These are lecture notes for an introductory course on probability taught as part of the first year's program at École des Ponts ParisTech. This course is currently under the direction of Aurélien Alfonsi (CERMICS) and based on the following textbook:

B. Jourdain, Probabilités et statistique, Ellipses 2009, 2nd edition 2016.

Chapter 3

Continuous random variables

3.1 Continuous random variables

3.1.1 Definition

We saw in the previous chapter the concept of discrete random variable, which models a value which depends on the outcome of a random experiment. However, this concept reaches its limits when it comes to a whole continuum of possible outcomes, typically an interval of $\mathbb{R}$. Suppose for instance that you expect someone that could arrive at any moment before an hour. Let T be his arrival time. Modelling T be a discrete random variable would be inadequate since it could not catch all the uncountably many possible values in the interval $[0,1]$. Indeed, say that any time of arrival between 0 and 1 is equally likely to happen, so that there would exist $\varepsilon > 0$ such that for all $t \in [0,1]$, $\mathbb{P}(\{T = t\}) = \varepsilon$. Let $n \in \mathbb{N}$ be such that $\varepsilon > 1/10^n$. Then by σ-additivity,

$$1 = \mathbb{P}(\{T \in [0,1]\}) \geq \mathbb{P}\left(\bigcup_{1 \leq k \leq 10^n} \{T = k/10^n\}\right) = \sum_{k=1}^{10^n} \mathbb{P}\left(\{T = k/10^n\}\right) = 10^n \varepsilon > 1,$$

which is nonsense. In a effort to address this shortcoming we introduce the notion of *continuous random variable*.

Definition 3.1.1 (Continuous random variable)**.** *A* real-valued random variable *is a (measurable) map $X : \Omega \to \mathbb{R}$.*

Let $p : \mathbb{R} \to \mathbb{R}_+$ be a (measurable) nonnegative map such that $\int_{\mathbb{R}} p(x)\,dx = 1$. We say that a real-valued random variable X has a probability density function *(often abbreviated* PDF*) p if*

$$\forall a, b \in \mathbb{R} \text{ such that } a \leq b, \quad \mathbb{P}(\{a < X \leq b\}) = \int_a^b p(x)\,dx. \tag{3.1}$$

We call continuous random variable *any real-valued random variable which has a PDF.*

Remark 3.1.2. Definition 3.1.1 remains unchanged if one replaces (3.1) with one of the three following assertions:

(i) $\forall a, b \in \mathbb{R}$ such that $a \leq b$, $\quad \mathbb{P}(\{a \leq X \leq b\}) = \int_a^b p(x)\,dx$;

(ii) $\forall a, b \in \mathbb{R}$ such that $a \leq b$, $\quad \mathbb{P}(\{a < X < b\}) = \int_a^b p(x)\,dx$;

(iii) $\forall a, b \in \mathbb{R}$ such that $a \leq b$, $\quad \mathbb{P}(\{a \leq X \leq b\}) = \int_a^b p(x)\,dx$.

The following proposition illustrates a major contrast between discrete and continuous random variables.

Proposition 3.1.3. *Let X be a continuous random variable and $E \subset \mathbb{R}$ be an at most countable subset of $\mathbb{R}$. Then*

$$\mathbb{P}(\{X \in E\}) = 0.$$

In particular, for all $x \in \mathbb{R}$, $\mathbb{P}(\{X = x\}) = 0$.

Proof. Let p be the PDF of X and $x \in \mathbb{R}$. By definition of a probability density function, for all $n \in \mathbb{N}^*$,

$$\mathbb{P}(\{X = x\}) \leq \mathbb{P}\left(x - \frac{1}{n} < X \leq x\right) = \int_{\mathbb{R}} \mathbb{1}_{]x-\frac{1}{n},x]}(y)p(y)\,dy.$$

By the dominated convergence theorem, the right-hand side converges to 0 when $n \to +\infty$, hence $\mathbb{P}(\{X = x\}) = 0$. By σ-additivity, we have

$$\mathbb{P}(\{X \in E\}) = \sum_{x \in E} \mathbb{P}(\{X = x\}) = 0.$$

□

3.1.2 Common continuous probability distributions

3.1.2.1 The continuous uniform distribution

Definition 3.1.4. *Let $a, b \in \mathbb{R}$ be such that $a < b$. We say that a continuous random variable X follows the* continuous uniform distribution *on $[a, b]$ if X has the PDF p defined by*

$$\forall x \in \mathbb{R}, \quad p(x) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(x) = \begin{cases} \frac{1}{b-a} & \textit{if} \quad a \leq x \leq b \\ 0 & \textit{else} \end{cases}.$$

In that case, we denote $X \sim \mathcal{U}([a, b])$.

The continuous uniform distribution on $[a, b]$ models a situation in which all intervals of $[a, b]$ of same length are equally likely to happen.

3.1.2.2 The exponential distribution

Definition 3.1.5. *Let $\lambda > 0$. We say that a continuous random variable X follows the* exponential distribution *with parameter λ if X has the PDF p defined by*

$$\forall x \in \mathbb{R}, \quad p(x) = \lambda e^{-\lambda x} \mathbb{1}_{\{x>0\}} = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

In that case, we denote $X \sim \mathcal{E}(\lambda)$.

3.1.2.3 The normal distribution

Definition 3.1.6. *Let $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. We say that a continuous random variable X follows the* normal *(or* Gaussian*)* distribution *with parameters μ and σ^2 if X has the PDF p defined by*

$$\forall x \in \mathbb{R}, \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

In that case, we denote $X \sim \mathcal{N}_1(\mu, \sigma^2)$. The particular case $\mathcal{N}_1(0,1)$ is called the standard normal distribution*.*

3.1.2.4 The Cauchy distribution

Definition 3.1.7. *Let $a > 0$. We say that a continuous random variable X follows the* Cauchy distribution *with parameter a if X has the PDF p defined by*

$$\forall x \in \mathbb{R}, \quad p(x) = \frac{a}{\pi(x^2 + a^2)}.$$

In that case, we denote $X \sim \mathcal{C}(a)$.

3.1.3 Cumulative distribution function

Definition 3.1.8 (Cumulative distribution function)**.** *Let X be a real-valued random variable. We call* cumulative distribution function *of X, often abbreviated* CDF*, the map F_X defined by*

$$\forall x \in \mathbb{R}, \quad F_X(x) = \mathbb{P}(\{X \leq x\}).$$

The next proposition clarifies the connection between PDF and CDF.

Proposition 3.1.9. *Let X be a real-valued random variable.*

(i) If X has the PDF p, then F_X is the antiderivative of p which converges to 0 at $-\infty$ and 1 at $+\infty$, that is

$$F_X : x \in \mathbb{R} \mapsto \int_{-\infty}^{x} p(y)\, dy.$$

In particular, F_X is continuous.

(ii) Conversely, if F_X is continuous and piecewise continuously differentiable, then X has the PDF F'_X.

Proof. (i) Let $x \in \mathbb{R}$. For all $n \in \mathbb{N}^*$,

$$\mathbb{P}(\{x - n < X \le x\}) = \int_{\mathbb{R}} \mathbb{1}_{]x-n,x]}(y) p(y)\, dy.$$

For $n \to +\infty$, the left-hand side converges to $\mathbb{P}(\{X \le x\})$ and the right-hand side converges to $\int_{\mathbb{R}} \mathbb{1}_{\{y \le x\}} p(y)\, dy$, hence $F_X(x) = \int_{-\infty}^{x} p(y)\, dy$.

(ii) Let $a, b \in \mathbb{R}$ be such that $a \le b$. Then $\{X \le b\} = \{X \le a\} \cup \{a < X \le b\}$, where the union is disjoint. So $\mathbb{P}(\{X \le b\}) = \mathbb{P}(\{X \le a\}) + \mathbb{P}(\{a < X \le b\})$. Therefore,

$$\mathbb{P}(\{a < X \le b\}) = F_X(b) - F_X(a).$$

Since F_X is continuous and piecewise continuously differentiable, we have $F_X(b) - F_X(a) = \int_a^b F'_X(x)\, dx$. We deduce that $\mathbb{P}(\{a < X \le b\}) = \int_a^b F'_X(x)\, dx$, which means by definition that F'_X is the PDF of X.

□

Proposition 3.1.10. *Two continuous random variables X and Y have the same PDF iff they have the same CDF. In that case, we denote $X \overset{d}{=} Y$.*

Remark 3.1.11. This equivalence also holds in the discrete setting, namely two discrete random variables are equal in distribution iff their CDF are equal.

3.2 Expected value and variance

Definition 3.2.1 (Expected value)**.** *Let X be a continuous random variable with PDF p. We say that X is* integrable *and denote $X \in L^1$ if*

$$\int_{\mathbb{R}} |x| p(x)\, dx < +\infty.$$

In that case, the expected value *of X is denoted $\mathbb{E}[X]$ and defined by*

$$\mathbb{E}[X] = \int_{\mathbb{R}} x p(x)\, dx.$$

Definition 3.2.2 (Variance)**.** *Let X be a continuous random variable with PDF p. We say that X is* square-integrable *and denote $X \in L^2$ if*

$$\int_{\mathbb{R}} x^2 p(x)\, dx < +\infty.$$

In that case, the variance *of X is denoted $\operatorname{Var} X$ and defined by*

$$\operatorname{Var} X = \mathbb{E}[(X - \mathbb{E}[X])^2].$$

The square root of the variance of X is called the standard deviation *of X.*

All the properties of the expected value (linearity, positivity, non-degeneracy, monotonicity) and the variance seen in the previous chapter hold in the continuous setting.

Proposition 3.2.3. *Let X and Y be two integrable continuous random variables.*

(i) Linearity: For all $\lambda \in \mathbb{R}$, $\lambda X + Y$ is an integrable random variable and

$$\mathbb{E}[\lambda X + Y] = \lambda \mathbb{E}[X] + \mathbb{E}[Y];$$

(ii) Positivity and non-degeneracy: If $\mathbb{P}(X \geq 0) = 1$, then $\mathbb{E}[X] \geq 0$. If in addition $\mathbb{E}[X] = 0$, then $\mathbb{P}(X = 0) = 1$.

(iii) Monotonicity: If $\mathbb{P}(X \leq Y) = 1$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Remark 3.2.4. In fact, the random variable $\lambda X + Y$ may not have a PDF (take for instance $\lambda = -1$ and $X = Y$). Then the equality $\mathbb{E}[\lambda X + Y] = \lambda \mathbb{E}[X] + \mathbb{E}[Y]$ can be used as a definition of the expected value of the random variable $\lambda X + Y$.

3.3 Characterisations of a continuous probability distribution

3.3.1 The method of transformations

Theorem 3.3.1. *Let X be a real-valued random variable and $p : \mathbb{R} \to \mathbb{R}_+$ be a (measurable) nonnegative map such that $\int_{\mathbb{R}} p(x)\,dx = 1$.*

(i) If X has the PDF p, then for all (measurable) map $f : \mathbb{R} \to \mathbb{R}$,

$$f(X) \in L^1 \iff \int_{\mathbb{R}} |f(x)| p(x)\,dx < +\infty.$$

In that case,

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x) p(x)\,dx. \tag{3.2}$$

(ii) Conversely, if (3.2) holds for any (measurable) bounded map $f : \mathbb{R} \to \mathbb{R}$, then p is the PDF of X.

Remark 3.3.2. Once again, $f(X)$ may not have a PDF (take for instance $f : x \mapsto 0$). Then (3.2) can be used as a definition of the expected value of the random variable $f(X)$.

The use of Theorem 3.3.1 in order to determine the probability distribution of a random variable is called *method of transformations*.

Exercise 3.3.3. Let $U \sim \mathcal{U}([0,1])$, $\lambda > 0$ and $X = -\frac{1}{\lambda} \ln U$. Find the probability distribution of X.

Exercise 3.3.4. Let $\mu \in \mathbb{R}$, $\sigma^2 > 0$ and $X \sim \mathcal{N}_1(\mu, \sigma^2)$.

1. Find the law of $Y = \frac{X-\mu}{\sigma}$.

2. Deduce the value of $\mathbb{E}[X]$ and $\operatorname{Var} X$.

3.3.2 Characteristic function and Laplace transform

Definition 3.3.5 (Characteristic function)**.** *Let X be a real-valued random variable. We call* characteristic function *of X and denote Φ_X the map*

$$\Phi_X \ : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{C} \\ u & \mapsto & \mathbb{E}[e^{iuX}] \end{array}.$$

Notice that $\Phi_X(0) = 1$ and for all $u \in \mathbb{R}$, $\Phi_X(-u) = \overline{\Phi_X(u)}$. Moreover, the Fourier inversion theorem implies the following proposition.

Proposition 3.3.6. *Let X be a real-valued random variable. If Φ_X is integrable on $\mathbb{R}$, that is $\int_{\mathbb{R}} |\Phi_X(u)|\, du < +\infty$, then X has the PDF p defined by*

$$\forall x \in \mathbb{R}, \quad p(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \Phi_X(u)\, du.$$

The characteristic function characterises the probability distribution of a real-valued random variable.

Proposition 3.3.7. *Let X and Y be two real-valued random variables. Then*

$$X \stackrel{d}{=} Y \iff \forall u \in \mathbb{R}, \quad \Phi_X(u) = \Phi_Y(u).$$

Definition 3.3.8 (Laplace transform)**.** *Let $X : \Omega \to \mathbb{R}_+$ be a nonnegative-valued random variable. We call* Laplace transform *of X and denote L_X the map*

$$L_X \ : \begin{array}{ccc} \mathbb{R}_+ & \to & \mathbb{R}_+ \\ \lambda & \mapsto & \mathbb{E}[e^{-\lambda X}] \end{array}.$$

The Laplace transform characterises the probability distribution of a nonnegative-valued random variable.

Proposition 3.3.9. *Let $X : \Omega \to \mathbb{R}_+$ and $Y : \Omega \to \mathbb{R}_+$ be two nonnegative-valued random variables. Then*

$$X \stackrel{d}{=} Y \iff \forall \lambda \in \mathbb{R}_+, \quad L_X(\lambda) = L_Y(\lambda).$$

3.3.3 Summary of characterisations of a probability distribution

We saw several propositions of the form

$$X \stackrel{d}{=} Y \iff \forall f \in \mathcal{C}, \quad \mathbb{E}[f(X)] = \mathbb{E}[f(Y)], \tag{3.3}$$

where $\mathcal{C}$ is a class of functions from $\mathbb{R}$ to $\mathbb{R}$.

- When $\mathcal{C}$ is the set of (measurable) bounded functions from $\mathbb{R}$ to $\mathbb{R}$, (3.3) is the method of transformations (Theorem 3.3.1).

– When $\mathcal{C} = \{x \mapsto \mathbb{1}_{\{x \leq a\}} \mid a \in \mathbb{R}\}$, (3.3) is the characterisation by CDF (Proposition 3.1.10).

– When $\mathcal{C} = \{x \mapsto \mathrm{e}^{iux} \mid u \in \mathbb{R}\}$, (3.3) is the characterisation by characteristic functions (Proposition 3.3.7).

– When $\mathcal{C} = \{x \mapsto \mathrm{e}^{-\lambda x} \mid \lambda \in \mathbb{R}_+\}$ and X and Y are nonnegative-valued, (3.3) is the characterisation by Laplace transforms (Proposition 3.3.9).

– When $\mathcal{C} = \{x \mapsto s^x \mid s \in [-1, 1]\}$ and X and Y are nonnegative integer-valued, (3.3) is the characterisation by moment-generating functions (see the previous chapter).

3.4 Review of common probability distributions

We end the present chapter by enumerating the main properties of the probability distributions given in Section 3.1.2.

Proposition 3.4.1. *Let $a, b \in \mathbb{R}$ be such that $a < b$, $\lambda > 0$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$, $\alpha > 0$, $x \in \mathbb{R}$, $u \in \mathbb{R}$, $t \in \mathbb{R}_+$ and X be a continuous random variable.*

(i) If $X \sim \mathcal{U}([a, b])$, then

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad \operatorname{Var} X = \frac{(b-a)^2}{12}, \quad F_X(x) = \mathbb{1}_{[a,b]}(x)\frac{x-a}{b-a} + \mathbb{1}_{(b,+\infty)}(x)$$
$$\text{and} \quad \Phi_X(u) = \frac{e^{iub} - e^{iua}}{iu(b-a)} = \frac{\sin((b-a)u/2)}{(b-a)u/2} e^{iu\frac{a+b}{2}}.$$

(ii) If $X \sim \mathcal{E}(\lambda)$, then

$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \operatorname{Var} X = \frac{1}{\lambda^2}, \quad F_X(x) = (1 - e^{-\lambda x})\mathbb{1}_{\{x>0\}},$$
$$\Phi_X(u) = \frac{\lambda}{\lambda - iu} \quad \text{and} \quad L_X(t) = \frac{\lambda}{\lambda + t}.$$

(iii) If $X \sim \mathcal{N}_1(\mu, \sigma^2)$, then

$$\mathbb{E}[X] = \mu, \quad \operatorname{Var} X = \sigma^2 \quad \text{and} \quad \Phi_X(u) = e^{iu\mu - \frac{1}{2}\sigma^2 u^2}.$$

(iv) If $X \sim \mathcal{C}(\alpha)$, then

$$X \notin L_1, \quad F_X(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{a}\right) \quad \text{and} \quad \Phi_X(u) = e^{-\alpha|u|}.$$

Proof. (i) Suppose $X \sim \mathcal{U}([a,b])$. Then

$$\mathbb{E}[X] = \int_a^b \frac{x}{b-a}\,dx = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2};$$

$$\mathbb{E}[X^2] = \int_a^b \frac{x^2}{b-a}\,dx = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3};$$

$$\operatorname{Var} X = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12};$$

$$F_X(x) = \int_a^b \frac{\mathbb{1}_{\{y\le x\}}}{b-a}\,dy = \begin{cases} 0 & \text{if} \quad x < a \\ \frac{x-a}{b-a} & \text{if} \quad a \le x \le b \\ 1 & \text{if} \quad x > b \end{cases};$$

$$\Phi_X(u) = \int_a^b \frac{\mathrm{e}^{iux}}{b-a}\,dx = \frac{\mathrm{e}^{iub}-\mathrm{e}^{iua}}{iu(b-a)} = \mathrm{e}^{iu(b+a)/2}\frac{\mathrm{e}^{iu(b-a)/2}-\mathrm{e}^{-iu(b-a)/2}}{iu(b-a)};$$

$$= \mathrm{e}^{iu(b+a)/2}\frac{\sin((b-a)u/2)}{(b-a)u/2}.$$

(ii) Suppose $X \sim \mathcal{E}(\lambda)$. Then

$$\mathbb{E}[X] = \int_0^{+\infty} x\lambda\mathrm{e}^{-\lambda x}\,dx = \int_0^{+\infty} \frac{d\left(-x\mathrm{e}^{-\lambda x} - \frac{1}{\lambda}\mathrm{e}^{-\lambda x}\right)}{dx}\,dx = \left[-x\mathrm{e}^{-\lambda x} - \frac{1}{\lambda}\mathrm{e}^{-\lambda x}\right]_0^{+\infty} = \frac{1}{\lambda};$$

$$\mathbb{E}[X^2] = \int_0^{+\infty} x^2\lambda\mathrm{e}^{-\lambda x}\,dx = \int_0^{+\infty} \frac{d\left(-\frac{2}{\lambda}x\mathrm{e}^{-\lambda x} - \frac{2}{\lambda^2}\mathrm{e}^{-\lambda x} - x^2\mathrm{e}^{-\lambda x}\right)}{dx}\,dx$$

$$= \left[-\frac{2}{\lambda}x\mathrm{e}^{-\lambda x} - \frac{2}{\lambda^2}\mathrm{e}^{-\lambda x} - x^2\mathrm{e}^{-\lambda x}\right]_0^{+\infty} = \frac{2}{\lambda^2};$$

$$\operatorname{Var} X = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2};$$

$$F_X(x) = \int_0^{+\infty} \mathbb{1}_{\{y\le x\}}\lambda\mathrm{e}^{-\lambda y}\,dy = \begin{cases} 0 & \text{if} \quad x \le 0 \\ 1-\mathrm{e}^{-\lambda x} & \text{if} \quad x > 0 \end{cases}.$$

For $z \in (i\mathbb{R}) \cup \mathbb{R}_-$,

$$\mathbb{E}[\mathrm{e}^{zX}] = \int_0^{+\infty} \mathrm{e}^{zx}\lambda\mathrm{e}^{-\lambda x}\,dx = \frac{\lambda}{z-\lambda}[\mathrm{e}^{(z-\lambda)x}]_0^{+\infty} = \frac{\lambda}{\lambda-z}.$$

We deduce $\Phi_X(u) = \frac{\lambda}{\lambda-iu}$ and $L_X(t) = \frac{\lambda}{\lambda+t}$.

(iii) Suppose $Y \sim \mathcal{N}_1(0,1)$. Then

$$\mathbb{E}[Y] = \int_{\mathbb{R}} \frac{x}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,dx = -\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{d(\mathrm{e}^{-x^2/2})}{dx}\,dx = -\frac{1}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{-\infty}^{+\infty} = 0;$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} \frac{x^2}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left(\mathrm{e}^{-x^2/2} - \frac{d(x\mathrm{e}^{-x^2/2})}{dx}\right)\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, dx - \frac{1}{\sqrt{2\pi}} [x\mathrm{e}^{-x^2/2}]_{-\infty}^{+\infty} = 1 - 0 = 1;\\
\operatorname{Var} Y &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 1 - 0 = 1;\\
\Phi_Y(u) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{iux - x^2/2}\, dx.
\end{aligned}
$$

Leibniz's rule for differentiation under the integral sign yields

$$
\begin{aligned}
\Phi_Y'(u) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{d\left(\mathrm{e}^{iux - x^2/2}\right)}{du}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ix\mathrm{e}^{iux - x^2/2}\, dx\\
&= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(iu\mathrm{e}^{iux - x^2/2} - \frac{d\left(\mathrm{e}^{iux - x^2/2}\right)}{dx} \right) dx\\
&= -\frac{u}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{iux - x^2/2}\, dx - \frac{i}{\sqrt{2\pi}} [\mathrm{e}^{iux - x^2/2}]_{-\infty}^{+\infty} = -u\Phi_Y(u).
\end{aligned}
$$

We deduce that there exists $C \in \mathbb{R}$ such that $\Phi_Y(u) = C\mathrm{e}^{-u^2/2}$. Then $\Phi_Y(0) = C = 1$, hence $\Phi_Y(u) = \mathrm{e}^{-u^2/2}$.

Suppose now $X \sim \mathcal{N}_1(\mu, \sigma^2)$. Let $Y = (X - \mu)/\sigma$. Then $Y \sim \mathcal{N}_1(0, 1)$ (see Exercise 3.3.4) and $X = \sigma Y + \mu$. Based on the above equalities, we get

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\sigma Y + \mu] = \sigma \times 0 + \mu = \mu;\\
\operatorname{Var} X &= \operatorname{Var}(\sigma Y + \mu) = \sigma^2 \operatorname{Var} Y = \sigma^2 \times 1 = \sigma^2;\\
\Phi_X(u) &= \mathbb{E}[\mathrm{e}^{iuX}] = \mathbb{E}[\mathrm{e}^{iu\sigma Y + iu\mu}] = \mathrm{e}^{iu\mu}\Phi_Y(\sigma u) = \mathrm{e}^{iu\mu}\mathrm{e}^{-\sigma^2 u^2/2} = \mathrm{e}^{iu\mu - \frac{1}{2}\sigma^2 u^2}.
\end{aligned}
$$

(iv) Suppose $X \in \mathcal{C}(\alpha)$. Then $|x|\frac{\alpha}{\pi(x^2+\alpha^2)} \sim_{|x|\to+\infty} \frac{\alpha}{\pi|x|}$, which is not integrable on $\mathbb{R}$. Therefore, $X \notin L^1$. In addition,

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{\alpha}{\pi(y^2 + \alpha^2)}\, dy = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\,(\arctan(y/\alpha))}{dy}\, dy = \frac{1}{\pi}[arctan(y/\alpha)]_{-\infty}^{x}\\
&= \frac{1}{\pi}(\arctan(x/\alpha) - (-\pi/2)) = \frac{1}{2} + \frac{1}{\pi} \arctan(x/\alpha).
\end{aligned}
$$

Let Y be a continuous random variable with PDF p defined by $p : y \in \mathbb{R} \mapsto \frac{\alpha}{2}\mathrm{e}^{-\alpha|y|}$. We have

$$
\begin{aligned}
\Phi_Y(u) &= \int_{\mathbb{R}} \mathrm{e}^{iuy} \frac{\alpha}{2} \mathrm{e}^{-\alpha|y|}\, dy = \frac{\alpha}{2} \left(\int_{-\infty}^{0} \mathrm{e}^{iuy + \alpha y}\, dy + \int_{0}^{+\infty} \mathrm{e}^{iuy - \alpha y}\, dy \right)\\
&= \frac{\alpha}{2} \left(\left[\frac{1}{iu + \alpha} \mathrm{e}^{iuy + \alpha y} \right]_{-\infty}^{0} + \left[\frac{1}{iu - \alpha} \mathrm{e}^{iuy - \alpha y} \right]_{0}^{+\infty} \right) = \frac{\alpha}{2} \left(\frac{1}{iu + \alpha} - \frac{1}{iu - \alpha} \right)\\
&= \frac{\alpha^2}{\alpha^2 + u^2}.
\end{aligned}
$$

The map Φ_Y satisfies $\int_{\mathbb{R}} |\Phi_Y(u)|\, du < +\infty$, so by Proposition 3.3.6, for all $y \in \mathbb{R}$,

$$p(y) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|y|} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iuy}\Phi_Y(u)\, du = \frac{\alpha}{2}\int_{\mathbb{R}} \mathrm{e}^{-iuy}\frac{\alpha}{\pi(u^2+\alpha^2)}\, du = \frac{\alpha}{2}\Phi_X(-y).$$

We deduce that $\Phi_X(u) = \mathrm{e}^{-\alpha|u|}$.

$\square$

We also complete the previous chapter by giving the expressions of the CDF, characteristic function and Laplace transform of common discrete probability distributions.

Proposition 3.4.2. *Let $p \in [0,1]$, $n \in \mathbb{N}^*$, $\lambda > 0$, $x \in \mathbb{R}$, $u \in \mathbb{R}$, $t \in \mathbb{R}_+$ and X be a discrete random variable.*

(i) If $X \sim \mathcal{B}(p)$, then

$$F_X(x) = (1-p)\mathbb{1}_{[0,1)}(x) + \mathbb{1}_{[1,+\infty)}(x), \quad \Phi_X(u) = 1-p+pe^{iu} \quad \text{and} \quad L_X(t) = 1-p+pe^{-t}.$$

(ii) If $X \sim \mathcal{B}(n,p)$, then

$$F_X(x) = \sum_{k=0}^{\lfloor x \rfloor}\binom{n}{k}p^k(1-p)^{n-k}\mathbb{1}_{[0,n)}(x) + \mathbb{1}_{[n,+\infty)}(x), \quad \Phi_X(u) = (1-p+pe^{iu})^n$$
$$\text{and} \quad L_X(t) = (1-p+pe^{-t})^n.$$

(iii) If $X \sim \mathcal{P}(\lambda)$, then

$$\Phi_X(u) = e^{\lambda(e^{iu}-1)} \quad \text{and} \quad L_X(t) = e^{\lambda(e^{-t}-1)}.$$

(iv) If $p \neq 0$ and $X \sim \mathcal{G}eo(p)$, then

$$F_X(x) = (1-(1-p)^{\lfloor x \rfloor})\mathbb{1}_{\{x \geq 1\}}, \quad \Phi_X(u) = \frac{pe^{iu}}{1-(1-p)e^{iu}} \quad \text{and} \quad L_X(t) = \frac{pe^{-t}}{1-(1-p)e^{-t}}.$$

Proof. (i) Suppose $X \sim \mathcal{B}(p)$. Then

$$F_X(x) = \mathbb{P}(\{X \leq x\}) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \mathbb{P}(\{X = 0\}) = 1-p & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}.$$

For $z \in (i\mathbb{R}) \cup \mathbb{R}_-$, we have

$$\mathbb{E}[\mathrm{e}^{zX}] = (1-p)\mathrm{e}^0 + p\mathrm{e}^z = 1-p+p\mathrm{e}^z.$$

We deduce that $\Phi_X(u) = 1-p+p\mathrm{e}^{iu}$ and $L_X(t) = 1-p+p\mathrm{e}^{-t}$.

(ii) Suppose $X \sim \mathcal{B}(n,p)$. Then

$$\begin{aligned} F_X(x) &= \mathbb{P}(\{X \leq x\} \\ &= \begin{cases} 0 & \text{if} \quad x < 0 \\ \mathbb{P}(\{X \in \{0, \cdots, \lfloor x \rfloor\}\}) = \sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1-p)^{n-k} & \text{if} \quad 0 \leq x < n \\ 1 & \text{if} \quad x \geq n \end{cases}. \end{aligned}$$

Let $X_1, \cdots, X_n$ be i.i.d. discrete random variables, each having a Bernoulli distribution with parameter p. Then $X \stackrel{d}{=} X_1 + \cdots + X_n$, so by independence of $X_1, \cdots, X_n$ and (i), we get

$$\begin{aligned} \Phi_X(u) &= \mathbb{E}[\mathrm{e}^{iu(X_1 + \cdots X_n)}] = \mathbb{E}[\mathrm{e}^{iuX_1}] \cdots \mathbb{E}[\mathrm{e}^{iuX_n}] = (1 - p + p\mathrm{e}^{iu})^n; \\ L_X(t) &= \mathbb{E}[\mathrm{e}^{-t(X_1 + \cdots + X_n)}] = \mathbb{E}[\mathrm{e}^{-tX_1}] \cdots \mathbb{E}[\mathrm{e}^{-tX_n}] = (1 - p + p\mathrm{e}^{-t})^n. \end{aligned}$$

(iii) Suppose $X \sim \mathcal{P}(\lambda)$. Then for $z \in (i\mathbb{R}) \cup \mathbb{R}_-$, we have

$$\mathbb{E}[\mathrm{e}^{zX}] = \sum_{n=0}^{+\infty} \mathrm{e}^{zn} \mathrm{e}^{-\lambda} \frac{\lambda^n}{n!} = \mathrm{e}^{-\lambda} \sum_{n=0}^{+\infty} \frac{(\lambda \mathrm{e}^z)^n}{n!} = \mathrm{e}^{-\lambda} \mathrm{e}^{\lambda \mathrm{e}^z} = \mathrm{e}^{\lambda(\mathrm{e}^z - 1)}.$$

We deduce that $\Phi_X(u) = \mathrm{e}^{\lambda(\mathrm{e}^{iu} - 1)}$ and $L_X(t) = \mathrm{e}^{\lambda(\mathrm{e}^{-t} - 1)}$.

(iv) Suppose $p \neq 0$ and $X \sim \mathcal{G}eo(p)$. Then

$$\begin{aligned} F_X(x) &= \mathbb{P}(\{X \leq x\} \\ &= \begin{cases} 0 & \text{if} \quad x < 1 \\ \mathbb{P}(\{X \in \{1, \cdots, \{x\}\}\}) = \sum_{k=1}^{\lfloor x \rfloor} p(1-p)^{k-1} = 1 - (1-p)^{\lfloor x \rfloor} & \text{if} \quad x \geq 1 \end{cases}. \end{aligned}$$

For $z \in (i\mathbb{R}) \cup \mathbb{R}_-$, we have

$$\mathbb{E}[\mathrm{e}^{zX}] = \sum_{n=1}^{+\infty} \mathrm{e}^{zn} p(1-p)^{n-1} = p\mathrm{e}^z \sum_{n=0}^{+\infty} ((1-p)\mathrm{e}^z)^n = \frac{p\mathrm{e}^z}{1 - (1-p)\mathrm{e}^z}.$$

We deduce that $\Phi_X(u) = \frac{p\mathrm{e}^{iu}}{1-(1-p)\mathrm{e}^{iu}}$ and $L_X(t) = \frac{p\mathrm{e}^{-t}}{1-(1-p)\mathrm{e}^{-t}}$.

□