TD 1: Hilbert Spaces and Applications

Generalities

Exercise 1 (Generalized Parallelogram law)**.** Let $(H, \langle , \rangle)$ be a Hilbert space. Let $n \in [|2, +\infty|[$. Let $(x_1, \dots x_n) \in H^n$. Prove that

$$\sum_{i=1}^{n} ||x_i||^2 = \frac{1}{2^n} \sum_{(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n} ||\varepsilon_1 x_1 + \dots + \varepsilon_n x_n||^2.$$

Exercise 2. We consider $\mathbb{R}[X]$, endowed with the following scalar product: $\langle P, Q\rangle = \int_0^1 P(x)Q(x)dx$.

1. Prove that this is indeed a scalar product.
2. Find a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ that converges uniformly to exp on $[0, 1]$.
3. Deduce that $(P_n)_{n\in\mathbb{N}} \to \exp$ for the norm associated to the scalar product $\langle , \rangle$.
4. What can we deduce on $(\mathbb{R}[X], \langle , \rangle)$?

Exercise 3 (Fréchet-von Neumann-Jordan Theorem)**.** Let E be a real Banach space endowed with a norm $|| \cdot ||$ that verifies the parallelogram law

$$||x + y||^2 + ||x - y||^2 = 2\left(||x||^2 + ||y||^2\right), \forall (x, y) \in E^2.$$

We introduce

$$\langle x, y\rangle := \frac{1}{2}\left(||x + y||^2 - ||x||^2 - ||y||^2\right), \forall (x, y) \in E^2.$$

We propose to verify that this expression defines a scalar product that verifies moreover $\langle x, x\rangle = ||x||^2$.

1. Prove that $\langle x, y\rangle = \langle y, x\rangle$, $\langle -x, y\rangle = -\langle x, y\rangle$, and $\langle x, 2y\rangle = 2\langle x, y\rangle$, $\forall (x, y) \in E^2$.
2. Prove that $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\forall (x, y, z) \in E^3$.
3. Prove that $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$, $\forall (\lambda, x, y) \in (\mathbb{R} \times E^2)$. One may first treat the case $\lambda \in \mathbb{N}$, then $\lambda \in \mathbb{Z}$ and $\lambda \in \mathbb{Q}$.
4. Conclude.

Exercise 4 (Complexification of a real Hilbert space)**.** Let $(H, \langle , \rangle)$ be a Hilbert space. We consider the product linear space $H^{\mathbb{C}} := H \times H$, where we define, for $(x, y) \in H \times H$, $i.(x, y) := (-y, x)$.

1. Find an addition and an extern multiplication law that enable to endow $H^{\mathbb{C}}$ with structure of complex linear space. Prove that the restriction of this structure for real numbers coincide with the usual structure of product linear space on $H \times H$.
2. We identify H and $H \times \{0\}$. Identifty iH to a real subspace of $H^{\mathbb{C}}$. Prove that as a real linear space, $H^{\mathbb{C}}$ is the direct sum of H and iH.
3. Let $(z_1, z_2) \in H^{\mathbb{C}} \times H^{\mathbb{C}}$, that we decompose in a unique way as $z_j = x_j + iy_j$, $j = 1, 2$. We define

   $$\langle z_1, z_2\rangle_{H^{\mathbb{C}}} := \langle x_1, x_2\rangle + \langle y_1, y_2\rangle + i\left(\langle x_1, y_2\rangle - \langle x_2, y_1\rangle\right).$$

   Prove that it is the unique hermitian product that extend the scalar product on H.

The space $H^{\mathbb{C}}$ is called the *complexified linear space* of H.

Projection on a closed convex set.

Exercise 5. We consider the space $E = (C^0([-1,1],\mathbb{R}), ||\cdot||_\infty)$, and F be the linear subspace of odd functions whose integral is zero on $[0,1]$. Let $\varphi : t \in [-1,1] \mapsto t$.

1. Prove that F is closed.
2. Prove that $d(\varphi, F) \geqslant 1/2$.
3. Does there exist some $\psi \in F$ such that $||\varphi - \psi||_\infty = 1/2$?
4. Prove that $d(\varphi, F) = 1/2$. *Hint: try to "approximate" $t - 1/2$ by some appropriate functions in F.*

   Comment.

Exercise 6. We consider the space $E = (C^0([-1,1],\mathbb{R}), ||\cdot||_\infty)$ and D the straight line generated by $x \mapsto 1 - x$. Prove that the distance between D and the function $x \mapsto 1$ is reached at several points. Comment.

Exercise 7. Let $(H, \langle,\rangle)$ be a Hilbert space. Compute the projection on the closed unit ball.

Exercise 8. Let $(H, \langle,\rangle)$ be a Hilbert space. Prove that any closed convex set admits a unique element of minimal norm.

Exercise 9. Let $H = l^2(\mathbb{N},\mathbb{R})$, endowed with the canonical scalar product. We introduce $C := \{(x_n)_{n\in\mathbb{N}} | x_n \geqslant 0\}$. Prove that C is a closed convex set and compute the orthogonal projection on C.

Exercise 10 (Nested closed convex sets and projection, I)**.** Let $(C_i)_{i\in\mathbb{N}^*}$ be a sequence of closed convex sets $(H, \langle,\rangle)$ such that $C_1 \supset C_2 \ldots \supset C_n \supset \ldots$. We introduce $C_\infty = \bigcap_{n=1}^{\infty} C_n$ and we assume that $C_\infty \neq 0$. We denote by P_i the orthogonal projection on C_i.

1. Prove that C_∞ is a closed convex set.Let P_∞ denote the orthogonal projection on C_∞.
2. We fix some $h \in H$, and we introduce $a_i = P_i(h)$. Prove that the sequence $||h - a_i||_{i\in\mathbb{N}^*}$ is increasing and bounded from above.
3. Prove that $a_i \to P_\infty(h)$ as $i \to \infty$. *Hint: first prove that $(a_i)_{\in\mathbb{N}}$ is a Cauchy sequence by using question 2.*

Exercise 11 (Nested closed convex sets and projection, II)**.** Let $(C_i)_{i\in\mathbb{N}^*}$ be a sequence of nonempty closed convex sets $(H, \langle,\rangle)$ such that $C_1 \subset C_2 \ldots \subset C_n \subset \ldots$. We introduce $C_\infty = \bigcup_{n=1}^{\infty} C_n$ and we assume that $C_\infty \neq 0$.We denote by P_i the orthogonal projection on C_i.

1. Prove that C_∞ is a closed convex set.Let P_∞ denote the orthogonal projection on C_∞.
2. We fix some $h \in H$, and we introduce $a_i = P_i(h)$. Prove that $a_i \to P_\infty(h)$ as$i \to \infty$. One may begin with proving that $(a_i)_{\in\mathbb{N}}$ is a Cauchy sequence.

Exercise 12 (Geometric Hahn-Banach Theorem, Hilbert version)**.** Let C be a closed convex set of a real Hilbert space $(H, \langle,\rangle)$.

1. Let $x \in H$. Prove that there exists $f \in H'$ and $\alpha \in \mathbb{R}$ such that

   $$f(x) < \alpha < f(y), \forall y \in C.$$

   What does it mean from a geometrical point of view?
2. Deduce that any proper closed convex set (i.e. different from H) can be written as an intersection of closed half-spaces.
3. Prove that any proper linear subspace can be written as an intersection of closed hyperplanes.
4. Let $\widehat{C}$ another closed convex set, assumed to be compact, such that $C \cap \widehat{C} = \emptyset$. Prove that there exists $f \in H'$ such that

   $$\sup_{x\in C} f(x) < \inf_{y\in\widehat{C}} f(y).$$

   What does it mean from a geometrical point of view?

   Does this property still hold true if $\widehat{C}$ is assumed to be only closed?

Exercise 13 (Conditional expectation)**.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{G}$ a sub $\sigma-$algebra of $\mathcal{A}$. Let $X \in L^1(\Omega, \mathbb{R})$. Prove the following properties of the conditional expectation:

1. If Z is $\mathcal{G}-$measurable and essentially bounded, then $\mathbb{E}[ZE[X|\mathcal{G}]] = Z\mathbb{E}[X|\mathcal{G}]$.
2. $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.
3. If $\mathcal{H}$ a sub $\sigma-$algebra $\mathcal{G}$, alors $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$.
4. If Z is $\mathcal{G}-$measurable and essentially bounded, then $\mathbb{E}[XZ|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$.
5. If X is $\mathcal{G}-$measurable, then $\mathbb{E}[X|\mathcal{G}] = X$.
6. If φ is a convex function such that $\varphi(X)$ is integrable, then $E[\varphi(X)|\mathcal{G}] \geqslant \varphi(E[X|\mathcal{G}])$.

Riesz Representation Theorem.

Exercise 14. Let E be the space of complex sequences $(u_n)_{n\in\mathbb{N}^*}$ that vanishes identically after some index, endowed with the usual hermitian product.

1. Prove that the following application $\varphi : u = (u_n) \in E \mapsto \sum_{n=1}^{\infty} \frac{u_n}{n}$ is a linear continuous form on E.
2. Does there exists an element $a \in E$ such that $\varphi(u) = \langle a, u\rangle$? What can we deduce on E?

Exercise 15 (Radon-Nykodym Theorem, weak version)**.** Let $(X, \mathcal{T})$ be a measurable space, let μ and ν two finite measures on X. We assume that for any measurable set A, we have $\nu(A) \leqslant \mu(A)$.

1. Prove that for any non-negative measurable function h, we have $\int_X h(x)d\nu(x) \leqslant \int_X h(x)d\mu(x)$.
2. Deduce that $L^2(\mu) \subset L^2(\nu)$ and that for $g \in L^2(\mu)$, we have $||g||_{L^2(\nu)} \leqslant ||g||_{L^2(\mu)}$.
3. Prove that $\varphi : g \in L^2(\mu) \mapsto \int_X g(x)d\nu(x)$ is a linear continuous form.
4. Deduce that there exists a measurable function f such that ν is the density measure of f with respect to μ, i.e. for any measurable set A, we have $\nu(A) = \int_A f(x)d\mu(x)$.

Exercise 16 (Universality of the convolution)**.** Let f a function from R^N ($N \in \mathbb{N}^*$) with real values. For $x \in \mathbb{R}^N$, we introduce $\tau_x f : y \mapsto f(y - x)$. An operator T acting on the function of $L^2(\mathbb{R}^N)$ is said to be invariant by translation if $T(\tau_x f) = \tau_x(Tf)$.

We now consider $T : L^2(\mathbb{R}^N) \to \mathcal{C}_b(\mathbb{R}^N)$ a linear continuous operator, assumed to be invariant by translation.

1. Prove that $f \mapsto Tf(0) \in L^2(\mathbb{R}^N)'$.
2. Writing $Tf(x) = \tau_{-x}(Tf)(0)$, deduce the existence of some $g \in L^2(\mathbb{R}^N)$ such that $Tf(x) = f * g(x)$.

Exercise 17 (Adjoint operator)**.** Let u be a continuous linear endomorphism on a real or complex Hilbert space $H, \langle,\rangle$, and let $y \in H$.

1. Prove that there exists a unique $z \in H$ such that for any $x \in H$, we have $\langle u(x), y\rangle = \langle x, z\rangle$.

We consider $u^* : y \mapsto z$ for z defined as in the previous question. u^* is called the adjoint operator of u.

2. Prove that u^* is a linear continuous operator such that $|||u^*||| \leqslant |||u|||$.
3. Prove that $u^{**} = u$.
4. Prove that $u \in \mathcal{L}_c(U) \mapsto u^* \in \mathcal{L}_c(U)$ is anti-linear, continous, bijective with continuous inverse.
5. Let $v \in \mathcal{L}_c(H)$. Prove that $(u \circ v)^* = v^* \circ u^*$.
6. Prove that $|||u^*||| = |||u|||$.
7. Prove that $|||u^* \circ u||| = |||u|||^2 = |||u \circ u^*|||$.
8. Prove that $\mathrm{Ker(u^*)} = \mathrm{Im(u)}^\perp$. Describe also $\mathrm{Ker(u^*)}^\perp$.

Exercise 18 (Computation of adjoint operators)**.** 1. We consider $H = l^2(\mathbb{N}, \mathbb{C})$ and $(a_n)_{n\in\mathbb{N}}$ a bounded sequence of complex numbers. We consider $T : (u_n)_{n\in\mathbb{N}} \mapsto (a_n u_n)_{n\in\mathbb{N}}$. Prove that T is linear continuous from H to H and compute its adjoint operator.

2. We consider $H = l^2(\mathbb{N}, \mathbb{C})$. We consider $S : (u_n)_{n\in\mathbb{N}} = (u_0, u_1, \ldots) \mapsto (v_n)_{n\in\mathbb{N}} := (0, u_0, u_1, \ldots)$. Prove that S is linear continuous from H to H and compute its adjoint operator.

3. We consider $H = L^2([0,1], C)$ and $K : [0,1] \times [0,1] \to \mathbb{C}$ a continuous function. We consider $U : f \in H \mapsto \left(x \mapsto \int_0^1 K(x,y)f(y)dy\right)$. Prove that U is linear continuous from H to H and compute its adjoint operator.

Exercise 19 (Canonical Gelfand triple)**.** Let $(\Omega, \mathcal{T}, \mu)$ be a measure space.

We consider $H = L^2(\Omega, \mathbb{R})$. Let m be some measurable function with real values such that there exists $\delta > 0$ such that for any $x \in \Omega$, one has $m(x) \geqslant \delta$, and $m(x) < \infty$ almost everywhere. We consider $V := \{f \text{ measurable function} \mid \int_\Omega m f^2 d\mu < \infty\}$.

1. Prove that $V \subset H$ with continuous inclusion, and that V is dense in H.

2. Identify V' with a simple space.

Orthogonality and Hilbert basis.

Exercise 20. Let E be the space of complex sequences $(u_n)_{n\in\mathbb{N}^*}$ that vanishes identically after some index, endowed with the usual hermitien product. We consider $\varphi : u = (u_n) \in E \mapsto \sum_{n=1}^{\infty} \frac{u_n}{n}$, which is a linear continuous form on E.

1. Prove that $Ker(\varphi)$ is a close hyperplane, and that $Ker(\varphi)^\perp = \{0\}$.

2. More generally, if $(H, \langle,\rangle)$ is a (real or complex) non-complete pre-Hilbert space, prove that there exists some closed hyperplane whose orthogonal is reduced to $\{0\}$. *Hint: we admit that H can be included in a Hilbert space $(\widehat{H}, \langle,\rangle)$, and that H is dense in $\widehat{H}$.*

Exercise 21. Let $H = l^2(\mathbb{N}, \mathbb{C})$. We fix some $n \in \mathbb{N}^*$ and we consider $M = \{(x_k)_{k\in\mathbb{N}} \in H | \sum_{k=0}^{n} M = 0\}$.

1. Prove that M is a closed subspace of H.

2. Find the orthogonal complement of M.

3. Compute the distance between $(1, 0, 0, \ldots)$ and M.

Exercise 22 (Characterization of orthogonal projections)**.** Let $(H, \langle,\rangle)$ be a Hilbert space and $p \in \mathcal{L}_c(H)$ a projection on H, i.e. a linear continuous operator verifying $p \circ p = p$.

1. Prove that H can be written as the direct sum of $Range(p)$ and $Ker(p)$.

2. Prove that p is an orthogonal projection $\Leftrightarrow$ p is 1-Lipschitz. *Hint: for $\Leftarrow$, make a picture and consider some $x \in (Ker(p))^\perp$.*

Exercise 23. Let $E = C^0([0,1], \mathbb{R})$, endowed with the L^2-scalar product. Let $C = \{f \in E | f(0) = 0\}$. Prove that $F^\perp = \{0\}$. Comment.

Exercise 24. let $(H, \langle,\rangle)$ be a separable Hilbert space. Prove that any orthonormal family can be extended as a Hilbert basis of H.

Exercise 25. 1. Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - ax^2 - bx - c|^2 dx,$$

and find

$$\max \int_{-1}^{1} x^3 g(x) dx$$

amongst all $g \in L^2(-1,1)$ satisfying the constraints

$$\int_{-1}^{1} g(x)dx = \int_{-1}^{1} xg(x)dx = \int_{-1}^{1} g(x)dx = 0; \int_{-1}^{1} |g(x)|^2 dx = 1.$$

2. Let $(H, \langle , \rangle)$ a Hilbert space. let $x_0 \in H$ and M a closed subspace of H, find a maximization problem associated to the minimization problem $\min_{x \in M} ||x_0 - x||$ as in the previous question.

Exercise 26. Let $H = L^2(\mathbb{R})$, and $V = \{f \in C_c^\infty(\mathbb{R}) / \int_{\mathbb{R}} f(t)dt = 0\}$.
Recall that $C_c^\infty(\mathbb{R})$ is the set of C^∞ functions with compact support (i.e. vanishing identically outside of their support), that $L^2(\mathbb{R})$ is a Hilbert space for the usual scalar product and that $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

1. let $\phi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi(t)dt = 1$. Let

$$\forall (n,t) \in \mathbb{N}^* \times \mathbb{R}, \phi_n(t) = \frac{1}{n}\phi\Big(\frac{t}{n}\Big).$$

   (a) Prove that

$$\phi_n \underset{n \to +\infty}{\to} 0 \text{ in } H.$$

   (b) Let $g \in C_c^\infty(\mathbb{R})$. Let

$$h = g - \Big(\int_{\mathbb{R}} g(t)dt\Big)\phi_n.$$

   Prove that h is in the subspace V.

2. Deduce that

$$V^\perp \subset C_c^\infty(\mathbb{R})^\perp.$$

3. Conclude that V is dense is H.

Exercise 27 (Shannon Sampling Theorem)**.** Let

$$BL^2 = \{f \in L^2(\mathbb{R}) | \widehat{f} = 0 \text{ on } \mathbb{R} \setminus [-1/2, 1/2]\}.$$

1. Prove that BL^2 is a closed subspace of L^2. We will now consider that BL^2 is a Hilbert space for the L^2 scalar product.
2. Prove that any $f \in BL^2$ is continuous and that $||f||_\infty \leqslant ||f||_2$.
3. Compute the Fourier transform of the characteristic function of $[-1/2, 1/2]$. We will call this function sinc.
4. Let $\tau_x f : y \mapsto f(y - x)$. Prove that $\{\tau_k(\text{sinc}) | \text{k} \in \mathbb{Z}\}$ is a Hilbert basis of BL^2. Give an interpretation of this result. In this case, what gives the Parseval identity?
5. We decompose $f \in BL^2$ as $f(x) = \sum_{k \in \mathbb{Z}} f_k \tau_k(\text{sinc})(\text{x})$. Prove that the convergence is uniform.
6. Prove that any $f \in BL^2$ is en entire function.

Exercise 28 (Orthogonal polynomials)**.** Let I be an interval of $\mathbb{R}$. We call weight function a measurable function ρ, positive, such that $\int_I |x|^n \rho(x) < \infty$, $\forall n \in \mathbb{N}$. We denote by $L^2(I, \rho)$ the space of square integrable functions with density measure ρ, endowed with the canonical scalar product

$$\langle f, g \rangle := \int_I \rho(x) f(x) \bar{g}(x) dx.$$

Recall that it is a Hilbert space.

1. Prove that $L^2(I, \rho)$ contains all polynomials.
2. Prove that there exists a unique unitary family of orthogonal polynomials $(P_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $deg(P_n) = n$.
3. Prove that the zeros of P_n are distinct, reals, and all in the interval I. One can introduce the polynomials $S(x) = \prod_{i=1}^m (x - x_i)$, where x_i are the points in I where P_n changes sign.
4. Prove that for any $n \geqslant 1$ we have the following induction formula $P_{n+1} = (X - A_n)P_n - B_n P_{n-1}$, where

$$A_n = \frac{\langle XP_n, P_n \rangle}{\langle P_n, P_n \rangle} \text{ and } B_n = \frac{\langle XP_n, P_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle}.$$

5. Prove by induction that for any $n \in \mathbb{N}$, we have
$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0.$$
Deduce that between (in a strict sense) each root of P_n, there is exactly one root of P_{n+1}.

6. Example (Hermite polynomials): let $I = \mathbb{R}$ and $\rho(x) = e^{-x^2}$ (it is a weight function).
   (a) Compute P_0, P_1, P_2, P_3.
   (b) Prove that for any $n \in \mathbb{N}^*$, we have
   $$P_n(X) = \frac{(-1)^n}{2^n} e^{x^2} \left(\frac{\mathrm{d}^\mathrm{n}}{\mathrm{dx}^\mathrm{n}} e^{-x^2} \right).$$
   (c) Prove that $||P_n||_2 = \pi^{1/4}\sqrt{n!}$.

   Let $\xi \in \mathbb{R}$, $n \in \mathbb{N}$ and
   $$\forall x \in \mathbb{R}, P_n^\xi(x) := \sum_{k=0}^{n} \frac{(-i\xi x)^k}{k!}.$$
   (d) Prove that $\forall x \in \mathbb{R}, P_n^\xi(x) \underset{n\to+\infty}{\to} e^{-i\xi x}$.
   (e) Prove that $\forall (n,x) \in \mathbb{N} \times \mathbb{R}, |P_n^\xi(x)| \leq e^{|\xi x|}$.
   (f) Let $f \in L^2(\mathbb{R})$. We assume that : $\forall n \in \mathbb{N}, \langle P_n, f \rangle = 0$. Let $g(x) = e^{-\frac{x^2}{2}} f(x)$. Prove that $g \in L^1(\mathbb{R})$.
   (g) Prove that
   $$\forall \xi \in \mathbb{R}, \hat{g}(\xi) = \int_\mathbb{R} e^{-i\xi x} g(x) dx = 0.$$
   Hint. One can compute the integrals $\int_\mathbb{R} P_n^\xi(x) g(x) dx$.
   (h) Deduce that f identically vanishes.
   (i) Give a Hilbert basis of $L^2(\mathbb{R}, e^{-x^2})$.

7. Deduce from the above example a Hilbert basis of $L^2(\mathbb{R})$.

Other applications.

Exercise 29. let$H = \ell^2(\mathbb{N}, \mathbb{R})$. Recall that H is a Hilbert space for the scalar product
$$\forall (x,y) \in H^2, < x, y >_H = \sum_{n=0}^{+\infty} x_n y_n.$$
Are the following bilinear forms continuous and coercive?

(i) $a(x,y) = \sum_{n=0}^{+\infty} x_n y_{n+1}$, (ii) $b(x,y) = \sum_{n=0}^{+\infty} x_{n+1} y_{n+1}$,

(iii) $c(x,y) = x_0 y_0 + \sum_{n=0}^{+\infty} (x_{n+1} - x_n)(y_{n+1} - y_n)$, (iv) $d(x,y) = \sum_{n=0}^{+\infty} \Big(x_{n+1} y_{n+1} + 2x_n y_{n+1} + 2x_n y_n \Big)$.

Exercise 30 (Babuska-Lax-Milgram Theorem, 1971)**.** We consider two real Hilbert spaces $(U, \langle,\rangle)$ and $(V, (|))$, and $a : U \times V$ a bilinear continuous form, i.e. $\exists \beta > 0$ such that for all $(u,v) \in U \times V$, we have $|a(u,v)| \leqslant \beta ||u||_U ||v||_V$. We assume moreover that a is weakly coercive in the following sense: there exists $\alpha > 0$ such that $\sup_{||v||=1} |a(u,v)| \geqslant c||u||$ and for all $v \neq 0 \in H$, one has $\sup_{||u||=1} |a(u,v)| > 0$. Let $l \in V'$.

Prove that there exists a unique $u \in U$ such that $a(u,v) = l(v), \forall v \in V$.

Exercise 31 (Schrödinger equation)**.** Let $L > 0$. We consider the following equation, called Schrödinger equation with Dirichlet boundary conditions

$$\begin{cases} i\partial_t u + \partial_{xx} u = 0 & \text{in }]0, +\infty[\times]0, L[, \\ u(0, x) = u_0(x) & \in C^2[0, L], \\ u(t, x = 0) = u(t, x = L) = 0 & \text{on } [0, \infty[. \end{cases} \tag{1}$$

A solution (1) is a function $u \in C^1([0, T] \times [0, L]$ such that for every $t \geqslant 0$, we have $x \mapsto u(t, x) \in C^2([0, L])$, verifying (1).

1. Find the solutions of this equation that have separated variables.
2. Prove the existence of a solution.
3. Prove that any solution u to (1) is such that the energy is conserved:

$$\int_0^L u^2(t)dt = \int_0^L u_0^2(x)dx.$$

   One can multiply the first line of (1) by $\bar{u}$, integrate on $[0, L]$, then do an integration by parts.

4. Deduce the uniqueness of the solution.
5. Prove that u is in fact defined in $\mathbb{R} \times [0, L]$. This property is called reversibility in time.

Exercise 32 ($2D$ heat equation)**.** Let $L_1 > 0$ and $L_2 > 0$. We consider the following equation, called heat equation on a rectangle with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \partial_{xx} u - \partial_{yy} u = 0 & \text{in }]0, +\infty[\times]0, L_1[\times]0, L_1[, \\ \lim_{t \to 0^+} u(0, x) = u_0(x) & \in L^2([0, L_1] \times [0, L_2]), \\ u(t, 0, y) = u(t, L_1, y) = 0 & \text{in }]0, \infty[\times[0, L_2], \\ u(t, x, 0) = u(t, x, L_2) = 0 & \text{in }]0, \infty[\times[0, L_2]. \end{cases}$$

Mimicking what was done during the course, prove that there exists a unique solution (in appropriate class) of this equation. *Hint: prove that functions with separated variables x and y are dense in $L^2(]0, L_1[\times]0, L_2[)$.*

Exercise 33 (Orthonormal basis of Haar wavelets)**.** Let

$$\psi(x) : \begin{cases} x \in [0, 1/2[\mapsto 1 \\ x \in [-1/2, 1] \mapsto -1 \\ x \notin [0, 1] \mapsto 0. \end{cases}$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, we introduce

$$\psi_{j,k}(x) := \frac{1}{2^{\frac{j}{2}}} \psi(\frac{x - 2^j k}{2^j}).$$

The goal of this exercise is to prove that the $\psi_{j,k}$ form a Hilbert basis of $L^2(\mathbb{R})$. In what follows, W_j denotes the space of functions of $L^2(\mathbb{R})$ that are constants on intervals of the form $2^j + z$ with $z \in \mathbb{Z}$ and of null mean value.

1. Prove that the $\psi_{j,k}$ form an orthonormal family of $L^2(\mathbb{R})$.
2. Find a base of W_j.
3. Prove that the functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with null mean value are dense in $L^2(\mathbb{R})$.

   Hint: for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, one can consider, for $R > 0$, the function $f - \frac{\int_{\mathbb{R}} f}{R} \chi_{[0,R]}$.

4. Deduce that the $\psi_{j,k}$ form an orthonormal basis of $L^2(\mathbb{R})$.