solution, thanks to the *algebraic solvability* procedure that we explained in Section 1.2. This means that we will express $(\tilde{y}_1, \tilde{y}_2, \tilde{u})$ as a linear combination of the source term $(\widehat{u}_1, \widehat{u}_2)$ and its derivatives. For the moment, we are only interested in $(2.13)_{1-2}$ (the fact that $(2.13)_{3-4}$ are verified will be a consequence of our construction and will be explained at the end of the reasoning). We rewrite $(2.13)_{1-2}$ in an abstract way as

$$\mathcal{L}\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{u}\end{pmatrix} = f, \tag{2.14}$$

where $\mathcal{L} : C^\infty([0,T],\mathbb{R}^3) \to C^\infty([0,T],\mathbb{R}^2)$ is the ordinary differential operator given by

$$\mathcal{L}\begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{u}\end{pmatrix} = \begin{pmatrix}\tilde{y}_1'(t) - a_{11}\tilde{y}_1(t) - a_{12}\tilde{y}_2(t)\\ \tilde{y}_2'(t) - a_{21}\tilde{y}_1(t) - a_{22}\tilde{y}_2(t) - \tilde{u}(t)\end{pmatrix},$$

and the source term f is given by

$$f = \begin{pmatrix}\widehat{u}_1\\ \widehat{u}_2\end{pmatrix}.$$

We can reformulate in an equivalent way our question as follows: find some *right inverse* to $\mathcal{L}$, *i.e.* some linear differential operator $\mathcal{M} : C^\infty([0,T],\mathbb{R}^2) \to C^\infty([0,T],\mathbb{R}^3)$ such that

$$\mathcal{L}\circ\mathcal{M} = \mathrm{Id}_{C^\infty([0,T],\mathbb{R}^2)}. \tag{2.15}$$

Indeed, the above equality applied to the source term gives $\mathcal{L}(\mathcal{M}(f)) = f$, so that if we define

$$\mathcal{M}(f) = \begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{u}\end{pmatrix},$$

we have found a solution to (2.14) by definition. Hence, our goal is now to solve (2.15).

From (2.15), we get

$$\mathcal{M}^*\circ\mathcal{L}^* = \mathrm{Id}_{C^\infty([0,T],\mathbb{R}^2)}. \tag{2.16}$$

Here, * correspond to the formal adjoint with respect to the L^2-norm introduced in Definition 1.1. According to this Definition, its expression is given here by

$$\mathcal{L}^* = \begin{pmatrix} -\partial_t - a_{11}\mathrm{Id} & -a_{21}\mathrm{Id} \\ -a_{12}\mathrm{Id} & -\partial_t - a_{22}\mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix},$$

i.e.

$$\mathcal{L}^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\varphi_1' - a_{11}\varphi_1 - a_{21}\varphi_2 \\ -\varphi_2' - a_{12}\varphi_1 - a_{22}\varphi_2 \\ -\varphi_2 \end{pmatrix}. \tag{2.17}$$

Here, Id means the identity on $C^\infty([0,T],\mathbb{R})$.

Now, let us go back to our goal: we want to find some linear differential operator $\mathcal{M}^*$ such that (2.16) holds. It means that we want to differentiate and make linear combinations on the lines of (2.17) in order to recover φ_1 and φ_2. In fact, in $(2.17)_3$, φ_2 is already there. Hence, we just have to recover φ_1. In order to reach this aim, we combine $(2.17)_2$ with $(2.17)_3$ in order to recover φ_1, by applying a differential operator of order 1 to $(2.17)_3$:

$$\varphi_1 = \frac{-(-\varphi_2' - a_{12}\varphi_1 - a_{22}\varphi_2) + (\partial_t + a_{22})(-\varphi_2)}{a_{12}}, \tag{2.18}$$

were we have used our hypothesis $a_{12} \neq 0$. Taking into account this computation, it is natural to introduce $\mathcal{M}^*$ as

$$\mathcal{M}^* \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} \frac{-g_2 + g_3' + a_{22}g_3}{a_{12}} \\ -g_3 \end{pmatrix}.$$

By construction, using (2.17) and (2.18), we have

$$\mathcal{M}^* \left(\mathcal{L}^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

From Definition 1.1 and the fact that * is an involution, it is easy to go back to $\mathcal{M}(= (\mathcal{M}^*)^*)$. Indeed, $\mathcal{M}^*$ can be written in matricial form as

$$\mathcal{M}^* = \begin{pmatrix} 0 & -\frac{\mathrm{Id}}{a_{12}} & \frac{\partial_t + a_{22}\mathrm{Id}}{a_{12}} \\ 0 & 0 & -\mathrm{Id} \end{pmatrix},$$

so that

$$\mathcal{M} = \begin{pmatrix} 0 & 0 \\ -\frac{\mathrm{Id}}{a_{12}} & 0 \\ \frac{-\partial_t + a_{22}\mathrm{Id}}{a_{12}} & -\mathrm{Id} \end{pmatrix}.$$

$\mathcal{M}$ is here a linear differential operator of order 1. Let us verify for the sake of security that such a $\mathcal{M}$ answers the initial problem $(2.13)_{1-2}$. Indeed, from (2.14), we introduce

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{u} \end{pmatrix} = \mathcal{M}\begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\widehat{u}_1}{a_{12}} \\ -\widehat{u}_2 + \frac{-\widehat{u}_1' + a_{22}\widehat{u}_1}{a_{12}} \end{pmatrix}.$$

Let us verify that $(2.13)_{1-2}$ holds. Indeed, we remark that:

- By definition of $\tilde{y}_1$, we have
$$\tilde{y}_1' = 0,$$
and by definition of $\tilde{y}_1$ and $\tilde{y}_2$, we also have
$$a_{12}\tilde{y}_2 + \widehat{u}_1 + a_{11}\tilde{y}_1 = 0.$$
We deduce that $(2.13)_1$ is verified.
- By definition of $\tilde{y}_2$, we have
$$\tilde{y}_2' = -\frac{\widehat{u}_1'}{a_{12}},$$
whereas by definition of $\tilde{y}_1$ and $\tilde{y}_2$, we have
$$a_{21}\tilde{y}_1 + a_{22}\tilde{y}_2 + \widehat{u}_2 + \tilde{u} = -\frac{a_{22}\widehat{u}_1}{a_{12}} + \widehat{u}_2 - \widehat{u}_2 + \frac{-\widehat{u}_1' + a_{22}\widehat{u}_1}{a_{12}} = -\frac{\widehat{u}_1'}{a_{12}}.$$
We deduce that $(2.13)_2$ is verified.

Remark that the source term $f = (\widehat{u}_1, \widehat{u}_2)$ is compactly supported in $(0,T)$. Hence, this is also the case of $(\tilde{y}_1, \tilde{y}_2, \tilde{u})$: by our construction, these functions only involve linear combinations and derivative (of order 1 here) of our source term f. This implies that the support of $(\tilde{y}_1, \tilde{y}_2, \tilde{u})$ is included in the support of f. This is in fact a crucial point and the main interest of the method. Hence, the initial and final conditions $(2.13)_{3-4}$ are verified automatically. This concludes the algebraic part.

2.2.3. *Conclusion.* We combine both the analytic and algebraic part. We introduce:

- $y_1 = \widehat{y}_1 - \tilde{y}_1$,
- $y_2 = \widehat{y}_2 - \tilde{y}_2$,
- $u = -\tilde{u}$.

Then, from (2.11) and (2.13), we have:

- $y_1(0) = \widehat{y}_1(0) - \tilde{y}_1(0) = y_1^0 - 0 = y_1^0$, since $\tilde{y}_1$ is compactly supported in $(0, T)$ (so that it vanishes on a neighbourhood of 0).
- $y_2(0) = \widehat{y}_2(0) - \tilde{y}_2(0) = y_2^0 - 0 = y_2^0$, for the same reasons.
- $y_1(T) = \widehat{y}_1(T) - \tilde{y}_1(T) = 0 - 0 = 0$, since $\widehat{y}_1$ is controlled to 0 at time T and $\tilde{y}_1$ is compactly supported in $(0, T)$ (so that it vanishes on a neighbourhood of T).
- $y_2(T) = \widehat{y}_2(T) - \tilde{y}_2(T) = 0 - 0 = 0$, for the same reasons.
- By linearity,

$$\begin{aligned} y_1' = \widehat{y}_1' - \tilde{y}_1' &= a_{11}\widehat{y}_1 + a_{12}\widehat{y}_2 + \widehat{u}_1 - a_{11}\tilde{y}_1 - a_{12}\tilde{y}_2 - \widehat{u}_1 \\ &= a_{11}\left(\widehat{y}_1 - \tilde{y}_1\right) + a_{12}\left(\widehat{y}_2 - \tilde{y}_2\right) \\ &= a_{11}y_1 + a_{12}y_2 \end{aligned}$$

and

$$\begin{aligned} y_2' = \widehat{y}_2' - \tilde{y}_2' &= a_{21}\widehat{y}_1 + a_{22}\widehat{y}_2 + \widehat{u}_2 - a_{21}\tilde{y}_1 - a_{22}\tilde{y}_2 - \widehat{u}_2 - \tilde{u} \\ &= a_{21}\left(\widehat{y}_1 - \tilde{y}_1\right) + a_{22}\left(\widehat{y}_2 - \tilde{y}_2\right) - \tilde{u} \\ &= a_{21}y_1 + a_{22}y_2 + u. \end{aligned}$$

Hence, (y_1, y_2, u) solves the initial control problem (2.10).

We remark that the "fictitious control" $(\widehat{u}_1, \widehat{u}_2)$ has disappeared. This justifies our terminology. □

2.3. **The scope of the method.** Let us enumerate some advantages and drawbacks of the method:

- We can deal both with constant and non-constant coefficients (in time and also in space, if we deal with PDEs). In fact, the non-constant case is in some sense richer, because one can get some help from the non-commutativity between the coefficients and the differential operators (which of course does not happen in the case of constant coefficients), that will add some extra coupling terms in our differentiation procedure. This has notably be strongly used in [11].
- It can be also used for the study of nonlinear systems, because it combines very well with the *return method* of Coron (see [9]). This method relies on a linearization procedure around some particular trajectories that we can choose in different ways (notably so that the algebraic solvability procedure described above works). This is exactly the spirit of the computations developed in [11].
- The method is purely *local*. It can be seen if we modify a little bit the example developed in Section 2.2. Assume that we change the constant coefficient a_{12} to a time-varying coefficient $a_{12}(t)$ that we choose to be equal to 0 on $[\frac{T}{3}, \frac{2T}{3}]$. Then, since we are working locally on $[\frac{T}{3}, \frac{2T}{3}]$

in the algebraic part, we do not "see" the coupling term a_{12} on this time interval and then the algebraic solvability fails in this context. Of course, this example is quite artificial (instead of controlling on $[\frac{T}{3}, \frac{2T}{3}]$, one can choose to control on a time interval where a_{12} is nonzero), but in more complex situations (notably in the context of partial differential equations, where the coupling region can be disjoint from the control region, notably in the space variable), it can become a real problem. Notably, we know situations where the algebraic solvability fails whereas the system is controllable (see *e.g.* [2]).

- This method relies on a differentiation procedure, hence it consumes a lot of regularity: the control created by this procedure is less regular than it "should" be. To be more precise, let us consider the following example: assume that A is a differential operator and the initial condition of the free system (2.7) is very regular. Then, in general, the solution of (2.7) on $[0, T] \times H$ will also be very regular, and will remain very regular if we add some source term f that is regular enough. However, when we use our notion of algebraic solvability, we will need to differentiate a system of the form "$y' = Ay + f$", so that the control obtained in the algebraic solvability procedure will be less regular than f and y. Hence, even if we want to control a system like (2.8) with controls in $L^2((0, T), U)$ for instance, we will need to take initial data that are not in the natural energy space H but that will be much more regular (in the domain of some power of A).

  For parabolic systems like systems of heat equations, this drawback can sometimes be avoided: we have a regularizing effect that ensure smoothness as soon as we are away from the initial condition and we have regular enough coefficients (see *e.g.* [17, Theorem 7, Page 367]). Hence, even if the initial condition is in H, after a short time on which we does not control, it becomes very regular and we can use our algebraic solvability procedure. However, for hyperbolic or dispersive systems (like the wave equation or the Schrödinger equation), the regularizing effect does not exist, and it means that even if we want to control a system with a low regularity control, we need to take very regular initial data (see the following Section 3 and notably system (3.21) and Theorem 3.6: the initial condition is very regular but the control is only in L^2). Hence, we lose sharpness in the results, in the sense that the of initial data that can be controlled is not the entire energy space H and has to be "artificially" reduced to make our argument work.

Let us mention that this method has been successfully used in different contexts in [1, 10, 11, 13, 14, 15, 26, 31].

3. System of Schrödinger equations

Now, we move from ordinary differential equations to an application of the fictitious control method to systems of coupled partial differential equations. The forthcoming example is coming from [26].

3.1. Controllability of the scalar Schrödinger equation.

For the sake of simplicity (in order to avoid problems with the boundary), we will consider the following setting. We consider (M, g) a smooth compact Riemannian manifold, *without boundary*. We denote by Δ the Laplace-Beltrami operator on M. Since $-\Delta$ is a selfadjoint and positive operator with compact resolvent, we can introduce its eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}^*}$, (note that each eigenvalue has finite multiplicity in this context), ordered such that $0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, each eigenvalue λ_k being associated with a corresponding eigenfunction e_k. We can construct the family $\{e_k\}_{k\in\mathbb{N}^*}$ in such a way that $\{e_k\}_{k\in\mathbb{N}^*}$ forms a Hilbert basis of $L^2(M)$. Each function $f \in L^2(M)$ can then be uniquely decomposed as

$$f = \sum_{k\in\mathbb{N}^*} a_k e_k,$$

where $(a_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Moreover, Parseval's identity gives

$$||f||^2_{L^2(M)} = \sum_{k=1}^{\infty} |a_k|^2.$$

Now, let us consider some $s \in \mathbb{R}^+$. We introduce the Sobolev space $H^s(M)$ as

$$H^s(M) := \{f = \sum_{k\in\mathbb{N}^*} a_k e_k \in L^2(M) \text{ s.t. } \sum_{k=1}^{\infty} \left(1 + \lambda_k^{2s}\right) |a_k|^2 < \infty\}.$$

$H^s(M)$ can alternatively be defined as the domain of the operator $(-\Delta)^{\frac{s}{2}}$. For more informations on Riemannian geometry, we refer to [22].

Example 3.1. The simplest example of such a manifold is given by $M = \mathbb{T}^d$ for some $d \in \mathbb{N}^*$, *i.e.* the $d-$dimensional torus. From the point of view of partial differential equations, this would correspond to the unit hypercube $[-1, 1]^d$ in $\mathbb{R}^d$ endowed with the flat metric, with periodic boundary conditions. In $\mathbb{T}^d$ seen as the unit hypercube in $\mathbb{R}^d$, the Laplace-Beltrami operator is just the usual Laplace operator given by

$$\Delta = \sum_{i=1}^{d} \partial^2_{x_i^2}.$$

In this case, for $s \in \mathbb{N}^*$, $H^s(M)$ is the space of periodic function $f \in L^2((-1, 1)^d)$ verifying

$$\partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_d}_{x_d} f \in L^2((-1, 1)^d),$$

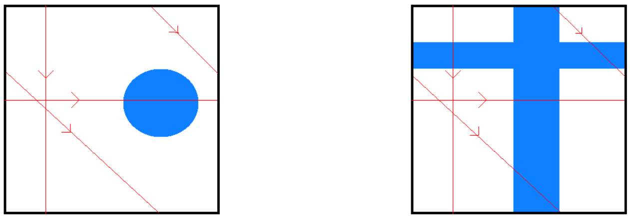

Figure 1. Illustration of GCC. We represent the 2-dimensional torus, the black lines have to be thought as periodic boundary conditions. ω is represented in blue. One horizontal, vertical and oblique ray are drawn in red. On the left picture, GCC is not verified since neither the vertical ray nor the oblique ray never encounter the control region. Conversely, on the right picture, the control region verifies GCC.

for any multi-index $(\alpha_1, \alpha_d) \in \mathbb{N}^d$ such that $\alpha_1 + \alpha_2 \ldots + \alpha_d \leqslant s$.

We will need the following condition.

Definition 3.2. Let ω be an open subset of M. We say that ω verifies the *geometric control condition* (shortened in GCC in what follows) if there exists $T > 0$ such that every geodesic starting from any point of M traveled at speed 1 enters ω before the time T.

This condition is strongly related to the controllability of the wave equation. It is notably proved in [30] and [3] that it is a sufficient condition of controllability, respectively for a manifold without and with boundary (in this last case, some reasonable additional properties on the boundary are also needed). Conversely, this condition is also necessary for the controllability of the wave equation, see [29] and [4].

Let $T > 0$, ω an open subset of M verifying GCC, and $u \in L^2((0,T), M)$. We will be interested in the following evolution equation, called the linear Schrödinger equation:

$$\begin{cases} i\partial_t y + \Delta y = \mathbf{1}_\omega(x) u, \\ y(0) = y^0 \in L^2(M). \end{cases} \tag{3.19}$$

If $u = 0$, this equation describes the evolution in time of the wave function associated to a free (*i.e.* without potential) quantic particle that moves into the manifold.

Equation (3.19) is a linear control system in the sense of Definition 2.1. Moreover, as in Section 2.1, it can be proved that for any $y^0 \in L^2(M)$ and any $u \in L^2((0,T),U)$, there exists a unique solution y to (3.19) (see for instance [8, Section 2.3] or [32, Chapter 4] for more explanations). As in the previous Section, our goal is to obtain the following null-controllability property.

Definition 3.3. We say that (3.19) is *null-controllable* if, for any $y^0 \in L^2(M)$, there exists $u \in L^2((0,T)\times\omega)$ such that the corresponding solution y to (3.19) satisfies $y(T) = 0_{L^2(M)}$.

Concerning controllability results, we have the following sufficient condition.

Theorem 3.4. *Assume that ω verifies GCC for some time $T' > 0$. Then,* (3.19) *is null-controllable for* any *time $T > 0$.*

This theorem was first proved in [27], in the case of a bounded domain of $\mathbb{R}^n$ and boundary control (but the same strategy applies for an internal control). In a more elementary way, it can also be seen as a consequence of the abstract result given in [32, Theorem 6.7.5], taking into account the already mentioned result on the wave equation given in [30].

Note that contrary to the wave equation, in the case of (3.19), GCC is in general far from being necessary, but it depends strongly and in a nontrivial way on the global geometry (*i.e.* on M). For example, if $M = \mathbb{T}^d$, then any nonempty open subset ω can be chosen to obtain the null-controllability of (3.19) (see [21]).

We will also need the following refinement.

Proposition 3.5. *Let $n \in \mathbb{N}^*$. For any open subset Ω of M, we define the following space:*

(3.20)
$$H_0^{n-1}((0,T),L^2(\Omega)) := \{u \in H^{n-1}((0,T),L^2(\Omega)) | u^k_{t\ldots t}(0) = u^k_{t\ldots t}(T) = 0, \forall k \in [|0,n-2|]\}.$$

Assume moreover that:

- $y^0 \in H^{n-1}(M)$ *for some* $n \in \mathbb{N}^*$,
- $\mathbf{1}_\omega$ *is replaced by some regularized version* $\widehat{\mathbf{1}}_\omega \in C^\infty(M)$, *which is supported on some* $\tilde{\omega}$ *containing* ω *with* $\widehat{\mathbf{1}}_\omega = 1$ *on* ω.

Then, there exists a control u such that

$$u \in H_0^{n-1}((0,T), L^2(\tilde{\omega})) \times \bigcap_{k=0}^{n-1} C^k([0,T], H^{2n-2k-2}(\tilde{\omega}))$$

and the solution of (3.19) *verifies* $y(T) = 0$.

This proof of this result can be found in [18] (see also [12]).

3.2. **A system version.** Let $(n,m) \in (\mathbb{N}^*)^2$, with $m < n$. Let $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n,m}(\mathbb{R})$. Let $U = (U_1, \dots U_m) \in L^2((0,T) \times \Omega)^m$. We consider the following system of coupled Schrödinger equations

$$\begin{cases} i\partial_t Y + \Delta Y = AY + BU\widehat{\mathbf{1}}_\omega, \\ \quad Y(0) = Y^0 \in H^{n-1}(M)^n. \end{cases} \tag{3.21}$$

Here and in what follows, for $Y = (Y_1, \dots, Y_n)$, we will write

$$\partial_t Y = \begin{pmatrix} \partial_t Y_1 \\ \dots \\ \partial_t Y_n \end{pmatrix} \text{ and } \Delta Y = \begin{pmatrix} \Delta Y_1 \\ \dots \\ \Delta Y_n \end{pmatrix}.$$

This is a *system* of n Schrödinger equations, that are coupled through the matrix A. Remind that $\widehat{\mathbf{1}}_\omega$ has been introduced in Proposition 3.5.

Since $m < n$, we have less controls than equations, meaning that as in Section 2.2, we have an *underactuated system*. The typical situations are:

(1)
$$B \begin{pmatrix} U_1 \\ \dots \\ U_m \end{pmatrix} = \begin{pmatrix} U_1 \\ \dots \\ U_m \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

Some equations are directly controlled (the m first ones here) and some are not controlled at all: we hope that the coupling terms will help in order to control the equations. This situation is called *indirect controllability*, similarly to the case studied in Section 2.2.

(2)
$$B(U_1) = \begin{pmatrix} U_1 \\ \dots \\ U_1 \end{pmatrix}.$$

Here, all equations are controlled, with the same control on each equation. This situation (which is very different from the previous one) is called *simultaneous controllability.*

We have the following characterization.

Theorem 3.6. *We introduce the following matrix* $[A|B] \in \mathcal{M}_{n,nm}(\mathbb{R})$*, called the Kalman matrix:*

$$[A|B] = [B, AB, A^2B, \dots A^{n-1}B].$$

(3.21) *is null-controllable with* $U \in L^2((0,T)\times\Omega)$ *if and only if* $[A|B]$ *is of full rank* n*.*

This condition is exactly the celebrated *Kalman rank condition* for the controllability of linear ordinary differential equations (see [25] and [23]). Note that by the Cayley-Hamilton Theorem, it is useless to include the matrices A^kB for $k \geqslant n+1$.

Remark 3.7.
- The regularity of the initial condition is very high (but it is necessary in our proof for reasons that will be clear later on). However, it is likely that this is a purely technical assumption. One should be able to obtain a result for initial condition in L^2, but this is an open problem.
- In the case where the Kalman matrix is not of full rank, we are also able to express the initial conditions that can be controlled (that are the ones in $[A|B](H^{n-1}(M))$).

Example 3.8. Consider

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Here, $n = 2$ and $m = 1 < 2$. Then, the Kalman matrix $[A|B] \in \mathcal{M}_2(\mathbb{R})$ is given by

$$[A|B] = [B, AB] = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}.$$

This matrix is of rank 2 if and only if $a_{12} \neq 0$. We find back the necessary and sufficient condition of controllability given in Section 2.2 for system (2.10).

Proof of Theorem 3.6 We only prove the "if" part (the "only if" part is easier, and is similar to the finite-dimensional case, see *e.g.* [8, Proof of Theorem 1.16]). The strategy is the same as in Section 2.2. Let us describe it briefly.

(1) Firstly, we control our system of equations with one control on each equation, *i.e.* we solve the control problem

$$\begin{cases} i\partial_t \widehat{Y} + \Delta \widehat{Y} = A\widehat{Y} + \widehat{U}\widehat{\mathbf{1}}_\omega, \\ Y(0) = Y^0, \\ Y(T) = 0, \end{cases}$$

where the fictitious control $\widehat{U} = (u_1, \dots u_n)$ is acting on each equation. This can be done by using a clever change of unknowns that enables to decouple the system and apply the scalar result of Theorem 3.6 on each equation.

(2) Secondly, we solve the auxiliary control problem

$$\begin{cases} i\partial_t \tilde{Y} + \Delta \tilde{Y} = A\tilde{Y} + B\tilde{U} + \widehat{U}\widehat{\mathbf{1}}_\omega, \\ \tilde{Y}(0) = \tilde{Y}(T) = 0. \end{cases}$$

The unknowns are $(\tilde{Y}, \tilde{U})$ and $\widehat{U}$ is the control created in the previous analytic part, seen as a source term. We apply our algebraic solvability procedure, which will make naturally the Kalman matrix $[A|B|$ appear. The Kalman rank condition will then be used to find a right inverse to $[A|B|$.

(3) To conclude, we investigate the system that is verified by $Y = \tilde{Y} - \widehat{Y}$, and we remark that there exists a control $U \in L^2((0,T),\Omega)^m$ such that Y verifies (3.21) together with $Y(T) = 0$.

3.2.1. *First step: analytic part.* We look at the fictitious control problem

$$\begin{cases} i\partial_t \widehat{Y} + \Delta \widehat{Y} = A\widehat{Y} + \widehat{U}\widehat{\mathbf{1}}_\omega, \\ Y(0) = Y^0, \\ Y(T) = 0, \end{cases} \tag{3.22}$$

where the fictitious control $\widehat{U} = (\widehat{U}_1, \dots, \widehat{U}_n)$ is acting on each equation. We introduce the following change of unknowns: $Z = e^{itA}\widehat{Y}$. Then, since $\widehat{Y}$ verifies (3.22), we have

$$\begin{aligned} i\partial_t Z &= i\left(e^{itA}\partial_t \widehat{Y} + iAe^{itA}\widehat{Y}\right) \\ &= ie^{itA}\partial_t \widehat{Y} - Ae^{itA}\widehat{Y} \\ &= e^{itA}\left(-\Delta \widehat{Y} + \widehat{U}\widehat{\mathbf{1}}_\omega\right) \\ &= \left(-\Delta Z + W\widehat{\mathbf{1}}_\omega\right), \end{aligned}$$

where we have used $\Delta e^{itA} = e^{itA}\Delta$ and we have introduced the new control $W = e^{itA}\widehat{U}$. Moreover, we have $Z(0) = Y^0$. Remark that Z verifies a new control problem where the coupling term A has disappeared, *i.e.* the system is *uncoupled*. Hence, we can apply Proposition 3.5 on each of the components of Z and we obtain that

$$W \in H_0^{n-1}((0,T), L^2(\tilde{\omega})) \times \bigcap_{k=0}^{n-1} C^k([0,T], H^{2n-2k-2})(\tilde{\omega}).$$

Going back to the original variables $(\widehat{Y}, \widehat{U})$, we obtain that $(\widehat{Y}, \widehat{U})$ verifies (3.22) and moreover

$$\widehat{U} \in H_0^{n-1}((0,T), L^2(\tilde{\omega})) \times \bigcap_{k=0}^{n-1} C^k([0,T], H^{2n-2k-2}(\tilde{\omega})). \tag{3.23}$$

3.2.2. *Second step: algebraic part.* We look at the following auxiliary control problem:

$$\begin{cases} i\partial_t \tilde{Y} + \Delta \tilde{Y} = A\tilde{Y} + B\tilde{U} + \widehat{U}\widehat{\mathbf{1}}_\omega, \\ \tilde{Y}(0) = \tilde{Y}(T) = 0. \end{cases} \tag{3.24}$$

The unknowns are $(\tilde{Y}, \tilde{U})$ and $\widehat{U}$ is the control appearing in (3.22), seen as a source term. Remark that we have $n+m$ unknowns and n equations, so that the system is *underdetermined*.

We rewrite this system as

$$\mathcal{L}(\tilde{Y}, \tilde{U}) = \widehat{U}\widehat{\mathbf{1}}_\omega,$$

where

$$\mathcal{L}(\tilde{Y}, \tilde{U}) = i\partial_t \tilde{Y} + \Delta \tilde{Y} - A\tilde{Y} - B\tilde{U}.$$

We forget the regularity issues for the moment and we consider $\mathcal{L}$ as an operator acting on smooth functions.

We want to find some differential operator

$$\mathcal{M} : (C^\infty((0,T) \times M))^n \to (C^\infty((0,T) \times M))^{n+m}$$

such that

$$\mathcal{L} \circ \mathcal{M} = \mathrm{Id}_{C^\infty((0,T)\times M)^n}. \tag{3.25}$$

Applying this identity to our source term $\widehat{U}\widehat{\mathbf{1}}_\omega$, we deduce that

$$\mathcal{L}(\mathcal{M}(\widehat{U}\widehat{\mathbf{1}}_\omega)) = \widehat{U}\widehat{\mathbf{1}}_\omega.$$

Hence, $(\tilde{Y},\tilde{U}) := \mathcal{M}(\widehat{U}\widehat{\mathbf{1}}_\omega)$ will answer the question. Remark that here, we only need to work "locally" in space on $\tilde{\omega}$. Using the same strategy as before, we pass to the formal adjoint and we rewrite the problem as

$$\mathcal{M}^* \circ \mathcal{L}^* = \mathrm{Id}_{C^\infty((0,T)\times M)^n}.$$

We remark that

$$\mathcal{L} : (C^\infty((0,T)\times M))^{n+m} \to (C^\infty((0,T)\times M))^n$$

can be written in a matricial form as

$$\mathcal{L} = \begin{pmatrix} i\partial_t + \Delta - A, & -B \end{pmatrix}.$$

Hence, using Definition 1.1 (that still apply for partial differential operators), we can compute its formal adjoint (using that Δ is self-adjoint), which is given by

$$\mathcal{L}^* = \begin{pmatrix} -i\partial_t + \Delta - A^* \\ -B^* \end{pmatrix}.$$

In other words, if $\varphi = (\varphi_1, \ldots, \varphi_n) \in C^\infty((0,T), M)^n$, we have

$$\mathcal{L}^*\varphi = \begin{pmatrix} -i\partial_t\varphi + \Delta\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}.$$

Our goal is to make some linear combinations of the lines of $\mathcal{L}^*\varphi$ and its derivatives, in order to recover φ. The trick is the following:

- the m last lines of $\mathcal{L}^*\varphi$ are $-B^*\varphi$. We multiply it by -1 to recover $B^*\varphi$.
- We would like to recover $B^*A^*\varphi$. We use the first n lines of $\mathcal{L}^*\varphi$, and we apply some appropriate differential operators to the m last lines. We have

  $$-B^*(-i\partial_t\varphi + \Delta\varphi - A^*\varphi) + (i\partial_t - \Delta)(-B^*\varphi) = B^*A^*\varphi,$$

  where we have used the commutativity relations

  $$B^*(i\partial_t\varphi) = i\partial_t(B^*\varphi) \text{ and } B^*(\Delta\varphi) = \Delta(B^*\varphi).$$

- Assume that we have recovered $B^*\varphi, B^*A^*\varphi, \ldots, B^*(A^*)^k\varphi$. Then, we can recover $B^*(A^*)^{k+1}\varphi$. Indeed, it is easy to see that

  $$\begin{aligned} B^*(A^*)^{k+1}\varphi = &-B^*(A^*)^k(-i\partial_t\varphi + \Delta\varphi - A^*\varphi) \\ &+ (i\partial_t - \Delta)(-B^*(A^*)^k\varphi). \end{aligned}$$

We can summarize the previous sequence of transformations in the following differential $\mathcal{N}$:

$$\mathcal{N}(\mathcal{L}^*\varphi) = \begin{pmatrix} B^*\varphi \\ B^*A^*\varphi \\ \dots \\ B^*(A^*)^{n-1}\varphi \end{pmatrix}, \tag{3.26}$$

with

$$\mathcal{N}\begin{pmatrix} \psi_1 \\ \dots \\ \psi_n \\ \psi_{n+1} \\ \dots \\ \psi_{n+m} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} -\psi_{n+1} \\ \dots \\ -\psi_{n+m} \end{pmatrix} \\ -B^*\begin{pmatrix} \psi_1 \\ \dots \\ \psi_n \end{pmatrix} + (i\partial_t - \Delta)\begin{pmatrix} \psi_{n+1} \\ \dots \\ \psi_{n+m} \end{pmatrix} \\ \dots \\ \sum_{j=0}^{k-2}(-1)^{1+j}(i\partial_t - \Delta)^j B^*(A^*)^{k-2-j}\begin{pmatrix} \psi_1 \\ \dots \\ \psi_n \end{pmatrix} \\ +(-1)^k(i\partial_t - \Delta)^{k-1}\begin{pmatrix} \psi_{n+1} \\ \dots \\ \psi_{n+m} \end{pmatrix} \\ \dots \end{pmatrix} \tag{3.27}$$

(we have written the generic term corresponding on the k-th line, for $k \in [|2, n|]$). Coming back to the definition of the Kalman matrix $[A|B]$, we can rewrite (3.26) as

$$\mathcal{N}(\mathcal{L}^*\varphi) = [A|B]^*\varphi.$$

Hence, we have

$$\mathcal{N} \circ \mathcal{L}^* = [A|B]^*.$$

Passing one more time to the formal adjoint and using the fact that * is an involution, we have for the moment

$$\mathcal{L} \circ \mathcal{N}^* = [A|B].$$

$\mathcal{N}^*$ involves only linear combinations of derivatives of the form $(-i\partial t - \Delta)^j$, for $j \in [|1, n-1|]$. Now, we introduce $[A|B]^{-1} \in \mathcal{M}_{n,nm}(\mathbb{R})$ as any right inverse to $[A|B]$ (such a right inverse exists, since $[A|B] \in \mathcal{M}_{nm,n}(\mathbb{R})$ and $[A|B]$ is of full rank). Hence, we have by linearity of all the operators

$$\mathcal{L} \circ \left(\mathcal{N}^*[A|B]^{-1}\right) = \mathrm{Id}_{C^\infty((0,T)\times M)^n}.$$

Hence, (3.25) is verified with $\mathcal{M} = \mathcal{N}^*[A|B]^{-1}$.

Now, let us go back to the problem of the regularity. Taking into account that (3.23) holds, we see that $\mathcal{M}(\widehat{U}\widehat{\mathbf{1}}_\omega) \in L^2((0,T)\times M)$ since $\mathcal{M}$ involves only linear combinations of derivatives of the form $(-i\partial t - \Delta)^j$, for $j \in [|1, n-1|]$.

Moreover, $(\tilde{Y}, \tilde{U}) := \mathcal{M}(\widehat{U}\widehat{\mathbf{1}}_\omega)$ is a solution of $(3.24)_1$, that is supported in space in $\tilde{\omega}$ (since it is made of linear combinations of the derivatives of $\widehat{U}\widehat{\mathbf{1}}_\omega$).

To finish, we have to verify that the initial and final conditions in time $(3.24)_2$ are also verified. Indeed, we have to look a little bit deeper at the structure of $\mathcal{N}^*$: for $\psi_1, \ldots, \psi_n$, we apply differential operators up to order $n-1$ in time but, for $\psi_{n+1}, \ldots, \psi_{n+m}$, we apply differential operators up the order $n-2$ in time only. Hence, when we go back to the operator $\mathcal{N}$, it can be seen that, for $g \in C^\infty((0,T)\times M)^{nm}$, $\mathcal{N}(g)$ (which is of size $n+m$) is such that on the n first lines, we only have derivatives up to order $n-2$ in time of g. From this fact, one easily deduce that $\tilde{Y}$ (which is of size n) also involves derivatives of the source term $\widehat{U}$ up to order $n-2$ in time only (this would not be the case for $\tilde{U}$). Hence, from the definition given in (3.20) and the conditions on the end points of $\widehat{U}$ given in (3.23), it is clear that $\widehat{Y}$ verifies $(3.24)_2$. Hence, we have found a solution to our algebraic problem (3.24).

3.2.3. *Conclusion.* As in Section 2.2, we set

$$Y = \tilde{Y} - \widehat{Y},\ U = -\tilde{U}.$$

Then, using (3.22) and (3.24), we conclude easily that (Y, U) is a solution to the initial control problem (3.21) by similar arguments. □

4. Conclusion

In these lecture notes, our goal was to present the fictitious method coupled with the algebraic solvability procedure, in order to derive results on underactuated systems of ordinary differential equations or partial differential equations. This method has the advantage of being quite systematic. We decouple the problems into two simple ones: one "analytic part" where we prove a controllability result with as many controls as equations, and one "algebraic part" where we take benefit of the structure of the coupling terms to perform the algebraic solvability procedure.

As highlighted before, this method has already been fruitful to obtain different controllability results.

- In [11], the local null-controllability of the Navier-Stokes system with a control only on one equation is obtained. The corresponding linearized system around the trajectory 0 (*i.e.* the Stokes system) is not controllable. Hence, we need to perform a different linearization procedure around particular families of trajectories. The first step is to establish a controllability result with controls on each equation and in a specific form. Then, we use a slightly different version of the previous algebraic solvability procedure in order to make the fictitious control disappear. To finish, we go back to the nonlinear system by a standard inverse mapping theorem. A similar procedure is used in [10] in order to study a system of three coupled semilinear heat equations, the coupling terms being in cascade form and cubic. In both cases, the choice of the family of linearized trajectories is crucial for the algebraic part.

- In [13] and [14], some coupled systems of two heat equations with zero and one order coupling terms is explored. In the case of constant coupling coefficients, it is possible to recover a necessary and sufficient condition. If we have non-constant coupling terms, the algebraic solvability procedure naturally gives a generic condition in order to ensure controllability. The case of constant coefficients was partially generalized in [31] in the case of more than 2 equations and constant coupling coefficients.
- In [1], the authors study the case of quasi-linear hyperbolic systems, using a linearization procedure. Due to the difference between the regularity of the initial data and the control explained in Section 2.3 and highlighted in Section 3, going back to the quasi-linear system requires the use of a Nash-Moser type theorem.
- In [15], a result of controllability to the non-zeros trajectories on the Fokker-Planck equation is derived. One more time, a linearization procedure is needed, since the control system is bilinear in the control and the state. Here we have a scalar equation, but the control (which in some sense acts on the gradient of the solution) has d components if we are working on $\mathbb{R}^d$. The main novelty is that the algebraic solvability is not performed on the control problem but on some dual problem, in order to reduce the number of controls needed.
- Finally, in [26], we derive some Kalman-like conditions for abstract systems of group of operators with application to Schrödinger and wave systems, using ideas very similar to Section 3.

It is likely that this method can also be applied successfully to many other problems. For instance, it would be interesting to investigate the following problems.

- Larger classes of semilinear parabolic systems of reaction-diffusion type, coming from real-life models. For instance, one could investigate the model studied in [24] concerning reaction-diffusions systems arising in chemistry, where results on local controllability around particular classes of equilibriums. Is it possible to use the algebraic solvability

method in order to get rid of the technical conditions present in this article?
- Other models coming from fluid mechanics, for instance micropolar fluids as in [7], or the Boussinesq system as in [5], in order to reduce the number of controls.
- Give an extension of the results of [26] to general systems of semi-groups, with constant or time-dependant coupling terms, under the hypothesis that the "scalar" equation that is reproduced (for instance the Schrödinger equation in Section 3) is controllable.

From a more theoretical point of view, the following issues to address would be very promising.

- Is it possible to modify the algebraic solvability procedure in order to handle the problem of the loss of derivatives that is in general artificial? One can think to try to both differentiate and integrate in the solvability procedure. However, integration does not preserve the support, so that it is not clear if such a procedure will give satisfying results.
- In some sense, the algebraic solvability procedure gives "more" than one we need: the support of the resulting solution and control is included in the support of the fictitious control. However, in many situations, we have a little bit of latitude: it will be not a problem if the support of the resulting solution and control is "slightly larger" than the support of the fictitious control. Is it possible to modify the method in such a way?

Both of these two perspectives are challenging and may require to develop new and different tools.

Acknowledgments

I would like to warmly thank the organizers of the 13th International Young Researchers Workshop on Geometry, Mechanics and Control, for having given me the opportunity to deliver these lectures, and also for the kind welcome at Coimbra and the very efficient organization of the Workshop. I also would like to thank the reviewers of these lecture notes for their numerous and relevant remarks and suggestions, that helped me to improve drastically the overall quality of the lecture notes.

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