Positive and negative results on the internal controllability of parabolic equations coupled by zero and first order terms

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Abstract

This paper is devoted to studying the null and approximate controllability of two linear coupled parabolic equations posed on a smooth domain Ω of $\mathbb{R}^N$ ($N \geqslant 1$) with coupling terms of zero and first orders and one control localized in some arbitrary nonempty open subset ω of the domain Ω. We prove the null controllability under a new sufficient condition and we also provide the first example of a not approximately controllable system in the case where the support of one of the nontrivial coupling terms intersects the control domain ω.

Keywords: Controllability; Parabolic systems; Fictitious control method; Algebraic solvability.

MSC Classification: 93B05; 93B07; 35K40.

1 Introduction

1.1 Presentation of the problem and main results

Let $T > 0$, let Ω be a bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and let ω be an arbitrary nonempty open subset of Ω. Let $Q_T := (0,T) \times \Omega$, $q_T := (0,T) \times \omega$ and $\Sigma_T := (0,T) \times \partial\Omega$. We consider the following system of two parabolic linear equations with variable coefficients and coupling terms of order zero and one

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $y^0 \in L^2(\Omega)^2$ is the initial condition and $u \in L^2(Q_T)$ is the control.

The zero and first order coupling terms $(a_{ij})_{1 \leqslant i,j \leqslant 2}$ and $(g_{ij})_{1 \leqslant i,j \leqslant 2}$ are assumed (for the moment) to be in $L^\infty(Q_T)$ and in $L^\infty(Q_T)^N$, respectively. For $l \in \{1,2\}$, the second order elliptic self-adjoint operator $\operatorname{div}(d_l \nabla)$ is given by

$$\operatorname{div}(d_l \nabla) = \sum_{i,j=1}^N \partial_i (d_l^{ij} \partial_j),$$

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with

$$\begin{cases} d_l^{ij} \in L^\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{cases}$$

where the coefficients d_l^{ij} satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{N} d_l^{ij}\xi_i\xi_j \geqslant d_0|\xi|^2 \text{ in } Q_T,\ \forall \xi \in \mathbb{R}^N,$$

for a constant $d_0 > 0$.

It is well-known (see for instance [25, Th. 3-4, p. 356-358]) that for every initial data $y^0 \in L^2(\Omega)^2$ and every control $u \in L^2(Q_T)$, System (1.1) admits a unique solution y in $W(0,T)^2$, where

$$W(0,T) := L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)) \hookrightarrow \mathcal{C}^0([0,T]; L^2(\Omega)).$$

In this article, we are concerned with the approximate or null controllability of System (1.1). Let us recall the precise definitions of these notions. We say that System (1.1) is

- *approximately controllable* on $(0,T)$ if for every initial condition $y^0 \in L^2(\Omega)^2$, every target $y^1 \in L^2(\Omega)^2$ and every $\varepsilon > 0$, there exists a control $u \in L^2(Q_T)$ such that the corresponding solution y to System (1.1) satisfies

$$\|y(T,\cdot) - y^1\|_{L^2(\Omega)^2} \leqslant \varepsilon.$$

- *null controllable* on $(0,T)$ if for every initial condition $y^0 \in L^2(\Omega)^2$, there exists a control $u \in L^2(Q_T)$ such that the corresponding solution y to System (1.1) satisfies

$$y(T,\cdot) = 0 \text{ in } \Omega.$$

It is well-known that if a parabolic system like (1.1) is null controllable on $(0,T)$, then it is also approximately controllable on $(0,T)$ (this is an easy consequence of usual results of backward uniqueness for parabolic equations as given for example in [11]).

We recall that the case $a_{21} \neq 0$ and $g_{21} = 0$ in $(t_0,t_1) \times \omega_0 \subset q_T$ has already been studied in [27]. In the present paper, we study the following case: There exists $t_0, t_1 \in (0,T)$ satisfying $t_0 < t_1$ and a nonempty open subset ω_0 of ω such that

$$g_{21} \neq 0 \text{ in } (t_0,t_1) \times \omega_0. \tag{1.2}$$

As we will see in Section 2, it is possible, with the help of appropriate change of variable and unknown (we lose a little bit of regularity on the coefficients though, see Section 2), to replace the coupling operator $g_{21}\cdot\nabla + a_{21}$ by the simpler coupling operator ∂_{x_1} (where x_1 is the first direction in space), at least locally on some subset of q_T. Hence, without loss of generality, we will work under the following condition:

Condition 1.1. There exists $t_0, t_1 \in (0,T)$ satisfying $t_0 < t_1$ and a nonempty open subset $\mathcal{O}$ of ω_0 such that

$$g_{21}\cdot\nabla + a_{21} = \partial_{x_1} \text{ on } \mathcal{O}_T := (t_0,t_1) \times \mathcal{O}.$$

For a nonempty open set $\omega_T \subset \mathbb{R}^{N+1}$, let us denote by $\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)$ the subset of $\mathcal{C}^0(\overline{\omega}_T)$ composed by the functions depending only on the variables $t,\ x_2,\ x_3,\ \dots,\ x_N$. For some functions $a_0,\dots,a_R : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, we denote by $\langle a_1,\dots,a_R\rangle_{\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)}$ the $\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)$-module generated by $a_1,\dots,a_R$, *i.e.* the set composed by the functions $\sum_{i=1}^R \alpha_i a_i$ with $\alpha_i \in \mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)$.

Additionally to Condition 1.1, we will assume the following condition:

Condition 1.2. We assume that $d_i^{kl} \in \mathcal{C}^{N^2+3}(\overline{\omega}_T) \cap W^1_\infty(Q_T)$, $g_{ij}^k \in \mathcal{C}^{N^2+3}(\overline{\omega}_T) \cap L^\infty(0,T;W^1_\infty(\Omega))$ and $a_{ij} \in \mathcal{C}^{N^2+2}(\overline{\omega}_T)$ for every $i,j \in \{1,2\}$ and $k,l \in \{1,...,N\}$. Moreover, there exists a nonempty open set $\omega_T \subset (t_0,t_1) \times \mathcal{O}$ such that

$$\left\{ \begin{array}{l} \widetilde{a}_{22} \text{ is not an element of the } \mathcal{C}^0_{t,x_2,...,x_N}(\overline{\omega}_T)\text{-module} \\ \langle 1, \widetilde{g}_{22}^2, ..., \widetilde{g}_{22}^N, d_2^{22}, ..., d_2^{NN} \rangle_{\mathcal{C}^0_{t,x_2,...,x_N}(\overline{\omega}_T)}, \end{array} \right. \tag{1.3}$$

where

$$\left\{ \begin{array}{l} \widetilde{g}_{22}^i := g_{22}^i - \sum\limits_{j=1}^N \partial_{x_j} d_2^{ij}, \\ \widetilde{a}_{22} := -a_{22} + \operatorname{div}(g_{22}). \end{array} \right. \tag{1.4}$$

Remark 1. Condition 1.2 will be crucial in our following results, but is closely related to the particular form for the coupling term given in Condition 1.1. We assume Condition 1.1, since the general form (1.2) would make Condition 1.2 impossible to write down explicitly.

Our main result is the following:

Theorem 1. *Assume that Conditions 1.1 and 1.2 hold. Then System* (1.1) *is null controllable on* $(0,T)$. *Moreover, the corresponding control* u *satisfies*

$$\|u\|_{L^2(Q_T)} \leqslant C \|y^0\|_{L^2(\Omega)^2}, \tag{1.5}$$

where $C > 0$ *does not depend on* y^0.

Remark 2. Condition 1.2 is a generalization to the N-dimensional case of Theorem 2 in [24]. More precisely, the condition in Item (e) of Remark 1 in [24] is exactly (1.3) in the one dimensional case. Condition 1.2 is clearly technical since it does not even cover the case of constant coefficients studied in [24], the result proved in [12] under some assumption on the control domain or the one-dimensional result given in [23].

Remark 3. For some studies of control problems (see for instance [24, Th. 2] or [21]), the application of the fictitious control method combined with the algebraic solvability requires the use of a computer, so that the obtained coupling conditions cannot be written with a general form available for any space dimension (see [21]). This remark has already be done in Item (f) of Remark 1 in [24]. In the present paper, we propose an explicit strategy without these constraints, that seems to the authors to be sharp with respect to the technique used (see the proof of Theorem 1).

Remark 4. Theorem 1 is stated and will be proved in the case of two coupled parabolic equations and one control. However, as in [24], it is possible to extend Theorem 1 to systems of n parabolic equations controlled by $n-1$ controls for arbitrary $n \geqslant 2$. More precisely, consider the system

$$\left\{ \begin{array}{ll} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + \sum_{i=1}^n g_{1i} \cdot \nabla y_i + \sum_{i=1}^n a_{1i} y_i + \mathbb{1}_\omega u_1 & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + \sum_{i=1}^n g_{2i} \cdot \nabla y_i + \sum_{i=1}^n a_{2i} y_i + \mathbb{1}_\omega u_2 & \text{in } Q_T, \\ \vdots & \\ \partial_t y_{n-1} = \operatorname{div}(d_{n-1} \nabla y_{n-1}) + \sum_{i=1}^n g_{(n-1)i} \cdot \nabla y_i + \sum_{i=1}^n a_{(n-1)i} y_i + \mathbb{1}_\omega u_{n-1} & \text{in } Q_T, \\ \partial_t y_n = \operatorname{div}(d_n \nabla y_n) + \sum_{i=1}^n g_{ni} \cdot \nabla y_i + \sum_{i=1}^n a_{ni} y_i & \text{in } Q_T, \\ y_1 = \ldots = y_n = 0 & \text{on } \Sigma_T, \\ y_1(0,\cdot) = y_1^0, \ldots, \ y_n(0,\cdot) = y_n^0 & \text{in } \Omega, \end{array} \right. \tag{1.6}$$

where $y^0 := (y_1^0, \ldots, y_n^0) \in L^2(\Omega)^n$ is the initial data and $u := (u_1, \ldots, u_{n-1}) \in L^2(Q_T)^{n-1}$ is the control. Let us suppose that there exists $i \in \{1,...,n\}$, t_0, $t_1 \in (0,T)$ satisfying $t_0 < t_1$ and a nonempty open subset ω_0 of ω such that $g_{ni}(t,x) \neq 0$ on $(t_0,t_1) \times \omega_0$. As explained in Section 2, we can suppose that the operator $g_{ni} \cdot \nabla + a_{ni}$ is equal to ∂_{x_1} in $(t_0, t_1) \times \mathcal{O}$ with $\mathcal{O} \subset\subset \omega_0$. Assume that there exists an open set $\omega_T \subset (t_0, t_1) \times \mathcal{O}$ such that

$$\left\{ \begin{array}{l} \widetilde{a}_{nn} \text{ is not an element of the } \mathcal{C}^0_{t,x_2,...,x_N}(\overline{\omega}_T)\text{-module} \\ \langle 1, \widetilde{g}^2_{nn}, ..., \widetilde{g}^N_{nn}, d^{22}_2, ..., d^{NN}_2 \rangle_{\mathcal{C}^0_{t,x_2,...,x_N}(\overline{\omega}_T)}, \end{array} \right.$$

where

$$\left\{ \begin{array}{l} \widetilde{g}^i_{nn} := g^i_{nn} - \sum\limits_{j=1}^{N} \partial_{x_j} d^{ij}_{nn}, \\ \widetilde{a}_{nn} := -a_{nn} + \operatorname{div}(g_{nn}). \end{array} \right.$$

Then we can adapt the proof of Theorem 1 to prove that System (1.6) is null controllable on $(0, T)$ under suitable regularity conditions on the coefficients.

One question that naturally arises is whether we can expect the null controllability to be true in general that is without the extra Condition 1.2 on the coefficients or not. The next result explains that null controllability may fail in some particular cases when Condition 1.2 is not satisfied. Hence, the establishment of a simple necessary and sufficient condition on the coupling terms for the null controllability of System (1.1) remains an open problem.

Theorem 2. *Consider the following system*

$$\left\{ \begin{array}{ll} \partial_t y_1 = \partial_{xx} y_1 + \mathbb{1}_\omega u & \text{in } (0,T) \times (0,\pi), \\ \partial_t y_2 = \partial_{xx} y_2 + a y_2 + \partial_x y_1 & \text{in } (0,T) \times (0,\pi), \\ y(\cdot, 0) = y(\cdot, \pi) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,\pi). \end{array} \right. \tag{1.7}$$

There exists a coefficient $a \in \mathcal{C}^\infty([0,\pi])$ such that:

1. *There exists an open interval $\omega \subset\subset (0,\pi)$ such that, for all $T > 0$, System (1.7) is null controllable (then approximatively controllable) on $(0,T)$.*

2. *There exists an open interval $\omega \subset\subset (0,\pi)$ such that, for all $T > 0$, System (1.7) is not approximatively controllable (then not null controllable) on $(0,T)$.*

Remark 5. Let us mention that Theorem 2 is the first negative result for the controllability of System (1.1) when the support of the coupling term intersects the control domain in the case of distributed controls (concerning boundary controls, a complete characterization in the case of cascade coupling terms of order 0 or 1 has already been obtained in [31, Theorem 3.3] in the one-dimensional case).

Remark 6. Theorem 2 tells us that for some well-constructed potential a, that there exists one control domain on which System (1.7) is not approximately controllable (hence not null controllable) and another control domain on which System (1.7) is null controllable (hence approximatively controllable), highlighting the fact that some *geometrical conditions* on the control domain has to be imposed in order to obtain a controllability result, as already remarked in [31] and [17]. The authors want to emphasize the fact that the coupling operator is different of zero (constant) in the whole domain and nevertheless the system can be controllable or not following the localisation of the control domain, which is an unexpected phenomenon.

Remark 7. The proof of Theorem 2 can be adapted to construct, in the N dimensional case, not approximately controllable systems with a non empty intersection of the support of the coupling terms and the control region. An example of such systems is given in Proposition 4.1.

1.2 State of the art

Many models of interest involve (linear or non-linear) coupled equations of parabolic systems, notably in medicine (see e.g. [26]), chemistry (see e.g. [16]), ecology (see e.g. [18]), etc., and this explains why during the past years, the study of the controllability properties of linear or nonlinear parabolic systems has been an increasing subject of interest (see for example the survey [7]). The main issue is what is called the *indirect* controllability, that is to say one wants to control many equations with less controls than equations, by acting indirectly on the equations where no control term appears thanks to the coupling terms appearing in the system. This notion is fundamental, since in some complex systems only some quantities can be effectively controlled. Here, we will concentrate on the previous results concerning the null or approximate controllability of linear parabolic systems with distributed controls, but there are also results concerning boundary controls or other classes of systems like hyperbolic systems.

First of all, in the case of zero order coupling terms, the case of constant coefficients is now completely treated and we refer to [5] and [6] for parabolic systems having constant coupling coefficients (with diffusion coefficients that may depend on the space variable though) and for some results in the case of time-dependent coefficients. In the case of zero and one order coupling terms and constant coefficients, a necessary and sufficient condition in the case of m equations and $m-1$ controls for constant coefficients is provided in [24] by the authors.

The case of space-varying coefficients remains still widely open despite many new partial results these last years. In the case where the support of the coupling terms intersects the control domain, a general result is proved in [27] for parabolic systems in cascade form with one control force (and possibly one order coupling terms). We also mention [4], where a result of null controllability is proved in the case of a system of two equations with one control force, with an application to the controllability of a nonlinear system of transport-diffusion equations. In the situation where the coupling regions do not intersect the control domain, the situation is still not very well-understood and there are only partial results. Some general results in one dimension has been obtained in [17] and [8]. Partial results has also been obtained under some strong technical conditions on the coupling terms (see [1] and [3]) or geometrical conditions, notably on the control domain (see [32]).

Let us mention that in this case, there might appear a minimal time for the null controllability of System (1.1) (see [9]), which is a very surprising phenomenon for parabolic equations, because of the infinite speed of propagation of the information.

Concerning the case of first order coupling terms, we mention [27] which gives some controllability results when the coefficient g_{21} is equal to zero on the control domain. Let us also mention the recent work [12], which concerns 2×2 and 3×3 systems. The authors of [12] suppose that the control domain contains a part of the boundary $\partial\Omega$. Recently, in [23], the first author studied a particular cascade system with space dependent coefficients and in dimension one thanks to the moment method, and obtained necessary and sufficient conditions on the coupling terms of order 0 and 1 for the null controllability. To conclude, let us also mention another result given in [24] by the authors, which provides a sufficient condition for null controllability in dimension one for space and time-varying coefficients under some technical conditions on the coefficients, which turns out to be exactly equivalent to Condition 1.2 under Condition 1.1 (but with more regularity than in Condition (1.2)). Hence, Theorem 1 can be seen as a generalization in the multi-dimensional case of the one-dimensional result given in [24]. For a more detailed state of the art concerning this problem, we refer to [24].

Hence, the present paper improves the previous results in the following sense:

- Contrary to [12, 30, 23, 24], we prove in Theorem 1 the null controllability of System (1.1) with a condition on a_{22} but for space/time dependent coefficients, in any space dimension and without any condition on the control domain.

- In the previous results, it was surprising to have some very different sufficient conditions for the null controllability of System (1.1) in the case of first order coupling terms, for example on one hand constant coupling coefficients and on the other hand a region of control which intersects the boundary of the domain. Through the example of a not approximately controllable system given in Proposition 4.1 and Theorem 2, we can now better understand why such different conditions appeared since the expected general condition for the null controllability of System (1.1) with space and time-varying coefficients (i.e. it is sufficient that the control and coupling region intersect) may be false in general if $\omega \subset\subset \Omega$.

This paper is organized as follows: in the first section, we explain how to replace (1.2) by Condition 1.1 thanks to an appropriate change of variable. The second section is devoted to the proof of Theorem 1. Finally, in the last section, we prove Theorem 4.1 and give an example of N dimensional system which is not approximately controllable and with a non empty intersection of the support of the coupling terms and the control region.

2 Simplification of the coupling term

In this section, we will prove that it is possible to replace locally the coupling operator $g_{21} \cdot \nabla + a_{21}$ by ∂_{x_1}, where x_1 is the first direction in space. Let us remark that the regularities stated in Lemma 2.1 are higher than the one stated in Theorem 1 due to technical reasons appearing in the proofs of Lemmas 2.1 and 2.2.

Lemma 2.1. *Let d_i^{kl}, g_{ij}^k, $a_{ij} \in \mathcal{C}^{N^2+4}([t_0,t_1] \times \overline{\omega}_0)$ for every $i,j \in \{1,2\}$ and $k,l \in \{1,...,N\}$. Suppose that Condition (1.2) is verified. Then, there exist a nonempty open subset U of $\mathbb{R}^{N-1}$, a positive real number ε and a $\mathcal{C}^{N^2+3}$-diffeomorphism Λ from $U_\varepsilon := (t_0,t_1) \times (0,\varepsilon) \times U$ to an open set $(t_0,t_1) \times \mathcal{O} \subset (t_0,t_1) \times \omega_0$ that keeps t invariant and such that if we call $\widetilde{y}_1 := y_1 \circ \Lambda$ and $\widetilde{y}_2 := y_2 \circ \Lambda$, then there exist a matrix $\widetilde{d}_2 \in \mathcal{M}_N(\mathcal{C}^{N^2+3}(U_\varepsilon))$, a vector $\widetilde{g}_{22} \in (\mathcal{C}^{N^2+3}(U_\varepsilon))^N$ and coefficients $\widetilde{a}_{21}$, $\widetilde{a}_{22} \in \mathcal{C}^{N^2+3}(U_\varepsilon)$ such that locally on U_ε one has*

$$\partial_t \widetilde{y}_2 = \operatorname{div}(\widetilde{d}_2 \nabla \widetilde{y}_2) + \widetilde{g}_{22} \cdot \nabla \widetilde{y}_2 + \widetilde{a}_{22} \widetilde{y}_2 + \partial_{x_1} \widetilde{y}_1 + \widetilde{a}_{21} \widetilde{y}_1 \text{ in } U_\varepsilon. \tag{2.1}$$

This kind of simplification has already been used in [12, Lemma 2.6] for example, and we refer to this article for a more detailed proof (see also [23]).

Proof of Lemma 2.1

Let us consider some open hyper-surface γ of class $\mathcal{C}^\infty$ included in ω_0 on which $g_{21} \cdot \nu < 0$, where ν is the normalized outward normal on γ (this can always be done since $g_{21} \neq 0$ on $(t_0,t_1) \times \omega_0$ and is at least continuous), small enough such that it can be parametrized by a local diffeomorphism

$$F : s_0 := (s_2, \dots, s_N) \in U \subset \mathbb{R}^{N-1} \mapsto F(s_0) \in \gamma,$$

where U is a nonempty open set. We call $\gamma_T := (t_0,t_1) \times \gamma$. Let us consider some $\mathcal{C}^{N^2+4}$ extension of g_{21} (that exists thanks to the regularity of γ and g_{21}) that we denote by $g_{21}^T : (t,x) \in \mathbb{R}^{N+1} \mapsto (0, g_{21}(t,x)) \in \mathbb{R}^{N+1}$. Using the Cauchy-Lipschitz Theorem, we infer that for every $(t,\sigma) \in \gamma_T$, there exists a unique global solution to the Cauchy Problem

$$\begin{cases} \frac{d}{ds}\Phi(t,s,\sigma) = g_{21}^T(\Phi(t,s,\sigma)), \\ \Phi(t,0,\sigma) = (t,\sigma). \end{cases}$$

Since Φ is continuous and $g_{21} \cdot \nu < 0$ on γ_T, we deduce that there exists some $\varepsilon > 0$ such that $\Phi(t,s,\sigma) \in (t_0,t_1) \times \omega_0$ for every $s \in (0,\varepsilon)$ and every $(t,\sigma) \in \gamma_T$. We define

$$\Lambda : (t,s,z) \in (t_0,t_1) \times (0,\varepsilon) \times U \mapsto \Phi(t,s,F(z)).$$

Then, by the inverse mapping theorem, Λ is a $\mathcal{C}^{N^2+4}$-diffeomorphism from U_ε to $(t_0,t_1)\times\mathcal{O}:=\Lambda(U_\varepsilon)$ with $\mathcal{O}\subset\omega_0$. Let us call $\widetilde{y}_1(t,s,z):=y_1(\Lambda(t,s,z))$ and $\widetilde{y}_2(t,s,z):=y_2(\Lambda(t,s,z))$, then it is clear that

$$\partial_t\widetilde{y}_i(t,s,z)=(\partial_t y_i)\circ\Lambda(t,s,z)\text{ for }i=1,2\ \text{ and }\partial_s\widetilde{y}_2(t,s,z)=(g_{21}\cdot\nabla y_2)\circ\Lambda(t,s,z),$$

and hence we obtain (2.1) and the regularities wished for the new coefficients by writing down the equation verified by $\widetilde{y}$. ■

Let us now perform a second useful reduction.

Lemma 2.2. *There exists an open subset $\mathcal{O}_T$ of U_ε and a function $\theta\in\mathcal{C}^{N^2+4}(\Omega)$ such that $|\theta(x)|\geqslant C$ for some constant $C>0$ and if*

$$\overline{y}_1(t,x):=\theta^{-1}(t,x)\widetilde{y}_1(t,x)$$

and

$$\overline{y}_2(t,x):=\theta^{-1}(t,x)\widetilde{y}_2(t,x),$$

then there exists some coefficients $\overline{a}_{22}\in\mathcal{C}^{N^2+2}(\mathcal{O}_T)$ and $\overline{g}_{22}\in\mathcal{C}^{N^2+3}(\mathcal{O}_T)^N$ such that locally on $\mathcal{O}_T$ one has

$$\partial_t\overline{y}_2=\operatorname{div}(\widetilde{d}_2\nabla\overline{y}_2)+\partial_{x_1}\overline{y}_1+\overline{g}_{22}\cdot\nabla\overline{y}_2+\overline{a}_{22}\overline{y}_2\ \textit{in}\ \mathcal{O}_T. \tag{2.2}$$

Proof of Lemma 2.2
Let us consider a function $\theta\in\mathcal{C}^{N^2+4}(\overline{\Omega})$ such that $|\theta(x)|\geqslant C$ for some constant $C>0$, and consider the change of unknowns

$$\begin{cases}\overline{y}_1(t,x):=\theta^{-1}(x)\widetilde{y}_1(t,x),\\ \overline{y}_2(t,x):=\theta^{-1}(x)\widetilde{y}_2(t,x).\end{cases}$$

Using equation (2.1), we infer that $\overline{y}_2$ verifies

$$\partial_t\overline{y}_2=\operatorname{div}(\widetilde{d}_2\nabla\overline{y}_2)+\overline{g}_{22}\cdot\nabla\overline{y}_2+\overline{a}_{22}\overline{y}_2+\partial_{x_1}\overline{y}_1+\theta^{-1}(\partial_{x_1}\theta+\widetilde{a}_{21}\theta)\overline{y}_1,$$

where $\overline{g}_{22}:=2\theta^{-1}\widetilde{d}_2\nabla\theta+\widetilde{g}_{22}$ and $\overline{a}_{22}:=\theta^{-1}\operatorname{div}(\widetilde{d}_2\nabla\theta)+\theta^{-1}\widetilde{g}_{22}\nabla\theta+\widetilde{a}_{22}$. Hence, if we choose $\theta\in\mathcal{C}^{N^2+4}(\overline{\Omega})$ satisfying $\partial_{x_1}\theta+\widetilde{a}_{21}\theta=0$ and $|\theta(x)|\geqslant C$ in Q_T, which is always possible, then $\overline{y}_1$ and $\overline{y}_2$ verify (2.2) and we have $\overline{a}_{22}\in\mathcal{C}^{N^2+2}(\mathcal{O}_T)$ and $\overline{g}_{22}\in\mathcal{C}^{N^2+3}(\mathcal{O}_T)^N$.

■

3 Proof of Theorem 1

The goal here is to prove Theorem 1. We first explain in Section 3.1 the global strategy. Then, in Section 3.2, we solve the algebraic problem which is the key point of the proof of Theorem 1. Finally, we conclude in Section 3.3. During all these Sections, we always assume that Conditions 1.1 and 1.2 are satisfied.

3.1 Strategy : Fictitious control method

The fictitious control method, which was introduced in [19] in the context of affine control systems of ordinary differential equations without drift, has already been used for instance in [28], [21], [2], [20] and [24]. Roughly, the method is the following: we first control the equations with two controls (one on each equation) and we try to eliminate the control on the last equation thanks to algebraic manipulations locally on the control domain. For more details, see for example [24, Section 1.3]. Let us be more precise and decompose the problem into three different steps:

(i) **Analytic Problem: Null controllability by two forces**
Find a solution $(\widehat{y}, \widehat{u})$ in an appropriate space to the control problem by two controls

$$\begin{cases} \partial_t \widehat{y}_1 = \operatorname{div}(d_1 \nabla \widehat{y}_1) + g_{11} \cdot \nabla \widehat{y}_1 + g_{12} \cdot \nabla \widehat{y}_2 + a_{11} \widehat{y}_1 + a_{12} \widehat{y}_2 + \widehat{u}_1 & \text{in } Q_T, \\ \partial_t \widehat{y}_2 = \operatorname{div}(d_2 \nabla \widehat{y}_2) + g_{21} \cdot \nabla \widehat{y}_1 + g_{22} \cdot \nabla \widehat{y}_2 + a_{21} \widehat{y}_1 + a_{22} \widehat{y}_2 + \widehat{u}_2 & \text{in } Q_T, \\ \widehat{y} = 0 & \text{on } \Sigma_T, \\ \widehat{y}(0, \cdot) = y^0, \ \widehat{y}(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where the controls $\widehat{u}_1$ and $\widehat{u}_2$ are regular enough and with a support strongly included in ω_T (remind that ω_T was introduced in Condition 1.2). Solving Problem (3.1) is easier than solving the null controllability on $(0, T)$ of System (1.1), because we control System (3.1) with one control on each equation. The important point is that the control has to be regular enough, so that it can be differentiated a certain amount of times with respect to the space and/or time variables (see the next section about the algebraic resolution).

Proposition 3.1. *Let $s \in \mathbb{N}^*$. Suppose that d_i^{kl}, $g_{ij}^k \in \mathcal{C}^{s+2}(\overline{\omega}_T)$ and $a_{ij} \in \mathcal{C}^{s+1}(\overline{\omega}_T)$ for every $i, j \in \{1, 2\}$ and $k, l \in \{1, ..., N\}$. Then there exists a constant $C_s > 0$ such that for every initial condition $y^0 \in L^2(\Omega)^2$, one can find a control $u \in \mathcal{C}^s(Q_T)^2$ verifying moreover* $\operatorname{Supp}(u) \subset\subset \omega_T$ *for which the solution to System* (3.1) *is equal to zero at time T and the following estimate holds:*

$$\|u\|_{\mathcal{C}^s(Q_T)^2} \leqslant C_s \|y^0\|_{L^2(\Omega)^2}.$$

The controllability of parabolic systems with regular controls is nowadays well-known. For a proof of Proposition 3.1, one can adapt the strategy developed in [13, 14, 15, 28] where the authors prove the controllability of parabolic systems with L^∞ controls thanks to the fictitious control method and the local regularity of parabolic equations. For more details, we refer to [22, Chap. I, Sec. 2.4]. It is also possible to use the Carleman estimates (see for instance [10] and [24, Section 2.3]), however this will impose the coefficients of System (3.1) to be regular in the whole space Q_T (and would require higher regularity on Ω).

(ii) **Algebraic Problem: Null controllability by one force**
For given $\widehat{u}_1, \widehat{u}_2$ with $\operatorname{Supp}(\widehat{u}_1, \widehat{u}_2)$ strictly included in ω_T, find (z, v), in an appropriate space, satisfying the following control problem:

$$\begin{cases} \partial_t z_1 = \operatorname{div}(d_1 \nabla z_1) + g_{11} \cdot \nabla z_1 + g_{12} \cdot \nabla z_2 + a_{11} z_1 + a_{12} z_2 + \widehat{u}_1 + v & \text{in } \omega_T, \\ \partial_t z_2 = \operatorname{div}(d_2 \nabla z_2) + \partial_{x_1} z_1 + g_{22} \cdot \nabla z_2 + a_{22} z_2 + \widehat{u}_2 & \text{in } \omega_T, \end{cases} \tag{3.2}$$

with $\operatorname{Supp}(z, v)$ strictly included in ω_T, so that $z(0, \cdot) = (T, \cdot) = 0$ on Ω and $z = 0$ on $\partial\Omega$.

We recall that $g_{21} \cdot \nabla + a_{21}$ is equal to ∂_{x_1} in ω_T. We will solve this problem using the notion of *algebraic resolvability* of differential systems, which is based on ideas coming from [29, Section 2.3.8] and was already used in some different contexts in [21], [2], [24] or [20]. The idea is to write System (3.2) as an *underdetermined* system in the variables z and v and to see $\widehat{u}$ as a source term. More precisely, we remark that System (3.2) can be rewritten as

$$\mathcal{L}(z, v) = f, \tag{3.3}$$

where $f := \widehat{u}$ and

$$\mathcal{L}(z, v) := \begin{pmatrix} \partial_t z_1 - \operatorname{div}(d_1 \nabla z_1) - g_{11} \cdot \nabla z_1 - g_{12} \cdot \nabla z_2 - a_{11} z_1 - a_{12} z_2 - v \\ \partial_t z_2 - \operatorname{div}(d_2 \nabla z_2) - \partial_{x_1} z_1 - g_{22} \cdot \nabla z_2 - a_{22} z_2 \end{pmatrix}.$$

The goal in Section 3.2 will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\mathcal{L} \circ \mathcal{M} = Id \text{ in } \omega_T. \tag{3.4}$$

Thus to solve control problem (3.2), it suffices to take

$$(z, v) := \mathcal{M}(f).$$

When (3.4) is satisfied, we say that System (3.3) is *algebraically solvable.*

(iii) **Conclusion**
If we are able to solve the analytic and algebraic problems, then it is easy to check that $(y, u) := (\widehat{y} - z, -v)$ will be a solution to System (1.1) in an appropriate space and will satisfy $y(T, \cdot) \equiv 0$ in Ω (for more explanations, see [21, Prop. 1] and the proof of Theorem 1 in the next section).

3.2 Algebraic solvability of the control problem

The goal of this section is to solve algebraic problem (3.3). We will use the following lemma:

Lemma 3.1. *Let ω be a nonempty open subset of $\mathbb{R}^n$ $(n \geqslant 1)$ and let $R \in \mathbb{N}^*$. Consider two differential operators $\mathcal{L}_1$ and $\mathcal{L}_2$ defined for every $\varphi \in \mathcal{C}^\infty(\overline{\omega})$ by*

$$\mathcal{L}_1\varphi := \partial_{x_1}\varphi \text{ and } \mathcal{L}_2\varphi := a_0\varphi + \sum_{i=1}^{R} a_i D^{\alpha_i}\varphi,$$

where, for $\alpha_i = (\alpha_i^2, ..., \alpha_i^n)$, $D^{\alpha_i} := \partial_{x_2}^{\alpha_i^2} \cdots \partial_{x_n}^{\alpha_i^n}$. If $a_i \in \mathcal{C}^M(\overline{\omega})$ for every $i \in \{0, ..., R\}$ where

$$M := \sum_{j=1}^{R} \beta_j \text{ with } \beta_j \text{ the order of the operator } \sum_{i=j}^{R} a_i D^{\alpha_i}$$

and, for a nonempty open subset $\widetilde{\omega}$ of ω, a_0 is not an element of the $\mathcal{C}^0_{x_2,...,x_n}(\overline{\widetilde{\omega}})$-module generated by $a_1, ..., a_R$, i.e.

$$a_0 \in \langle a_1, ..., a_R \rangle_{\mathcal{C}^0_{x_2,...,x_n}(\overline{\widetilde{\omega}})}, \tag{3.5}$$

then there exists two differential operators $\mathcal{M}_1$ and $\mathcal{M}_2$ such that

$$\mathcal{M}_1 \circ \mathcal{L}_1 + \mathcal{M}_2 \circ \mathcal{L}_2 = Id \text{ in } \mathcal{C}^\infty(\overline{\widetilde{\omega}}). \tag{3.6}$$

We refer to the introduction for the definition of a module.
Proof of Lemma 3.1
The proof is divided into two steps:

- In a first step, we will build two differential operators $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ and a function $f \in \mathcal{C}^0(\overline{\omega})$ such that
$$\widetilde{\mathcal{M}}_1 \circ \mathcal{L}_1 + \widetilde{\mathcal{M}}_2 \circ \mathcal{L}_2 = f Id. \tag{3.7}$$
- In a second step, we will prove that f is invertible in a subset $\tilde{\omega}$ of ω under condition (3.5). Thus, multiplying (3.7) by f^{-1}, we will obtain (3.6).

Step 1:
The goal is to apply some differential operators $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ to $\mathcal{L}_1\varphi$ and $\mathcal{L}_2\varphi$ in order to obtain $f\varphi$, with f a function in $\mathcal{C}^0(\overline{\omega})$. So, since φ is not appearing in $\mathcal{L}_1\varphi$, we would like to eliminate all the derivatives $D^{\alpha_i}\varphi$ in the expression of $\mathcal{L}_2\varphi$ by differentiations and linear combinations. If $a_0 \neq 0$ and $a_i = 0$ in ω for every $i \in \{1, ..., R\}$, then we obtain (3.7) with $\widetilde{\mathcal{M}}_1 = 0$, $\widetilde{\mathcal{M}}_2 = Id$ and

$$f := a_0.$$

If not, let k_1 be the smallest number of $\{1,....,R\}$ such that there exists a nonempty open subset ω_1 of ω where $|a_{k_1}|>\delta>0$. Then we consider $\mathcal{L}_3$ the commutator of $\mathcal{L}_1$ and $a_{k_1}^{-1}\mathcal{L}_2$:

$$\mathcal{L}_3\varphi := [\mathcal{L}_1, a_{k_1}^{-1}\mathcal{L}_2]\varphi = \partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)\varphi + \sum_{i=k_1+1}^{R}\partial_{x_1}\left(\frac{a_i}{a_{k_1}}\right)D^{\alpha_i}\varphi.$$

Again, if for every $i\in\{1,...,k_1\}$, we have $\partial_{x_1}\left(\frac{a_i}{a_{k_1}}\right)=0$ in ω, then we obtain (3.7) with $\widetilde{\mathcal{M}}_1=-\mathcal{L}_2$, $\widetilde{\mathcal{M}}_2=\mathcal{L}_1$ and

$$f := \partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right).$$

If not, let k_2 be the smallest number of $\{1,...,k_1\}$ such that there exists a nonempty open subset ω_2 of ω_1 where $|\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)|>\delta>0$. Then we consider $\mathcal{L}_4$ the commutator of $\mathcal{L}_1$ and $\left[\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)\right]^{-1}\mathcal{L}_3$:

$$\mathcal{L}_4\varphi := \left[\mathcal{L}_1, \left[\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)\right]^{-1}\mathcal{L}_3\right]\varphi = \partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)\varphi + \sum_{i=k_2+1}^{R}\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_i}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)D^{\alpha_i}\varphi.$$

Again, if, for every $i\in\{1,...,k_2\}$, we have $\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_i}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)=0$ in ω_2, then we obtain (3.7) with

$$f := \partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right).$$

If not, we continue the same reasoning that will stop at some point since there is only a finite order of derivatives R. Hence, we obtain, for a $m\in\{1,...,R\}$, the equality (3.7) for a nonempty open subset $\widetilde{\omega}$ of ω and

$$f := \partial_{x_1}\left(\frac{\partial_{x_1}\left(\cdots\frac{\partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)}{\vdots}\right)}{\partial_{x_1}\left(\cdots\frac{\partial_{x_1}\left(\frac{a_{k_m}}{a_{k_1}}\right)}{\vdots}\right)}\right). \tag{3.8}$$

Moreover, f is obtained by making iterated commutators of operators involving only $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence it is clear that there exists two linear partial differential operators $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ such that (3.7) holds.

Step 2:
In view of (3.8), we will have the desired conclusion as soon as the coefficient in the right-hand side in (3.8) is different from zero. Let us explain into more details what this condition exactly means. For the sake of clarity, let us assume that $m=3$ (but the following reasoning can be extended to any $m\in\{1,\ldots,R\}$). We remark that

$$\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)}{\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_{k_3}}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)}\right)=0 \tag{3.9}$$

holds only if, for some $\lambda_3 \in \mathcal{C}^0_{x_2,...,x_n}(\overline{\widetilde{\omega}})$, we have

$$\frac{\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_0}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)}{\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_{k_3}}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right)} = \lambda_3.$$

The last expression can be rewritten as

$$\partial_{x_1}\left(\frac{\partial_{x_1}\left(\frac{a_0-\lambda_3 a_{k_3}}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)}\right) = 0. \tag{3.10}$$

Again, (3.10) holds only if, for some $\lambda_2 \in \mathcal{C}^0_{x_2,...,x_n}(\overline{\widetilde{\omega}})$, we have

$$\frac{\partial_{x_1}\left(\frac{a_0-\lambda_3 a_{k_3}}{a_{k_1}}\right)}{\partial_{x_1}\left(\frac{a_{k_2}}{a_{k_1}}\right)} = \lambda_2,$$

or, equivalently,

$$\partial_{x_1}\left(\frac{a_0 - \lambda_3 a_{k_3} - \lambda_2 a_{k_2}}{a_{k_1}}\right) = 0.$$

Thus (3.9) is satisfied only if, for some $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathcal{C}^0_{x_2,...,x_n}(\overline{\widetilde{\omega}})$, we have

$$a_0 = \lambda_3 a_{k_3} + \lambda_2 a_{k_2} + \lambda_1 a_{k_1}.$$

Hence, we find back condition (1.2) and the proof of Lemma 3.1 is achieved.

■

We are now able to prove the algebraic solvability of (3.3).

Proposition 3.2. *Suppose that* $d_i^{kl},\ g_{ij}^k,\ a_{ij} \in \mathcal{C}^{N^2}(\overline{\omega}_T)$ *for every* $i,j \in \{1,2\}$ *and* $k,l \in \{1,...,N\}$. *Then, under Condition 1.2, System* (3.3) *is algebraically solvable with an operator* $\mathcal{M}$ *of order* N^2.

Proof of Proposition 3.2

Let us remark that the first equation of System (3.3) can be rewritten locally on ω_T as

$$v = \partial_t z_1 - \operatorname{div}(d_1 \nabla z_1) - g_{11}\cdot\nabla z_1 - g_{12}\cdot\nabla z_2 - a_{11}z_1 - a_{12}z_2 - f_1,$$

hence one can always solve algebraically first the second equation of System (3.3), v will then be given with respect to z_1, z_2 and f_1. Hence, solving (3.3) is equivalent to solving

$$\mathcal{L}_0 z = f_2,$$

where

$$\mathcal{L}_0 z := \partial_t z_2 - \operatorname{div}(d_2\nabla z_2) - \partial_{x_1} z_1 - g_{22}\cdot\nabla z_2 - a_{22}z_2 \text{ in } \omega_T.$$

Hence, finding a differential operator $\mathcal{M}$ such that (3.4) is satisfied is now equivalent to finding a differential operator $\mathcal{M}_0$ such that

$$\mathcal{L}_0 \circ \mathcal{M}_0 = Id. \tag{3.11}$$

We can remark that equality (3.11) is formally equivalent to

$$\mathcal{M}_0^* \circ \mathcal{L}_0^* = Id,$$

where the formal adjoint $\mathcal{L}_0^*$ of the operator $\mathcal{L}_0$ is given for every $\varphi \in \mathcal{C}^\infty(\overline{\omega}_T)$ by

$$\mathcal{L}_0^*\varphi \quad := \quad \begin{pmatrix} \mathcal{L}_1\varphi \\ \mathcal{R}_2\varphi \end{pmatrix} = \begin{pmatrix} \partial_{x_1}\varphi \\ -\partial_t(\varphi) - \operatorname{div}(d_2\nabla(\varphi)) + \operatorname{div}(g_{22}\varphi) - a_{22}\varphi \end{pmatrix}.$$

Operator $\mathcal{R}_2$ can be rewritten as

$$\mathcal{R}_2\varphi = -\partial_t\varphi - \sum_{i,j=1}^N d_2^{ij}\partial_{x_ix_j}\varphi + \sum_{i=1}^N \widetilde{g}_{22}^i\partial_{x_i}\varphi + \widetilde{a}_{22}\varphi,$$

where $\widetilde{g}_{22}^i$ and $\widetilde{a}_{22}$ are given in (1.4). Let us first consider the following linear combination of $\mathcal{L}_1$ and $\mathcal{R}_2$:

$$\begin{aligned} \mathcal{L}_2\varphi &= \mathcal{R}_2\varphi - [-d_2^{11}\partial_{x_1} - 2\sum_{i=2}^N d_2^{i1}\partial_{x_i} + \widetilde{g}_{22}^1]\mathcal{L}_1\varphi \\ &= -\partial_t\varphi - \sum_{i,j=2}^N d_2^{ij}\partial_{x_ix_j}\varphi + \sum_{i=2}^N \widetilde{g}_{22}^i\partial_{x_i}\varphi + \widetilde{a}_{22}\varphi. \end{aligned}$$

Lemma 3.1 leads to the algebraic resolvability of System (3.3) under Condition 1.2.

Concerning the order of $\mathcal{M}$, if we follow the proof of Lemma 3.1 step by step, we apply at most $N \times (N-1)/2$ operators of order two to eliminate the terms $d_2^{ij}\partial_{x_ix_j}$ with $i,j \in \{2,...,N\}$ (thanks to the symmetry property of d_2), then at most $N-1$ operators of order one for the term $\widetilde{g}_{22}^i\partial_{x_i}$ with $i \in \{2,...,N\}$ and finally an operator of order at most one for ∂_t. Thus the operator $\mathcal{M}$ is of order at most $N \times (N-1) + (N-1) + 1 = N^2$. ∎

3.3 Conclusion

We have now all the tools to prove Theorem 1. We can follow the strategy described in Section 3.1.

Proof of Theorem 1.

We apply Proposition 3.1 with $k := N^2+1$ and obtain the existence of a constant $C > 0$ such that for every initial condition $y^0 \in L^2(\Omega)^2$ one can find a control $\widehat{u} \in \mathcal{C}^{N^2+1}(\overline{Q_T})^2$ verifying $\operatorname{Supp}(\widehat{u}) \subset\subset \omega_T$ for which the solution $\widehat{y}$ to System (3.1) is equal to zero at time T and the following estimate holds:

$$\|\widehat{u}\|_{\mathcal{C}^{N^2+1}(Q_T)^2} \leqslant C\|y^0\|_{L^2(\Omega)^2}, \tag{3.12}$$

where $C > 0$ does not depend on y^0.

Now, using Proposition 3.2, locally on ω_T there exists a solution $(z, v) \in \mathcal{C}^1(\overline{Q}_T)^3 \subset W(0,T)^2 \times L^2(Q_T)$ to the following control problem:

$$\begin{cases} \partial_t z_1 = \operatorname{div}(d_1\nabla z_1) + g_{11}\cdot\nabla z_1 + g_{12}\cdot\nabla z_2 + a_{11}z_1 + a_{12}z_2 + \widehat{u}_1 + v & \text{in } \omega_T, \\ \partial_t z_2 = \operatorname{div}(d_2\nabla z_2) + \partial_{x_1}z_1 + g_{22}\cdot\nabla z_2 + a_{22}z_2 + \widehat{u}_2 & \text{in } \omega_T, \end{cases}$$

with $(\widehat{u}_1, \widehat{u}_2) := \widehat{u}$. Moreover, since $\operatorname{Supp}(z) \subset\subset \omega_T$, we have $z(0,\cdot) = z(T,\cdot) = 0$ in Ω.

We conclude by remarking that $(y, u) := (\widehat{y} - z, -v)$ is a solution to System (1.1) which satisfies $y(T,\cdot) \equiv 0$ in Ω and estimate (3.12) leads to (1.5).

∎

4 Negative results

In this section, we first give an example of N dimensional system which is not approximately controllable (hence not null controllable) and with a non empty intersection of the support of the coupling terms and the control region. We then prove Theorem 2.

Proposition 4.1. *Let us assume that $\omega \subset\subset \Omega$. Let ω_1 be a nonempty regular open set satisfying $\omega \subset\subset \omega_1 \subset\subset \Omega$. and consider a function $\theta \in \mathcal{C}^\infty(\overline{\Omega})$ satisfying*

$$\begin{cases} \theta = 1 \ \text{in } \omega, \\ \text{Supp}(\theta) \subset \overline{\omega}_1, \\ \theta > 0 \ \text{in } \omega_1. \end{cases}$$

Then there exists $a \in \mathcal{C}^\infty(\overline{\Omega})$ such that the system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a y_2 + \partial_{x_1}(\theta y_1) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0,\cdot) = y^0 & \text{in } \Omega \end{cases}$$

is not approximately controllable (hence not null controllable) on $(0,T)$.

Proof of Proposition 4.1.
Let ω_1 be a nonempty regular open set satisfying $\omega \subset\subset \omega_1 \subset\subset \Omega$. Let θ be a function of $\mathcal{C}^\infty(\overline{\Omega})$ satisfying

$$\begin{cases} \theta = 1 \text{ in } \omega, \\ \text{Supp}(\theta) \subset \omega_1, \\ \theta > 0 \text{ in } \omega_1. \end{cases}$$

Consider the following system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a y_2 + \partial_{x_1}(\theta y_1) & \text{in } Q_T, \\ y = 0 & \text{on } \partial\Omega, \\ y(0,\cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $u \in L^2(Q_T)$ is the control and $a \in L^\infty(\Omega)$ will be specified later. If we can control approximately System (4.1), then it implies that we are also able to control approximately the following equation:

$$\begin{cases} \partial_t z = \Delta z + a z + \partial_{x_1}(\theta v) & \text{in } Q_T, \\ z = 0 & \text{on } \partial\Omega, \\ z(0,\cdot) = y_2^0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

where $v \in L^2((0,T), H^1(\Omega))$ is the control. We recall that $\theta > 0$ on ω_1. It is well known that the approximate controllability of System (4.1) on $(0,T)$ implies the following property, called the Fattorini-Hautus test: for every $s \in \mathbb{C}$ and every $\varphi \in D(\Delta)$,

$$\left.\begin{array}{ll} -\Delta\varphi - a\varphi = s\varphi & \text{in } \Omega \\ \partial_{x_1}\varphi = 0 & \text{in } \omega_1 \end{array}\right\} \Rightarrow \varphi = 0. \tag{4.3}$$

In the rest of the proof, we will construct a coefficient $a \in \mathcal{C}^\infty(\overline{\Omega})$, a complex number s and a function $\varphi \in D(\Delta)$ for which implication (4.3) does not hold.

Since $\omega_1 \subset\subset \Omega$, then there exists an open set ω_2 such that $\omega_1 \subset\subset \omega_2 \subset\subset \Omega$. The first eigenfunction φ_1 of $-\Delta$ is well-known to be positive in Ω, so we can define a function $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ satisfying

$$\begin{cases} \varphi = \varphi_1 & \text{in } \Omega \backslash \omega_2, \\ \varphi = 1 & \text{in } \omega_1, \\ \varphi > \delta > 0 & \text{in } \omega_2. \end{cases} \tag{4.4}$$

For instance, in the one dimensional case, if $\Omega := (0,\pi)$, $\omega_1 := (2\pi/5, 3\pi/5)$ and $\omega_2 := (\pi/5, 4\pi/5)$, as in Figure 1, we may construct a function $\varphi \in \mathcal{C}^2([0,\pi])$ satisfying

$$\begin{cases} \varphi(x) = \sin(x) & \text{for every } x \in \Omega\backslash\omega_2 = [0,\pi/5] \cup [4\pi/5,\pi], \\ \varphi(x) = 1 & \text{for every } x \in \omega_1 = [2\pi/5, 3\pi/5], \\ \varphi > \delta > 0 & \text{in } \omega_2 = [\pi/5, 4\pi/5]. \end{cases}$$

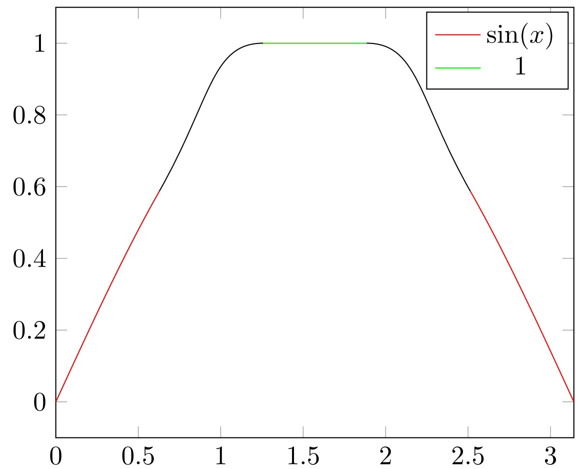

Figure 1: Example of function φ on $[0,\pi]$

Consider

$$a := \frac{-\Delta\varphi - \varphi}{\varphi}. \tag{4.5}$$

Thanks to the definition of φ, a is well defined in $\overline{\Omega}$ and is an element of $\mathcal{C}^\infty(\overline{\Omega})$. We remark that φ satisfies

$$\begin{cases} -\Delta\varphi - a\varphi = \varphi & \text{in } \Omega, \\ \partial_{x_1}\varphi = 0 & \text{in } \omega_1, \\ \varphi \neq 0. \end{cases}$$

For $s := 1$, φ given in (4.4) and a given in (4.5), implication (4.3) is not satisfied. Thus System (4.2) is not approximately controllable on $(0,T)$.

■

Remark 8. Let us emphasize that in this case, as expected, Condition 1.2 is not verified: on ω we have by definition $a_{22} = 1$, $g_{22} = 0$ and $d_2^{ij} = \delta_{ij}$ for every $i,j \in \{1,\dots,N\}$, which implies that $\tilde{a}_{22} = -1$ on ω and $\widetilde{g}_{22} = 0$, hence

$$\begin{cases} \widetilde{a}_{22} \text{ is an element of the } \mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)\text{-module} \\ \left\langle 1, \widetilde{g}^2_{22}, \dots, \widetilde{g}^N_{22}, d_2^{22}, \dots, d_2^{NN} \right\rangle_{\mathcal{C}^0_{t,x_2,\dots,x_N}(\overline{\omega}_T)}. \end{cases}$$

This will also be the case for the potential constructed in the first part of the proof of Proposition 2.

Let us now prove Theorem 2.

Proof of Theorem 2.
Let $\Omega := (0,\pi)$ and $\omega := (7\pi/15, 8\pi/15)$. Consider the following system

$$\left\{\begin{array}{ll}\partial_t y_1 = \partial_{xx} y + \mathbb{1}_\omega u & \text{in } Q_T,\\ \partial_t y_2 = \partial_{xx} y_2 + a y_2 + \partial_x y_1 & \text{in } Q_T,\\ y = 0 & \text{on } \Sigma_T,\\ y(0,\cdot) = y^0 & \text{in } \Omega,\end{array}\right. \tag{4.6}$$

where $u \in L^2(Q_T)$ is the control and $a \in C^\infty(\overline{\Omega})$ will be specified later.

As in the previous section, it is well-known that the approximate controllability on $(0,T)$ of System (4.2) implies the following property: for every $s \in \mathbb{C}$ and every $\varphi \in D(\partial_{xx})$,

$$\left.\begin{array}{ll}-\partial_{xx}\varphi - \partial_x\psi = s\varphi & \text{in } \Omega\\ -\partial_{xx}\psi - a\psi = s\psi & \text{in } \Omega\\ \varphi = 0 & \text{in } \omega\end{array}\right\} \Rightarrow (\varphi,\psi) = (0,0).$$

The rest of the present proof involves constructing three functions φ, ψ, $a \in \mathcal{C}^\infty(\overline{\Omega})$ satisfying

$$\left\{\begin{array}{ll}-\partial_{xx}\varphi - \partial_x\psi = 9\varphi & \text{in } \Omega,\\ -\partial_{xx}\psi - a\psi = 9\psi & \text{in } \Omega,\\ \varphi(0) = \varphi(\pi) = \psi(0) = \psi(\pi) = 0, & \\ \varphi = 0 & \text{in } \omega,\\ \varphi \neq 0,\ \psi \neq 0 & \text{in } \Omega.\end{array}\right. \tag{4.7}$$

The idea will be to construct the function ψ as a perturbation of $x \mapsto \sin(3x)$. Consider ψ a function of $\mathcal{C}^\infty(\overline{\Omega}) \cap D(\partial_{xx})$ satisfying

$$\left\{\begin{array}{ll}\psi(x) = \sin(3x) + C_1\theta_1 + C_2\theta_2 + C_3\theta_3 & \text{for all } x \in \overline{\Omega},\\ \psi(x) = \sin(7\pi/5) & \text{for all } x \in \overline{\omega},\\ |\psi(x) - \sin(3x)| < \varepsilon & \text{for all } x \in [6\pi/15, 7\pi/15] \cup [8\pi/15, 9\pi/15],\end{array}\right. \tag{4.8}$$

where $\theta_1,\ \theta_2,\ \theta_3$ are three nontrivial functions of $\mathcal{C}^\infty(\overline{\Omega})$ satisfying

$$\left\{\begin{array}{l}\mathrm{Supp}(\theta_1) \subset (\pi/12, \pi/6),\\ \mathrm{Supp}(\theta_2) \subset (9\pi/12, 5\pi/6),\\ \mathrm{Supp}(\theta_3) \subset (5\pi/6, 11\pi/12),\\ \theta_1,\ \theta_2,\ \theta_3 \geqslant 0 \text{ in } \Omega,\end{array}\right.$$

$\varepsilon > 0$ small enough and $C_1,\ C_2,\ C_2$ are three positive constants to determined (See Figure 2 for some examples of function ψ). Let us remark that, for a constant $\alpha \in \mathbb{R}$ to determined, the function $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ defined for all $x \in \overline{\Omega}$ by

$$\varphi(x) \quad := \quad \alpha\sin(3x) - \tfrac{1}{3}\int_0^x \sin(3(x-y))\partial_x\psi(y)dy$$

is solution to the first equation of (4.7). In order to satisfy (4.7), let us first prove that C_1 and α can be chosen such that $\varphi = 0$ in ω. Since $\psi = \sin(7\pi/5)$ in ω,

$$\begin{aligned}\varphi(x) \quad = \quad & \left[\alpha - \tfrac{1}{3}\cos(7\pi/5)\sin(7\pi/5) - \int_0^{7\pi/15} \sin(3y)\psi(y)dy\right]\sin(3x)\\ & + \left[\tfrac{1}{3}\sin(7\pi/5)^2 - \int_0^{7\pi/15}\cos(3y)\psi(y)dy\right]\cos(3x),\end{aligned}$$

for all $x \in \omega$. Since $\cos(3x) > 0,\ \sin(3x) > 0$ for all x in $(\pi/12, \pi/6)$ and

$$\frac{1}{3}\sin(7\pi/5)^2 - \int_0^{7\pi/15} \cos(3y)\sin(3y)dy > 0,$$

then, according to the last line of (4.8), for ε small enough, it is possible to choose $C_1 > 0$ in order to obtain

$$\frac{1}{3}\sin(7\pi/5)^2 - \int_0^{7\pi/15} \cos(3y)\psi(y)dy = 0.$$

Thus, for α given by

$$\alpha := \frac{1}{3}\cos(7\pi/5)\sin(7\pi/5) + \int_0^{7\pi/15} \sin(3y)\psi(y)dy,$$

we obtain $\varphi = 0$ in ω. By definition of φ, we have $\varphi(0) = 0$. Let us now prove that for some appropriate C_2 and C_3, we have $\varphi(\pi) = 0$. We remark that

$$\varphi(\pi) = \frac{1}{3}\int_0^{\pi} \cos(3y)\psi(y)dy.$$

Let us distinguish two cases:

1. If
$$\frac{1}{3}\int_0^{2\pi/3} \cos(3y)\psi(y)dy + \frac{1}{3}\int_{2\pi/3}^{\pi} \cos(3y)\sin(3y)dy \tag{4.9}$$
is negative, then, using the fact that $\sin(3x),\ \cos(3x) > 0$ for all $x \in (9\pi/12, 5\pi/6)$, one can choose $C_3 := 0$ and find some $C_2 > 0$ such that $\varphi(\pi) = 0$.

2. If now the quantity (4.9) is positive, since $\sin(3x) > 0$ and $\cos(3x) < 0$ for all $x \in (5\pi/6, 11\pi/12)$, one can choose $C_2 := 0$ and find some $C_3 > 0$ such that $\varphi(\pi) = 0$.

The function ψ will have one of the two following forms

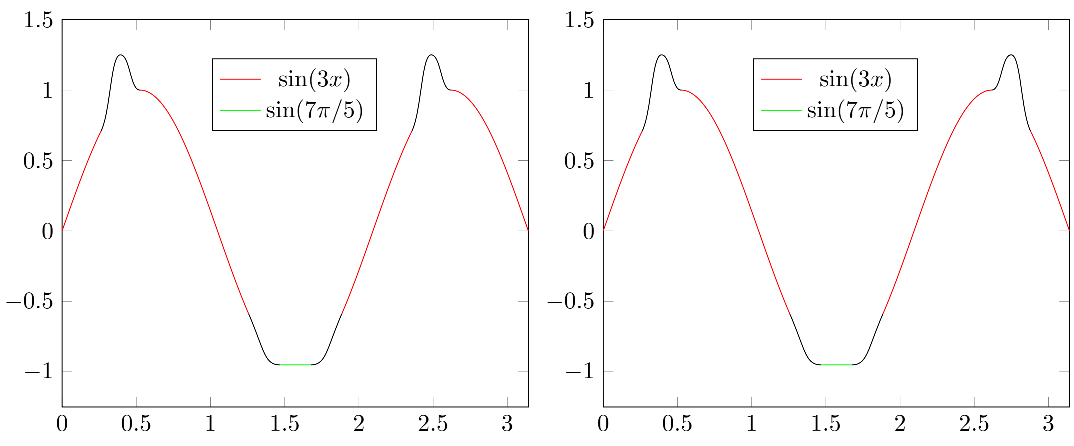

Figure 2: Examples of function ψ on $[0, \pi]$

To satisfy the second equality in (4.7), we define the function $a \in \mathcal{C}^\infty(\overline{\Omega})$ as follows

$$a := \frac{-\partial_{xx}\psi - 9\psi}{\psi}.$$

This function a is bounded since at each point where ψ is null, i.e. at 0, $\pi/3$, $2\pi/3$ and π, there exists a neighbourhood in which $\psi(x)$ is equal to $\sin(3x)$. The constructed φ, ψ and a verify (4.7). Thus System (4.6) is not approximately controllable on $(0,T)$.

Let us now prove the second item of Theorem 2. We remark that it is possible to chose $\theta_1 = \exp$ in $\omega_1 \subset\subset (\pi/12, \pi/6)$ with ω_1 small enough. Then a is defined in ω_1 for all $x \in \omega_1$ by

$$a(x) = \frac{-10C_1 \exp(x)}{\sin(3x) + C_1 \exp(x)}.$$

Thus a satisfies

$$\partial_x a \neq 0 \text{ in } \omega_1.$$

Item (ii) of Theorem 2 in [24] applies for $\omega := \omega_1$ (see Item (e) of Remark 1 in [24]). Theorem 1 of the present paper can also be used in the particular case $N := 1$. Thus System (1.7) is null controllable on $(0,T)$ for $\omega := \omega_1$ and the coefficient a built above.

■

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Conflict of Interest

The authors declare that they have no conflict of interest.

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