TD 2: Sobolev spaces and $1D$ linear elliptic problems

Exercise 1. 1. Let $I=]-1,1[$. Are the following functions in the space $H^1(I)$?

(i) $a(x)=|x|$, (ii) $b(x)=0$, if $x\leq 0$, 1 otherwise, (iii) $c(x)=x^\alpha$ if $x\geq 0$, 0 otherwise, where $\alpha\in\mathbb{R}$.

2. Let $I=]0,1[$, and $1\leq p\leq+\infty$. For which p are the following functions in the space $W^{1,p}(I)$?

(i) $d(x)=|2x-1|$, (ii) $e(x)=x^\beta$, where $\beta\in\mathbb{R}$, (iii) $f(x)=|\ln(x)|^\gamma$, where $\gamma\in\mathbb{R}$.

Exercise 2. Let I and J be two open intervals of $\mathbb{R}$ and $p\in[1,\infty]$. Prove that $J\subset I\Rightarrow W^{1,p}(I)\subset W^{1,p}(J)$.

Exercise 3 (A characterization of $H^1(I)$)**.** Let $I:=]a,b[$ be an open interval of $\mathbb{R}$ with $-\infty<a<b<+\infty$. For any $0<\alpha<(b-a)/2$, we introduce $I_\alpha:=]a+\alpha;b-\alpha[$.

1. (i) Prove that if $u\in\mathcal{C}^1([a;b])$, then, for any α as above,

$$|u(x+h)-u(x)|^2\leq h^2\int_0^1|u'(x+sh)|^2ds\qquad\forall x\in I_\alpha,\ h\in\mathbb{R},\ |h|<\alpha\,.$$

(ii) Deduce that for any function $u\in H^1(I)$, any interval I_α and any $h\in\mathbb{R}$ such that $|h|<\alpha$, we have

$$\left\|\frac{\tau_h u-u}{h}\right\|_{L^2(I_\alpha)}\leq\|u'\|_{L^2(I)},$$

where $\tau_h u(x)=u(x+h)$.

2. Conversely, we assume that $u\in L^2(I)$ is such there exists a constant $C>0$ such that for any interval I_α and for any $h\in\mathbb{R}$ such that $|h|<\alpha$, we have

$$\left\|\frac{\tau_h u-u}{h}\right\|_{L^2(I_\alpha)}\leq C\,.$$

(i) Let $\phi\in\mathcal{C}_c^1(I)$ and $\alpha>0$ such that ϕ is supported in I_α. Prove that for $|h|<\alpha$, we have

$$\int_{I_\alpha}(u(x+h)-u(x))\phi(x)dx=\int_I u(x)(\phi(x-h)-\phi(x))dx\,.$$

Deduce that

$$\left|\int_I u(x)\phi'(x)dx\right|\leq C\|\phi\|_2\,.$$

We denote $T(\phi)=\displaystyle\int_I u(x)\phi'(x)dx$ for $\phi\in\mathcal{C}_c^1(I)$.

(ii) Let $\phi\in L^2(I)$. Prove that if $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}_c^1(I)$ converging to ϕ, then the sequence $(T(\phi_n))_{n\in\mathbb{N}}$ converges.

(iii) By using the density of $\mathcal{C}_c^1(I)$ in $L^2(I)$, prove that there exists a unique continuous linear form Φ on $L^2(I)$ such that

$$\Phi(v)=\int_I u(x)v'(x)dx\qquad\forall v\in\mathcal{C}_c^1(I)\,.$$

(iv) Deduce that $u\in H^1$.

Exercise 4. Let I be an open interval of $\mathbb{R}$, let $p>1$ and p' its conjugated exponent. Prove that for any $u\in W^{1,p}(I)$, there exists $C>0$ such that $\forall(x,y)\in\overline{I}^2$, we have

$$|u(x)-u(y)|\leqslant C|x-y|^{\frac{1}{p'}}.$$

(such a function is called a $1/p'-$Hölder function.)

Exercise 5. Let I be an open interval of $\mathbb{R}$.

1. Prove that if $u \in W^{1,\infty}(I)$ then u is bounded and Lipschitz.
2. Conversely, assume that u is bounded and Lipschitz. prove that $u \in W^{1,\infty}(\mathbb{R})$. *Hint: one may use without proof that any Lipschitz function is differentiable almost everywhere, and study its derivative.*

Exercise 6 (Dual space of H_0^1 on a bounded interval)**.** We consider two real numbers $a < b$ and $I =]a,b[$. We identify $L^2(a,b)$ with its dual. We endow $H_0^1(a,b)$ with the norm $||u||_{H_0^1(a,b)} = ||u'||_{L^2(a,b)}$.

1. Prove that $H_0^1(a,b) \subset L^2(a,b)$ with continuous inclusion, and that $H_0^1(a,b)$ is dense in $L^2(a,b)$.
2. We denote by $H^{-1}(a,b)$ the dual space of $H_0^1(a,b)$ with pivot space $L^2(a,b)$. Prove that for any $F \in H^{-1}(a,b)$, there exists a unique (almost everywhere) $f \in L^2(a,b)$, called the representant of F, such that $F(v) = \int_a^b fv', \forall v \in H_0^1(a,b)$. Prove that $|||F||| = ||f||_{L^2(a,b)}$.
3. We endow $H^{-1}(a,b)$ with the following scalar product: if f_1 is the representant of F_1 and f_2 the one of F_2, then $\langle F_1, F_2\rangle_{H^{-1}(a,b)} = \int_a^b f_1 f_2$. Prove that $L^2(a,b) \subset H^{-1}(a,b)$ with continuous injection,and that $L^2(a,b)$ is dense in $H^{-1}(a,b)$.

Exercise 7. We consider the space $H^1(0,\infty)$.

1. Prove that for any $v \in H^1(0,\infty)$, we have $v(x) \to 0$ as $x \to \infty$. *Hint: one can apply the Cauchy criterium for functions at $+\infty$.*
2. We consider the application $v \in H^1(0,\infty) \mapsto ||v'||_{L^2(0,\infty)}$. Prove that it is a norm. Is it equivalent to the usual H^1-norm?

Exercise 8 (Poincaré and Poincaré-Wirtinger inequalities)**.** We consider $a < b$ two real numbers, $I =]a,b[$, and $p \in [1,\infty]$. For $u \in L^1(a,b)$, we introduce $mv(u) := \frac{1}{b-a}\int_a^b u(t)dt$.

1. For $p = 2$, prove the following Poincaré inequality: for any $u \in H_0^1(0,\pi)$, we have
$$||u||_{L^2(0,\pi)} \leqslant ||u'||_{L^2(0,\pi)}.$$
One can consider a well-chosen Hilbert basis of $L^2(0,\pi)$. Prove that this inequality is optimal in the sense that there exist some functions for which this inequality is an equality, and find all functions verifying this equality.
2. Prove a similar result on the interval $I =]a,b[$.
3. Prove the Poincaré-Wirtinger inequality: there exists $C > 0$ such that for any $u \in W^{1,p}(I)$, we have
$$||u - vm(u)||_{L^p(a,b)} \leqslant C||u'||_{L^p(a,b)}.$$

Exercise 9 (Nash inequality, 1958)**.** We want to prove the Nash inequality: there exists $C > 0$ such that for any $f \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, we have
$$||f||^3_{L^2(\mathbb{R})} \leqslant C||f||_{L^1(\mathbb{R})}||f'||^2_{L^2(\mathbb{R})}.$$
We denote by $\widehat{f}$ the Fourier transform of f. We consider some $f \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$.

1. Prove that $\widehat{f} \in L^\infty(\mathbb{R})$ and that
$$||\widehat{f}||_\infty \leqslant ||f||_{L^1(\mathbb{R})}.$$
2. Express $||f'||^2_{L^2(\mathbb{R})}$ by using $\widehat{f}$.
3. Let $R > 0$. Prove that
$$||\widehat{f}||^2_{L^2(\mathbb{R})} \leqslant 2R||\widehat{f}||_{L^\infty(\mathbb{R})} + \frac{1}{R^2}||f'||^2_{L^2(\mathbb{R})}.$$
4. By choosing an adequate $R > 0$, conclude. *Hint: one can choose R as a function of $||f||_{L^1(\mathbb{R})}$ and $||f'||_{L^2(\mathbb{R})}$ to some power, and adjust the powers in an adequate way.*

Exercise 10. Let $p \geqslant 1$.We wonder if it is possible to prove the existence of some $q \geqslant 1$ and some $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R})$,we have

$$||f||_{L^q(\mathbb{R})} \leqslant C||f'||_{L^q(\mathbb{R})}.$$

Assume that this inequality is true.

Apply this inequality to $g(x) = f(\lambda x)$, for some $\lambda > 0$. Deduce a necessary condition on the value of q. Conclude.

Exercise 11. Let $\varepsilon > 0$.We consider $f_\varepsilon(t) := \sqrt{t^2 + \varepsilon^2} - \varepsilon$ for $t \geqslant 0$, and $f_\varepsilon(t) = 0$ otherwise.

1. Prove that $f_\varepsilon \in C^1(\mathbb{R})$ and $f(0) = 0$.

Let I an open interval of $\mathbb{R}$ and let $u \in H^1(I)$.

2. Prove that $f_\varepsilon(u) \to u^+$ as $\varepsilon \to 0$, where $u^+ = \max(u, 0)$.
3. Using the definition of the weak derivative of $f_\varepsilon(u)$, prove that $u^+ \in H^1(I)$ and that $(u^+)' = 1_{u>0}u$.
4. Deduce that $|u| \in H^1(I)$ and compute the weak derivative of $|u|$ as a function of the weak derivative of u.

Exercise 12. Let L^2_{per} be the set of measurable functions from $\mathbb{R}$ into $\mathbb{C}$, 2π-periodic and square-root integrable on $(0, 2\pi)$. We consider the Sobolev space H^1_{per} of the functions in L^2_{per} that admit a weak derivative in L^2_{per}. H^1_{per} is then a Hilbert space for the scalar product

$$\forall (f, g) \in (H^1_{\text{per}})^2, < f, g >_{H^1_{\text{per}}} = < f, g >_{L^2_{\text{per}}} + < f', g' >_{L^2_{\text{per}}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(t)\overline{g(t)} + f'(t)\overline{g'(t)} \right) dt.$$

We remind that the family $\{e^n\}_{n\in\mathbb{Z}}$ (defined by $e^n(t) = e^{int}$ for any $t \in \mathbb{R}$) is a Hilbert basis of L^2_{per}. We define the Fourier coefficients $c_n(f)$ as

$$\forall n \in \mathbb{Z}, c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt.$$

1. Let $f \in H^1_{\text{per}}$. We remind that f admits a representant that is a continuous function on $\mathbb{R}$ and verifies

$$\forall (x_0, x) \in \mathbb{R}^2, f(x) = f(x_0) + \int_{x_0}^{x} f'(t) dt.$$

a. Let $\mathcal{P}$ be the space of the 2π-periodic trigonometric polynomials. prove that there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of $\mathcal{P}$ such that

$$P_n \underset{n\to+\infty}{\to} f' \text{ in } L^2_{\text{per}}.$$

b. Deduce that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{P}$ such that

$$f_n \underset{n\to+\infty}{\to} f \text{ in } H^1_{\text{per}}.$$

c. Conclude that $\mathcal{P}$ is dense in H^1_{per}.

2. Let $f \in L^2_{\text{per}}$.

a. We assume that f is in $\mathcal{P}$. Prove that

$$\forall n \in \mathbb{Z}, c_n(f') = inc_n(f).$$

b. Deduce that this formula remains true if f is in H^1_{per}.

3. Let $H = \{f \in L^2_{\text{per}}, \sum\limits_{n=-\infty}^{+\infty} n^2|c_n(f)|^2 < +\infty\}$. We endow H with the following scalar product

$$\forall (f, g) \in H^2, < f, g >_H = \sum_{n=-\infty}^{+\infty} (1 + n^2)c_n(f)\overline{c_n(g)}.$$

a. Verify that H is a Hilbert space.

b. We are going to prove that H^1_{per} and H are equal.

(i) Prove that H^1_{per} is a closed subspace of H, and that

$$\forall (f, g) \in (H^1_{\text{per}})^2, < f, g >_{H^1_{\text{per}}} = < f, g >_H .$$

Hint: one may use the sequential characterization of closed sets.
(ii) Prove that the orthogonal of H^1_{per} for the scalar product $<\ ,\ >_H$ is equal to $\{0\}$.
(iii) Conclude.

Exercise 13. Let $f \in L^2(0,1)$ and $a(u,v) = \int_0^1 u'v' + (\int_0^1 u)(\int_0^1 v)$.

1. Prove that there exists a unique $u \in H^1(0,1)$ such that $a(u,v) = \int_0^1 fv, \forall v \in H^1(0,1)$.
2. Prove that $u' \in H^1(0,1)$ and find an interpretation of the differential problem that is solved (i.e. find the differential equation and the boundary conditions satisfied by u).

Exercise 14. Let $V = \{u \in H^1(0,1) | u(1/2) = 0\}$.

1. Prove that V is a closed subspace of $H^1(0,1)$ and that $v \in V \mapsto ||v'||_{L^2(0,1)}$ is a norm on V equivalent to the H^1-norm.
2. Prove that there exists a unique $u \in V$ such that $\int_0^1 u'v' = v(0), \forall v \in V$.
3. Find an interpretation of the differential problem that is solved, and determine explicitly u. Do we have $u' \in H^1(0,1)$?

Exercise 15. We consider the following Dirichlet problem on $]a,b[$

$$-u''(x) + u(x) = f(x) \text{ in }]a,b[, \qquad u(a) = 0,\ u(b) = 0,$$

where $f \in L^\infty(a,b)$.

1. Recall the results given in the course. Prove that $u' \in H^1(a,b)$.
2. Let $G \in C^1(\mathbb{R})$ such that $G = 0$ on $\mathbb{R}^-$ and G is strictly increasing on $\mathbb{R}^+$. We denote $K = ||f||_\infty$. Prove that $G(u-K) \in H^1_0(I)$.
3. Deduce that $||u||_{L^\infty(a,b)} \leqslant ||f||_{L^\infty(a,b)}$.

Exercise 16 (Sturm-Liouville problem with Dirichlet boundary conditions)**.** Let $I =]0,1[$. We consider p and q two functions in $L^\infty(I)$. We assume that there exists some $\alpha > 0$ such that

$$\text{a.e.in } x \in I, p(x) \geq \alpha \text{ and } q(x) \geq 0.$$

1. We consider

$$\forall (u,v) \in H^1_0(I)^2, a(u,v) = \int_0^1 \big(p(t)u'(t)v'(t) + q(t)u(t)v(t)\big)dt.$$

a. Prove that the bilinear form a is continuous and coercive on $H^1_0(I)$.
b. Let $f \in L^2(I)$. Deduce that there exists $u \in H^1_0(I)$ such that

$$\forall v \in H^1_0(I), a(u,v) = \int_0^1 f(t)v(t)dt.$$

2.a. Prove that the function pu' is in the space $H^1(I)$ and that

$$-(pu')' + qu = f.$$

b. We consider some v in $H^1_0(I)$, such that pv' is in the space $H^1(I)$ and verifies

$$-(pv')' + qv = f.$$

Prove that

$$u = v.$$

3. We assume moreover that p is of class $\mathcal{C}^1$ on I, and that q and f are continuous on I. Prove that the function u is of class $\mathcal{C}^2$ on I, and verifies the equation

$$-pu'' - p'u' + qu = f,\ u(0) = u(1) = 0.$$

Exercise 17 (Neumann problem)**.** Let $I =]0,1[$. We consider p and q two functions in $L^\infty(I)$. We assume that there exists some $\alpha > 0$ such that

$$\text{a.e.in } x \in I, p(x) \geq \alpha \text{ and } q(x) \geq 0.$$

1. We consider

$$\forall (u,v) \in H^1(I)^2, a(u,v) = \int_0^1 \big(p(t)u'(t)v'(t) + q(t)u(t)v(t)\big)dt.$$

a. Prove that the bilinear form a is continuous and coercive on $H_0^1(I)$.
b. Let $f \in L^2(I)$. Deduce that there exists a unique $u \in H^1(I)$ such that

$$\forall v \in H^1(I), a(u,v) = \int_0^1 f(t)v(t)dt.$$

2.a.Prove that pu' is in the space $H_0^1(I)$ and that

$$-(pu')' + qu = f.$$

b. We consider some v in $H^1(I)$, such that pv' is in the space $H_0^1(I)$ and verifies

$$-(pv')' + qv = f.$$

Prove that

$$u = v.$$

3. We assume moreover that p is of class $\mathcal{C}^1$ on I, and that q and f are continuous on I. Prove that the function u is of class $\mathcal{C}^2$ on I, and verifies the equation

$$-pu'' - p'u' + qu = f, \ u'(0) = u'(1) = 0.$$

Exercise 18 (Non-homogeneous Neumann boundary conditions)**.** Let $I =]a,b[$ be a bounded open interval of $\mathbb{R}$. We want to solve

$$-u''(x) + u(x) = f(x) \text{ in }]a,b[, \qquad u'(a) = \alpha, \ u'(b) = \beta$$

where f is a continuous function on $[a,b]$ and $(\alpha,\beta) \in \mathbb{R}^2$.

1. Prove that, if u is a solution of class $\mathcal{C}^2$, then for any function in $v \in H^1(I)$ we have

$$\int_I u'(x)v'(x) + u(x)v(x) \ dx = \int_I f(x)v(x) \ dx + \beta v(b) - \alpha v(a) \ .$$

2. Prove that the linear form $\Phi(v) = \displaystyle\int_I f(x)v(x) \ dx + \beta v(b) - \alpha v(a)$ is continuous on $H^1(I)$.

3. Deduce the existence of a unique function $u \in H^1(I)$ such that

$$\int_I u'(x)v'(x) + u(x)v(x) \ dx = \int_I f(x)v(x) \ dx + \beta v(b) - \alpha v(a) \qquad \forall v \in H^1(I) \ .$$

4. Prove finally that u is of class $\mathcal{C}^2$ and verifies the desired equation.