Interest rate modelling in insurance: Jacobi stochastic volatility in the Libor Market Model

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Introduction

- Problem: efficiently calibrate a given stochastic volatility version of the LIBOR Market Model (LMM)
  - Insurance context;
  - Replicate $\approx$ 300 swaptions prices.
- Common practices:
  - exploit the explicit knowledge of the characteristic function of the model to get an analytical price (widely used, see for instance Wu and Zhang (2006));
  - successive suited approximations (Piterbarg (2003)).
- Lately: *Gram-Charlier expansion* (Devineau et al. (2017)).

Standard model

Goal: compute swaptions prices (call options on swap rate)
$m, n, K \mapsto P(m,n,K) = \mathbb{E}\left[(S^{m,n}_{T_m} - K)_+\right]$ using the following dynamics

$$\begin{cases} dS^{m,n}_t = \sqrt{V_t}\Big(\rho(t)\|\lambda^{m,n}(t)\|dW_t + \sqrt{1-\rho(t)^2}\lambda^{m,n}(t)\cdot dW^{S,*}_t\Big), \\ dV_t = \kappa\big(\theta - \xi^0(t)V_t\big)dt + \epsilon\sqrt{V_t}dW_t. \end{cases} \quad (1)$$

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- Efficient computation using Gram-Charlier expansion (Devineau et al. (2017)):

$$P(m,n,K) = P_{Bachelier}(m,n,K) + \mu_2 \exp(-K^2/2)\left(\frac{\mu_3}{6}K + \frac{\mu_4 - 3}{24}(K^2-1)\right).$$

$\rightsquigarrow$ Convergence ?

Gram-Charlier and stochastic volatility models (1/2)

$$X := \sqrt{V} \times G$$

with $G \sim \mathcal{N}(0, \sigma^2)$ and $V \sim \chi^2(d)$, G and V being independent.

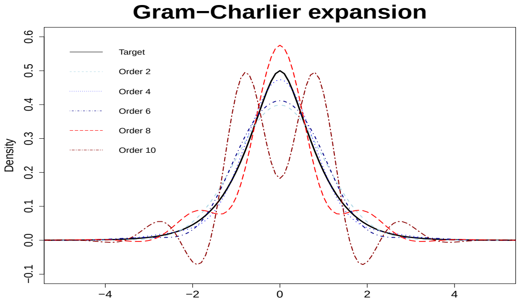

Figure: Gram-Charlier expansion of the density of X up to order 10 - $\sigma^2 = 0.25$ & $d = 4$

Gram-Charlier and stochastic volatility models (2/2)

$$X^{(m)} := \sqrt{\min(V, m)} \times G$$

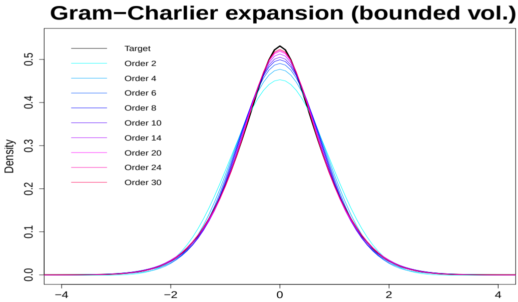

Figure: Gram-Charlier expansion of the density of $X^{(m)}$ up to order 30 - $m = 4$

Sufficient condition: $\sigma^2 m < 2$

Jacobi volatility factor

Take $0 \leqslant v_{min} < v_{max} \leqslant \infty$, $Q : v \mapsto \frac{1}{C}(v_{max} - v)(v - v_{min})$, and consider (following the work of Ackerer et al. (2018))

$$\begin{cases} dS_t^{m,n} = \rho(t)\sqrt{Q(V_t)}\|\lambda^{m,n}(t)\|dW_t + \sqrt{V_t - \rho(t)^2 Q(V_t)}\lambda^{m,n}(t)\cdot dW_t^{S,*}, \\ dV_t = \kappa\big(\theta - \xi^0(t)V_t\big)dt + \epsilon\sqrt{Q(V_t)}dW_t. \end{cases} \quad (2)$$

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- $\mathbb{P}(\forall t \geqslant 0 : v_{min} < V_t < v_{max}) = 1$ under Feller condition.
- Gram-Charlier is theoretically allowed if:

$$v_{max} T \max_t \|\lambda^{m,n}(t)\|^2 < 2\sigma_r^2.$$

- (2) converges in a strong sense towards (1) as $(v_{min}, v_{max}) \to (0, \infty)$.

Gram-Charlier approximating prices (1/2)

$$\sigma_r^2 > \frac{v_{max} T}{2} \max_{t \leqslant T} \|\lambda^{m,n}(t)\|^2$$

is sharp to ensure the convergence of Gram-Charlier series.

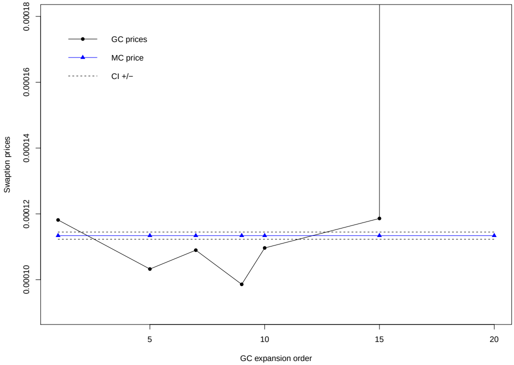

Figure: Divergence of Gram-Charlier expansion.

Gram-Charlier approximating prices (2/2)

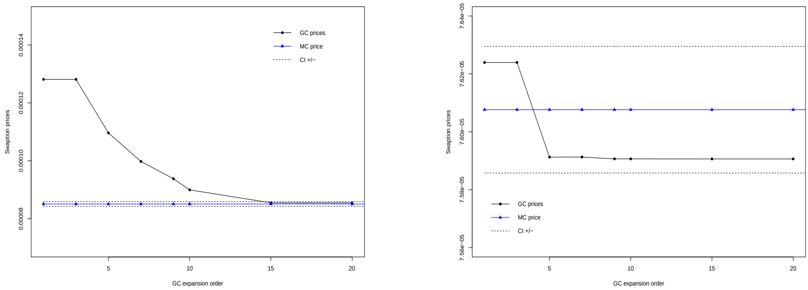

Figure: Exemple of convergence of approximating prices to empirical ones: using a given Gaussian density as reference (left) and an adapted Gaussian distribution (matching first two moments) as reference (right).

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End of presentation

Thank you!

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