1 : Ordinary Differential Equations (ODEs)

Exercise 1. Let $\alpha > 0$. Let $x_0 \geqslant 0$. We consider the following Initial Value Problem (IVP):

$$x'(t) = \max(x(t), 0)^\alpha, \; x(0) = x_0,$$

defined on $\Omega := \mathbb{R}^2$.

1. Find some function f such that the equation can be written as $x'(t) = f(t, x(t))$. Verify that f is well-defined on Ω. Is it a linear equation? An autonomous equation?
2. Find α such that the Cauchy-Peano-Arzela Theorem can be applied.
3. Find α such that the Cauchy-Lipschitz Theorem (local version) can be applied.
4. Find α such that the Cauchy-Lipschitz Theorem (global version) can be applied.
5. Assume that $\alpha > 1$. Prove that there exist non-global solutions.
6. Assume that $\alpha < 1$. Prove that the IVP has infinitely many solutions if $x_0 = 0$.

Exercise 2. We consider the following scalar IVP

$$\begin{aligned} x'(t) &= f(t, x(t)), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where f is defined on the whole place $\mathbb{R}^2$ and is globally Lipschitz with respect to the second variable, its Lipchitz constant being called Λ. We are interested in approximating the solution of this Problem on the time interval $[0, 1]$. Let $N \in \mathbb{N}^*$ destined to tend to $+\infty$, and the usual mesh with constant step given by $t_n = n/N = nh$, where $h = 1/N$ is the discretization step. We consider the following approximation scheme:

$$x_{n+1} = x_n + hf(t_{n+1}, x_{n+1}). \tag{2}$$

We remind the following theorem, which is called Banach Fixed Point Theorem: Let E be a Banach space and $f : E \to E$ a contraction mapping (i.e. a Lipschitz function with Lipschitz constant $k < 1$), then f admits a unique fixed point.

1. Is it a single-step method? An explicit method? What are the theoretical and practical difficulties to implement this method? From which quadrature method (for numerical integration) does this method come from?
2. Find a sufficient condition on h such that for every given x_n, there exists a unique solution x_{n+1} to the equation (**??**). Prove that this condition is also "necessary" in the sense that there exist some f for which the existence and uniqueness of solutions is not true if this condition is not satisfied. One can consider $f(t, x) := \Lambda x$. What are the practical difficulties coming from this condition?
3. Follow the computations done during the course for the forward Euler method, and prove the convergence of this method.

This method is called *backward Euler Method*.

Exercise 3. We consider the following IVP:

$$\begin{aligned} y'(t) &= 3y - 1, \\ y(0) &= 1/3. \end{aligned} \tag{3}$$

1. Find the exact solution of this IVP.
2. We assume that we pertub the initial condition with some $\varepsilon > 0$. Find the solution of the perturbed IVP. Compare the two solutions at time $t = 10$.

3. For $\varepsilon = 10^{-10}$, can we consider that the exact solution is a little bit or a lot perturbed? What are the consequences you can draw concerning the numerical approximation of this problem?

Such a problem is *numerically ill-posed*: a small perturbation of the initial value (for example round-off errors) may cause large perturbations on the solution of the IVP.

Exercise 4. We consider the following IVP

$$\begin{aligned} x'(t) &= f((t, x(t)), \\ x(0) &= x^0, \end{aligned}$$

where f is globally Lipschitz with Lipschits constant Λ, and the following approximation method:

$$x_{n+1} = x_n + h/2(f(t_n, x_n) + f(t_{n+1}, x_{n+1})).$$

1. Is it an explicit or implicit method?
2. Find a condition on h ensuring that this method makes sense (one can use what was done in Exercice 2). From now on, we will assume that this condition is verified.
3. Prove that this method is convergent. We admit that in this case, we can still use the notions and theorems introduced for the explicit methods.
4. Find the order of the method. Comment.

Exercise 5.

We consider $\lambda \in \mathbb{R}$ and the following IVP:

$$\begin{aligned} x'(t) &= \lambda x(t), \\ x(0) &= 1. \end{aligned} \tag{4}$$

1. Prove that applying an explicit Runge-Kutta method with s stages for the numerical resolution of this problem lead to write $x_{n+1} = P(h)x_n$ with P a polynomial of degree less or equal to s.
2. Deduce that the order of such a method cannot be larger than s.
3. For $s = 1, 2$, prove that there exists at least one method of order s.

Remark: this property is still true for $s = 3, 4$, but is becoming false as soon as $s \geqslant 5$: for $s = 5, 6, 7$, the maximal order is $s - 1$, for $s = 8, 9$, the maximal order is $s - 2$, for $s \geqslant 10$ the maximal order is less or equal to $s - 2$.

Exercise 6. We consider the following ODE $x'(t) = f(t, x(t))$, associated with some initial condition $x(0) = x_0 \in \mathbb{R}$, where f is assumes to be regular enough (for example of class C^∞. In what follows, we will denote by h some discretization step, and $t_n := nT/h$. We consider the scheme defined by

$$x_{n+1} = x_n + h\Phi(t_n, x_n, h).$$

Prove that this scheme is of order p if and only if, for every $k \leqslant p - 1$:

$$\frac{\partial^k}{\partial h^l}\Phi(t, x, 0) = \frac{1}{k+1} f^{[k]}(t, x),$$

where $f^{[l]}$ is defined by induction thanks to the following formula: $f^{[0]} = f$ and $f^{[l+1]} = \frac{\partial f^{[l]}}{\partial t} + f\frac{\partial f^{[l]}}{\partial x}$.

Hint: prove $x^{(k)}(t) = f^{[k-1]}(t, x(t))$ and use the definition of the consistency error.

Exercise 7. We consider the following explicit Runge-Kutta method with 4 stages, also called *"Classical" Runge-Kutta method of order 4* or RK4, whose Butcher Tableau is the following:

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}.$$

1. Prove that this method converges. Write explicitly the algorithm describing the method. What is the interest of this method?
2. For each step, identify the quadrature method for numerical integration it comes from.
3. Find the order of this method. One can use Exercise 6.

Exercise 8. 1. Prove that the region of absolute stability of an explicit Runge-Kutta method is bounded. Can such a scheme be A-stable?

2. Find the region of absolute stability of the method given in Exercise 4. Is this method A-stable?

Exercise 9. We consider the following multi-step scheme, called the Nyström method: $x_{n+1} = x_{n-1} + 2hf(t_n, x_n)$.

1. Explain with a figure where this scheme comes from.
2. Give the values of α_i, β_i.
3. What is the order of the scheme?
4. Is this scheme stable?
5. Prove that this scheme is not A-stable.

Exercise 10 (Exam of september 2015)**.** Let f be a function defined on $[0,T] \times \mathbb{R}$, supposed to be globally Lipschitz with Lipschitz constant Λ with respect to the second variable, and smooth enough (for example of class C^∞ with all derivatives bounded). We consider the following EDO:

$x'(t) = f(t, x(t))$, associated to the initial condition $x(0) = x_0 \in \mathbb{R}$. We will denote by h the discretization step, and $t_n := nT/h$. Let α, β, γ three numbers between 0 and 1. We consider the following scheme:

$$x_{n+1} = x_n + h\Phi(t_n, x_n, h)$$

with

$$\Phi(t, x, h) := \alpha f(t, x) + \beta f(t + \frac{h}{2}, x + \frac{h}{2} f(t, x)) + \gamma f(t + h, x + hf(t, x)).$$

1. Is it a scheme of Runge-Kutta type? Why?
2. For which value of α, β, γ do we identify the Euler scheme? The middle-point scheme?
3. Prove rigorously that this scheme is stable.
4. Which relation(s) have to verify α, β, γ so that the scheme is consistant? Convergent? Of order 1?
5. Prove that
$$f(t_n + \frac{h}{2}, x(t_n + \frac{h}{2})) = f(t_n, x(t_n)) + \frac{h}{2} x''(t_n) + O(h^2)$$
and
$$f(t_n + h, x(t_n + h)) = f(t_n, x(t_n)) + hx''(t_n) + O(h^2).$$
6. Prove that
$$x(t_n + \frac{h}{2}) - x(t_n) - hf(t_n, x(t_n)) = O(h^2).$$
Deduce that
$$\beta hf(t_n + \frac{h}{2}, x(t_n) + \frac{h}{2} f(t, x(t_n))) - \beta hf(t_n + \frac{h}{2}, x(t_n + \frac{h}{2})) = O(h^3).$$
7. Prove likewise that
$$\gamma hf(t_n + h, x + hf(t, x(t_n))) - \gamma hf(t_n + h, x(t_n + h)) = O(h^3).$$
8. Using the previous questions, find a condition on α, β, γ such that the scheme is of order at least 2. Give an example of α, β, γ (different from the middle-point method) such that the scheme is of order 2.
9. Compute the stability region of this scheme and identify a geometrical objet. Is this scheme A-stable?